1 Mathematical Foundation

Definition 1.1 (Causal Set)**.** A **causal set** is a pair $(C, \leq)$ where:

- C is a countable set (the "events")
- $\leq$ is a partial order on C that is locally finite

We denote the cardinality of C as $|C|$. For finite C with C_n we write C_n. We denote $\mathscr{C}_n$ as the set of all (unlabeled) causal sets with n elements

Definition 1.2 (Causal Automorphism)**.** An order-preserving map $\phi : C \to C'$ is a function such that:

- $X \leq y$ in $C \quad \Longrightarrow \quad \phi(x) \leq \phi(y)$ in C'

The set $\mathrm{Aut}(C)$ of bijective order-preserving maps $C \to C$ forms a group (the automorphism group). The set $\mathrm{End}(C)$ of all order-preserving maps $C \to C$ forms a monoid under composition.

Definition 1.3 (Labeled vs Unlabeled Causal Sets)**.**
- A **labeled** causal set has distinguishable elements (elements have names)
- An **unlabeled** causal set is an isomorphism class under Aut

For physics, unlabeled is more natural (events don't have intrinsic labels), but for mathematics, labeled is easier to work with.

2 The Semigroup of Dynamics

Definition 2.1 (Markov Growth Operator)**.** Fix $N \in \mathbb{N}$. Let $\mathscr{C}_{\leq N} = \cup_{n=0}^{N} \mathscr{C}_n$ be the set of all (unlabeled) causal sets with at most N elements.

A **Markov growth operator** is a map that acts on probability distributions:

$$G : \Delta(\mathscr{C}_{\leq N}) \to \Delta(\mathscr{C}_{\leq N})$$

defined by its transition kernel $G(C \to C')$, which gives the probability of transitioning from causal set C to causal set C'. Where $\Delta(\mathscr{C}_{\leq N})$ is the space of probability distributions on $\mathscr{C}_{\leq N}$. And we have the **action on distributions** for a probability distribution $\mu \in \Delta(\mathscr{C}_{\leq N})$

$$(G_\mu)(C') = \sum_{C \in \mathscr{C}_{\leq N}} \mu(C) \cdot G(C \to C')$$

such that:

- $G(C \to C') \geq 0$ for all C, C'
- $\sum_{C'} G(C \to C') = 1$ for each C
- Causality: $G(C \to C') > 0$ only if C' can be obtained from C by adding ≤ 1 element

Alternative notation: we can also write G as acting on individual causets by:

$$G(C) := G(\delta_C)$$

where δ_C is the point mass at C. Then $G(C)$ is a probability distribution on $\mathscr{C}_{\leq N}$

Physically we just interpret G as describing a stochastic rule for how causal sets evolve/grow.

Definition 2.2 (Composition of Growth Operators)**.** For Markov growth operators G_1, G_2, define their composition:

$$(G_1 \circ G_2)\mu = G_1(G_2\mu)$$

or in terms of transition kernels:

$$(G_1 \circ G_2)(C \to C'') = \sum_{C' \in \mathscr{C}_{\leq N}} G_1(C \to C') \cdot G_2(C' \to C'')$$

This is the Chapman-Kolmogorov equation

Proposition 2.3 (Semigroup Structure)**.** *The set $\mathcal{G}_N$ of all growth operators on $\mathscr{C}_{\leq N}$ forms a semigroup under composition*

Proof.

- Composition of Markov operators is associative
- The identity operator $I(C \to C') = \delta_{C,C'}$ is the identity element
- so this is a monoid and hence a semigroup

□

Example 2.4 (Uniform Growth Operator)**.** Define the uniform growth operator $G_{uniform}$:

For C with $|C| < N$:

$$G_{\text{uniform}}(C \to C') = \begin{cases} \frac{1}{M(C)} & \text{if } C' = C \cup x \text{ with valid causal relations} \\ 0 & \text{otherwise} \end{cases}$$

where $M(C)$ = number of ways to add one element to C. If $|C| = N$ (at capacity):

$$G_{uniform}(C \to C') = \delta_{C,C'}$$

Meaning we randomly add one element in all possible causally-consistent ways, or stay put if at max size.

