Useful formulae

Integration by parts: $\int_a^b dx\, f(x)\frac{\partial g(x)}{\partial x} = -\int_a^b dx\, \frac{\partial f(x)}{\partial x} g(x) + \left. f(x)g(x)\right|_a^b$

Commutators: $[A, B] \equiv AB - BA$.

Gaussian Integrals:

$$\int_{-\infty}^{\infty} e^{-\lambda x^2} dx = \sqrt{\frac{\pi}{\lambda}}, \qquad \int_{-0}^{\infty} x e^{-\lambda x^2} dx = \frac{1}{2\lambda}, \qquad \int_{-\infty}^{\infty} x^2 e^{-\lambda x^2} dx = 2\sqrt{\pi/\lambda^3}$$

Probability: $\text{Prob}(a \le x \le b \text{ at time } t) = \int_a^b \Psi^*(x,t)\Psi(x,t)dx$

Probability Current: $\frac{\partial}{\partial t}|\Psi|^2 = -\frac{\partial}{\partial x} J(x,t)$ for $J(x,t) = -\frac{i\hbar}{2m}\left(\Psi^* \frac{\partial \Psi}{\partial x} - \frac{\partial \Psi^*}{\partial x}\Psi\right)$

Operators: $\hat{x} = x$ and $\hat{p} = -i\hbar\frac{\partial}{\partial x}$ (in position space) and $[\hat{x}, \hat{p}] = i\hbar$

Expectation Values: $\langle \hat{Q}(x,p)\rangle = \int_{-\infty}^{\infty} \Psi^*(x,t)\hat{Q}(x, -i\hbar\frac{\partial}{\partial x})\Psi(x,t)dx$

Uncertainties: $\sigma_x \equiv \sqrt{\langle x^2\rangle - \langle x\rangle^2}$, $\sigma_p \equiv \sqrt{\langle p^2\rangle - \langle p\rangle^2}$.

Schrödinger Equation: $i\hbar\frac{\partial}{\partial t}\Psi(x,t) = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\Psi(x,t) + V(x,t)\Psi(x,t)$

Hamiltonian: $\hat{H} = \frac{\hat{p}^2}{2m} + V = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + V$

Time-independent Schrödinger Equation: $\hat{H}\psi(x) = E\psi(x)$ for separable solutions $\psi(x,t) = e^{-iEt/\hbar}\psi(x)$.

Orthonormality and completeness: $\int \psi_m^* \psi_n dx = \delta_{mn}$ for orthonormality; completeness implies $\Psi(x,0) = \sum_n c_n \psi_n(x)$ with $c_n = \int \psi_n^* \Psi(x,0)dx$ for discrete spectrum of ψ_n.

Infinite square well: For $\hat{H} = \frac{\hat{p}^2}{2m} + V$ with $V = 0$ for $0 \le x \le a$, ∞ otherwise. Stationary solutions $\psi_n(x) = \sqrt{\frac{2}{a}}\sin(n\pi x/a)$, $E_n = \frac{n^2\pi^2\hbar^2}{2ma^2}$, $n = 1, 2, 3, \ldots$

Harmonic oscillator: $\hat{H} = \frac{\hat{p}^2}{2m} + \frac{1}{2}m\omega^2\hat{x}^2$

$$a_\pm = \frac{1}{\sqrt{2m\hbar\omega}}(m\omega\hat{x} \mp i\hat{p}), \qquad \hat{x} = \sqrt{\frac{\hbar}{2m\omega}}(a_+ + a_-), \qquad \hat{p} = i\sqrt{\frac{\hbar m\omega}{2}}(a_+ - a_-)$$

$$\hat{H} = \hbar\omega(a_+ a_- + 1/2) \qquad E_n = \hbar\omega(n + 1/2), n = 0, 1, \ldots \qquad 1 = [a_-, a_+]$$

$$a_+\psi_n = \sqrt{n+1}\psi_{n+1} \qquad a_-\psi_n = \sqrt{n}\psi_{n-1} \qquad \psi_0(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} e^{-\frac{m\omega}{2\hbar}x^2}$$