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HW Set 3
MATH 111A

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Problem 1

We want to show that the product of an even and an odd permutation in S_n is an odd permutation. Let α be an even permutation and let β be an odd permutation. We then know that the parity of their product is the sum of their parities.

$$\text{Parity}(\alpha\beta) = \text{Parity}(\alpha) + \text{Parity}(\beta) = 0 + 1 = 1$$

So the product necessarily is odd.

Problem 2

We want to show that

$$H \leq S_n, |H| \text{ is odd} \implies H \leq A_n$$

For the sake of contradiction assume that H is not a subgroup of A_n. Then we know that H must include some odd permutation (since A_n is all the even permutations). And we know that the elements of odd parity and even parity must be in one-to-one correspondence in H. Thus H must be of even order and so we have a contradiction. So our assumption that H is not a subgroup of A_n must be incorrect.

Problem 3

We want to show that

$$A_n \text{ is non abelian} \iff n \geq 4$$

Let us start by showing that for all $n \geq 4$ we have that A_n is non-abelian.

$$(123)(234) = (12)(34), (234)(123) = (13)(24)$$

$$\implies (123)(234) \neq (234)(123)$$

And since these elements are in all A_n with $n \geq 4$ we have that none of these groups are abelian.
Now we show that A_1, A_2, A_3 are all abelian. A_1, A_2 are both the trivial group and so must be abelian. So we now want to show that A_3 is abelian. Note that $A_3 = \{e, (123), (132)\}$. And $(123)(132) = e = (132)(123)$ and so the group is abelian.

Problem 4

We want to show that

$$\sigma \in S_n \text{ is a permutation of odd order} \implies \sigma \text{ is an even permutation}$$

Since σ is a permutation of odd order we know we can decompose its cycle into something like

$$\sigma = \sigma_1\sigma_2 \dots \sigma_n$$

Further more we know that each disjoint cycle must have an odd length since the order of σ is the least common multiple of the lengths of the σ_i. We then know that we can decompose each σ_i into an even number of transpositions

$$\sigma_i = (a_1a_2 \dots a_n) = (a_1a_n) \dots (a_1a_2)$$

And since n is odd we know the number of transpositions is even. And since each σ_i is even, their composition will also be even.

Problem 5

We want to show that

$$(ab),(cd) \text{ commute } \iff (ab),(cd) \text{ are disjoint}$$

$\Longleftarrow$. Assume that $(ab),(cd)$ are disjoint. Then we can simply check what both combinations of them map things to. First look at $(ab)(cd)$

$$\begin{aligned} a &\to a \to b \\ b &\to b \to a \\ c &\to d \to d \\ d &\to c \to c \end{aligned}$$

Now look at $(cd)(ab)$

$$\begin{aligned} a &\to b \to b \\ b &\to a \to a \\ c &\to c \to d \\ d &\to d \to c \end{aligned}$$

And clearly these two map the same so we know they are equal and thus commute
$\Longrightarrow$. Let us do proof by contrapositive. So we must assume that $(ab),(cd)$ are not disjoint. Without loss of generality we can do this by saying that $a = c$. So all we need show is that $(ab),(ad)$ do not commute. First look at $(ab)(ad)$:

$$a \to d \to d$$

Now look at $(ad)(ab)$:

$$a \to b \to b$$

And since these two do not act on a the same obviously $(ab)(ad) \neq (ad)(ab)$ so they do not commute.

Problem 6

We are given

$$\gamma = (124)(425)(64)$$

We can feed each element through the permutation to create a new cycle

$$\begin{aligned} 1 &\to 1 \to 1 \to 2 \\ 2 &\to 2 \to 5 \to 5 \\ 5 &\to 5 \to 4 \to 1 \end{aligned}$$

And since we have reached 1 again we know this is a disjoint cycle. So

$$\gamma = (125)\sigma$$

Let's continue with our methodology starting with the first number not in our cycle but in γ, which is 4

$$\begin{aligned} 4 &\to 6 \to 6 \to 6 \\ 6 &\to 4 \to 2 \to 4 \end{aligned}$$

And thus we have found our next and last disjoint cycle, so we can write gamma

$$\gamma = (125)(64)$$

Next, we want to figure out if gamma is an even or odd permutation, we can do this by breaking it down into transpositions using the method stated in problem 4

$$\gamma = (12)(15)(64)$$

And so γ is an odd permutation.