Definition 2.5 (Action-weighted Growth)**.** Given an **action functional** $S : \mathscr{C}_{\leq N} \to \mathbb{R}$ define the Boltzmann growth operator:

$$G_S(C \to C') = \begin{cases} \frac{1}{Z(C)} e^{-S(C')} & \text{if } C' \text{ obtained from } C \text{ by adding one element} \\ 0 & \text{otherwise} \end{cases}$$

where $Z(C)$ is a normalization constant.

Physically this means: Adding an element with low action is favored. This is like a Metropolis algorithm or path integral weight.

3 Idempotents and Equilibria

Definition 3.1 (Support)**.** For a probability distribution $\mu \in \Delta(\mathscr{C}_{\leq N})$, the support is:

$$\text{supp}(\mu) = \{C \in \mathscr{C}_{\leq N} : \mu(C) > 0\}$$

Meaning: The set of causal sets that have non-zero probability under μ.

Definition 3.2 (Idempotent Growth Operator)**.** A growth operator G is **idempotent** if:

$$G \circ G = G$$

That is $\forall\, C \in \mathscr{C}_{\leq N}$,

$$(G \circ G)(C) = G(G(C)) = G(C)$$

Proposition 3.3 (Meaning of Indempotence)**.** *G is idempotent if and only if:*

$$\text{Im}(G) = \text{Fix}(G)$$

where:

- $\mathrm{Im}(G) = \{\nu : \nu = G(\mu)$ *for some* $\mu\}$ *(image of* G*)*
- $\mathrm{Fix}(G) = \{\nu : G(v) = \nu\}$ *(fixed points of* G*)*

Proof. ($\Rightarrow$) If $G^2 = G$, then for any $\nu \in G(\mu)$ in the image:

$$G(\nu) = G(G(\mu)) = G(\mu) = \nu$$

So ν is a fixed point. Hence $\mathrm{Im}(G) \subseteq \mathrm{Fix}(G)$. Conversely, if $\nu \in \mathrm{Fix}(G)$, then $\nu = G(\nu)$, so $\nu \in \mathrm{Im}(G)$. Hence $\mathrm{Fix}(G) \subseteq \mathrm{Im}(G)$.
($\Leftarrow$) If $\mathrm{Im}(G) = \mathrm{Fix}(G)$, then for any μ:

$$G(G(\mu)) = G(\nu) \text{ where } \nu = G(\mu) \in \mathrm{Im}(G) = \mathrm{Fix}(G)$$

so $G(\nu) = \nu = G(\mu)$. Thus $G^2 = G$. $\square$

Physically think of this as saying: An idempotent operator is a projection onto its fixed point set. After one application, you're in equilibrium.

Corollary 3.4 (Characterization of Idempotent Fixed Points)**.** *If* G *is idempotent, then:*

- G *is a projection:* G *projects any distribution onto* $\mathrm{Fix}(G)$
- *Not all* $G(C)$ *are the same: Different initial* C *can give different equilibria*
- *Each* $G(C)$ *is a fixed point, i.e.,* $G(G(C)) = G(C)$

Example 3.5 (Two-Basin Idempotent)**.** Consider $\mathscr{C}_{\leq 2} = \{\emptyset, \bullet, \bullet\bullet_{parallel}, \bullet\bullet_{ordered}\}$ where:

- $\emptyset$ = empty causet
- $\bullet$ = one element
- $\bullet\bullet_{parallel}$ = two unrelated elements (spacelike)
- $\bullet\bullet_{ordered}$ = two elements with $x < y$ (timelike)

Define G by:

$$G(\emptyset \to \bullet\bullet_{parallel}) = 1$$
$$G(\bullet \to \bullet\bullet_{parallel}) = 1$$
$$G(\bullet\bullet_{parallel} \to \bullet\bullet_{parallel}) = 1$$
$$G(\bullet\bullet_{ordered} \to \bullet\bullet_{ordered}) = 1$$

