Non-Standard Discretization Methods

for Some Biological Models

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2018B4PS1039G

Mathematical Modelling Project

Introduction

In this project, I will be presenting various numerical schemes that produce differential equations whose dynamics remains unchanged from the underlying continuous equations.

They are used to transform the initially continuous problem which has an infinite number of degrees of freedom into a discrete problem where the degree of freedom is inevitably limited.

Outline

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Three types of relationships between two species

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Modelling the Lotka-Volterra Differential Equations

Lotka-Volterra System

$$\left.\begin{aligned} \frac{dX}{dt} &= X(t)[r_1 - a_1 x(t) - by(t)] \\ \frac{dY}{dt} &= Y(t)[r_2 - cx(t) - a_2 y(t)] \end{aligned}\right\} \tag{1}$$

where r_1 and r_2 represent the intrinsic growth/decay rate of X and Y, a_1 and a_2 are effects caused within the same species,
b is the growth effect of X from Y and
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Clearly $a_1 \geqslant 0$ and $a_2 \geqslant 0$.

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  - Presence of X (prey) has a positive impact on the growth of Y (predator) while the presence of Y has a negative impact on X.
  - $b > 0$, $c > 0$

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Stability Analysis for the Lotka-Volterra Model

For System (1) to possess a equilibrium point (X^*, Y^*), it must satisfy:

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Solving equations (2) and (3) for X^* and Y^*, we get:

$$X^* = \frac{r_1 a_2 - r_2 b}{a_1 a_2 - bc}, Y^* = \frac{r_2 a_1 - r_1 c}{a_1 a_2 - bc} \tag{4}$$

Stability Analysis for the Lotka-Volterra Model

Classification of an Equilibrium Point

The equilibrium point (X^*, Y^*) is said to be **stable** if for any open neighbourhood U of (X^*, Y^*) $\exists$ an open neighbourhood V of (X^*, Y^*) such that if $(X_0, Y_0) \in$ V, then (X(t,X_0),Y(t,Y_0)) $\in$ U $\forall$ t $\geqslant$ 0.

If (X^*, Y^*) is stable and $\lim_{t\to\infty}$ (X(t,X_0),Y(t,Y_0)) $= (X^*, Y^*)$ $\forall$ (X_0, Y_0) in an open neighbourhood W of (X^*, Y^*), then (X^*, Y^*) is **asymptotically stable**.

If W $= \mathbb{R}^2$, then (X^*, Y^*) is **globally asymptotically stable**.

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Stability means that the solution of the differential equation will not leave the ϵ-ball, but asymptotic stability means that the solution does not leave the ϵ-ball and goes to the equilibrium point.

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Using Taylor series expansion and neglecting higher order terms,

Forward Euler Method

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$$\frac{dX}{dt} = \frac{X(t+h) - X(t)}{h} \; and \; \frac{dY}{dt} = \frac{Y(t+h) - Y(t)}{h} \tag{5}$$

where h is the step size of the Euler method.

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where h is the step size of the Euler method.

This results in the Forward Euler Method having a *local truncation error* of $O(h^2)$, and therefore is a first order technique.

Replacing System (1) with Equations (5), and taking n $= \frac{t}{h}$, $X(t) = X(nh) = X(n)$, and $Y(t) = Y(nh) = Y(n)$, we arrive at the difference system:

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Difference System for Lotka-Volterra Model

$$\left.\begin{aligned} X(n+1) &= X(n)[1 + r_1 h - a_1 hX(n) - bhY(n)] \\ Y(n+1) &= Y(n)[1 + r_2 h - chX(n) - a_2 hY(n)[\end{aligned}\right\} \tag{6}$$

Plotting the Lotka-Volterra System

Predator-Prey Model

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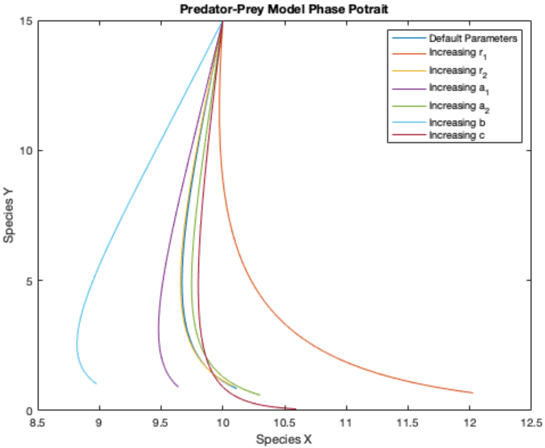
- **Case I:** Increasing Parameters



