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Homework Assignment 1
MATH 118C

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Problem 1 (Rudin Ch9p1)

Let $S = \{x_i\}_{i \in I}$. Recall Span$(S) = \{\sum_i^n a_i x_i : x_i \in S, a_i \in K\}$.
We want to show that Span(S) is a vector space. Clearly $0 \in$ Span(S) since $0 = \sum_i^n 0x_i$. And say $a, b \in$ Span(S) with $a = \sum_i^n a_i x_i$ and $b = \sum_i^n b_i x_i$. Then $a+b \in$ Span(S) since $a+b = \sum_i^n a_i x_i + \sum_i^n b_i x_i = \sum_i^n (a_i+b_i)x_i$. Now let $k \in K$ and $a \in$ Span(S) we want to show that $ka \in$ Span(S). Clearly so, since $ka = k\sum_i^n a_i x_i = \sum_i^n k a_i x_i$. And since all these are linear combinations of vector from the space X, we have sufficiently shown that Span(S) is a subspace, where the other axioms of a vector space (associativity, ...) follow from X.

Problem 2 (Rudin Ch9p4)

Let $T : X \to X'$ be a linear transformation. We want to show that both the range and the null space are vector spaces.

We start with the null space. Say $x, y \in X$ such that $T(x) = T(y) = 0$. Then $T(x+y) = T(x) + T(y) = 0$ so $x + y$ is also in the null space. Let $k \in K$, then $T(kx) = kT(x) = k \cdot 0 = 0$ and so kx is also in the null space.

Now say with the range that $x', y' \in Y$ such that $T(x) = x', T(y) = y'$ for some $x, y \in X$, ie they are in the range. Then $x' + y' = T(x) + T(y) = T(x+y)$ and since $x + y \in X$ then $x' + y'$ is in the range. Similarly say $k \in K$ then $kx' = kT(x) = T(kx)$ and $kx \in X$ so kx' is in the range.

Problem 3 (Rudin Ch9p5)

We want to show that for ever $A \in L(\mathbb{R}^n, \mathbb{R})$ corresponds a unique $y \in \mathbb{R}^n$ such that $Ax = x \cdot y$. And also that $||A|| = |y|$.
Starting with existence and uniqueness of y. We say $y \in \mathbb{R}^n$ such that $y_i = A(e_i)$. Now for $x \in \mathbb{R}^n$ we have

$$x = \sum_{i=1}^n x_i e_i.$$

and by linearity of A

$$A(x) = A\left(\sum_{i=1}^n x_i e_i\right) = \sum_{i=1}^n x_i A(e_i) = \sum_{i=1}^n x_i y_i = x \cdot y.$$

Then if we suppose that $x \cdot y = x \cdot z$ for all $x \in \mathbb{R}^n$ then $x \cdot (y - z) = 0$ and so $y - z = 0$ and $y = z$.
Now for the second part, by definition

$$\|A\| = \sup_{|x|=1} |A(x)| = \sup_{|x|=1} |x \cdot y|.$$

By the Cauchy-Schwarz inequality

$$|x \cdot y| \le |x|\,|y|$$

and since $|x| = 1$

$$|x \cdot y| \le |y|.$$

So

$$\|A\| \le |y|.$$

Problem 4 (Rudin Ch9p7)

We want to show that if $E \subset \mathbb{R}^n$ and that $f : E \to \mathbb{R}$ with it's partial derivatives bounded then f is continuous.
Say $x, y \in E$. Define the line segment between them

$$\gamma(t) = x + t(y - x) \quad t \in [0, 1]$$

So we have $\gamma(t) \subset E$. Now define $\phi(t) = f(\gamma(t))$ so by the fundamental theorem of calculus

$$\phi(1) - \phi(0) = f(y) - f(x) = \int_0^1 \phi'(t)dt$$

and we know by chain rule

$$\phi'(t) = \nabla f(\gamma(t))\gamma'(t) = \nabla f(\gamma(t)) \cdot (y - x)$$

and so

$$f(y) - f(x) = \int_0^1 \nabla f(\gamma(t)) \cdot (y - x)dt$$

Now aiming for Lipschitz continuity we have

$$|f(y) - f(x)| = \int_0^1 \nabla f(\gamma(t)) \cdot (y - x)dt \leq \int_0^1 |\nabla f(\gamma(t)) \cdot (y - x)|dt \leq \int_0^1 \|\nabla f(\gamma(t))\| \cdot \|(y - x)\|dt$$

And since partial derivatives are bounded by some value M then $\|\nabla f(\gamma(t))\| \leq \sqrt{n}M$. And so

$$|f(y) - f(x)| \leq \sqrt{n}M\|y - x\|$$

Problem 5 (Rudin Ch9p8)

We want to show if f is a differentiable real function on the open set $E \subset \mathbb{R}^n$ and that f has a local maximum at a point $x \in E$, prove that $f'(x) = 0$.

Define the function $\phi(t) = f(x + tv)$ for some $|v| = 1$ and $t \in (-\epsilon, \epsilon)$ where ϵ is small enough. Now observe that $\phi(t) : \mathbb{R} \to \mathbb{R}$. So since f has a local max at x then ϕ has a local max at $t = 0$ so $\phi'(0) = 0$. Now

$$\phi'(0) = 0 = \nabla f(x) \cdot v \quad \forall |v| = 1 \quad \implies \quad \nabla f(x) = 0$$

Problem 6 (Rudin Ch9p11)

We want to show the product rule for $f, g : \mathbb{R}^n \to \mathbb{R}$

$$\nabla(fg)(x) = \left(\frac{\partial}{\partial x_1}(fg), \ldots, \frac{\partial}{\partial x_n}(fg)\right).$$

and for each component

$$\frac{\partial}{\partial x_i}(fg)(x) = \frac{\partial}{\partial x_i}(f(x)g(x)) = f(x)\frac{\partial g}{\partial x_i}(x) + g(x)\frac{\partial f}{\partial x_i}(x),$$

and so, summing these over all x_i we get

$$\nabla(fg)(x) = f(x)\nabla g(x) + g(x)\nabla f(x).$$

Now let $h(x) = \frac{1}{f(x)}$. By the chain rule

$$\frac{\partial h}{\partial x_i}(x) = \frac{\partial}{\partial x_i}\left(f(x)^{-1}\right) = -f(x)^{-2} \cdot \frac{\partial f}{\partial x_i}(x).$$

And so once again summing over all x_i we have

$$\nabla\left(\frac{1}{f}\right)(x) = \left(-\frac{1}{f(x)^2}\frac{\partial f}{\partial x_1}, \ldots, -\frac{1}{f(x)^2}\frac{\partial f}{\partial x_n}\right) = -\frac{1}{f(x)^2}\nabla f(x).$$