Invariants associated to models of curves over discrete valuation rings

by

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Submitted to the Department of Mathematics
in partial fulfillment of the requirements for the degree of

Doctor of Philosophy

at the

MASSACHUSETTS INSTITUTE OF TECHNOLOGY

June 2016

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May 2, 2016

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Abstract

This thesis consists of two parts. In part one of this thesis, we study the relationship between the Artin conductor and the minimal discriminant of a hyperelliptic curve defined over the fraction field K of a discrete valuation ring. The Artin conductor and the minimal discriminant are two measures of degeneracy of the singular fiber in a family of hyperelliptic curves. In the case of elliptic curves, the Ogg-Saito formula shows that (the negative of) the Artin conductor equals the minimal discriminant. In the case of genus two curves, Liu showed that equality no longer holds in general, but the two invariants are related by an inequality. We extend Liu's inequality to hyperelliptic curves of arbitrary genus, assuming rationality of the Weierstrass points over K.

In part two of this thesis, we study the sizes of component groups and Tamagawa numbers of Néron models of Jacobians using matrix tree theorems from combinatorics. Raynaud gave a description of the component group of the special fiber of the Néron model of a Jacobian, in terms of the multiplicities and intersection numbers of components in the special fiber of a regular model of the underlying curve. Bosch and Liu used this description, along with some Galois cohomology computations to provide similar descriptions of Tamagawa numbers. We use various versions of the matrix-tree theorem from combinatorics to make Raynaud's and Bosch and Liu's descriptions more explicit in terms of the combinatorics of the dual graph and the action of the absolute Galois group of the residue field on it. We then use these explicit descriptions to provide a new geometric condition for obtaining a uniform bound on the size of the component group of a Jacobian. We also prove a certain periodicity property of the component group of a Jacobian under contraction of connecting chains of specified lengths in the dual graph. We also obtain an alternate proof of one of the key steps in Halle and Nicaise's proof of the rationality of the Néron component series for Jacobians.

Thesis Supervisor: Bjorn Poonen
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Acknowledgments

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Contents

Chapter 1

Introduction

Let R be a discrete valuation ring with fraction field K and residue field k. Let X be a nice (smooth, projective and geometrically intergral) K-variety. The variety X is said to have good reduction if there exists a smooth and proper R-scheme $\mathscr{X}$ whose generic fiber $\mathscr{X}_K$ is isomorphic to X. In his 1967 paper [Ogg67], Ogg proved that an elliptic curve E (a nice group variety of dimension 1) defined over K has good reduction if and only if the natural action of the inertia group I_K on the ℓ-adic Tate module of E is trivial. This criterion (the *Néron-Ogg-Shafarevich criterion*) was later generalized by Serre and Tate [ST68] to abelian varieties of arbitrary dimension. The non-triviality of the inertia action on the Tate module of an abelian variety is captured by the non-vanishing of a certain numerical invariant, called (the exponent of) the conductor of the abelian variety (See Chapter 2 for the definition). For an abelian variety defined over a number field, the local conductor at various primes appear in the conjectured functional equation for the L-function of the abelian variety.

Elliptic curves occupy a special place in the study of nice varieties, since they straddle the world of algebraic curves (nice varieties of dimension 1) and abelian varieties (nice group varieties). One can ask if the Néron-Ogg-Shafarevich criterion extends to curves of arbitrary genus $g \geq 2$, with $H^1(X_{\overline{K}}, \mathbb{Q}_\ell)$ in place of the the ℓ-adic Tate module, but this turns out to be false [Oda95]. The correct generalization of the conductor also takes into account the dimensions of the cohomology groups of the special fiber of the minimal proper regular model of the algebraic curve, and this is encoded in another numerical invariant called the Artin conductor (See Chapter 2 for the definition). The Artin conductor vanishes exactly when the minimal proper regular model of the curve is smooth over R. For a nice curve defined over a number field, the Artin conductor at various primes in turn appear in the conjectured functional equation for the L-function associated to a global regular model of X over the ring of integers of the number field [Blo87]. The (negative of the) Artin conductor is an upper bound for the conductor of the I_K-representation $H^1(X_{\overline{K}}, \mathbb{Q}_\ell)$, and the difference of the two conductors is one less than the number of the components in the special fiber of the minimal proper regular model of X.

Elliptic curves over K also have integral Weierstrass equations over R. An integral Weierstrass equation is an equation of the form $F(x,y,z) = y^2z + a_1xyz + a_3yz^2 + x^3 + a_2x^2z + a_4xz^2 + a_6z^3$ that cuts out the elliptic curve as a cubic hypersurface in $\mathbb{P}^2_K$, with $\{a_1, a_2, \ldots, a_6\} \subset R$. Any such equation has an associated (valuation of) discriminant[1] (See Chapter 2 for a precise definition), which is a non-negative integer that measures how far the corresponding closed subscheme of $\mathbb{P}^2_R$ is from being smooth over R. An integral Weierstrass equation that minimizes the value of the discriminant is called a minimal Weierstrass equation, and the corresponding discriminant is called the minimal discriminant. It can be shown that an elliptic curve over K has good reduction if and only if it has an integral Weierstrass equation with discriminant 0.

Since elliptic curves have these two measures of failure of good reduction, namely the Artin conductor and the minimal discriminant, it is quite natural to ask how these two invariants are related. In [Ogg67], Ogg showed that the minimal discriminant of an elliptic curve over K equals the (negative) of the Artin conductor. He attributes this to a result of Tate from 1960 in the case where $\operatorname{char} k \neq 2, 3$ (the *tame* case), and remarks that the results in his paper are essentially about filling in the two remaining cases. However, the arguments in his paper fall short of handling the mixed characteristic 2 case, i.e., when $\operatorname{char} K = 0$ and $\operatorname{char} k = 2$. The gap in his proof was finally filled in after 20 years by Saito in [Sai88]. The proof given by Ogg proceeds by a lengthy case by case analysis, and uses

[1]Our definitions of the discriminant and the conductor come from taking the valuation of the usual discriminant and conductor; the definitions we use are better suited to our situation since we are interested in studying the local behaviour at a *single* prime.

the Kodaira-Néron classification of the possible special fibers of regular models of elliptic curves. Why does it fail in mixed char 2? Has something to do with Newton's method of solving polynomial equations... Saito's result, on the other hand, is far more general, and holds for regular models of *arbitrary* curves, not just elliptic curves. Saito proved that the (negative of) the Artin conductor of a regular model $\mathscr{X}$ equals a certain discriminant that he defines in [Sai88], using a canonical isomorphism given by Deligne between pushforwards of powers of the relative dualizing sheaf of $\mathscr{X}$ over Spec R (See Chapter 2 for the definition). This canonical isomorphism has its roots in a relation in the Picard group of the Deligne-Mumford compactification $\overline{\mathcal{M}_g}$ of the moduli space of genus g curves that was first observed by Mumford [Mum77, Theorem 5.10]. This new discriminant, which we call the Deligne discriminant, coincides with the minimal discriminant in the case when $\mathscr{X}$ is the minimal proper regular model of an elliptic curve. In the case of genus 2 curves, Saito relates his result to an explicit formula given by Ueno for the Deligne discriminant when $\operatorname{char} k = 0$ or when $\operatorname{char} k > 6$, in terms of yet another notion of discriminant special to genus 2 curves, and data pertaining to the geometry of the special fiber of a minimal regular model for a genus 2 curve [Uen88]. The explicit classification of special fibers of regular models of genus 2 curves that was used by Ueno already has over 120 different types! For a general genus g curve, the Deligne discriminant is very hard to explicitly compute in practice.

For a general genus g hyperelliptic curve, it is possible to define a minimal discriminant that is quite similar to the minimal discriminant for elliptic curves (See Chapter 2 for the definition). In [Liu94], Liu proved that the minimal discriminant for a genus 2 hyperelliptic curve is an upper bound for the (negative of the) Artin conductor. For genus 2 curves, unlike elliptic curves, the minimal discriminant and the (negative of the) Artin conductor are sometimes different. In his paper, Liu gives an exact formula for the difference, that can be computed quite explicity in terms of the aforementioned classification of fibers of genus 2 curves when $\operatorname{char} k \neq 2$. When the hyperelliptic curve has semistable reduction over K, Kausz [Kau99] (when $\operatorname{char} k \neq 2$), and Maugeais [Mau03] (for all residue characteristics) prove theorems that relate the Deligne discriminant to yet another discriminant. We are not sure if the minimal discriminant that we define always coincides with the discriminant that is used by Kausz and Maugeais. One of the main results in this thesis is the following.

Theorem 1.0.1. *Let R be a discrete valuation ring with perfect residue field k. Assume that $\operatorname{char} k \neq 2$. Let K be the fraction field of R. Let K^{sh} denote the fraction field of the strict Henselization of R. Let C be a hyperelliptic curve over K of genus g. Let $\nu\colon K \to \mathbb{Z} \cup \{\infty\}$ be the discrete valuation on K. Assume that the Weierstrass points of C are K^{sh}-rational. Let $S = \operatorname{Spec} R$ and let $\mathcal{X}/S$ be the minimal proper regular model of C. Let $\nu(\Delta)$ denote the minimal discriminant of C. Then,*

$$-\operatorname{Art}(\mathcal{X}/S) \leq \nu(\Delta).$$

The method of proof is different from the one adopted by Liu in the case of genus 2 curves. Liu compares the Deligne discriminant of the minimal proper regular model and the minimal discriminant by comparing both of them to a third discriminant that he defines, that is specific to genus 2 curves following a definition given by Ueno [Liu94, p.56, Definition 1, p.52, Theoreme 1 and p.53, Theoreme 2].

We instead proceed by constructing an explicit proper regular model for the curve C (Section 3.1). We can immediately reduce to the case where R is a Henselian discrete valuation ring with algebraically closed residue field. We may then write a minimal Weierstrass equation for our curve of the form $y^2 - f(x)$ where f is a monic polynomial in $R[x]$ that splits completely. If the Weierstrass points of C specialize to distinct points of the special fiber, then the usual compactification of the plane curve $y^2 - f(x)$ in weighted projective space over R is already regular. In the general case, we iteratively blow up $\mathbb{P}^1_R$ until the Weierstrass points have distinct specializations. After a few additional blow-ups, we take the normalization of the resulting scheme in the function field of the curve C. This gives us a proper regular model for the curve C (Theorem 3.1.4) (not necessarily minimal).

We have the relation $-\operatorname{Art}(X/S) = n(X_s) - 1 + \tilde{f}$ for a regular model X of the curve C, where $n(X_s)$ is the number of components of the special fiber of X and $\tilde{f}$ is an integer that depends only on the curve C and not on the particular regular model chosen. This tells us that to bound $-\operatorname{Art}(X/S)$ for the minimal proper regular model from above, it suffices to bound $-\operatorname{Art}(X/S)$ for some regular model for the curve from above.

In Section 3.2, we give an explicit formula for the Deligne discriminant for the model we have constructed. After a brief interlude on dual graphs in Section 3.3, we restate the formula for the Deligne discriminant using dual graphs. This formula tells us that the Deligne discriminant decomposes as a sum of local terms, indexed by the vertices of the dual graph of the special fiber of the regular model we constructed (Section 3.4). In Section 3.5, we give a description of the rest of the strategy to prove the main theorem using this formula. The additional ingredients that are necessary are a decomposition of the minimal discriminant into a sum of local terms (Section 3.6) and explicit formulae for the local terms in the Deligne discriminant in terms of dual graphs (Section 3.7). In Section 3.8, we show how to compare the Deligne discriminant for the model we have constructed and the minimal discriminant locally. To finish the proof, we sum the inequalities coming from all the local terms to obtain $-\operatorname{Art}(X/S) \leq \nu(\Delta)$. As a corollary, we obtain upper bounds on the number of components in the special fiber of the minimal proper regular model (Corollary 3.8.8). This has applications to Chabauty's method of finding rational points on curves of genus at least 2 [PS14].

It might be possible to adapt the same strategy to extend the results to the case of non-rational Weierstrass points. The main difficulties in making this approach work are in understanding the right analogues of the results in Sections 3.6 and 3.7.

In the second half of this thesis, we study the component group scheme attached to the special fiber of the Néron model of a Jacobian. The Néron model of an abelian variety $\mathcal{A}$ is proper if and only if the abelian variety has good reduction. In general, the special fiber of the Néron model is an algebraic group that may not even be connected. The special fiber $\mathcal{A}_s$ admits a filtration

$$0 \to \mathcal{A}_s^0 \to \mathcal{A}_s \to \Phi \to 0,$$

where $\mathcal{A}_s^0$ is a connected group scheme and Φ is a finite étale group scheme, called the component group scheme. The component group is the set of points of the component group scheme over an algebraic closure, and the Tamagawa number is the size of the group $\Phi(k)$.

Computing the sizes of component groups and Tamagawa numbers have arithmetic applications. Bounds on the size of the component group can quite often be used to give bounds on the size of the torsion subgroup of $A(K)$ for an abelian variety A defined over a number field K. This fact was used by Mazur in his paper [Maz77], where he proved that the size of the torsion subgroup of $E(\mathbb{Q})$ for an elliptic curve $E/\mathbb{Q}$ is bounded above by 16. The local Tamagawa numbers are some of the invariants that appear in the statement of the full Birch and Swinnerton-Dyer conjecture; explicit verification of the full BSD conjecture for certain specific abelian varieties would require explicit computation of Tamagawa numbers.