3.1 Detailed Balance and Stationary Distributions

Definition 3.6 (Detailed Balance)**.** A growth operator G satisfies detailed balance with respect to measure μ if:

$$\mu(C) \cdot G(C \to C') = \mu(C') \cdot G(C' \to C) \quad \forall C, C'$$

Proposition 3.7 (Unique Stationary Distribution)**.** *If G satisfies detailed balance with respect to μ and G is irreducible and aperiodic, then:*

1. *μ is the unique stationary distribution: $G\mu = \mu$*

2. *For any initial μ_0, we have $G^n\mu_0 \to \mu$ as $n \to \infty$*

Lemma 3.8 (Boltzmann Growth and Detailed Balance)**.** *The action-weighted growth operator G_S satisfies detailed balance with respect to the Boltzmann distribution:*

$$\mu_S(C) = \frac{1}{Z} e^{-\beta S(C)}$$

if and only if the growth dynamics is reversible...

4 Compactness and Ellis-Numakura

Proposition 4.1 (Finite State Space)**.** *$\mathscr{C}_{\leq N}$ is finite (for fixed N).*

Proof.

- Number of labeled causets with n elements is bounded by 2^{n^2} (at most n^2 possible order relations)
- Number of unlabeled causets is smaller (divide by $n!$)
- Total for all $n \leq N$ is finite

□

Corollary: $\Delta(\mathscr{C}_{\leq N})$ is a simplex in finite-dimensional space, hence compact.

Proposition 4.2 (Continuity)**.** *The composition operation $(G_1, G_2) \mapsto G_1 \circ G_2$ is continuous on $\mathcal{G}_N$ (with appropriate topology).*

Proof. Probability measure composition is continuous in weak topology □

Theorem 4.3 (Ellis-Numakura Applies). *Under suitable topology on $\mathcal{G}_N$, if $\mathcal{G}_N$ is a closed subsemigroup, then $\mathcal{G}_N$ contains at least one idempotent element.*

Physically, there exists at least one growth dynamics G with a universal equilibrium configuration.

4.1 Topology on the Space of Growth Operators

Definition 4.4 (Weak Operator Topology). A sequence of growth operators G_n converges to G in the weak operator topology if:

$$\langle \mu, G_n \nu \rangle \to \langle \mu, G\nu \rangle \quad \forall \mu, \nu \in \Delta(\mathscr{C}_{\leq N})$$

where $\langle \mu, \nu \rangle = \sum_C \mu(C)\nu(C)$.

Definition 4.5 (Strong Operator Topology). $G_n \to G$ in the strong operator topology if:

$$\|G_n \mu - G\mu\|_{TV} \to 0 \quad \forall \mu \in \Delta(\mathscr{C}_{\leq N})$$

where $\| \cdot \|_{TV}$ is the total variation norm.

Proposition 4.6 (Compactness of Closed Subsets). *Let $\mathcal{G}_N^{bd} \subset \mathcal{G}_N$ be the set of growth operators with uniformly bounded transition rates. Then $\mathcal{G}_N^{bd}$ is compact in the weak operator topology.*

5 Connection to Metric Reconstruction

Theorem 5.1 (Malament-Hawking). *In a Lorentzian manifold (M, g), the causal structure (i.e., which events are causally related) determines the metric g up to a conformal factor. More precisely: If (M, g_1) and (M, g_2) have the same causal structure, then $g_2 = \Omega^2 g_1$ for some smooth positive function Ω.*

Physically, causality determines light cones, which determines the metric up to "time dilation" (conformal factor).

Conjecture 5.2 (Discrete Malament-Hawking). *For a causal set C that is well-approximated by a Lorentzian manifold (M, g):*

- *The causal structure of C determines g up to conformal factor*

- *The density of elements in C determines the conformal factor (via volume)*

Thus, a causal set encodes the full metric.

Proposition 5.3 (Volume from Counting)**.** *In causal set theory, spacetime volume is fundamental:*

$$\text{Volume}(R) = \lim_{\rho\to\infty} \frac{|C \cap R|}{\rho}$$

where ρ is the fundamental discreteness scale.

