Finiteness theorems for reductions of Hecke orbits

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Outline

1. Lifting isogenies and endomorphisms of abelian varieties
2. A Galois-theoretic criterion for finiteness of Hecke orbits
3. Verifying Galois-theoretic criterion for supersingular $\overline{A}$
4. Applications to CM-lifting theorems

Lifting p-isogenies from characteristic p to characteristic 0

$$
\begin{aligned}
K\colon &\quad \text{finite extension of } \mathbb{Q}_p \\
A/K\colon &\quad \text{Abelian variety over } K \text{ with good reduction} \\
\overline{A}/\mathbb{F}_q\colon &\quad \text{Reduction of } A \\
I_p(A)\colon &\quad \{B/K' \mid [K' : K] < \infty,\ B \text{ is } p\text{-power isogenous to } A\} \\
I_p(\overline{A})\colon &\quad \{\overline{B}/\mathbb{F}_{q'} \mid [\mathbb{F}_{q'} : \mathbb{F}_q] < \infty,\ \overline{B} \text{ is } p\text{-power isogenous to } \overline{A}\}
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A/K:	Abelian variety over K with good reduction
$\overline{A}/\mathbb{F}_q$:	Reduction of A
$I_p(A)$:	$\{B/K' \mid [K' : K] < \infty,\ B$ is p-power isogenous to $A\}$
$I_p(\overline{A})$:	$\{\overline{B}/\mathbb{F}_{q'} \mid [\mathbb{F}_{q'} : \mathbb{F}_q] < \infty,\ \overline{B}$ is p-power isogenous to $\overline{A}\}$

Note: All abelian varieties in $I_p(A)$ also have good reduction.
$\overline{I_p(A)} :=$ Reductions of all the abelian varieties in $I_p(A)$.

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Main Question 1 (Lifting p-isogenies):
How large is the subset $\overline{I_p(A)}$ of $I_p(\overline{A})$?

Definition

Let A be a g-dimensional abelian variety over a characteristic 0 local field K. We say that A is a CM-abelian variety if there is an embedding

$$F \hookrightarrow \mathrm{End}(A) \otimes_{\mathbb{Z}} \mathbb{Q}$$

of a commutative, semisimple $\mathbb{Q}$-algebra F of dimension $2g$.

Main Question 2 (Existence of CM-lifts):
For which $\overline{A}/\overline{F_p}$ does there exist a CM-abelian variety A over a characteristic 0 local field with reduction $\overline{A}$?

History of lifting problems

- Honda-Tate (Lifting up to isogeny)
  Every $\overline{A}/\overline{\mathbb{F}_p}$ is isogenous to a $\overline{B}/\overline{\mathbb{F}_p}$ with a CM-lift.

- Serre-Tate (Canonical lifts for *ordinary* abelian varieties)
  Every ordinary abelian variety $\overline{A}/\overline{\mathbb{F}_p}$ admits a CM-lift A.
  All isogenies of such $\overline{A}$ lift to isogenies of the canonical lift A.

- Oort/Conrad-Chai-Oort (Non-existence of CM lifts)
  There are *supersingular* abelian varieties $\overline{A}/\overline{\mathbb{F}_p}$ with *no CM lifts*.

Theorem (Kisin, Lam, Shankar, S.)

Fix a lift A/K of $\overline{A}/\overline{\mathbb{F}_p}$ to a characteristic 0 local field. Assume that $\overline{A}$ is supersingular or that the p-adic Galois representation $\rho\colon G_K \to \mathrm{GL}(T_p(A))$ has reductive monodromy. Then, the subset $\overline{I_p(A)}$ of $I_p(\overline{A})$ is finite.

Theorem (Kisin, Lam, Shankar, S.)

1. *Only finitely many supersingular*[a] *abelian varieties $\overline{A}/\overline{\mathbb{F}_p}$ of a given dimension admit CM-lifts.*
2. *Only finitely many supersingular K3 surfaces $\overline{X}/\overline{\mathbb{F}_p}$ admit CM-lifts when $p \geqslant 5$.*

[a] We also prove a common generalization of the results for ordinary/supersingular strata to other Newton strata.

