Nathan Madsen
PERM: 5679642
madsen@ucsb.edu

HW Set 2
MATH 111A

2024-10-17

Problem 1

We want to show

$$\text{a group } G \text{ has } p \text{ elements of order } p \implies G \text{ is not cyclic}$$

We proceed by contrapositive. Assume the group G is cyclic and generated by g. We know that by Lagrange's theorem that if there are any elements of order p, p must divide the order of the group so $|G| = pn$. And since G is assumed to be cyclic the only elements that have order p will be $g^n, g^{2n}, \dots g^{(p-1)n}$. But there are only $p - 1$ of these elements.

Problem 2

We want to show that $|Perm(\mathbb{N})| = \infty$ and that $Perm(\mathbb{N})$ is not a cyclic group.

For the sake of contradiction, assume $|Perm(\mathbb{N})| = n$. Then consider the transpositions $(1, 2), (3, 4), \dots (2n - 1, 2n), (2n + 1, 2n + 2)$ since all these are transpositions of natural numbers they will be in the group, and we have found $n + 1$ of them, contradicting our original claim that the order of the group was finite.
Similarly we can show that $Perm(\mathbb{N})$ is not cyclic because cylic groups must always be abelian (since powers of elements commute with themselves), but $(12)(23) \neq (23)(12)$.

Problem 3

We want to show

$$G \text{ is infinite, cyclic} \implies G \text{ contains no finite subgroups}$$

Let's assume for the sake of contradiction that $G = \langle g \rangle, |G| = \infty$ and assume that $H < G, |H| = n < \infty$. H must contain some g^m. Then by closure $e, g^{2m}, g^{3m}, \dots, g^{nm} \in H$ (and for all $\neq e$ since $|G| = \infty$), which violates our assumption that $|H| = n$ because we have found $n + 1$ elements in H.

Problem 4

We want to show

$$\text{a finite group is the union of proper subgroups} \iff \text{it is not cyclic}$$

We start by showing that the LHS $\implies$ RHS using contrapositive, ie we show that a group being cyclic implies that the group is not the union of proper subgroups. We know that every cyclic group of order n is generated by $\phi(n)$ elements, so any subgroup containing any one of those elements would contain all elements of the group, so there is no subgroup containing it.
Next we show that LHS $\impliedby$ RHS. Assume our group G is not cyclic, by definition of not cyclic there is no such element $g \in G : \langle g \rangle = G$. Then we can take an element $g_1 \in G$ and we know $\langle g_1 \rangle \subset G$. We can then choose a $g_2 \in G \backslash \langle g_1 \rangle$ and we know that $\langle g_2 \rangle \subset G$. Since G is finite we know this process must terminate and we can write $G = \cup \langle g_i \rangle$.

Problem 5

We want to show that in some group

$$|a| \text{ relatively prime to } |b| \implies \langle a \rangle \cap \langle b \rangle = e$$

We again proceed by the contrapositive. Assume that $\langle a \rangle \cap \langle b \rangle$ contains some element $g \neq e$. Then we know that $\langle g \rangle \subseteq \langle a \rangle, \langle b \rangle$. And since by Lagrange's theorem we know that the order of a subgroup must divide the group and we know that $|\langle g \rangle| \neq 1$ because $g \neq e$, it must be that $|a|$ is not relatively prime to $|b|$.

Problem 6

We want to show that no group can contain exactly two elements of order 2.

We proceed by contradiction. Consider a group G containing exactly two unique elements of order 2, $|a| = |b| = 2$ or $a^2 = b^2 = e$ or $a = a^{-1}, b = b^{-1}$.
Take the case where the case where a, b commute. Then $ab \neq e$ (since $a \neq b$) and $(ab)^2 = abba = aa = e$, so ab also has order 2.
Now take the case where a, b do not commute. Then aba^{-1} will clearly have order 2. And $aba^{-1} \neq b$ since otherwise a, b would commute. And $aba^{-1} \neq a$ since otherwise $a = b$. And $aba^{-1} \neq e$ since otherwise $b = e$