This clearly gives us a conformal factor. We may hypothesize: If an idempotent G has fixed point μ^* concentrated on causets C^* with certain properties:

- C^* has causal structure approximating some (M, g)
- The density of C^* determines the volume element $\sqrt{|g|}$
- Therefore C encodes a full metric structure

So we may expect that C^* should satisfy a discrete version of Einstein's equations.

6 Connection to Einstein's Equations

6.1 Low Temperature Limit and Action Extremization

Lemma 6.1 (Concentration at Low Temperature)**.** *Let $S_{\min} = \min_{C\in\mathscr{C}_{\leq N}} S(C)$. As $\beta \to \infty$:*

$$\mu_{S,\beta}(C) \to \begin{cases} \frac{1}{|\mathscr{C}_{\min}|} & \text{if } S(C) = S_{\min} \\ 0 & \text{otherwise} \end{cases}$$

where $\mathscr{C}_{\min} = \{C : S(C) = S_{\min}\}$ is the set of global minima.

Proof. The Boltzmann distribution is:

$$\mu_{S,\beta}(C) = \frac{e^{-\beta S(C)}}{Z(\beta)} \quad \text{where} \quad Z(\beta) = \sum_{C'\in\mathscr{C}_{\leq N}} e^{-\beta S(C')}$$

Factor out the minimum action:

$$Z(\beta) = e^{-\beta S_{\min}} \sum_{C'} e^{-\beta(S(C')-S_{\min})} = e^{-\beta S_{\min}} \left(|\mathscr{C}_{\min}| + \sum_{C'\notin\mathscr{C}_{\min}} e^{-\beta(S(C')-S_{\min})} \right)$$

For $C' \notin \mathscr{C}_{\min}$, we have $S(C') - S_{\min} = \Delta S > 0$, so $e^{-\beta \Delta S} \to 0$ as $\beta \to \infty$.

Therefore:

$$\mu_{S,\beta}(C) = \frac{e^{-\beta S(C)}}{e^{-\beta S_{\min}}(|\mathscr{C}_{\min}| + o(1))} \to \begin{cases} \frac{1}{|\mathscr{C}_{\min}|} & \text{if } S(C) = S_{\min} \\ 0 & \text{otherwise} \end{cases}$$

□

Remark. This is analogous to ground state dominance in quantum mechanics: at zero temperature, only the minimum energy state survives. Here, at infinite inverse temperature, only minimum action configurations survive.

Proposition 6.2 (Idempotent Limit). *If $G_{S,\beta}$ converges to an idempotent operator G_∞ as $\beta \to \infty$, then:*

$$\operatorname{supp}(\operatorname{Fix}(G_\infty)) \subseteq \mathscr{C}_{\min}$$

Proof. Since G_∞ is idempotent, we have $\operatorname{Im}(G_\infty) = \operatorname{Fix}(G_\infty)$ by Proposition 3.3'.

For any distribution μ, we have:

$$G_\infty(\mu) = \lim_{\beta \to \infty} G_{S,\beta}(\mu)$$

The action of $G_{S,\beta}$ on a point mass δ_C gives:

$$G_{S,\beta}(\delta_C)(C') \propto e^{-\beta S(C')} \quad (\text{for } C' \text{ accessible from } C)$$

Taking $\beta \to \infty$, this concentrates on $\arg\min_{C':C \to C'} S(C')$, which must be in $\mathscr{C}_{\min}$ (since we can only increase or maintain the number of elements, and the action grows with structure).

Therefore, for any μ, $G_\infty(\mu)$ has support in $\mathscr{C}_{\min}$. Since $\operatorname{Fix}(G_\infty) = \operatorname{Im}(G_\infty)$, all fixed points have support in $\mathscr{C}_{\min}$. □

Remark. This proposition is conditional: it assumes $G_{S,\beta}$ converges to an idempotent. This is a strong assumption and may not hold for generic actions S. Understanding when this convergence occurs is an important open question.

6.2 The Benincasa-Dowker Action

We now introduce a specific action functional proposed in causal set theory.