In the special case where the abelian variety is the Jacobian of a nice K-curve X, there are multiple approaches for constructing the Néron model. Under relatively mild hypotheses, one can also construct the Néron model of a Jacobian using the theory of the relative Picard scheme of a proper, regular model of X over Spec R [BLR90, Chapter 9, Section 5, Theorem 4]. This method also leads to a description of the component group of the special fiber of the Néron model in terms of the multiplicities and intersection numbers of components of the special fiber of a proper, regular model of the curve [BLR90, Chapter 9, Section 6, Theorem 1]. The component group is given as the homology of a two term complex of free abelian groups, where the maps between the groups in the complex are given in terms of multiplicities and intersection numbers of the components in the special fiber of a regular model of $X \times K^{\mathrm{sh}}$, where K^{sh} is the fraction field of a strict henselization R^{sh} of R (Reference to 2). The free abelian groups in this complex also admit a natural action of the absolute Galois group G of k that commute with the maps in the complex; Bosch and Liu used this action to give a description of the Tamagawa number $\Phi(k)$, similar to the description of the component group Theorem 4.3.3, assuming that G is procyclic.

The multiplicities and the intersection numbers of components in the special fiber of a regular model can be encoded in a weighted graph, called the dual graph of the special fiber (Reference to 2). In Theorem 4.2.2, we give an explicit formula for the size of the component group that can be expressed in terms of the combinatorics of this dual graph, using the matrix-tree theorem 4.1.4. Formulas of this type for the component group are not new, and special cases are well-known; when the regular model is semi-stable, then the size of the component group equals the number of spanning trees in the dual graph, and when the dual graph is a tree, the size of the component group equals the product of the multiplicities of the components raised to certain exponents, which depend on the number of neighbours of the corresponding component in the dual graph. The formula in Theorem 4.2.2 can be viewed as a hybrid of the formula in these two special cases, where each spanning tree in the dual graph is assigned a weight, and the expression for the weight is formally similar to the expression for the size of the component group in the case when the dual graph is a tree. A version of this formula in fact appears embedded in [Lor89, p.489, Cor. 3.5]. Using a weighted version of the matrix-tree theorem 4.1.3, we obtain a similar formula for Tamagawa numbers expressed in terms of the combinatorics of a certain quotient graph.

Using our explicit formula in Theorem 4.2.2, we generalize the following fact about elliptic curves to curves of higher genus: the size of the component group for an elliptic curve having good/additive reduction is bounded above by 4. The condition of having good/additive reduction in genus 1 is replaced by a certain geometric condition on the -2 curves in the minimal regular model in higher genus (Theorem 4.2.8). To explain the nature of the condition that we impose for obtaining a uniform bound, we would first like to recall the following filtration on the connected component of the special fiber of the Néron model of any abelian variety $\mathcal{A}_s^0$.

$$0 \to U \times T \to \mathcal{A}_s^0 \to B \to 0.$$

In the filtration above, U is a unipotent algebraic group, T is a torus and B is an abelian variety. The dimensions of these groups are called the unipotent rank, toric rank and abelian rank respectively. The condition of having good/additive reduction for an elliptic curve is equivalent to the elliptic curve having vanishing toric rank. When the abelian variety is a Jacobian, the unipotent, abelian and toric ranks can be computed from the geometry of the special fiber of a regular model of the curve [Lor90, p.148]. In [Lor90], Lorenzini shows that there is a uniform bound on the size of component groups of Jacobians having toric rank 0. The condition that we impose is strictly weaker than requiring that $\mathcal{A}_s^0$ have toric rank 0, since we only disallow loops of -2 curves, and not loops of curves where at least one of the curves in the loop has geometric genus ≥ 1.

Even though the size of the component group for Jacobians is well-understood, the structure of the group remains quite mysterious. Inspired by the explicit formula and the explicit calculations of component groups for elliptic curves for the various Kodaira-Néron types, we prove a periodicity property of the component group under partial contraction of connecting chains (Theorem 4.2.10), generalizing [BN07, p.782, Corollary] from the case of unweighted graphs.

Néron models do not behave well with ramified extensions of the base: if (R', K') is a ramified extension of (R, K), the Néron model of $A \times_K K'$ does not in general equal $\mathcal{A} \times_R R'$. The Néron component series, which is a power series (Give reference in 2 for the definition) simultaneously records the changes in the size of the component group changes under all tamely ramified extensions of the base. Using our explicit formula (Theorem 4.2.2), we provide an alternate proof of the key step in Halle and Nicaise's proof of the rationality of the Néron component series for Jacobians, without having to resort to a reduction to the mixed characteristic case (Theorem 4.2.14).

Chapter 2

Background and Definitions

Chapter 3

Comparing conductors and discriminants

3.0.2 Notation

The invariants $-\operatorname{Art}(X/S)$ and $\nu(\Delta)$ are unchanged when we extend scalars to the strict Henselization. So from the very beginning, we let R be a Henselian discrete valuation ring with algebraically closed residue field k. Let K be its fraction field. Assume that $\operatorname{char} k \neq 2$. Let $\nu\colon K \to \mathbb{Z} \cup \{\infty\}$ be the discrete valuation on K. Let t be a uniformizer of R; $\nu(t) = 1$. Let $S = \operatorname{Spec} R$. Let C be a hyperelliptic curve over K with K-rational Weierstrass points and genus $g \geq 2$.

Let $y^2 - f(x) = 0$ be an integral Weierstrass equation for C, i.e., $f(x) \in R[x]$ and C is birational to the plane curve given by this equation. The discriminant of a Weierstrass equation d_f equals the discriminant of f considered as a polynomial of degree $2g + 2$. A minimal Weierstrass equation for C is a Weierstrass equation for C such that $\nu(d_f)$ is as small as possible amongst all integral Weierstrass equations for C. The minimal discriminant $\nu(\Delta)$ of C equals $\nu(d_f)$ for a minimal Weierstrass equation $y^2 - f(x)$ for C.

We will first show that we can find a minimal Weierstrass equation such that f is a monic, separable polynomial of degree $2g + 2$ in $R[x]$ that splits completely; $f(x) = (x - b_1)(x - b_2) \ldots (x - b_{2g+2})$ in $R[x]$. Let $y^2 - h(x)$ be any minimal Weierstrass equation for C. Let $H(x, z) = z^{2g+2} g(x/z)$. Choose a point $\tilde{P} \in \mathbb{P}^1(k)$ that is not a zero of H and let $P \in \mathbb{P}^1(R)$ be a lift of $\tilde{P}$; $P \bmod t = \tilde{P}$. Since $\mathrm{GL}_2(R)$ acts transitively on $\mathbb{P}^1(R)$, we can find $\varphi \in \mathrm{GL}_2(R)$ that sends P to $[1 : 0] \in \mathbb{P}^1_R$. Then, if $F(x, z) = \varphi \cdot H(x, z)$, then $F(x, 1)$

is of degree $2g+2$ and $u := F(1,0) \in R$ is a unit. Let $f(x) = u^{-1}F(x,1)$. Since $\operatorname{char} k \neq 2$ and R is Henselian with algebraically closed residue field, we can find a $u' \in R$ such that $u'^2 = u$. This tells us that by scaling y by u', we obtain a Weierstrass equation $y^2 - f(x)$ for C such that $f(x)$ is monic and separable of degree $2g+2$. Since $\det \varphi$ is a unit in R, and the discriminant of f differs from the discriminant of h by a power of $\det \varphi$, it follows that $y^2 - f(x)$ is a minimal Weierstrass equation for C. Fix such an equation.

For any proper regular curve Z over S, we will denote the special fiber of Z by Z_s, the generic fiber by Z_η and the geometric generic fiber by $Z_{\overline{\eta}}$. We will denote the fraction field of an integral scheme Z by $K(Z)$, the local ring at a point z of a scheme Z by $\mathcal{O}_{Z,z}$ and the unique maximal ideal in $\mathcal{O}_{Z,z}$ by $\mathfrak{m}_{Z,z}$. The reduced scheme attached to a scheme Z will be denoted Z_{red}.

3.1 Construction of the regular model

We first prove a lemma that gives sufficient conditions for the normalization of a regular 2-dimensional scheme in a degree 2 extension of its function field to be regular.

Lemma 3.1.1. *Let Y be a regular integral 2-dimensional scheme and let f be a rational function on Y that is not a square. Assume that the residue field at any closed point of Y is not of characteristic 2. (Weil divisors make sense on Y.) Let $(f) = \sum_{i \in I} m_i \Gamma_i$. Assume that*

(a) Any two Γ_i for which m_i is odd do not intersect.

(b) Any Γ_i for which m_i is odd is regular.

Then the normalization of Y in $K(Y)(\sqrt{f})$ is regular.

Proof. We will sketch the details of the proof. The construction of the normalization is local on the base. Therefore, it suffices to check that for every closed point y of Y, the normalization of the corresponding local ring $\mathcal{O}_{Y,y}$ in $K(Y)(\sqrt{f})$ is regular. There are two cases to consider.

The first case is when m_i is even for every Γ_i that contains y. In this case, since $\mathcal{O}_{Y,y}$ is a regular and hence a unique factorization domain, we can write $f = (c_1/c_2)^2 u$ for some $c_1, c_2 \in \mathcal{O}_{Y,y} \setminus \{0\}$ and a unit $u \in \mathcal{O}_{Y,y}$. Using the fact that 2 is a unit in $\mathcal{O}_{Y,y}$ for every y, a standard computation then shows that the normalization of $\mathcal{O}_{Y,y}$ in $K(Y)(\sqrt{f})$ is $\mathcal{O}_{Y,y}[z]/(z^2 - u)$. From this presentation, we conclude that the normalization is étale over $\mathcal{O}_{Y,y}$, and hence regular by [BLR90, p.49, Proposition 9].

The second case is when exactly one of the m_i is odd for the Γ_i that contain y. Let a be an irreducible element of the unique factorization domain $\mathcal{O}_{Y,y}$, corresponding to the unique Γ_i for which m_i is odd. In this case, $f = (c_1/c_2)^2 au$, where $c_1, c_2 \in \mathcal{O}_{Y,y} \setminus \{0\}$ and u is a unit in $\mathcal{O}_{Y,y}$ as before. One can then check that the normalization of $\mathcal{O}_{Y,y}$ in $K(Y)(\sqrt{f})$ is $\mathcal{O}_{Y,y}[z]/(z^2 - au)$. Since Γ_i is regular at y, we can find an element $b \in \mathcal{O}_{Y,y}$ such that a and b generate the maximal ideal of $\mathcal{O}_{Y,y}$. One can then check that z and b generate the unique maximal ideal of $\mathcal{O}_{Y,y}[z]/(z^2 - au)$. This implies that $\mathcal{O}_{Y,y}[z]/(z^2 - au)$ is regular. □

Remark 3.1.2. The construction of a regular model in Lemma 3.1.1 as the normalization of a regular scheme in a degree 2 extension of the function field, is special to hyperelliptic curves. A similar construction exists for tricyclic covers of the projective line, but it does not extend to other Galois covers [LL99].

In our example, $Y = \mathbb{P}^1_R$ and the rational function f is $(x - b_1)(x - b_2)\ldots(x - b_{2g+2})$. The divisor of f is just the sum of the irreducible principal horizontal divisors $(x - b_i)$, all appearing with multiplicity 1 in (f), and the divisor at ∞ (the closure of the point at ∞ on the generic fiber), with multiplicity $-(2g + 2)$. If the b_i belong to distinct residue classes modulo t, then the condition in the lemma is satisfied and we get the regular scheme $\operatorname{Proj} \frac{R[x,y,z]}{y^2 - z^{2g+2} f(x/z)}$. If some of the b_i belong to the same residue class, then the corresponding horizontal divisors would intersect at the closed point on the special fiber given by this residue class and we cannot apply the lemma directly with $Y = \mathbb{P}^1_R$. We will instead apply the lemma to the divisor of f on an iterated blow-up of $\mathbb{P}^1_R$. The generic fiber of this new Y is still $\mathbb{P}^1_K$, so the regular scheme that we obtain will still be a relative S-curve with generic fiber the hyperelliptic curve we started with.

We will need another lemma to show that we can resolve the issue discussed above by replacing $\mathbb{P}^1_R$ by an iterated blow-up of $\mathbb{P}^1_R$. The following lemma is a minor modification of [LL99, p.64, Lemma 1.4], where we consider irreducible divisors appearing in the divisor of an arbitrary rational function on a model (instead of the rational function t) and the order of vanishing of f along these divisors instead. We recover [LL99, p.64, Lemma 1.4] by taking f to be t.

Lemma 3.1.3. *Let Y/R be a regular model of a curve Y_η/K. Let f be a rational function on Y. Let C and D be irreducible divisors of Y that appear in the divisor of f, and let the order of vanishing of f along C and D be r_C and r_D respectively. Let $y \in Y$ be a closed point, and let Y' denote the model of Y_η obtained by blowing up Y at y. Let $E \subset Y'$ denote the exceptional divisor.*

(a) If y is a regular point of C that does not belong to any other irreducible divisor appearing in (f), then the order of vanishing of f along E equals r_C.

(b) If $y \in C \cap D$ and does not belong to any other divisors appearing in (f), and if C and D intersect transversally at y, then the order of vanishing of f along E is $r_C + r_D$.

