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HW Set 5
MATH 111A

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Problem 1

We want to find the number of distinct subgroups of order p of the direct product of n copies of $\mathbb{Z}_p$.

Consider that any subgroup of order p must be cyclic and therefore must be generated by an element of order p. It therefore suffices to find all such generators with taking measures to not overcount the subgroups since each subgroup contains $p-1$ many generators. Examine the element

$$a = (a_1, a_2, \dots a_n) \in \mathbb{Z}_p \times \cdots \times \mathbb{Z}_p$$

If not all $a_i = 0$ then this element clearly generates a subgroup of order p as the element itself has order p.

$$a^p = (a_1, a_2, \dots, a_n)^p = (a_1^p, a_2^p, \dots, a_n^p) = (0, 0, \dots, 0) = 0$$

We then know that there are $p^n - 1$ many such elements. There are p choices for each n entry and then we subtract one since the identity element is not a generator. Then since each subgroup of order p has $p-1$ generators we can say that there are in total $(p^n-1)/p-1$ many such subgroups.

Problem 2

We want to show that there exists an infinite group G which has a normal subgroup H with quotient group G/H such that H and G/H are both abelian but G is not abelian.

Let G be the set of all unit upper triangular matrices with entries in $\mathbb{Z}$. Then we define

$$H = \left\{ \begin{bmatrix} 1 & 0 & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix} : b, c \in \mathbb{Z} \right\}$$

We can easily show that H is abelian since

$$\begin{bmatrix} 1 & 0 & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & b' \\ 0 & 1 & c' \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & b+b' \\ 0 & 1 & c+c' \\ 0 & 0 & 1 \end{bmatrix}$$

And clearly the quotient group is characterized as

$$G/H = \left\{ \begin{bmatrix} 1 & a & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} : a \in \mathbb{Z} \right\}$$

And we can check again that this is abelian since

$$\begin{bmatrix} 1 & a & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & a' & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & a+a' & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Lastly we need to verify that H is normal

$$\begin{bmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & b' \\ 0 & 1 & c' \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -a & ac-b \\ 0 & 1 & -c \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & b+b' \\ 0 & 1 & c' \\ 0 & 0 & 1 \end{bmatrix}$$

And so we have a G of infinite order that is non abelian but found a normal H such that it and G/H are both abelian.

Problem 3

We want to show that there is a surjective group homomorphism $\phi : A_4 \to \mathbb{Z}_3$.

We know by the first isomorphism theorem that the kernel of ϕ must be a normal subgroup of A_4. There are three normal subgroups of A_4: The trivial group, the Klein four-group, and A_4 itself. We want to construct our homomorphism by quotienting out our normal subgroup, so clearly only V, the Klein four-group, will work.

$$V = \{e, (12)(34), (13)(24), (14)(23)\}$$

And since up to isomorphism there is one group of order 3 we know that $|A_4/V| = 3$ so then $A_4/V \cong \mathbb{Z}$

Problem 4

We want to show that if G is a finite group and H a normal subgroup then the order of gH in G/H divides the order of g in G.

Say that $g \in G$ has order n. Now construct a homomorphism

$$\begin{aligned} \phi : G &\to G/H \\ g &\mapsto gH \end{aligned}$$

Then we know $(gH)^n = \phi(g)^n = \phi(g^n) = \phi(e) = H$. This means that n must be a multiple of the order of gH

Problem 5

We can construct a group G such that it has two subgroups H, K where HK is normal but H and K are not. Let

$$\begin{aligned} G &= D_4 \\ H &= \{e, \tau\} \\ K &= \{e, \sigma, \sigma^2, \sigma^3\} \end{aligned}$$

We know that HK is normal in G since $HK = G$. Next we know that H is not normal since $\sigma\tau\sigma^{-1}(v_1) = \sigma\tau(v_n) = \sigma(v_2) = v_3$ whereas $\tau(v_1) = v_1$ and $e(v_1) = v_1$. So $\sigma\tau\sigma^{-1} \notin H$

Problem 6

I claim there is not a surjective group homomorphism from $\mathbb{R}$ to a non-trivial finite group (assuming the operation on $\mathbb{R}$ is addition, if it is multiplication we can map onto the group with 2 elements by mapping positives to 1 and negatives to -1). Suppose we have a homomorphism ϕ. Then we know that $\mathbb{R}/\ker(\phi) \cong G$, by the first isomorphism theorem. We also know that $ker(\phi)$ must be a normal subgroup of $\mathbb{R}$. But the only normal subgroups of $\mathbb{R}$ are $\{0\}$ which makes the image of the isomorphism non-finite, and $\mathbb{R}$ itself which makes G trivial.

Problem 7

We want to show that if G is an abelian group and H a subgroup consisting of all elements that have finite order then G/H has no element of finite order other than the identity element

For the sake of contradiction assume $q \notin H, |qH| = n$.

$$|qH| = n \implies (qH)^n = H \implies q^n H = H \implies q^n \in H$$

And since $q^n \in H$ then it must be that $|q^n| < \infty$ and thus $|q| < \infty$, so it must be in H which violates our original assumption.