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HW Set 7
MATH 111A

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Problem 1

We found all the conjugacy classes of D_4 as $\{e, r^2\}, \{s, sr^2\}, \{r, r^3\}, \{sr, sr^2\}$. And the conjugation classes of D_3 are $\{e\}, \{s, sr, sr^2\}, \{r, r^2\}$. Since $r^{-1}sr = ssr^{-1}sr = sr^2$, and likewise the other reflection can be obtained. And $srs = r^{-1} = r^2$.
And since conjugacy classes are determined by $(h_1, h_2)(g_1, g_2)(h_1^{-1}, h_2^{-2}) = (h_1 g_1 h_1^{-1}, h_2 g_2 h_2^{-1})$ so then (g_1, g_2) is conjugate to $(g_1', g_2') \in D_3 \times D_4$ iff g_1 is conjugate to g_1' in D_3 and g_2 is conjugate to g_2' in D_4. Thus the conjugacy classes are given by pairings of the conjugacy classes, so there should be 12 in total and are sufficiently described given this rule and the conjugacy classes of D_3 and D_4.

Problem 2

Recall that the Normalizer is defined as $N_G(S) = \{g \in G : gS = Sg\}$. We are given a subgroup $H \leq S_4$. We want to find its normalizer. $(12)(34) \in H, (12) \in S_4, (12)^{-1} = (12)$

$$(12)(12)(34)(12) = (12)(34) \in H \implies (12) \in N_G(S_4)$$

likewise

$$(34)(12)(34)(34) = (12)(34) \implies (34) \in N_G(S_4)$$

Similarly with the other 2 elements of the subgroup we can prove that the other transpositions are in the normalizer: $(13), (24), (14), (23) \in N_G(S_4)$. We know that $(12)(34) \in N_G(S_4)$ and we know $(123)^{-1} = (132)$.

$$(132)(12)(34)(132) = (14)(23) \in H \implies (123) \in N_G(S_4)$$

$$(124)(12)(34)(142) = (13)(24) \in H \implies (124) \in N_G(S_4)$$

Problem 3

Recall that the center of a group is $Z(G) = \{z \in G : \forall g \in G, zg = gz\}$. We know that A_5 is simple (this was stated as a fact in lecture). Since A_5 is non abelian it must be the case that $g \in A_5, g \notin Z(A_5)$. We know that $Z(A_5)$ is normal since every element of the center commutes with every elements of the group. Since the center is normal and not the entire group, but the group is simple so cannot contain any non-trivial normal subgroup it must be that the center is trivial.

Problem 4

Let's consider a subgroup of order 2 in A_5. It must contain an element of order 2 and these are all characterized by a set of two disjoint transpositions. Say $(ab)(cd) \in H$ (our hypothetical normal subgroup of order 2). But $(abc)(ab)(cd) \neq (ab)(cd)(abc)$ and so it cannot be normal.

Problem 5

We want to show that A_4 does not have a normal subgroup of order 2. Since it is a subgroup of order 2 it must contain the identity element and an element that is its own inverse, all these are $(12)(34), (13)(24), (14)(23)$ and the subgroups they generate.

$$(123)(12)(34)(123)^{-1} = (14)(23)$$

$$(123)(13)(24)(123)^{-1} = (12)(34)$$

$$(123)(14)(23)(123)^{-1} = (13)(24)$$

And so clearly none of these subgroups can be normal.

Problem 6

No. We have shown that $N_G(H)$ is the largest subgroup in which H is normal.
We can show this by saying, let K be a subgroup such that H is normal in it. Then for all $k \in K$ we have $kHk^{-1} \subseteq H$ and $k^{-1}Hk \subseteq H$ so $H = (kk^{-1})H(kk^{-1}) = k(k^{-1}Hk)k^{-1} \subseteq kHk^{-1} \subseteq H$. So then $kHk^{-1} = H$ and then $k \in N_G(H)$.
Since H is normal is $N_G(H)$ it will also be normal in $N_G(N_G(H))$, but the second group must be larger or equal in size. If it is larger we have then found a group in which H is normal larger that $N_G(H)$, which is a contraction. So it must be the same group and thus $N_G(H) = N_G(N_G(H))$

Problem 7

Yes. Since H is non-normal in G it is trivial that $N_G(H) \neq G$. And since the index of H in G is a prime number p we can say $|H| = n$ and $|G| = np$. Since the normalizer is a subgroup it must divide np by Lagrange's theorem, and since H is a subgroup of it, it must be divisible by n. But since p is prime the only such numbers are np and n, but it cannot be np since it cannot be the whole group as stated earlier, and since it must contain H and has the same order as H it must be H.