Definition 6.3 (Benincasa-Dowker Action)**.** The **Benincasa-Dowker action** is defined as:

$$S_{BD}(C) = \lambda|C| + \kappa \sum_{x,y \in C : x < y} f\left(\ell(x,y)\right)$$

where:

- $|C|$ is the number of elements (the "volume" term)
- λ is a coupling constant (related to cosmological constant Λ)
- $\ell(x,y) := |\{z \in C : x < z < y\}|$ is the number of elements between x and y
- $f : \mathbb{N} \to \mathbb{R}$ is a function encoding geometric information
- κ is a coupling constant (related to gravitational constant G)

Remark (Physical Interpretation).

- The first term $\lambda|C|$ acts as a cosmological constant, penalizing or favoring volume.
- The second term encodes curvature: in flat space, the number of elements between x and y scales as (*proper time*)^d where d is dimension. Deviations from this scaling indicate curvature.
- The function f is typically chosen to match the Einstein-Hilbert action in the continuum limit. A common choice is $f(\ell) = (\ell - \bar{\ell})^2$ where $\bar{\ell}$ is the expected value in flat space.

Example 6.4 (Simple Quadratic Action)**.** Consider:

$$f(\ell) = (\ell - \bar{\ell})^2$$

where $\bar{\ell}$ is tuned to the expected causal interval in flat spacetime. Then:

$$S_{BD}(C) = \lambda|C| + \kappa \sum_{x<y} (\ell(x,y) - \bar{\ell})^2$$

This penalizes deviations from flat space structure, making configurations with "flat-like" causal structure favorable.

6.3 Discrete Variational Calculus

To make sense of "critical points" for discrete structures, we need a discrete notion of variation.

Definition 6.5 (Local Variation)**.** A **local variation** of a causet C is a causet C' that differs from C by:

- Adding or removing one element, OR
- Changing one causal relation (adding or removing an edge in the Hasse diagram)

while maintaining the partial order properties (transitivity, acyclicity).

Definition 6.6 (Critical Point)**.** A causet C is a **critical point** of the action S if:

$$S(C) \leq S(C') \quad \text{for all local variations } C' \text{ of } C$$

That is, C is a local minimum under all allowed perturbations.

Remark. This is analogous to the continuum case where $\delta S/\delta g_{\mu\nu} = 0$. In the discrete setting, we don't have smooth variations, so we use discrete perturbations instead. The condition says: you cannot decrease the action by small changes to the causal structure.

Example 6.7 (2D Minkowski Space)**.** Consider a causet obtained by sprinkling points in 2D Minkowski space. For the Benincasa-Dowker action with appropriate f and λ:

- Adding extra causal relations (making it "more timelike") increases curvature deviation increases action
- Removing causal relations breaks the embedding may increase action due to loss of structure
- The original Minkowski sprinkling is (approximately) a critical point

6.4 Connection to Continuum Einstein Equations

We now state our main conjectures connecting discrete equilibria to continuum gravity.

Conjecture 6.8 (Idempotents Extremize Action)**.** *Let G_S be the Boltzmann growth operator for action S. If G_S is idempotent with fixed point μ^*, then:*

$$\mathrm{supp}(\mu^*) \subseteq \{C \in \mathscr{C}_{\leq N} : C \textit{ is a critical point of } S\}$$

Remark (Justification)*.* This conjecture is motivated by:

1. **Statistical mechanics analogy**: In equilibrium statistical mechanics, the stationary distribution extremizes free energy $F = \langle E \rangle - TS_{entropy}$. Here, we expect the fixed point μ^* to extremize an analogous functional.

2. **Detailed balance**: If G_S satisfies detailed balance (which would require reversibility), then its stationary distribution is the Boltzmann distribution $\mu_S(C) \propto e^{-\beta S(C)}$. At low temperature ($\beta \to \infty$), this concentrates on action minima.

3. **Variational characterization**: Idempotent operators are projections. The fixed point set can often be characterized as the minimizers of some functional, connecting equilibrium to extremization.

However, the precise conditions under which this holds remain to be established.

