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HW Set 1
MATH 111A

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Problem 1

(1) * is clearly a binary relation on $\mathbb{R}$ since both multiplication and addition are binary relations on $\mathbb{R}$.

(2) We already know that the operation is closed. It is also the case that it is associative (this is trivial to show, just tedious). Searching for an identity we see

$$\begin{aligned} a * e = a &= a + e + ae \\ 0 &= e(1 + a) \end{aligned}$$

And this is always true if $e = 0$. Now looking at inverses

$$\begin{aligned} a * b = 0 &= a + b + ab \\ -a &= b(1 + a) \\ \frac{-a}{1 + a} &= b \end{aligned}$$

So clearly we must exclude $a_0 = -1$ from the set, because it would not have an inverse.

(3) Examine $a * a = 2a + a^2$. If we plug in $-2 = a$ this clearly equals 0 so -2 has order 2.

(4)

$$\begin{aligned} 2 * x * 3 &= 7 \\ (2 + 3x) * 3 &= 7 \\ 14 + 12x &= 7 \\ x &= \frac{-7}{12} \end{aligned}$$

Problem 2

First we know that $|S_3| = 6$ so every subgroup should have an order that divides 6. The first group we can identify is the trivial group $\{e\}$. Next all the transpositions $\{e, (12)\}, \{e, 13\}, \{e, 23\}$. Then if we look at the cyclic groups generated by $(123), (132)$ we see we get the same group $\{e, (123), (132)\}$.

Problem 3

Let G be a group. Let $a \neq e$ be an element of G. Then consider the subgroup $\langle a \rangle \leq G$. But since G has no non-trivial subgroups by assumption we must have $\langle a \rangle = G$.

Problem 4

Let the group $G = \{z \in \mathbb{C} : |z| = 1\}$ and let the operation be multiplication. The group is clearly closed, associative, and the identity element $e = 1$. Consider some element $e^{2\pi i/n}$ will clearly have order n and any element e^{ix} where $x \notin \mathbb{Q}$ will have order ∞.

Problem 5

Let G be a finite group and let $a \in G$, I claim that $|a| \leq |G|$. Let's assume for the sake of contradiction that $|a| > n = |G|$. We then find that there is some element in $\{a, a^2, a^3, \ldots, a^n, a^{n+1}\}$ that is not in G so then G is not closed under the group operation and cannot be a group, so our assumption that the order of an arbitrary element can be greater than the order of the group must be wrong

Problem 6

Since $(ab)^{-1} = a^{-1}b^{-1}$. We have

$$\begin{aligned} e &= (ab)(ab)^{-1} \\ &= (ab)a^{-1}b^{-1} \\ b &= (ab)a^{-1} \\ ba &= ab \end{aligned}$$

Thus the group must be abelian since a, b are arbitrary and they commute with each other.