Proof. Omitted. This can be seen using explicit equations of the blow-up in a neighbourhood of y. □

We are now ready to construct the regular model X of C. A very similar construction already appears in [Kau99] under some additional simplifying hypotheses. The model that is obtained there turns out to be semi-stable. The regular model X that is constructed below is not necessarily semi-stable.

Let D_i be the irreducible principal horizontal divisor $(x - b_i)$ on $\mathbb{P}^1_R$. First blow-up $\mathbb{P}^1_R$ at those closed points on the special fiber where any two of the D_i intersect to obtain a new scheme $\mathrm{Bl}_1(\mathbb{P}^1_R)$. On this scheme, the strict transforms of any two divisors D_i and D_j for which the b_i agree mod t and not mod t^2 will no longer intersect. If some of the b_i agree mod t^2 as well, then continue to blow-up (that is, now blow up $\mathrm{Bl}_1(\mathbb{P}^1_R)$ at the closed points on the special fiber of $\mathrm{Bl}_1(\mathbb{P}^1_R)$ where any two of the strict transforms of the divisors $(x - b_i)$ intersect, and call the result $\mathrm{Bl}_2(\mathbb{P}^1_R)$). Since the b_i are pairwise distinct, we will eventually end up with a scheme $\mathrm{Bl}_n(\mathbb{P}^1_R)$ where no two of the irreducible horizontal divisors occuring in (f) intersect. We may hope to set Y equal to $\mathrm{Bl}_n(\mathbb{P}^1_R)$, but the divisor of the rational function f might now vanish along some irreducible components of the special fiber.

Lemma 3.1.3 now tells us that a single blow-up of $\mathrm{Bl}_n(\mathbb{P}^1_R)$ based at a finite set of closed points will ensure that no two components where f vanishes to odd order intersect. Do this as well and call the resulting scheme Y. Call an irreducible component of the special fiber of Y even if the order of vanishing of f along this component is even. Similarly define odd component. Similarly define odd and even components of $\mathrm{Bl}_n(\mathbb{P}^1_R)$.

Recall the notion of a good model as defined in [LL99, p.66,1.8]. A regular model $Y/\mathcal{O}_K$ of Y_η/K is good if it satisfies the following two conditions:

(a) The irreducible components of Y_s are smooth.

(b) Each singular point of Y_s belongs to exactly two irreducible components of Y_s and these components intersect transversally.

The blow-up of a good model at a closed point is again a good model.

The model Y we have constructed is a good model of $\mathbb{P}^1_K$ as it is obtained using a sequence of blow-ups starting from the good model $\mathbb{P}^1_R$ of $\mathbb{P}^1_K$. The model $\mathrm{Bl}_n(\mathbb{P}^1_R)$ is the model we would get using [LL99, p.66, Lemma 1.9] if we start with the model $\mathbb{P}^1_R$ and the divisor (f) on it. Set X to be equal to the normalization of Y in $K(Y)(\sqrt{f})$.

Theorem 3.1.4. *The scheme X/S is regular.*

Proof. The components of Y_s are smooth and the divisor (f) satisfies the conditions in the statement of Lemma 3.1.1. It follows that X is regular. $\square$

We will now prove that X is a good model of C and compute the multiplicities of the components of the special fiber of X. Let the divisor of t on X be $\sum m_i\Gamma_i$; here the sum runs over all irreducible components of the special fiber X_s and the Γ_i are integral divisors on X. Let ψ denote the map $X \to \mathrm{Bl}_n(\mathbb{P}^1_R)$.

Lemma 3.1.5.

(a) *The scheme X is a good model of C.*

(b) *Each m_i is 1 or 2. Furthermore, $m_i = 2$ if and only if either*

(i) $\psi(\Gamma_i)$ is an odd component of $(\mathrm{Bl}_n(\mathbb{P}^1_R))_s$, or,

(ii) $\psi(\Gamma_i) = \Gamma \cap \Gamma'$ for two distinct odd components Γ and Γ' of $(\mathrm{Bl}_n(\mathbb{P}^1_R))_s$.

Proof.

(a) Let S be the set of odd components of Y_s and let B be the divisor $\sum_{\Gamma \in S} \Gamma + \sum_{i=1}^{2g+2} \overline{\{b_i\}}$ where $\overline{\{b_i\}}$ is the horizontal divisor that is the closure of the point b_i on the generic fiber $\mathbb{P}^1_K$. Since the map $X \to Y$ is finite of degree 2, the image of an irreducible component of X_s is an irreducible component of Y_s, and there are at most two irreducible components of X_s mapping down to an irreducible component of Y_s. All the irreducible components of Y_s are isomorphic to $\mathbb{P}^1_k$. There are two irreducible components of X_s mapping down to a given component of Y_s only when the component of Y_s is an even component that does not intersect any of the irreducible divisors appearing in B. In this case the two components in X_s that map down to the given component of Y_s do not intersect, and are isomorphic to $\mathbb{P}^1_k$. In all other cases there is a unique component of X_s mapping down to a component of Y_s.

Since at most two irreducible components of Y_s pass through any given point of Y_s, we see that this implies that at most two irreducible components of X_s pass through any given point of X_s. The intersection point x of two irreducible components of X_s has to map to the intersection point y of two irreducible components of Y_s. If y is the intersection of two even components, then the map ψ is etale at x, so the intersection is still transverse. If y is the intersection of an even and odd component, because the intersection of these components is transverse, we can pick the function g in the proof of Lemma 3.1.1 to be a uniformizer for the even component. This shows that étale locally, the two components that intersect at x are given by the vanishing of $\sqrt{t_j u}$ and g and as these two elements generate the maximal ideal at x étale locally, the intersection is transverse once again. For a closed point x on X_s lying on exactly one component *Gamma* of X_s, the same argument shows that we can choose a system of parameters at the point such that one of them cuts out the component Γ of X_s. This shows that the irreducible components of X_s are smooth.

(b) A repeated application of [LL99, p.64, Lemma 1.4] tells us that the multiplicity of every irreducible component of $(\mathrm{Bl}_n(\mathbb{P}^1_R))_s$ is 1. The same lemma tells us that Y_s has a few additional components of multiplicity either 1 or 2 - If we blow up the closed point that is the intersection of an odd component of the special fiber of $\mathrm{Bl}_n(\mathbb{P}^1_R)$ with a horizontal divisor appearing in (f), then we get a component of multiplicity 1 in the special fiber and if we blow up the intersection of two odd components of the special fiber, we get a component of multiplicity 2. Since f vanishes to an even order along components of multiplicity 2 in Y_s, each m_i is either 1 or 2 - It is 1 if Γ_i maps down to an even component of Y_s and its image in $(\mathrm{Bl}_n(\mathbb{P}^1_R))_s$ does not equal the intersection point of two components of the special fiber and it is 2 otherwise. This is because $\mathcal{O}_{Y,\eta(\psi(\Gamma_i))} \to \mathcal{O}_{X,\eta(\Gamma_i)}$ is an extension of discrete valuation rings (here $\eta(C)$ for an integral curve C denotes its generic point), and the corresponding extension of fraction fields is of degree 2. t is a uniformizer in $\mathcal{O}_{Y,\eta(\psi(\Gamma_i))}$, so its valuation above is either 1 or 2 depending on whether the extension is ramified at (t) or not. The extension is not ramified if the image of Γ_i in Y is an even component. □

3.2 An explicit formula for the Deligne discriminant

The Deligne discriminant of the model X is $-\operatorname{Art}(X/S) := -\chi(X_{\overline{\eta}})+\chi(X_s)+\delta$, where δ is the Swan conductor associated to the ℓ-adic representation $\mathrm{Gal}\ (\overline{K}/K) \to \mathrm{Aut}_{\mathbb{Q}_\ell}\ (H^1_{\mathrm{et}}(X_{\overline{\eta}}, \mathbb{Q}_\ell))$ $(\ell \neq \mathrm{char}\ k)$ [Sai88, p.153].

Lemma 3.2.1.

$$-\operatorname{Art}(X/S) = -\chi(X_{\overline{\eta}}) + \chi(X_s) = \sum_i \left((1-m_i)\chi(\Gamma_i) + \sum_{j\neq i}(m_j - 1)\Gamma_i.\Gamma_j \right) + \sum_{i<j} \Gamma_i.\Gamma_j.$$

Proof. Since all irreducible components of X_s have multiplicity either 1 or 2 in the special fiber and char $k \neq 2$, [Sai87, p.1044, Theorem 3] implies that $\delta = 0$.

Using the intersection theory for regular arithmetic surfaces, for a canonical divisor K

on X, we have

$$
\begin{aligned}
-\chi(X_{\overline{\eta}}) &= 2p_a(X_{\overline{\eta}}) - 2 \\
&= 2p_a(X_s) - 2 \\
&= X_s.(X_s + K) \\
&= X_s.K \quad \text{(because } X_s \text{ is a complete fiber, } X_s.X_s = 0) \\
&= \sum_i m_i \Gamma_i.K \\
&= \sum_i m_i(-\chi(\Gamma_i) - \Gamma_i.\Gamma_i) \quad \text{(by the adjunction formula applied to the divisor } \Gamma_i) \\
&= \sum_i \left(-m_i\chi(\Gamma_i) + \sum_{j\neq i} m_j \Gamma_j.\Gamma_i \right).
\end{aligned}
$$

The last equality is obtained from $X_s.\Gamma_i = 0$.

Let $\lambda\colon \sqcup \Gamma_i \to (X_s)_{\mathrm{red}}$ be the natural map which is just the inclusion of each Γ_i into $(X_s)_{\mathrm{red}}$. Since the Γ_i are smooth, [Lor90, p.151, Theorem 2.6] tells us that $\chi(X_s) = \chi((X_s)_{\mathrm{red}}) = -\delta_{X_s} + \sum \chi(\Gamma_i)$ where $\delta_{X_s} = \sum_{P\in(X_s)_{\mathrm{red}}}(|\lambda^{-1}(P)| - 1)$. In our case δ_{X_s} is just the number of points where two components of X_s meet. Since the intersections in X_s are all transverse,

$$\delta_{X_s} = \sum_{i<j} \Gamma_i.\Gamma_j = \sum_i \sum_{j\neq i} \Gamma_i.\Gamma_j - \sum_{i<j} \Gamma_i.\Gamma_j.$$

Putting all this together, we can rewrite $\chi(X_s)$ in the following form

$$\chi(X_s) = \sum_i \left(\chi(\Gamma_i) - \sum_{j\neq i} \Gamma_i.\Gamma_j \right) + \sum_{i<j} \Gamma_i.\Gamma_j.$$

This expression, together with the formula above for $-\chi(X_{\overline{\eta}})$ gives

$$-\operatorname{Art}(X/S) = \sum_i \left((1-m_i)\chi(\Gamma_i) + \sum_{j\neq i}(m_j - 1)\Gamma_i.\Gamma_j \right) + \sum_{i<j} \Gamma_i.\Gamma_j. \qquad \square$$

Remark 3.2.2. The formula

$$-\chi(X_{\overline{\eta}}) + \chi(X_s) = \sum_i \left((1-m_i)\chi(\Gamma_i) + \sum_{j\neq i}(m_j - 1)\Gamma_i.\Gamma_j \right) + \sum_{i<j} \Gamma_i.\Gamma_j$$

holds for any regular S curve X with smooth, projective, geometrically integral generic fiber and whose special fiber is a strict simple normal crossings divisor (i.e., the components themselves might have multliplicities bigger than 1, but each of the components is smooth, and the reduced special fiber has at worst nodal singularities). We also recover the result that if X/S is regular and semi-stable, then $-\operatorname{Art}(X/S) = \sum_{i<j} \Gamma_i.\Gamma_j$, since in this case $m_i = 1$ for all i and $\delta = 0$ by [Sai87, p.1044, Theorem 3].

3.3 Dual graphs

By the construction of X we have a sequence of maps $X \to Y \to \mathrm{Bl}_n(\mathbb{P}^1_R) \to \mathbb{P}^1_R$. Let T_X be the dual graph of X_s, i.e., the graph with vertices the irreducible components of X_s, and an edge between two vertices with an edge if the corresponding irreducible components intersect. Let T_Y be the dual graph of Y_s and T_B the dual graph of $(\mathrm{Bl}_n(\mathbb{P}^1_R))_s$. For a vertex v of any of the graphs T_X, T_Y or T_B, the irreducible component corresponding to the vertex in the respective dual graph will be denoted Γ_v. Let ψ_1 denote the map $X \to Y$ and let ψ_2 the map $Y \to \mathrm{Bl}_n(\mathbb{P}^1_R)$. Let $\psi = \psi_2 \circ \psi_1$.

We will denote the vertices of a graph G by $V(G)$. For any $v \in V(G)$, let $N(v)$ (for neighbours of v) denote the set of vertices w for which there is an edge between v and w. If G is a directed graph and $v \in V(G)$, let $C(v)$ (for children of v) denote the set of vertices w for which there is an edge pointing from v to w.

The graph T_B naturally has the structure of a rooted tree (remembering the sequence of blow-ups, i.e., whether the component was obtained as a result of a blow-up at a closed point of the other component). The graph T_Y is obtained from the graph of T_B by attaching some additional vertices between two pre-existing vertices connected by an edge and some additional leaves, so T_Y is also a tree. By virtue of being rooted trees, the edges of T_B and T_Y can be given a direction (and we choose the direction that points away from the root).