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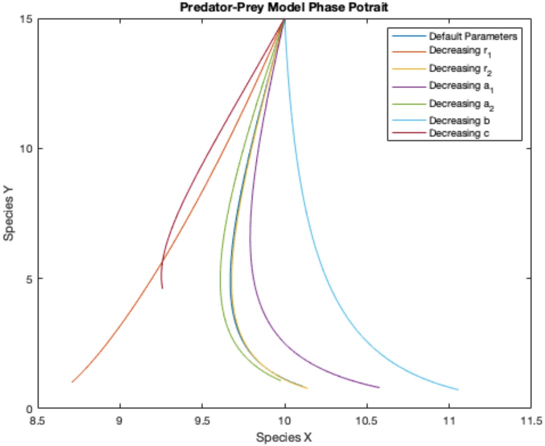

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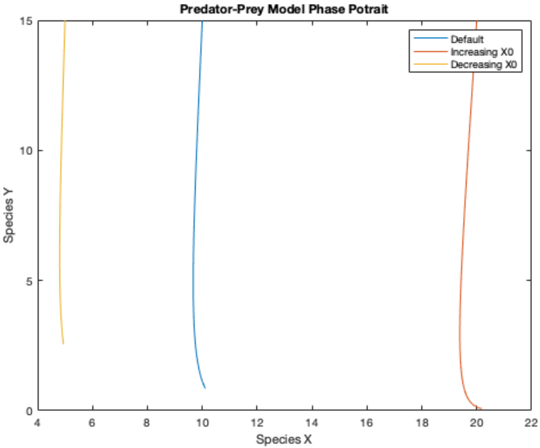
- **Case III:** Altering Initial Conditions: X0



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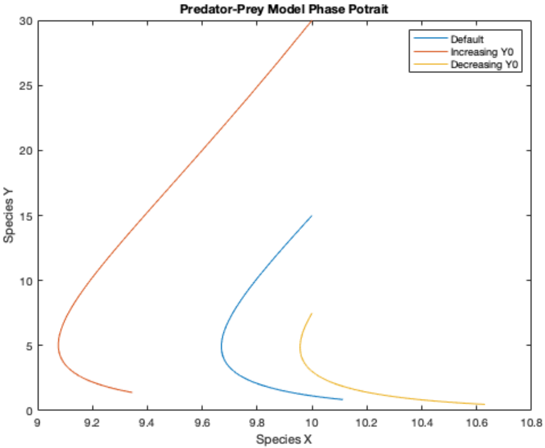
- **Case IV:** Altering Initial Conditions: Y0



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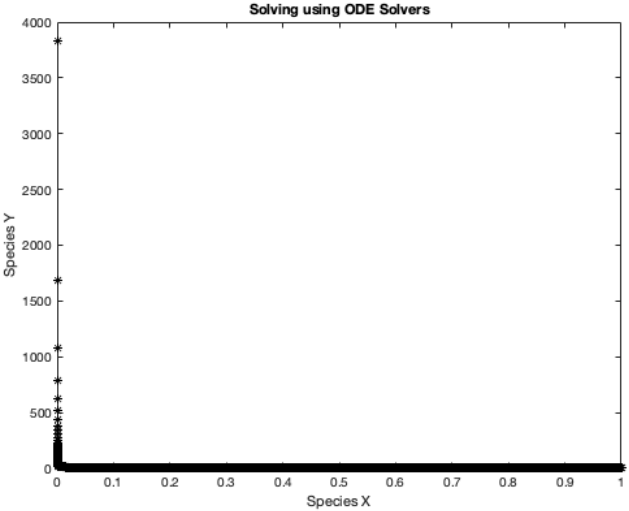
Predator-Prey Model

In all 4 cases, we see that the equilibrium point remains asymptotically stable, regardless of the modifications that are made, when $a_1 > 0$, $a_2 > 0$.