K: finite extension of $\mathbb{Q}_p$
G_K: absolute Galois group of K
I_K: inertia subgroup of G_K

A: abelian variety over K with good reduction
$\mathscr{G}$: p-divisible group over K with good reduction

V: rational p-adic Tate module of A or $\mathscr{G}$
ρ: p-adic Galois representation $G_K \to \mathrm{GL}(V)$

A Galois-theoretic criterion for finiteness

$$\rho\colon G_K \to \mathrm{GL}(V) \cong \mathrm{GL}_{2g}(\mathbb{Q}_p).$$

Proposition (*"Totally ramified up to finite index" criterion*)

If $\rho(I_K)$ has finite index in $\rho(G_K)$, then the reduction of the p-Hecke orbit of the corresponding A or $\mathscr{G}$ is finite.

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Proof sketch:

- All abelian varieties in the p-Hecke orbit of A are defined over the the fixed field K_ρ of $\ker(\rho)$.
- Assumption $\Rightarrow$ The residue field of K_ρ is a finite field.
- The existence of the moduli space $\mathcal{A}_g$ + Zarhin's trick $\Rightarrow$ there are only finitely many isomorphism classes of abelian varieties of a given dimension defined over a fixed finite field.

The unramified quotient T

$$\rho\colon G_K \to \mathrm{GL}(V).$$

Consider the exact sequence of algebraic groups by taking Zariski closures in $\mathrm{GL}(V)$.

$$1 \to \overline{\rho(I_K)} \to \overline{\rho(G_K)} \to T \to 1.$$

Sen $+ \epsilon \Rightarrow$ If T is finite, then $\rho(I_K)$ has finite index in $\rho(G_K)$.

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Goal: *Show T is finite if $\overline{A}$ is supersingular.*

V_T: a faithful $\mathbb{Q}_p$-representation of T

σ: image of a Frobenius element in $\mathrm{GL}(V_T)$, a generator for the image of T in $\mathrm{GL}(V_T)$.

We will show σ is semisimple and its eigenvalues are roots of unity.

T is finite if $\overline{A}$ is supersingular

$$1 \to \overline{\rho(I_K)} \to \overline{\rho(G_K)} \to T \to 1,$$

V_T a faithful repn. of T and $\langle \sigma \rangle = T \subset \mathrm{GL}(V_T)$ (Frobenius).

Proof sketch:

- V_T is in the Tannakian category generated by $V (= V_p(A))$.
- V_T is unramified by the definition of T and crystalline, so the eigenvalues of σ are p-adic units.
- Since $\overline{A}$ is *supersingular*, Frobenius acts semisimply on $\mathbb{D}(V) \otimes \mathbb{Q}_p$ with eigenvalues rational powers of p up to roots of unity.
- The only power of p that is a p-adic unit is 1.

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Only finitely many supersingular abelian varieties of a given dimension have CM-lifts

Proof strategy:

1. There are *only finitely many supersingular* $\overline{A}$ with a given p-divisible group $\mathscr{G} := \overline{A}[p^\infty]$. (Oort)
2. For fixed dimension, *finitely many choices* for the CM-subalgebra F of $\mathrm{End}(\mathscr{G}) \otimes \mathbb{Q}_p$.
   For fixed F, *only finitely many possibilities* for the p-adic CM type $\Phi \colon F \to \prod_{i=1}^{g} \overline{\mathbb{Q}_p} = \mathrm{End}_F(\mathrm{Lie}\, \mathscr{G}_{\overline{\mathbb{Q}_p}})$.
3. Upto unramified twists, there is *only one isogeny class* $\mathscr{G}_\Phi/K$ of p-divisible group over local field with CM type Φ. (Conrad-Chai-Oort)
4. Since $\mathscr{G}_\Phi$ has CM, the reduction of its p-Hecke orbit is *finite* by our reductive monodromy theorem.