There is a natural surjective map $\varphi_1 : V(T_X) \to V(T_Y)$: if the image of an irreducible component $\Gamma_{v''}$ of X_s under ψ_1 is an irreducible component $\Gamma_{v'}$ of Y_s then let $\varphi_1(v'') = v'$. If two vertices of T_X are connected by an edge, so are their images in T_Y. We can use this surjection to transfer the direction on the edges of T_Y to the edges of T_X; this makes T_X a directed graph. Call a vertex of T_B odd (respectively even) if the order of vanishing of f along the corresponding component is odd (respectively even). Similarly define odd and even vertices of T_Y. This definition is consistent with the earlier definition of odd and even components of Y and $\mathrm{Bl}_n(\mathbb{P}^1_R)$.

3.4 Deligne discriminant and dual graphs

The last term $\sum_{i<j} \Gamma_i.\Gamma_j$ in the Deligne discriminant can be thought of as the sum $\sum_{v''\in V(T_X)} \Big(\sum_{w''\in C(v'')} \Gamma_{v'}$
. We use this observation to decompose the Deligne discriminant as a sum over the vertices of the graph T_X. Let $m_{v''}$ be the multiplicity of $\Gamma_{v''}$ in X_s. We then have

$$-\operatorname{Art}(X/S) = \sum_{v''\in V(T_X)} \left((1 - m_{v''})\ \chi(\Gamma_{v''}) + \sum_{w''\in N(v'')} (m_{w''} - 1)\Gamma_{v''}.\Gamma_{w''} + \sum_{w''\in C(v'')} \Gamma_{v''}.\Gamma_{w''} \right).$$

3.5 Description of the strategy

To compare the discriminant d_f of the polynomial f with the valuation of the Deligne discriminant of the model X, it would be useful if we could decompose d_f as a sum of local terms. In the next section, we will show that there is a way to decompose the minimal discriminant as a sum over the vertices of T_B. There is a simple relation between the irreducible components of X_s and those of $(\mathrm{Bl}_n(\mathbb{P}^1_R))_s$ (which we will describe below), so we will be able to compare the two discriminants using this decomposition, by first comparing them locally.

The image of an irreducible component of Y_s under ψ_2 is either an irreducible component of $(\mathrm{Bl}_n(\mathbb{P}^1_R))_s$ or a point that lies on exactly one of the irreducible components of $(\mathrm{Bl}_n(\mathbb{P}^1_R))_s$ or the intersection point of two irreducible components of $(\mathrm{Bl}_n(\mathbb{P}^1_R))_s$. This induces a surjective map $\varphi_2 : V(T_Y) \to V(T_B)$ where the vertex corresponding to an irreducible component of Y_s

is mapped either to the vertex corresponding to the unique irreducible component that its image is contained in or to the smaller of the two vertices (by which we mean the vertex closer to the root) corresponding to the two irreducible components that its image is contained in. Let $\varphi = \varphi_1 \circ \varphi_2$.

We have written the Deligne discriminant as $\sum_{v'' \in V(T_X)} \cdots$ and we can rewrite this sum as $\sum_{v \in V(T_B)} (\sum_{v'' \in V(T_X), \varphi(v'')=v} \cdots)$, so the Deligne discriminant can be regarded as a sum over the vertices of T_B.

The discussion above implies the following lemma, which will be useful later on in an explicit computation of the Deligne discriminant.

Lemma 3.5.1. *Let $v'' \in V(T_X)$.*

(a) *If $w'' \in C(v'')$, then $\varphi_1(w'') \in C(\varphi_1(v''))$. In particular, if $w'' \in N(v'')$, then $\varphi_1(w'') \in N(\varphi_1(v''))$.*

(b) *Let $w'' \in C(v'')$. If $\psi(\Gamma_{w''})$ is a point, then $\varphi(w'') = \varphi(v'')$ and $\varphi(v'')$ is an odd vertex. Otherwise, $\varphi(w'') \in C(\varphi(v''))$.*

3.6 A decomposition of the minimal discriminant

To each vertex v of T_B, we want to associate an integer $d(v)$ such that the minimal discriminant equals $\sum_{v \in V(T_B)} d(v)$. We will now define $d(v)$ by inducting on the vertices of T_B.

For the base case, note that if the b_i belong to distinct residue classes modulo t, then $\mathrm{Bl}_n(\mathbb{P}^1_R) = \mathbb{P}^1_R$ and T_B is the graph with a single vertex v. The minimal discriminant is 0, so we set $d(v) = 0$.

The scheme $\mathrm{Bl}_n(\mathbb{P}^1_R)$ was obtained as an iterated blow-up of $\mathbb{P}^1_R$ while trying to separate the horizontal divisors $(x - b_i)$ corresponding to the linear factors of f. This can be done for any separable polynomial $g \in R[x]$ that splits completely – let $\mathrm{Bl}(g)$ denote the iterative blow up of $\mathbb{P}^1_R$ that one obtains while trying to separate the divisors corresponding to the linear factors of g. With this notation $\mathrm{Bl}(f)$ equals the scheme $\mathrm{Bl}_n(\mathbb{P}^1_R)$ we had above.

Let A be the set of residues of the b_i modulo t. For a residue $a \in A$, let the weight of the residue a ($:= \mathrm{wt}_a$), be the number of b_i belonging to the residue class of a. Observe that the subtrees of the root of T_B are in natural bijection with the residues of weight strictly larger than 1.

The minimal discriminant $\nu(\Delta)$ $(= \nu(d_f))$ can be decomposed as follows:

$$\begin{aligned}\nu(d_f) &= \sum_{\substack{a\in A\\ \mathrm{wt}_a>1}} \nu\left(\prod_{\substack{b_i \bmod t\, =\, a\\ b_j \bmod t\, =\, a\\ i\neq j}} (b_i - b_j)\right)\\ &= \sum_{\substack{a\in A\\ \mathrm{wt}_a>1}} \nu\left(t^{\mathrm{wt}_a(\mathrm{wt}_a - 1)} \prod_{\substack{b_i \bmod t\, =\, a\\ b_j \bmod t\, =\, a\\ i\neq j}} \left(\frac{b_i - b_j}{t}\right)\right)\\ &= \sum_{a\in A} \mathrm{wt}_a(\mathrm{wt}_a - 1) + \sum_{\substack{a\in A\\ \mathrm{wt}_a>1}} \nu\left(\prod_{\substack{b_i \bmod t\, =\, a\\ b_j \bmod t\, =\, a\\ i\neq j}} \left(\frac{b_i - b_j}{t}\right)\right).\end{aligned}$$

Set $d(\text{root of } T_B) = \sum_{a\in A} \mathrm{wt}_a(\mathrm{wt}_a - 1)$. Pick an element b_i belong to the residue class $a \in A$ of weight strictly bigger than 1. The subtree corresponding to the residue a can naturally be identified with the dual graph of $\mathrm{Bl}(g_a)_s$ for the polynomial $g_a = \prod\limits_{b_j \bmod t\, =\, a} (x - \frac{b_j - b_i}{t})$. Let d_a denote the discriminant of g_a. Then,

$$\nu(d_f) = \sum_{a\in A} \mathrm{wt}_a(\mathrm{wt}_a - 1) + \sum_{a\in A} \nu(d_a).$$

Now recursively decompose $\nu(d_a)$ as a sum over the vertices of the dual graph of $\mathrm{Bl}(g_a)_s$. Identifying the dual graph of $\mathrm{Bl}(g_a)_s$ with the corresponding subtree in T_B, this gives us a way to decompose the minimal discriminant as a sum over the vertices of T_B.

We will now prescribe a way to attach weights to the vertices of T_B and give an explicit formula for $d(v)$ in terms of these weights.

3.6.1 Weight of a vertex

Suppose $v \in V(T_B)$. Let T_v be the complete subtree of T_B with root v. The complete subtree of T_B with root v has as its set of vertices all those vertices of T_B whose path to the root crosses v. There is an edge between two vertices in this subtree if there is an edge between them when considered as vertices of T_B.

For each vertex v of T_B, define the weight of the vertex wt_v as follows: Let J be the set of all irreducible components of $(\mathrm{Bl}_n\mathbb{P}^1_R)_s$ corresponding to the vertices that are in T_v. Let wt_v equal the total number of irreducible horizontal divisors that occur in the divisor (f) in $\mathrm{Bl}_n(\mathbb{P}^1_R)$, not counting the divisor $\overline{\{\infty\}}$, that intersect any of the irreducible components in J. Thus, if Γ_v was obtained as the exceptional divisor in the blow-up of an intermediate iterated blow-up Z between $\mathrm{Bl}_n(\mathbb{P}^1_R)$ and $\mathbb{P}^1_R$ at a smooth closed point of the special fiber $z \in Z_s$, then wt_v is exactly the number of irreducible horizontal divisors that occur in (f) that intersect Z_s at z. This in turn implies the following:

Lemma 3.6.2. *If $v \in V(T_B)$, then $\mathrm{wt}_v \geq 2$.*

3.6.3 Local contribution and weights

Lemma 3.6.4. *For any vertex v of T_B,*

$$d(v) = \sum_{w \in C(v)} \mathrm{wt}_w(\mathrm{wt}_w - 1).$$

Proof. This will once again proceed through an induction on the number of vertices of the tree. For the base case, note that the tree T_B has only one vertex if and only if all the roots of the polynomial f belong to distinct residue classes $\bmod t$ and in this case $d(v) = 0$. Now for the general case. It is clear that the equality holds for the root — for a residue class $a \in A$ such that $\mathrm{wt}_a > 1$, the weight of the residue class as in the definition is just the weight of the subtree corresponding to the residue class. For any vertex v at depth 1 (by which we mean one of the nearest neighbours of the root) corresponding to a residue class a such that $\mathrm{wt}_a > 1$, we first observe that the set of roots of the polynomial $g_a = \prod_{b_j \bmod t \,=\, a} (x - \frac{b_j - b_i}{t})$

corresponding to the residue class a is in natural bijection with a subset of the horizontal divisors of (f) – namely the ones corresponding to the strict transforms of the divisors $(x-b_j)$ on P^1_R for $b_j \bmod t = a$. These are the divisors that intersect the special fiber at one of the irreducible components corresponding to the vertices in this subtree with root v. These horizontal divisors are also in bijection with the horizontal divisors of the function g_a different from $\overline{\{\infty\}}$ on $\mathrm{Bl}(g_a)$. The identification of horizontal divisors of $\mathrm{Bl}(g_a)$ and a subset of the horizontal divisors of $\mathrm{Bl}(f)$ is compatible with the identification of the subtree of T_B with the dual graph of $\mathrm{Bl}(g_a)_s$. By this we mean that the set of horizontal divisors intersecting the irreducible component corresponding to any given vertex match up. This tells us that the weight of a vertex of the dual graph of $\mathrm{Bl}(g_a)_s$ equals the weight of the corresponding vertex in T_B. Since the lemma holds for the complete subtree at vertex v by induction (where the weights to the vertices of $\mathrm{Bl}(g_a)_s$ are assigned using the horizontal divisors of $\mathrm{Bl}(g_a)$), we are done. □

3.7 A combinatorial description of the local terms in the Deligne discriminant

The goal of this section is to obtain explicit formulae (Theorem 3.7.22) for the local terms appearing in the Deligne discriminant in terms of the combinatorics of the tree T_B (Definition 3.7). This involves a careful analysis of the special fiber of X which we present as a series of lemmas.

Lemma 3.7.1.

(a) *The branch locus of the double cover $\psi_1 : X \to Y$ is the set of all odd components of Y_s along with the strict transforms of the horizontal divisors $(x - b_i)$ on $\mathbb{P}^1_R$.*

(b) *If Γ is an even component of Y_s and Γ' is an irreducible component of the branch locus that intersects Γ, then $\Gamma.\Gamma' = 1$.*

Proof.

(a) This is clear from the construction of X as outlined in Lemma 3.1.1.

(b) From (a), it follows that Γ does not belong to the branch locus and Γ' is either an odd component of Y_s or the strict transform of the horizontal divisor $(x - b_i)$ on $\mathbb{P}^1_R$ for some b_i.

Suppose Γ' is an odd component of Y_s. It follows from the construction of Y that if any two irreducible components of Y_s intersect, then they intersect transversally and there is at most one point in the intersection. This implies that $\Gamma.\Gamma' = 1$.