Now, we plot and compare the phase portraits for Systems (1) and (6) when $a_1 < 0$, $a_2 < 0$.

Phase Portrait of System (1) when $a_1 < 0$, $a_2 < 0$:



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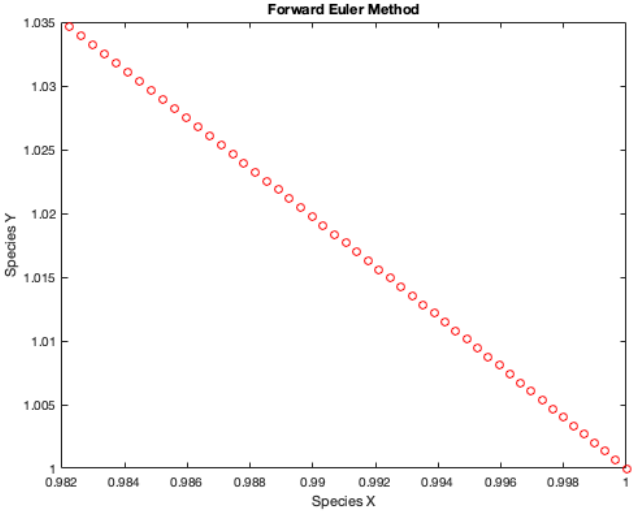
Phase Portrait of System (6) when $a_1 < 0$, $a_2 < 0$:



Limitations of Forward Euler Method

Alternative Numerical Schemes

As seen in the above slides, the dynamics of System (6) is different from that of System (1) and can cause chaotic behaviour. (Ushiki, 1982)

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An alternative well-known method is to let System (1) have piecewise constant arguments:

$$\left.\begin{aligned} \frac{dX}{dt} &= X(t)[r_1 - a_1 x(\lfloor t \rfloor) - by(\lfloor t \rfloor)] \\ \frac{dY}{dt} &= Y(t)[r_2 - cx(\lfloor t \rfloor) - a_2 y(\lfloor t \rfloor)] \end{aligned}\right\} \tag{7}$$

where $0 \leqslant n \leqslant t <$ n + 1, and $\lfloor t \rfloor$ is the greatest integer in t.

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System (8) has a unique positive equilbrium point (x^*, y^*) with

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The equilibrium point (x^*, y^*) is asymptotically stable $\forall$ parameters > 0.

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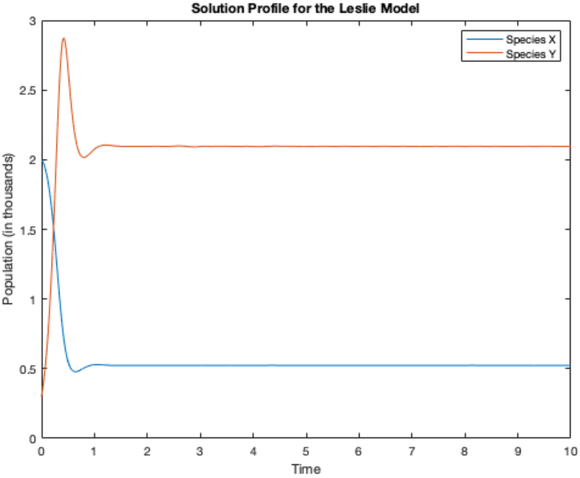

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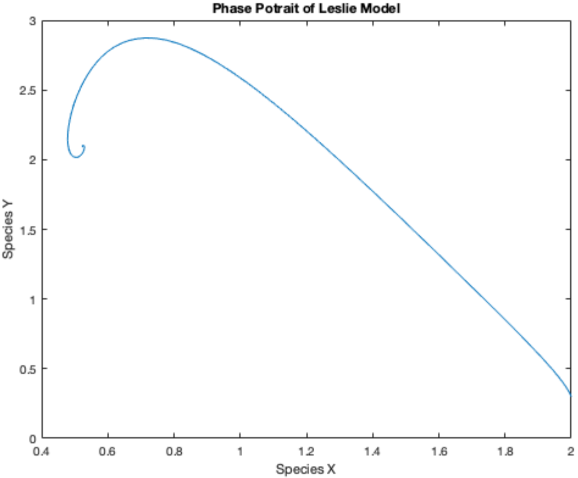

Schur-Cohn Criterion for Asymptotic Stability

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The *trace* of a n x n matrix A (tr A) is the sum of its diagonal elements.