Conjecture 6.9 (Critical Points are Einstein Solutions)**.** *Let S be the Benincasa-Dowker action with appropriately tuned parameters. Let $C^* \in \mathscr{C}_{\leq N}$ be a critical point of S. Then in the continuum limit ($N \to \infty$, $\rho \to \infty$ with N/ρ fixed), the causet C^* approximates a spacetime (M, g) satisfying Einstein's equations:*

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} + \Lambda g_{\mu\nu} = 0$$

where Λ is determined by the parameter λ in the action.

Remark (What "Approximates" Means)*.* More precisely, this conjecture claims:

1. There exists a Lorentzian manifold (M, g) such that C^* can be embedded in M with causal order preserved

2. The metric g can be reconstructed from C^* via Malament-Hawking (up to conformal factor) and volume counting (determines conformal factor)

3. The reconstructed metric g satisfies Einstein's equations to within $O(1/\sqrt{N})$ errors

4. The parameter λ relates to Λ via $\lambda \sim \Lambda \cdot \ell_{Planck}^{d-2}$ where d is dimension

Remark (Necessary Conditions for Conjecture). For this conjecture to hold, we need:

1. The action S_{BD} to reduce to the Einstein-Hilbert action $\int \sqrt{|g|}(R - 2\Lambda)d^d x$ in the continuum limit

2. The critical point equation $\delta S_{BD} = 0$ to reduce to $\delta S_{EH} = 0$, which gives Einstein's equations

3. The continuum limit to be well-defined (requires understanding how to take $N \to \infty$)

These are all non-trivial assumptions that require further investigation.

6.5 Implications and Open Questions

If the above conjectures hold, then we have a remarkable result:

Theorem 6.10 (Existence via Ellis-Numakura (Conditional))**.** *Assume Conjectures 6.1 and 6.2 hold. Then the Ellis-Numakura theorem guarantees the existence of causets C^* that approximate solutions to Einstein's equations in the continuum limit.*

This would provide a novel perspective on the existence of gravitational solutions: they emerge as equilibrium configurations in a stochastic dynamics on discrete causal structures.

Open Questions:

1. **When is G_S idempotent?** For which action functionals S does the growth operator have idempotent limits? Is there a characterization?

2. **Uniqueness**: Does Ellis-Numakura give a unique idempotent, or multiple? If multiple, do they correspond to different vacuum solutions (Minkowski, de Sitter, Schwarzschild, etc.)?

3. **Continuum limit**: How do we rigorously take $N \to \infty$? Can we prove convergence of the discrete theory to continuum GR?

4. **Matter coupling**: How do we extend this framework to include matter fields? Does the coupled system still admit idempotents?

5. **Computational verification**: Can we numerically compute idempotents for small N and check if their fixed points concentrate on "flat-like" causets?

6. **Comparison to path integral**: How does this relate to the causal set path integral $Z = \sum_C e^{iS(C)}$? Are idempotents related to saddle points of the Euclidean path integral?

6.6 Comparison to Other Approaches

Regge Calculus: In Regge calculus, spacetime is triangulated and the Einstein-Hilbert action is discretized on the triangulation. Critical points of the Regge action approximate Einstein solutions. Our approach is similar but uses causets (partial orders) rather than triangulations (simplicial complexes).

Causal Dynamical Triangulations (CDT): CDT also uses a path integral over geometries. The key difference is that CDT uses a full triangulation (metric structure), while causets use only causal order. Our focus on idempotents (equilibria) is also distinct from CDT's Monte Carlo sampling approach.

Classical Sequential Growth: The existing CSG approach (Rideout-Sorkin) grows causets sequentially with action-weighted probabilities. Our framework generalizes this by:

- Treating growth operators as elements of a semigroup
- Applying Ellis-Numakura to guarantee equilibria
- Connecting equilibria to Einstein's equations via extremal action

Asymptotic Safety: In asymptotic safety, gravity is defined by a fixed point of the renormalization group. Our idempotents play a similar role: fixed points of the growth dynamics. The connection deserves further exploration.