Suppose Γ' is the strict transform of the horizontal divisor $(x - b_i)$ on $\mathbb{P}^1_R$ for some b_i. Let $\pi : Y \to \mathbb{P}^1_R$ be the iterated blow-up map that we obtain from the construction of Y. Since π is an iterated blow-up morphism, $\operatorname{Pic}\mathbb{P}^1_R$ is a direct summand of $\operatorname{Pic} Y$, with a canonical projection map $\pi_* : \operatorname{Pic} Y \to \operatorname{Pic}\mathbb{P}^1_R$. Let B_i denote the Weil divisor $(x - b_i)$ on $\mathbb{P}^1_R$. Then $\pi_*\Gamma' = B_i$.

$$0 < \Gamma.\Gamma' \leq Y_s.\Gamma' = \pi^*(\mathbb{P}^1_R)_s.\Gamma' = (\mathbb{P}^1_R)_s.(\pi_*\Gamma') = (\mathbb{P}^1_R)_s.B_i = 1.$$

This implies that $\Gamma.\Gamma' = 1$. $\square$

Lemma 3.7.2. *Let $v \in V(T_B)$ and $w \in C(v)$. Then $w(f) = v(f) + \mathrm{wt}_w$. (Here $v(f)$ and $w(f)$ denote the valuation of f in the discrete valuation rings corrresponding to the irreducible divisors Γ_v and Γ_w of $\mathrm{Bl}_n(\mathbb{P}^1_R)$). In particular, if v is even, then w is odd if and only if wt_w is odd; if v is odd, then w is odd if and only if wt_w is even.*

Proof. The scheme $\mathrm{Bl}_n(\mathbb{P}^1_R)$ was constructed as an iterated blow-up of $\mathbb{P}^1_R$. There exist intermediate iterated blow-ups Z' and Z of $\mathbb{P}^1_R$ with iterated blow-up maps $\mathrm{Bl}_n(\mathbb{P}^1_R) \to Z'$, $Z' \to Z$ and $Z \to \mathbb{P}^1_R$ such that

(a) The scheme Z' is the blow-up of Z at a smooth closed point z of the special fiber Z_s.

(b) The divisor $\Gamma_v \subset \mathrm{Bl}_n(\mathbb{P}^1_R)$ is the strict transform of a vertical divisor D on Z under the morphism $\mathrm{Bl}_n(\mathbb{P}^1_R) \to Z$.

(c) $z \in D$.

(d) The divisor $\Gamma_w \subset \mathrm{Bl}_n(\mathbb{P}^1_R)$ is the strict transform of E under the morphism $\mathrm{Bl}_n(\mathbb{P}^1_R) \to Z'$, where E denotes the exceptional divisor of $Z' \to Z$.

The valuation of f along E equals the multiplicity $\mu_z(f)$ (that is, the largest integer m such that $f \in \mathfrak{m}_{Z,z}^m \setminus \mathfrak{m}_{Z,z}^{m+1}$). There are wt_w distinct irreducible horizontal divisors of (f) that intersect Z_s at z, and z is a smooth point on each of these divisors. This in particular implies that a uniformizer for each of the corresponding discrete valuation rings is in $\mathfrak{m}_{Z,z} \setminus \mathfrak{m}_{Z,z}^2$. From the factorization of f and the fact that $\mathcal{O}_{Z,z}$ is a regular local ring (in particular, a unique factorization domain), one can deduce that $w(f) = \mu_z(f) = v(f) + \mathrm{wt}_w$. This implies that $w(f)$ and wt_w have the same parity if $v(f)$ is even and have opposite parity if $v(f)$ is odd. □

Definition. Suppose $v \in V(T_B)$. Let r_v be the total number of children of v of odd weight, and let s_v be the total number of children of v of even weight. Let l'_v equal the number of horizontal divisors of (f) different from $\overline{\{\infty\}}$ passing through Γ_v and let $l_v = l'_v + r_v$. For a vertex v of T_B (or of T_Y) not equal to the root, let p_v denote the parent of v.

Since $\mathrm{Bl}_n(\mathbb{P}^1_R)$ was obtained by iteratively blowing up a regular scheme at smooth rational points on the special fiber, all the components of its special fiber are isomorphic to $\mathbb{P}^1_k$ and X_s is reduced. Similarly, all the components of the special fiber of Y are also isomorphic to $\mathbb{P}^1_k$, though Y_s may no longer be reduced.

Lemma 3.7.3. *Let $v \in V(T_B)$ be an even vertex. Then l_v is odd if and only if v has an odd parent. In particular, if v is the root, then l_v is even.*

Proof. Suppose $v \in V(T_B)$ is even. Then $\psi_2^{-1}(\Gamma_v)$ is a single irreducible component F of Y_s and ψ_2 is an isomorphism above a neighbourhood of Γ_v. Using Lemma 3.7.1(b) and the Riemann-Hurwitz formula, we see that the branch locus of ψ_1 has to intersect F at an even number of points. Since v is even, Lemma 3.7.1(a) and Lemma 3.7.2 imply that F intersects the branch locus at $l_v + 1$ points if v has an odd parent, and at l_v points otherwise. □

Lemma 3.7.4. *A component of Y_s is odd if and only if it is the strict transform of an odd component of $(\mathrm{Bl}_n(\mathbb{P}^1_R))_s$.*

Proof. The exceptional divisors that arise when we blow up $\mathrm{Bl}_n(\mathbb{P}^1_R)$ to obtain Y are all even by Lemma 3.1.3, as every point that is blown up in $\mathrm{Bl}_n(\mathbb{P}^1_R)$ is at the intersection of two odd components. □

Lemma 3.7.5.

(a) *Let* $v' \in V(T_Y)$. *Then* $\#\varphi_1^{-1}(v') = 1$ *if* $\Gamma_{v'}$ *intersects the branch locus of* ψ_1, *and* $\#\varphi_1^{-1}(v') = 2$ *otherwise. If* $\#\varphi_1^{-1}(v') = 2$, *then both irreducible components of* X_s *corresponding to vertices in* $\varphi_1^{-1}(v')$ *are isomorphic to* $\mathbb{P}^1_k$.

(b) *Suppose* $v \in V(T_B)$ *is an even vertex. Then* $\#\varphi^{-1}(v)$ *is either* 1 *or* 2. *It is* 1 *if and only if* $\psi_2^{-1}(\Gamma_v)$ *intersects the branch locus of* ψ_1. *If* $\#\varphi^{-1}(v) = 2$, *then both irreducible components of* X_s *corresponding to vertices in* $\varphi^{-1}(v)$ *are isomorphic to* $\mathbb{P}^1_k$.

(c) *Suppose* $v \in V(T_B)$ *is odd. Let* $v' \in V(T_Y)$ *be the vertex corresponding to the strict transform of* Γ_v *in* Y. *Let*

$$T_0 = \{v'\},$$

$$T_1 = \{u' \in \varphi_2^{-1}(v) \mid \psi_2(\Gamma_{u'}) = \Gamma_v \cap \Gamma_u \text{ for some odd } \ u \in C(v)\}, \text{ and,}$$

$$T_2 = \left\{ u' \in \varphi_2^{-1}(v) \;\middle|\; \begin{array}{l} \psi_2(\Gamma_{u'}) = \Gamma_v \cap H \text{ for some irreducible horizontal divisor} \\ H \neq \overline{\infty} \text{ appearing in the divisor of } (f). \end{array} \right\}$$

Let $S_0 = \varphi_1^{-1}(T_0), S_1 = \varphi_1^{-1}(T_1)$ *and* $S_2 = \varphi_1^{-1}(T_2)$. *Then*

(i) The sets T_0, T_1 *and* T_2 *form a partition of* $\varphi_2^{-1}(v)$. *Hence* $\{S_0, S_1, S_2\}$ *is a partition of* $\varphi^{-1}(v)$.

(ii) We have that $\#S_0 = \#T_0 = 1$. *Suppose* $S_0 = \{\tilde{v}\}$. *Then* v' *is odd,* $m_{\tilde{v}} = 2$, *and*

$$S_0 = \{v'' \in \varphi^{-1}(v) \mid \ \psi(\Gamma_{v''}) \text{ is not a point}\}.$$

(iii) We have that $\#S_1 = \#T_1 = s_v$. *If* $v'' \in S_1$, *then* $m_{v''} = 2$. *If* $u' \in T_1$, *then* u' *is not a leaf in* T_Y.

(iv) We have that $\#S_2 = \#T_2 = l'_v$. *If* $v'' \in S_2$, *then* $m_{v''} = 1$.

(v) We have that

$$T_2 = \{u' \in \varphi^{-1}(v) \mid u' \text{ is an even leaf of } T_Y\}.$$

(vi) The map φ_1 induces an isomorphism of graphs between $\varphi_2^{-1}(v)$ and $\varphi^{-1}(v)$.

(vii) The graph $\varphi_2^{-1}(v)$ is a tree with root v' and the graph $\varphi^{-1}(v)$ is a tree with root $\psi_1^{-1}(\Gamma_{v'})$.

(viii) If $v'' \in \varphi^{-1}(v)$, then $\Gamma_{v''} \cong \mathbb{P}^1_k$.

Proof.

(a) All the components of Y_s are isomorphic to $\mathbb{P}^1_k$. Let $v' \in V(T_Y)$. The vertices in $\varphi_1^{-1}(v')$ are the irreducible components of $\psi_1^{-1}(\Gamma_{v'})$. If v' is even, then Lemma 3.7.1(b) tells us that if $\Gamma_{v'}$ intersects the branch locus at all, it intersects it transversally. Since ramified double covers of $\mathbb{P}^1_k$ are irreducible, $\psi_1^{-1}(\Gamma_{v'})$ is irreducible if $\Gamma_{v'}$ intersects the branch locus. If $\Gamma_{v'}$ does not intersect the branch locus, as $\mathbb{P}^1_k$ has no connected unramified double covers, we see that $\psi_1^{-1}(\Gamma_{v'})$ has two irreducible components, both of which are isomorphic to $\mathbb{P}^1_k$. This implies that $\#\varphi_1^{-1}(v')$ is 1 if $\Gamma_{v'}$ intersects the branch locus of ψ_1 and is 2 otherwise.

(b) Suppose $v \in V(T_B)$ is even. Then $\psi_2^{-1}(\Gamma_v)$ is a single irreducible component F of Y_s and ψ_2 is an isomorphism above a neighbourhood of Γ_v. Let $v' \in V(T_Y)$ be such that $\Gamma_{v'} = F$. Then $\varphi_2^{-1}(v) = \{v'\}$ and $\varphi^{-1}(v) = \varphi_1^{-1}(v')$. Apply (a) to v'.

(c) (i) The component $\Gamma_{v'}$ of Y_s satisfies $\psi_2(\Gamma_{v'}) = \Gamma_v$ and it is the only component of Y_s with this property. It follows that $\varphi_2(v') = v$. The other components $\Gamma_{u'}$ of Y_s satisfying $\varphi_2(u') = v$ are the exceptional divisors of $\psi_2 : Y \to \mathrm{Bl}_n(\mathbb{P}^1_R)$ that get mapped to a point of Γ_v that does not also lie on Γ_{p_v}. Since Y is the blow-up of $\mathrm{Bl}_n(\mathbb{P}^1_R)$ at the finite set of points consistsing of the intersection of any two odd components of the special fiber and the intersection of an odd component of the special fiber with an irreducible horizontal divisor $H \neq \overline{\infty}$ appearing in (f), it follows that $\{T_0, T_1, T_2\}$ is a partition of $\varphi_2^{-1}(v)$. Since $\varphi^{-1}(v) = \varphi_1^{-1}(\varphi_2^{-1}(v))$, it follows that $\{S_0, S_1, S_2\}$ is a partition of $\varphi^{-1}(v)$.

(ii) Lemma 3.7.4 tells us $\Gamma_{v'}$ is odd, and Lemma 3.7.1(a) tells us that ψ_1 is ramified over $\Gamma_{v'}$ and therefore $\psi_1^{-1}(\Gamma_{v'})$ is irreducible, and isomorphic to $\mathbb{P}^1_k$. It follows that

$\#S_0 = \#T_0 = 1$. Since $\psi(\Gamma_{\tilde{v}}) = \psi_2(\Gamma_{v'}) = \Gamma_v$ and v is odd, Lemma 3.1.5(b) tells us that $m_{\tilde{v}} = 2$.

Since $\psi(\Gamma_{\tilde{v}}) = \Gamma_v$, it follows that $\psi(\Gamma_{\tilde{v}})$ is not a point. Conversely, suppose $v'' \in \varphi^{-1}(v)$ and $\psi(\Gamma_{v''})$ is not a point. Since $\{T_0, T_1, T_2\}$ is a partition of $\varphi_2^{-1}(v)$ by (a) and $\psi_2(\Gamma_{u'})$ is a point for $u' \in T_1 \cup T_2$, it follows that $v'' \in \varphi_1^{-1}(T_0) = S_0$.

(iii) For every odd $u \in C(v)$, there exists a unique exceptional curve E of the blow-up $Y \to \mathrm{Bl}_n(\mathbb{P}^1_R)$ such that if $u' \in V(T_Y)$ is the vertex such that $\Gamma_{u'} = E$, then $u' \in \varphi_2^{-1}(v)$ and $\psi_2(\Gamma_{u'}) = \Gamma_v \cap \Gamma_u$. This shows that

$$\#T_1 = \#\text{odd children of v} = s_v \text{ (by Lemma 3.7.2 since } v \text{ is odd).}$$

Suppose $u' \in T_1$. Let $w \in C(v)$ be an odd vertex such that $\psi_1(\Gamma_{u'}) = \Gamma_v \cap \Gamma_w$. Let $w' \in V(T_Y)$ be the vertex corresponding to the strict transform of Γ_w in Y. Then $u' \in C(v')$ and $w' \in C(u')$. In particular, u' is not a leaf. Since v' is odd, Lemma 3.7.1(a) and part (a) applied to u' imply that $\#\varphi_1^{-1}(u') = 1$. This tells us that $\#S_1 = \#T_1 = s_v$.

Suppose $v'' \in S_1$. Since v is odd and $\varphi_1(v'') \in T_1$, Lemma 3.1.5(b) implies that $m_{v''} = 2$.