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Theorem 1.1

For $|\lambda_i| < 0$, it is necessary and sufficient that for the Jacobian matrix J,

$$|tr\ J| < 1 + det\ J < 2$$

where det J is the determinant of J.

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Discretizing the Leslie Predator-prey Model

We propose the discrete model:

$$\left.\begin{aligned} \frac{x(t+h)-x(t)}{\varphi_1(h)} &= \gamma_1 x(t) - a_1 x(t)x(t+h) - bx(t+h)y(t) \\ \frac{y(t+h)-y(t)}{\varphi_2(h)} &= \gamma_2 y(t) - \frac{a_2 y(t)y(t+h)}{x(t)} \end{aligned}\right\} \tag{10}$$

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which gives us the difference system:

$$\left.\begin{array}{l} x(n+1) = \frac{x(n)(1+\gamma_1\varphi_1(h))}{1+a_1\varphi(h)x(n)+b\varphi_1(h)y(n)} = F(x(n), y(n)) \\ y(n+1) = \frac{x(n)y(n)(1+\gamma_2\varphi_2(h))}{x(n)+a_2\varphi_2(h)y(n)} = G(x(n), y(n)) \end{array}\right\} \quad (11)$$

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Through direct checking, we see that all 3 Schur-Cohn conditions are satisfied and hence, (x^*, y^*) is asymptotically stable.

Limitations of the Leslie Model

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- This causes an environmental imbalance, and is not considered in the Leslie Model.
- This model also does not take into consideration the **Allee Effect**
- An Allee Effect is a positive correlation between individual fitness and population size over some finite time interval (like mate limitation).

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- The model used above also does not consider non-topological equivalent behaviours.
- For instance, species that are endangered species have a low probability of locating responsive mates or a biased sex-ratio.

Recommendations for the Leslie Model

With a strong Allee effect on prey, zero, one or two equilibrium points can exist. (González-Olivares, Mena- Lorca, Rojas-Palma and Flores, 2007).

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To factor in the limitations previously mentioned, the Leslie-Grower Model is a more apt model, in which the Allee threshold is a parameter in the system.

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Simple Predator-Prey Model

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$$\left.\begin{aligned} \frac{dx}{dt} &= \alpha x + \beta xy \\ \frac{dy}{dt} &= \gamma y + \delta xy \end{aligned}\right\} \tag{12}$$

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If

$$\alpha < 0, \beta > 0, \gamma > 0, \delta < 0 \tag{13}$$

then $(-\alpha/\beta, -\gamma/\delta)$ is the only positive equilibrium point of system (12).

Implementing the Simple Predator-Prey Model in MATLAB

Taking $\alpha = -1, \beta = 0.01, \gamma = 0.5$ *and* $\delta = -0.01$, we plot the solution profile and phase portrait for System (12). Using these parameters, the only positive equilibrium point is (100,50).

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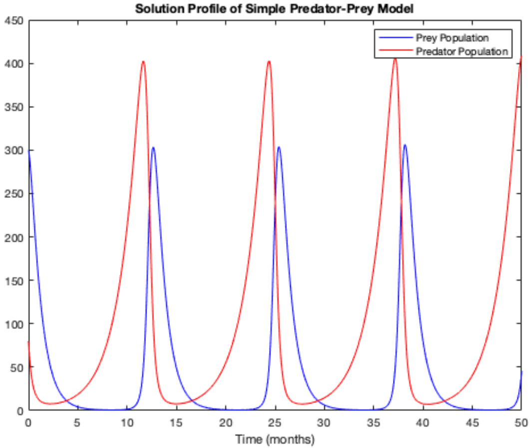

