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HW Set 6
MATH 111A

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Problem 1

We want to show that in D_n (with $n \geq 3$) a reflection followed by another reflection is a rotation.

Recall that D_n is a group of order $2n$ with the properties $r^n = e, s^2 = e, srs = r^{-1}$. Also recall we know that $r^i s = sr^{-i}$ for all $i \geq 0$. Now since any power of r is clearly a rotation, every "flip" must have the form $r^m s$. So we can look at combining two flips $r^m s, r^l s$.

$$(r^m s)(r^l s) = (sr^{-m})(r^l s) = s(r^{l-m} s) = s(sr^{m-l}) = s^2 r^{m-l} = r^{m-l}$$

And thus two flips is just a rotation.

Problem 2

We want to show that a reflection followed by a rotation or the other way around is indeed just a reflection

We will show this by showing it cannot be a reflection. Geometrically say we label the vertices in order, the act of rotating will preserve the order of the vertices but not the position of any (unless we rotate by 2π). Next we rotate, this will flip the order of the vertices and it might preserve the position of some vertices. So a rotation and a reflection or reflection and rotation will result in the order of the vertices being flipped so cannot be a rotation.

Problem 3

I claim that the subgroup of rotations in D_n is a normal subgroup. Since the group of rotations $\langle r \rangle$ has order n then $[D_n : n] = 2$ and we have stated in class that any group with an index of 2 is normal. We can also prove this without the claim about the index. Since we know that $srs = r^{-1} \implies rs = sr^{-}1$ we know that any element can be written in the form $r^i s^j$ with $0 \leq i \leq n-1$ and $j = 0, 1$. We know want to show that the conjugation of a rotation by anything is still a rotation

$$(r^i s^j) r^n (r^i s^j)^{-1} = (r^i s^j) r^n (r^i s^j)^{-1}$$

If $j = 0$ this is trivial so assume that $j = 1$

$$\begin{aligned}
(r^i s^j) r^n (r^i s^j)^{-1} &= (r^i s) r^n (sr^{-i}) \\
&= r^i (sr^n s) r^{-i} \\
&= r^i (r^{-n}) r^{-i} \in \langle r \rangle
\end{aligned}$$

Problem 4

I claim that for $n > 3$ it is not the case that $D_n \triangleleft S_n$. For $n = 3$ we know the dihedral group is a subgroup of the permutation group but we also know that they must have the same order so they must be the same group and is thus trivially normal. We can show that D_4 is not normal in S_4. We know that $(24) \in D_4$ (it is swapping across the diagonal). And we know that $(12) \in S_4$ and is a self inverse. So we compute $(12)(24)(12) = (14)$ which is not in D_4 and so it is not normal.

Problem 5

There is group action $\phi : \mathbb{R}^\times \times \mathbb{R}^n \to \mathbb{R}^n$ that is faithful. Recall an action is faithful if $g \cdot x = x \forall x \in X \implies g = e_G$. I claim that scalar multiplication is such an action. Let us first check that scalar multiplication is indeed a group action. Let $v \in \mathbb{R}^n$, $1v = (1v_1, 1v_2, \ldots 1v_n) = v \; \forall v \in \mathbb{R}^n$. Let $t_1, t_2 \in \mathbb{R}^\times$, then

$$t_1(t_2 v) = t_1(t_2 v_1, t_2 v_2, \ldots t_2 v_n) = (t_1 t_2 v_1, t_1 t_2 v_2, \ldots t_1 t_2 v_n) = (t_1 t_2) v$$

And so we have satisfied both properties. Lastly we must check that it is faithful. Say $tv = v$ for all $v \in \mathbb{R}^n$. Then $t(1, 0, 0\dots) = (1, 0, 0, \dots)$ so clearly $t = 1$.

Problem 6

I claim that the subgroup of invertible diagonal matrices in $GL_n(\mathbb{R})$ is not a normal subgroup of $GL_n(\mathbb{R})$. To illustrate say $a \neq b$ and we know that $Diag(a, b)$ is in our subgroup.

$$\begin{aligned} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}^{-1} &= \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix} \\ &= \begin{bmatrix} a & b-a \\ 0 & b \end{bmatrix} \end{aligned}$$