(iv) For every irreducible horizontal divisor $H \neq \overline{\infty}$ appearing in the divisor of (f) on $\mathrm{Bl}_n(\mathbb{P}^1_R)$, there exists a unique exceptional curve E of the blow-up $Y \to \mathrm{Bl}_n(\mathbb{P}^1_R)$ such that if $u' \in V(T_Y)$ is the vertex such that $\Gamma_{u'} = E$, then $u' \in \varphi_2^{-1}(v)$ and $\psi_2(\Gamma_{u'}) = \Gamma_v \cap H$. This shows that

$$\#T_2 = \#\left\{ \begin{array}{l} \text{irreducible horizontal divisors } H \neq \overline{\infty} \text{ appearing in} \\ (f) \text{ on } \mathrm{Bl}_n(\mathbb{P}^1_R) \text{ that intersect } \Gamma_v \end{array} \right\} = l'_v.$$

Suppose $u' \in T_2$. Then $u' \in C(v')$. Since v' is odd, Lemma 3.7.1(a) and part (a) applied to u' imply that $\#\varphi_1^{-1}(u') = 1$. This tells us that $\#S_2 = \#T_2 = l'_v$.

Suppose $v'' \in S_2$. Then $\varphi_1(v'') \in T_2$. This implies that $\psi(\Gamma_{v''})$ is a point lying on a unique odd component of $(\mathrm{Bl}_n(\mathbb{P}^1_R))_s$, namely Γ_v. Lemma 3.1.5(b) implies that

$m_{v''} = 1$.

(v) We already observed that v' is the unique vertex of T_0 and that it is odd (by Lemma 3.7.4). If $u' \in T_1$, then (iii) implies that u' is not a leaf. This shows

$$\{u' \in \varphi^{-1}(v) \mid u' \text{ is an even leaf of } T_Y\} \subset T_2.$$

If $u' \in T_2$, then Lemma 3.7.4 implies that u' is even. Since $\Gamma_{u'}$ is the exceptional curve that is obtained by blowing up the point of intersection of an odd component and a horizontal divisor, u' is a leaf. This shows the opposite inclusion.

(vi) Parts (ii),(iii),(iv) imply that $\#S_0 = \#T_0, \#S_1 = \#T_1$ and $\#S_2 = \#T_2$. Since φ_1 is a surjection and $\{T_0, T_1, T_2\}$ is a partition of $\varphi_2^{-1}(v)$, it follows that φ_1 induces a bijection between $\varphi^{-1}(v)$ and $\varphi_2^{-1}(v)$.

If $u' \in T_1 \cup T_2$, let $u'' \in \varphi^{-1}(v)$ be the unique vertex such that $\varphi_1(u'') = u'$. Let $\{\tilde{v}\} = S_0$. If $u' \in T_1 \cup T_2$, then $u' \in C(v')$.

If $u' \in T_1 \cup T_2$, then $\Gamma_{\tilde{v}} \cap \Gamma_{u''} = \psi_1^{-1}(\Gamma_{v'} \cap \Gamma_{u'}) \neq \emptyset$. This implies that $u'' \in N(\tilde{v})$ for any $u'' \in S_1 \cup S_2$. If $\tilde{v} \in C(u'')$ for some $u'' \in S_1 \cup S_2$, then Lemma 3.5.1(a) would imply $v' \in C(u')$. Since $u' \in C(v')$, it follows that $u'' \in C(\tilde{v})$.

If $u'_1, u'_2 \in T_1 \cup T_2$, then $\Gamma_{u'_1} \cap \Gamma_{u'_2} = \emptyset$. It now follows from Lemma 3.5.1(a) and the fact that $\varphi_1(u''_1), \varphi_1(u''_2) \in T_1 \cup T_2$ that if $u''_1, u''_2 \in S_1 \cup S_2$, then $\Gamma_{u''_1} \cap \Gamma_{u''_2} = \emptyset$.

Combining the previous three paragraphs, we get that φ_1 induces an isomorphism of graphs between $\varphi^{-1}(v)$ and $\varphi_2^{-1}(v)$.

(vii) The proof of (vi) shows that if $u' \in T_1 \cup T_2$, then $u' \in C(v')$ and that if $u'_1, u'_2 \in T_1 \cup T_2$, then $\Gamma_{u'_1}$ and $\Gamma_{u'_2}$ do not intersect. It follows that $\varphi_2^{-1}(v)$ is a tree with root v'. Since (vi) shows φ_1 induces an isomorphism of graphs between $\varphi^{-1}(v)$ and $\varphi_2^{-1}(v)$, it follows that $\varphi^{-1}(v)$ is a tree with root $\psi_1^{-1}(\Gamma_{v'})$.

(viii) We already observed in the proof of (ii) that if $\{\tilde{v}\} = S_0$, then $\Gamma_{\tilde{v}} \cong \mathbb{P}^1_k$.

Suppose $u'' \in S_1$. Let $u' = \varphi_1(u'')$. Then $u' \in T_1$. Let $w \in C(v)$ be an odd vertex such that $\psi_2(\Gamma_{u'}) = \Gamma_v \cap \Gamma_w$. Let w' be the vertex corresponding to the strict transform of Γ_w in Y. Then from the construction of Y, it follows that

$N(u') = \{v', w'\}$, $u' \in C(v')$ and $w' \in C(u')$. Lemma 3.7.4 implies that v' and w' are odd and u' is even. Since $\Gamma_{u'} \cong \mathbb{P}^1_k$ and $\Gamma_{u'}$ intersects the branch locus transversally at two points (the points of intersection with $\Gamma_{v'}$ and $\Gamma_{w'}$) by Lemma 3.7.1(a,b), the Riemann-Hurwitz formula implies that $\Gamma_{u''} = \psi_1^{-1}(\Gamma_{u'}) \cong \mathbb{P}^1_k$.

Suppose $u'' \in S_2$ and $u' = \varphi_1(u'')$. Then $u' \in T_2$. Like in the previous paragraph, we can argue that $\Gamma_{u'}$ intersects the branch locus at exactly two points, corresponding to the point of intersection of $\Gamma_{u'}$ with its odd parent $\Gamma_{v'}$ and the point of intersection of $\Gamma_{u'}$ with an irreducible horizontal divisor $H \neq \overline{\infty}$ appearing in the divisor of (f), and that these intersections are transverse. The Riemann-Hurwitz formula would once again imply $\Gamma_{u''} \cong \mathbb{P}^1_k$. Since (vi) implies that $\{S_0, S_1, S_2\}$ is a partition of $\varphi^{-1}(v)$, this completes the proof. □

We have the following restatement of Lemma 3.1.5(b) using φ and φ_1.

Lemma 3.7.6. *Suppose $v'' \in V(T_X)$. Then $m_{v''} = 2$ if and only if $\varphi(v'')$ is odd and $\varphi_1(v'')$ is not an even leaf. In particular, if $\varphi(v'')$ is even, then $m_{v''} = 1$.*

Proof. Lemma 3.1.5(b) tells us that $m_{v''} = 2$ if and only if $\psi(\Gamma_{v''})$ is an odd component, or, if $\psi(\Gamma_{v''}) = \Gamma_v \cap \Gamma_w$ for two odd vertices $v, w \in V(T_B)$. Let $v = \varphi(v'')$. If either of the conditions above hold, it follows from the definition of φ that the vertex v is odd. So now assume v is odd. Let $\{S_0, S_1, S_2\}$ be the partition of $\varphi^{-1}(v)$ as in Lemma 3.7.5(c). Lemma 3.7.5(c)(ii,iii,iv) imply that $m_{v''} = 2$ if and only if $v'' \notin S_2$. Lemma 3.7.5(c)(v) then tells us that $v'' \notin S_2$ if and only if $\varphi_1(v'')$ is not an even leaf.

Putting all this together, we get that $m_{v''} = 2$ if and only if $\varphi(v'')$ is odd and $\varphi_1(v'')$ is not an even leaf. □

Lemma 3.7.7.

(a) *Suppose $u'' \in V(T_X)$ and $\psi(\Gamma_{u''})$ is a point.*

   (i) We have that $\#N(u'') = 1$ if $\psi(\Gamma_{u''})$ belongs to a unique odd component of $(\mathrm{Bl}_n(\mathbb{P}^1_R))_s$, and $\#N(u'') = 2$ otherwise.

   (ii) If $\#N(u'') = 1$, then $\#C(u'') = 0$. If $\#N(u'') = 2$, then $\#C(u'') = 1$.

(iii) If $w'' \in N(u'')$, then $\varphi(w'')$ is an odd vertex.

(iv) If $w'' \in N(u'')$, then $m_{w''} = 2$.

(b) *Suppose $u'' \in V(T_X)$, $w'' \in N(u'')$, $\varphi(u'')$ is odd and $\varphi(w'')$ is even. Then $\psi(\Gamma_{u''})$ is not a point, and the component $\Gamma_{u''}$ is the inverse image under ψ_1 of the strict transform of $\Gamma_{\varphi(u'')}$.*

Proof.

(a) Let $v = \varphi(u'')$. Since $\psi(\Gamma_{u''})$ is a point, v is odd. Construct the partition S_0, S_1, S_2 of $\varphi^{-1}(v)$ as in Lemma 3.7.5(c). Since $\psi(\Gamma_{u''})$ is a point, Lemma 3.7.5(c)(ii) implies that $u'' \in S_1 \cup S_2$.

If $u'' \in S_1$, then $\psi(\Gamma_{u''}) = \Gamma_v \cap \Gamma_w$ for an odd vertex $w \in V(T_B)$. Let v', w' be the vertices in T_Y corresponding to the strict transforms of Γ_v and Γ_w respectively. Since v and w are odd, Lemma 3.7.4 tells us that v' and w' are odd. Then $N(\varphi_1(u'')) = \{v', w'\}$. By Lemma 3.7.5(a), the vertices $v', \varphi_1(u''), w'$ of T_Y each have exactly one preimage under under φ_1. Let $v'', w'' \in V(T_X)$ such that $\varphi_1(v'') = v'$ and $\varphi_1(w'') = w'$. The unique point $\Gamma_{v'} \cap \Gamma_{\varphi_1(u'')}$ has exactly one preimage under ψ_1 and therefore lies on both $\Gamma_{v''}$ and $\Gamma_{u''}$. Similarly, $\Gamma_{u''} \cap \Gamma_{w''}$ is nonempty. Lemma 3.5.1(a) now tells us that $N(u'') = \{v'', w''\}$. This implies that $\#N(u'') = 2$ and $\#C(u'') = 1$. We also have $\varphi(v'') = v$ and $\varphi(w'') = w$, and both v and w are odd vertices. Since $\varphi(v'')$ is odd and $\varphi_1(v'') = v'$ is odd, Lemma 3.7.6 tells us that $m_{v''} = 2$. Similarly, we can show $m_{w''} = 2$.

If $u'' \in S_2$, then Lemma 3.7.5(c)(v) implies that $u' := \varphi_1(u'')$ is an even leaf of T_Y. Lemma 3.7.5(c)(vii) shows u' has a parent. Let $v' = p_{\varphi_1(u'')}$ and $v = \varphi_2(v')$. Lemma 3.7.5(c)(ii,vii) imply that v' is an odd vertex corresponding to the strict transform of Γ_v in Y, and $\#\varphi_1^{-1}(v') = 1$. Let $v'' \in V(T_X)$ be such that $\varphi_1(v'') = v'$. Then the unique point in $\Gamma_{v'} \cap \Gamma_{u'}$ has exactly one preimage under ψ_1 and this preimage is contained in $\Gamma_{v''} \cap \Gamma_{u''}$. Lemma 3.5.1 now tells us that $\#N(u'') = 1$ and $\#C(u'') = 0$. Lemma 3.7.4 implies that $\varphi(v'') = \varphi_2(v') = v$ is odd. Since $\varphi(v'') = v$ is odd and $\varphi_1(v'') = v'$ is also odd, Lemma 3.7.6 implies that $m_{v''} = 2$.

The definitions of T_1, T_2, S_1, S_2 in Lemma 3.7.5(c) show that the vertices in S_1 are exactly the ones corresponding to irreducible components of X_s whose images under ψ are contained in two odd components of $(\mathrm{Bl}_n(\mathbb{P}^1_R))_s$ and the vertices in S_2 are the ones corresponding to irreducible components of X_s whose images under ψ are contained in exactly one odd component.

(b) Suppose $u'' \in V(T_X)$, $w'' \in N(u'')$, $\varphi(u'')$ is odd and $\varphi(w'')$ is even. Then part (a) of this lemma tells us that $\psi(\Gamma_{u''})$ is not a point. If S_0, S_1, S_2 is the partition of $\varphi^{-1}(\varphi(u''))$ as in Lemma 3.7.5(c), then Lemma 3.7.5(c)(ii) implies that $u'' \in S_0$ since $\psi(\Gamma_{u''})$ is not a point. As S_0 has a unique vertex, and this vertex corresponds to the inverse image under ψ_1 of the strict transform of $\Gamma_{\varphi(u'')}$, we are done. □

Lemma 3.7.8. *Let $v'', w'' \in V(T_X)$. Then $\Gamma_{v''}.\Gamma_{w''} \in \{0, 1, 2\}$. Let $v = \varphi(v''), w = \varphi(w''), v' = \varphi_1(v'')$ and $w' = \varphi_1(w'')$. Then $\Gamma_{v''}.\Gamma_{w''} = 2$ if and only if*

(i) both v and w are even,

(ii) the vertices v and w are neighbours of each other, and,

(iii) both $\Gamma_{v'}$ and $\Gamma_{w'}$ intersect the branch locus of ψ_1.