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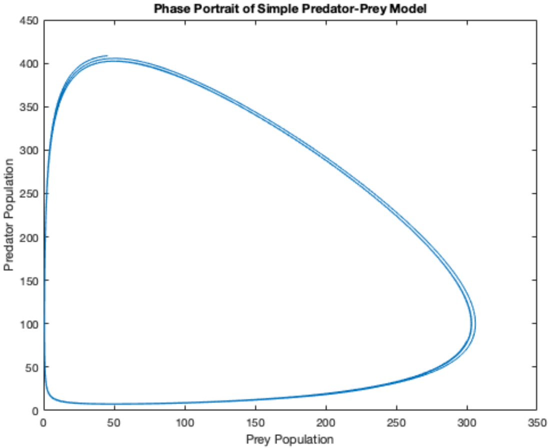

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$$\phi(h) = \mathrm{h} + \mathrm{O(h)},\ 0 < \phi(h) < 1,\ \mathrm{h} \rightarrow 0$$

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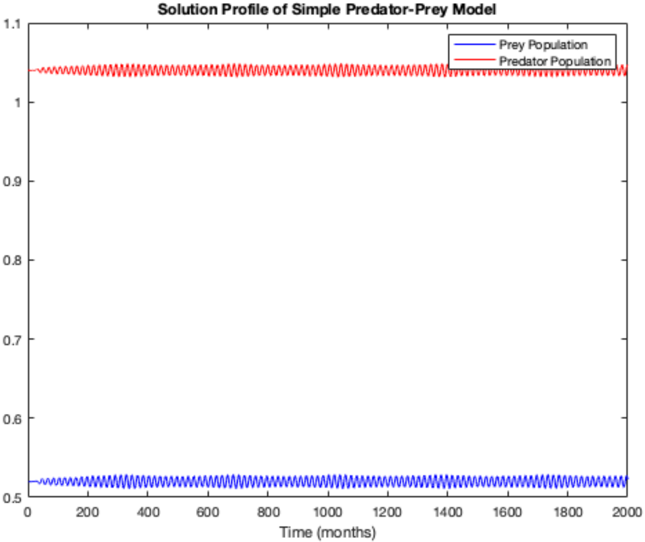
In the Mickens discretization scheme, this is taken one step further and the derivative is replaced with $\frac{y(t+h)-y(t)}{\phi(h)}$, where $\phi(h)$ is a continuous function of step size h, and is such that

$$\phi(h) = \mathrm{h} + \mathrm{O(h)},\ 0 < \phi(h) < 1,\ \mathrm{h} \to 0$$

If the Mickens discretization scheme is applied to the system (12), the solution either spirals in towards (x^*, y^*) or spirals out of the equilibrium point (x^*, y^*).

Implementing the Discretized Simple Predator-Prey Model in MATLAB



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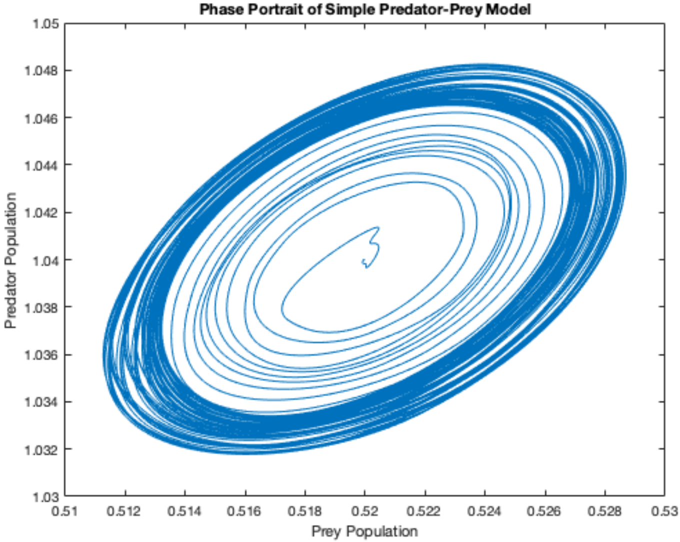

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Consider two species with population densities $x(t)$ and $y(t)$.

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$$\left.\begin{aligned} \frac{dx}{dt} &= r_1 x(t)\left(1 - \frac{x(t)}{\beta_1 + \alpha_1 y(t)}\right) \\ \frac{dy}{dt} &= r_2 y(t)\left(1 - \frac{y(t)}{\beta_2 + \alpha_2 x(t)}\right) \end{aligned}\right\} \tag{14}$$

where r_1,r_2, α_1, α_2, β_1, β_2 are positive numbers.