Proof. Lemma 3.1.5(b) tells us that all intersections in X_s are transverse, so the the number of points in the intersection of any two irreducible components in X_s equals their intersection number.

Let $v'', w'' \in V(T_X)$. Then $\Gamma_{v''} \cap \Gamma_{w''} \subset \psi_1^{-1}(\Gamma_{v'} \cap \Gamma_{w'})$. Since ψ_1 is finite of degree 2, any point of Y has at most two preimages under ψ_1 and therefore $\#\psi_1^{-1}(\Gamma_{v'} \cap \Gamma_{w'}) \leq 2\#\Gamma_{v'} \cap \Gamma_{w'}$. The set $\Gamma_{v'} \cap \Gamma_{w'}$ has at most one point since the dual graph T_Y of Y_s is a tree. This implies that $\#\Gamma_{v'} \cap \Gamma_{w'} \leq 1$. Putting these together, we get

$$\begin{aligned}\Gamma_{v''}.\Gamma_{w''} &= \#\Gamma_{v''} \cap \Gamma_{w''} \\ &\leq \#\psi_1^{-1}(\Gamma_{v'} \cap \Gamma_{w'}) \\ &\leq 2\ \#\Gamma_{v'} \cap \Gamma_{w'} \\ &\leq 2.1 \\ &= 2.\end{aligned}$$

It follows that $\Gamma_{v''}.\Gamma_{w''} \in \{0, 1, 2\}$.

Suppose that the three conditions in the lemma hold. Then, conditions (i) and (ii) imply that $\Gamma_v \cap \Gamma_w$ is nonempty and consists of a single point, say b. Then the strict transforms of Γ_v and Γ_w are $\Gamma_{v'}$ and $\Gamma_{w'}$ respectively and the map ψ_2 is an isomorphism above a neighbourhood of $\Gamma_v \cup \Gamma_w$. Let y be the unique point in $\Gamma_{v'} \cap \Gamma_{w'}$. As T_Y is a tree, the point y does not lie on any other component of Y_s except $\Gamma_{v'}$ and $\Gamma_{w'}$. Lemma 3.7.4 tells us that v' and w' are even. Lemma 3.7.1(a) now tells us that the point y has two preimages under ψ_1. Since $\Gamma_{v'}$ and $\Gamma_{w'}$ intersect the branch locus, their inverse images under ψ_1 are irreducible. This tells us $\Gamma_{v''} = \psi_1^{-1}(\Gamma_{v'})$ and $\Gamma_{w''} = \psi_1^{-1}(\Gamma_{w'})$. Then $\psi_1^{-1}(\Gamma_{v'} \cap \Gamma_{w'}) = \Gamma_{v''} \cap \Gamma_{w''}$.

$$\begin{aligned}\Gamma_{v''}.\Gamma_{w''} &= \#\Gamma_{v''} \cap \Gamma_{w''} \\ &= \#\psi_1^{-1}(\Gamma_{v'} \cap \Gamma_{w'}) \\ &= \#\psi_1^{-1}(y) \\ &= 2.\end{aligned}$$

Now assume $\Gamma_{v''}.\Gamma_{w''} = 2$. Since the intersections in X_s are transverse, the set $\Gamma_{v''} \cap \Gamma_{w''}$ has two points, say x_1 and x_2. Then, $\psi_1(x_1)$ and $\psi_1(x_2)$ must lie in $\Gamma_{v'} \cap \Gamma_{w'}$. Since any two components of Y_s cannot intersect at more than one point, this tells us that $\psi_1(x_1) = \psi_1(x_2)$. Call this point of intersection y. Since y has two preimages under ψ_1, it cannot lie on the branch locus of ψ_1. Lemma 3.7.1(a) tells us that v' and w' must both be even. Since $\psi(\Gamma_{v''}) = \Gamma_v$, it follows that $\psi(\Gamma_{v''})$ is not a point. Similarly $\psi(\Gamma_{w''}) = \Gamma_w$ is not a point.

Either $w'' \in C(v'')$ or $v'' \in C(w'')$, and Lemma 3.5.1(b) tells us that in both cases v and w are neighbours of each other. If $\Gamma_{v'}$ did not intersect the branch locus, then Lemma 3.7.5(a) implies that $\psi_1^{-1}(\Gamma_{v'})$ must have two disjoint irreducible components, one of which is the $\Gamma_{v''}$ we started with. Let $\tilde{v}'' \in V(T_X)$ be the other. Then there is exactly one point of $\psi_1^{-1}(y)$ in each $\Gamma_{v''}$ and $\Gamma_{\tilde{v}''}$. This contradicts the fact that $\Gamma_{v''}$ has both points of $\psi_1^{-1}(y)$. A similar argument shows that $\Gamma_{w'}$ intersects the branch locus. $\square$

We now make some definitions motivated by Sections 6 and 7. For $v'' \in V(T_X)$, define

$$\delta(v'') = (1 - m_{v''})\ \chi(\Gamma_{v''}) + \sum_{w'' \in N(v'')} (m_{w''} - 1)\Gamma_{v''}.\Gamma_{w''} + \sum_{w'' \in C(v'')} \Gamma_{v''}.\Gamma_{w''}.$$

Let $v \in V(T_B)$. Define

$$D(v) = \sum_{v'' \in \varphi^{-1}(v)} \delta(v'').$$

3.7.9 Computation of $D(v)$ for an even vertex v

Suppose $v \in V(T_B)$ is an even vertex. We define $D_0(v), D_1(v), D_2(v)$ as follows.

$$D_0(v) = \sum_{v'' \in \varphi^{-1}(v)} (1 - m_{v''})\ \chi(\Gamma_{v''}).$$

$$D_1(v) = \sum_{v'' \in \varphi^{-1}(v)} \ \sum_{w'' \in N(v'')} (m_{w''} - 1)\Gamma_{v''}.\Gamma_{w''}.$$

$$D_2(v) = \sum_{v'' \in \varphi^{-1}(v)} \ \sum_{w'' \in C(v'')} \Gamma_{v''}.\Gamma_{w''}$$

Then, $D(v) = D_0(v) + D_1(v) + D_2(v)$. We will now compute $D_i(v)$ for each $i \in \{0, 1, 2\}$ in terms of l_v, r_v and s_v.

Lemma 3.7.10. *Suppose $v \in V(T_B)$ is even. Then, $D_0(v) = 0$.*

Proof. Suppose v is an even vertex. Lemma 3.7.6 implies that $m_{v''} = 1$ for every $v'' \in \varphi^{-1}(v)$ and therefore,

$$D_0(v) = \sum_{v'' \in \varphi^{-1}(v)} (1 - m_{v''})\ \chi(\Gamma_{v''}) = 0. \qquad \square$$

Lemma 3.7.11. *Suppose $v \in V(T_B)$ is even. Let $v'' \in \varphi^{-1}(v)$ and $w'' \in N(v'')$. Let $v' = \varphi_1(v''), w' = \varphi_1(w'')$ and $w = \varphi(w'')$.*

(a) *The vertex v' is even and $\varphi_2^{-1}(v) = \{v'\}$.*

(b) *The multiplicity $m_{w''} = 2$ if and only if w is odd.*

(c) *If $v'' \in C(w'')$, then $v \in C(w)$. If $w'' \in C(v'')$, then $w \in C(v)$. In particular, $w \in N(v)$.*

(d) *If $r_v = 0$ and l_v is even, then every neighbour of v is even.*

(e) *The branch locus of ψ_1 intersects $\Gamma_{v'}$ at $l_v + (l_v \bmod 2)$ points, and all these intersections are transverse.*

(f) *If $l_v = 0$, then $\Gamma_{v'}$ does not intersect the branch locus of ψ_1 and $\#\varphi^{-1}(v) = 2$.*

(g) *If $l_v \neq 0$, then $\Gamma_{v'}$ intersects the branch locus of ψ_1, $\#\varphi^{-1}(v) = 1$ and $\varphi^{-1}(v) = \{v''\}$.*

(h) *If w is odd, then $\Gamma_{v''}.\Gamma_{w''} = 1$.*

(i) *Suppose $u \in N(v)$ is odd. Then there exists a unique $u'' \in \varphi^{-1}(u)$ such that $u'' \in N(v'')$. If $u \in C(v)$, then $u'' \in C(v'')$. If $v \in C(u)$, then $v'' \in C(u'')$.*

(j) *Suppose $l_v \neq 0$, $w'' \in C(v'')$ and w is even. Then, $\#\varphi^{-1}(w) \in \{1, 2\}$. If $\#\varphi^{-1}(w) = 1$, then $\Gamma_{v''}.\Gamma_{w''} = 2$. If $\#\varphi^{-1}(w) = 2$, then $\Gamma_{v''}.\Gamma_{w''} = 1$.*

(k) *Suppose $l_v \neq 0$ and $u \in C(v)$ is even. If $u'' \in \varphi^{-1}(u)$, then $u'' \in C(v'')$.*

(l) *If $l_v = 0$, then $\Gamma_{v''}.\Gamma_{w''} = 1$.*

(m) *Suppose $l_v = 0$ and $u \in C(v)$ is even. If $\varphi^{-1}(u) = \{u''\}$, then $u'' \in C(v'')$. If $\varphi^{-1}(u) = \{u_1'', u_2''\}$, then, after possibly interchanging u_1'' and u_2'', we have that $u_1'' \in C(v'')$ and $\Gamma_{v''}.\Gamma_{u_2''} = 0$.*

Proof.

(a) Since $\varphi_2(v') = \varphi(v'') = v$ and v is even, Lemma 3.7.4 tells us that v' is even.

(b) First assume w is odd. Since v' is even, Lemma 3.7.7(b) implies that $\Gamma_{w''}$ is the preimage under ψ_1 of the strict transform of Γ_w in Y. In particular, Lemma 3.7.4 tells us that w' is odd, and therefore not an even leaf. Lemma 3.7.6 applied to w'' then implies that $m_{w''} = 2$.

Conversely, assume $m_{w''} = 2$. Lemma 3.7.6 applied to w'' implies that w is odd.

(c) If w is odd, since v is even, Lemma 3.7.7(b) tells us that $\psi(\Gamma_{w''})$ is not a point. If w is even, then $\psi(\Gamma_{w''})$ is not a point. Since v is even, $\psi(\Gamma_{v''})$ is not a point. Since $w'' \in N(v'')$, either $v'' \in C(w'')$ or $w'' \in C(v'')$. Since both $\psi(\Gamma_{v''})$ and $\psi(\Gamma_{w''})$ are not points, Lemma 3.5.1(b) tells us that in the first case $v \in C(w)$ and in the second case $w \in C(v)$. Both of these imply $w \in N(v)$.

(d) Suppose $r_v = 0$ and l_v is even. Since v is even and $r_v = 0$, Lemma 3.7.2 implies that every child of v is even. Since l_v is even, Lemma 3.7.3 implies that v does not have an odd parent. Therefore every neighbour of v is even.

(e) Lemma 3.7.1(a) and Lemma 3.7.4 tell us that $\Gamma_{v'}$ does not belong to the branch locus since $\varphi_2(v') = v$, which is even. Lemma 3.7.1(b) tells us that any component of the branch locus that intersects $\Gamma_{v'}$, intersects it transversally.

- Lemma 3.7.1(a) tells us that the components of the branch locus are the odd components of Y_s and the irreducible horizontal divisors appearing in (f) different from $\overline{\infty}$.
- Lemma 3.7.4 tells us that the odd components of Y_s are the strict transforms of odd components of $(\mathrm{Bl}_n(\mathbb{P}^1_R))_s$.
- Since v is even, the map ψ_2 induces an isomorphism above a neighbourhood of Γ_v.

Therefore, the number of components of the branch locus intersecting $\Gamma_{v'}$ is the number of odd neighbours of v added to the number of horizontal divisors different from $\overline{\infty}$ appearing in the divisor of (f) that intersect Γ_v. The latter number is l'_v. Since v is even, Lemma 3.7.2 tells us that the number of odd children of v is r_v. Lemma 3.7.3 tells us that the number of odd parents of v is (l_v mod 2). Since $l'_v + r_v + (l_v \bmod 2) = l_v + (l_v \bmod 2)$, the branch locus intersects $\Gamma_{v'}$ at $l_v + (l_v \bmod 2)$ points.

(f) Suppose $l_v = 0$. Then $l_v + (l_v \bmod 2) = 0$. Part (e) tell us that $\Gamma_{v'}$ does not intersect the branch locus of ψ_1. Since v is even, Lemma 3.7.5(b) implies that $\#\varphi^{-1}(v) = 2$.

(g) Suppose $l_v \neq 0$. Then $l_v + (l_v \bmod 2) \neq 0$. Part (e) tells us that $\Gamma_{v'}$ intersects the branch locus of ψ_1. Lemma 3.7.5(b) then implies that $\#\varphi^{-1}(v) = 1$. It follows that $\varphi^{-1}(v) = \{v''\}$.

(h) Suppose w is odd. Since w is odd, Lemma 3.7.8 tells us that $\Gamma_{v''}.\Gamma_{w''} < 2$. On the other hand, since $w'' \in N(v'')$, it follows that $\Gamma_{v''}.\Gamma_{w''} \geq 1$.