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where r_1,r_2, α_1, α_2, β_1, β_2 are positive numbers.
If $\alpha_1\alpha_2 < 1$, then $\exists$ a positive equilibrium point (x^*, y^*) which satisfies the equations:

$$\left.\begin{aligned} -x^* + \alpha_1 y^* &= -\beta_1 \\ \alpha_2 x^* - y^* &= -\beta_2 \end{aligned}\right\} \tag{15}$$

Stability of Robert May's Model

The linearized system around (x^*, y^*) has the coefficient matrix:

$$\mathrm{B} = \begin{pmatrix} e^{-r_1} & \alpha_1(1 - e^{-r_1}) \\ \\ \alpha_2(1 - e^{-r_2}) & e^{-r_2} \end{pmatrix}$$

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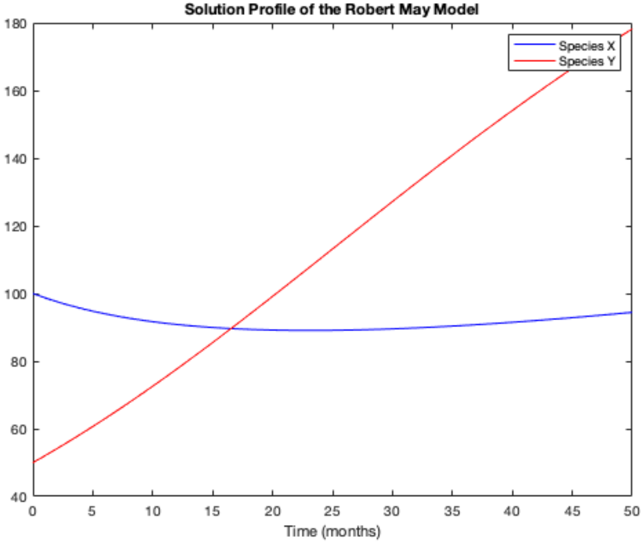
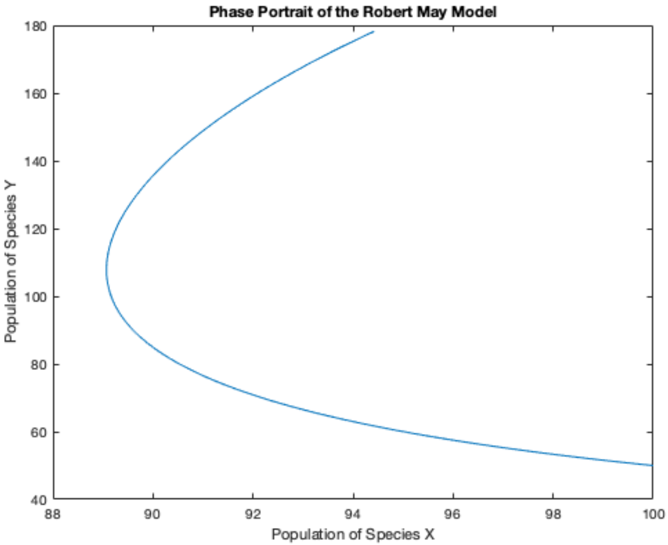
Through direct checking, we can see that matrix B satisfies the Schur-Cohn criterion and therefore, makes (x^*, y^*) a locally asymptotically stable equilibrium point.

Implementing the Robert May Model in MATLAB



Implementing the Robert May Model in MATLAB



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Concluding Statements

We have gone through and implemented the various non-standard discretization Lotka-Volterra models, classical discretization, the Leslie model, further numerical schemes and finally a Kolmogorov Model of cooperative systems, and discussed its local stability.

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We have gone through and implemented the various non-standard discretization Lotka-Volterra models, classical discretization, the Leslie model, further numerical schemes and finally a Kolmogorov Model of cooperative systems, and discussed its local stability.

Apart from the recommendations and limitations mentioned in my presentation, there is definitely more scope to make biological models that factor in more aspects of the environment of the species, making it more precise and reliable.

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Thank you!