(i) Suppose $u \in N(v)$ is odd. Let u' be the vertex corresponding to the strict transform of Γ_u in Y. As $u \in N(v)$ and ψ_2 is an isomorphism above a neighbourhood of Γ_v, it follows that $u' \in N(v')$. In fact, this shows that if $u \in C(v)$, then $u' \in C(v')$; if $v \in C(u)$, then $v' \in C(u')$.

Lemma 3.7.4 shows that u' is odd. Lemma 3.7.1(a) and Lemma 3.7.5(a) applied to u' show that there is a unique u'' in $V(T_X)$ such that $\varphi_1(u'') = u'$. Part (a) tells us that v' is even and $\varphi_2^{-1}(v) = \{v'\}$. Since $\Gamma_{v'}$ intersects $\Gamma_{u'}$ and u' is odd, Lemma 3.7.1(a) and Lemma 3.7.5(a) applied to the even vertex v' tell us that $\varphi^{-1}(v) = \varphi_1^{-1}(v') = \{v''\}$. Since $\psi_1^{-1}(\Gamma_{u'}) = \Gamma_{u''}$ and $\psi_1^{-1}(\Gamma_{v'}) = \Gamma_{v''}$, it follows that $\Gamma_{u''} \cap \Gamma_{v''} = \psi_1^{-1}(\Gamma_{u'} \cap \Gamma_{v'})$. Since ψ_1 is surjective and $\Gamma_{u'} \cap \Gamma_{v'}$ is nonempty, it follows that $u'' \in N(v'')$. We also have $\varphi(u'') = \varphi_2(u') = u$. This proves the existence of $u'' \in \varphi^{-1}(u)$ such that $u'' \in N(v'')$.

Suppose that we are given $u'' \in \varphi^{-1}(u)$ such that $u'' \in N(v'')$. Since v is even and u is odd, Lemma 3.7.7(b) forces u'' to be the inverse image under ψ_1 of the strict transform of Γ_u in Y. This proves uniqueness.

Lemma 3.5.1(a) tells us that if $v'' \in C(u'')$, then $v' \in C(u')$. If $u \in C(v)$, then $u' \in C(v')$ and therefore $u'' \in C(v'')$. Similarly, one can show that if $v \in C(u)$, then $v'' \in C(u'')$.

(j) Part (g) tells us that $\Gamma_{v'}$ intersects the branch locus of ψ_1. Since w is even, Lemma 3.7.5(b) implies that $\#\varphi^{-1}(w) \in \{1, 2\}$. Since v and w are even, $\#\psi^{-1}(\Gamma_v \cap \Gamma_w) = 2$. Since

$w'' \in C(v'')$, Lemma 3.7.8 tells us that $1 \leq \Gamma_{v''}.\Gamma_{w''} \leq 2$. We have that v and w are even, $w \in C(v)$ (by (c)) and that $\Gamma_{v'}$ intersects the branch locus; thus, Lemma 3.7.8 implies that $\Gamma_{v''}.\Gamma_{w''} = 2$ if $\Gamma_{w'}$ intersects the branch locus, and $\Gamma_{v''}.\Gamma_{w''} = 1$ if it does not. Lemma 3.7.5(b) applied to w tells us that this can be restated as follows: If $\#\varphi^{-1}(w) = 1$, then $\Gamma_{v''}.\Gamma_{w''} = 2$; if $\#\varphi^{-1}(w) = 2$, then $\Gamma_{v''}.\Gamma_{w''} = 1$.

(k) Let $u' \in V(T_Y)$ be the vertex corresponding to the strict transform of Γ_u in Y. Let $u'' \in \varphi^{-1}(u)$.

- Part (g) tells us that $\Gamma_{v'}$ intersects the branch locus of ψ_1 and $\varphi^{-1}(v) = \{v''\}$. Therefore $\psi_1^{-1}(\Gamma_{v'}) = \Gamma_{v''}$.
- Since ψ_2 is an isomorphism above a neighbourhood of Γ_v, we have that $u' \in C(v')$. In particular, $\Gamma_{u'} \cap \Gamma_{v'} \neq \emptyset$.
- The map ψ_1 restricts to a surjection $\Gamma_{u''} \to \Gamma_{u'}$.

These three facts together imply that $\Gamma_{u''} \cap \Gamma_{v''}$ is not empty. In particular, $u'' \in N(v'')$. If $v'' \in C(u'')$, then Lemma 3.5.1(a) would imply $v' \in C(u')$. Since $u' \in C(v')$, Lemma 3.5.1(a) implies that $u'' \in C(v'')$.

(l) Suppose $l_v = 0$. Part (f) tells us that $\Gamma_{v'}$ does not intersect the branch locus. Lemma 3.7.8 applied to the pair v'', w'' tells us $\Gamma_{v''}.\Gamma_{w''} < 2$. On the other hand, since $w'' \in N(v'')$, we have that $\Gamma_{v''}.\Gamma_{w''} \geq 1$. Therefore, $\Gamma_{v''}.\Gamma_{w''} = 1$.

(m) Let $u' \in V(T_Y)$ be the vertex corresponding to the strict transform of Γ_u in Y. Since ψ_2 is an isomorphism above a neighbourhood of Γ_v, we get that $u' \in C(v')$.

Suppose $\varphi^{-1}(u) = \{u''\}$. Since $\psi_1^{-1}(\Gamma_{u'}) = \Gamma_{u''}$ and ψ_1 restricts to a surjection $\Gamma_{v''} \to \Gamma_{v'}$, an appropriate modification of the argument in part(j) tells us that $u'' \in C(v'')$.

Suppose $\varphi^{-1}(u) = \{u''_1, u''_2\}$. Then, Lemma 3.7.5(a) implies that $\Gamma_{u'}$ does not intersect the branch locus. Part (f) implies that $\Gamma_{v'}$ does not intersect the branch locus. This implies that the map ψ_1 is étale above a neighbourhood of $\Gamma_{v'} \cup \Gamma_{u'}$. Since $\mathbb{P}^1_k$ has no connected étale covers, this implies that $\psi_1^{-1}(\Gamma_{v'} \cup \Gamma_{u'})$ has two connected components, each of which maps isomorphically on to $\Gamma_{v'} \cup \Gamma_{u'}$ via ψ_1. This finishes the proof. $\square$

Lemma 3.7.12. *Suppose $v \in V(T_B)$ is even. Then, $D_1(v) = (l_v \bmod 2) + r_v$. (Here and subsequently $l_v \bmod 2$ is an integer in $\{0, 1\}$. It is 0 if l_v is even and 1 if l_v is odd.)*

Proof. Suppose $v \in V(T_B)$ is even. We break up the computation of $D_1(v)$ into two cases:

$\mathbf{Case\ 1: l_v = 0}$

In this case,

$$\begin{aligned} D_1(v) &= \sum_{v'' \in \varphi^{-1}(v)} \sum_{w'' \in N(v'')} (m_{w''} - 1)\Gamma_{v''}.\Gamma_{w''} \\ &= \sum_{v'' \in \varphi^{-1}(v)} \sum_{\substack{w'' \in N(v'') \\ \varphi(w'') \text{ even}}} (m_{w''} - 1)\Gamma_{v''}.\Gamma_{w''} \quad \text{(by Lemma 3.7.11(d) since } \mathrm{r_v} = \mathrm{l_v} = 0) \\ &= \sum_{v'' \in \varphi^{-1}(v)} \sum_{\substack{w'' \in N(v'') \\ \varphi(w'') \text{ even}}} (1 - 1)\Gamma_{v''}.\Gamma_{w''} \quad \text{(by Lemma 3.7.11(b))} \\ &= 0 \\ &= (l_v \bmod 2) + r_v \quad \text{(since } l'_v \text{ and } r_v \text{ are nonnegative, } r_v = 0). \end{aligned}$$

$\mathbf{Case\ 2: l_v \neq 0}$

In this case, Lemma 3.7.11(g) implies that $\#\varphi^{-1}(v) = 1$. Let $\varphi^{-1}(v) = \{v''\}$. Then,

$$\begin{aligned}
D_1(v) &= \sum_{\tilde{v}'' \in \varphi^{-1}(v)} \sum_{w'' \in N(\tilde{v}'')} (m_{w''} - 1)\Gamma_{\tilde{v}''}.\Gamma_{w''} \\
&= \sum_{w'' \in N(v'')} (m_{w''} - 1)\Gamma_{v''}.\Gamma_{w''} \\
&= \sum_{\substack{w'' \in N(v'') \\ \varphi(w'') \text{ odd}}} (m_{w''} - 1)\Gamma_{v''}.\Gamma_{w''} + \sum_{\substack{w'' \in N(v'') \\ \varphi(w'') \text{ even}}} (m_{w''} - 1)\Gamma_{v''}.\Gamma_{w''} \\
&= \sum_{\substack{w'' \in N(v'') \\ \varphi(w'') \text{ odd}}} (2 - 1)\Gamma_{v''}.\Gamma_{w''} + \sum_{\substack{w'' \in N(v'') \\ \varphi(w'') \text{ even}}} (1 - 1)\Gamma_{v''}.\Gamma_{w''} \quad \text{(by Lemma 3.7.11(b))} \\
&= \sum_{\substack{w'' \in N(v'') \\ \varphi(w'') \text{ odd}}} 1 \quad \text{(by Lemma 3.7.11(h))} \\
&= \sum_{\substack{w \in N(v) \\ w \text{ odd}}} \sum_{\substack{w'' \in N(v'') \\ \varphi(w'') = w}} 1 \quad \text{(by Lemma 3.7.11(c))} \\
&= \sum_{\substack{w \in N(v) \\ w \text{ odd}}} 1 \quad \text{(by Lemma 3.7.11(i) with u = w)} \\
&= \begin{cases} 1 + \displaystyle\sum_{\substack{w \in C(v) \\ w \text{ odd}}} 1 & \text{if v has an odd parent} \\ \displaystyle\sum_{\substack{w \in C(v) \\ w \text{ odd}}} 1 & \text{otherwise} \end{cases} \\
&= (l_v \bmod 2) + r_v \quad \text{(by Lemma 3.7.3 and Lemma 3.7.2 since v is even).} \qquad \square
\end{aligned}$$

Lemma 3.7.13. *Suppose $v \in V(T_B)$ is even. Then, $D_2(v) = r_v + 2s_v$.*

Proof. We break up the computation of $D_2(v)$ into two cases:

Case 1: $\mathbf{l_v = 0}$

In this case, Lemma 3.7.11(f) tells us that $\#\varphi^{-1}(v) = 2$. Since l'_v and r_v are nonnegative, $r_v = 0$. Then,

$$
\begin{aligned}
D_2(v) &= \sum_{v''\in\varphi^{-1}(v)} \sum_{w''\in C(v'')} \Gamma_{v''}.\Gamma_{w''} \\
&= \sum_{\substack{w\in C(v)\\ w \text{ even}}} \sum_{v''\in\varphi^{-1}(v)} \sum_{\substack{w''\in C(v'')\\ \varphi(w'')=w}} \Gamma_{v''}.\Gamma_{w''} \\
&\qquad\qquad \text{(since Lemma 3.7.11(c,d) imply that } \varphi(w'')\in C(v) \text{ and is even)} \\
&= \sum_{\substack{w\in C(v)\\ w \text{ even}}} 2 \\
&\text{(by Lemma 3.7.11(l,m) since Lemma 3.7.5(b) implies that } \#\varphi^{-1}(w)\in\{1,2\}) \\
&= r_v + 2s_v \quad \text{(by Lemma 3.7.2 since v is even and } \mathrm{r_v}=0).
\end{aligned}
$$

Case 2: $\mathbf{l_v \neq 0}$

In this case, Lemma 3.7.11(g) implies that $\#\varphi^{-1}(v) = 1$. Let $\{v''\} = \varphi^{-1}(v)$. Then,

$$
\begin{aligned}
D_2(v) &= \sum_{w''\in C(v'')} \Gamma_{v''}.\Gamma_{w''} \\
&= \sum_{\substack{w''\in C(v'')\\ \varphi(w'') \text{ odd}}} 1 + \sum_{\substack{w''\in C(v'')\\ \varphi(w'') \text{ even}}} \Gamma_{v''}.\Gamma_{w''} \quad \text{(by Lemma 3.7.11(h))} \\
&= \sum_{\substack{w\in C(v)\\ w \text{ odd}}} \sum_{\substack{w''\in C(v'')\\ \varphi(w'')=w}} 1 + \sum_{\substack{w\in C(v)\\ w \text{ even}}} \sum_{\substack{w''\in C(v'')\\ \varphi(w'')=w}} \Gamma_{v''}.\Gamma_{w''} \quad \text{(by Lemma 3.7.11(c))} \\
&= \sum_{\substack{w\in C(v)\\ w \text{ odd}}} 1 + \sum_{\substack{w\in C(v)\\ w \text{ even}}} 2 \\
&\qquad\qquad \text{(by Lemma 3.7.11(i), (k) with } u = w \text{ and Lemma 3.7.11(j))} \\
&= r_v + 2s_v \quad \text{(by Lemma 3.7.2 since v is even).}
\end{aligned}
$$

□

Lemma 3.7.14. *Suppose $v \in V(T_B)$ is even. Then,*

$$D(v) = (l_v \bmod 2) + 2r_v + 2s_v.$$

Proof. Combine Lemmas 3.7.10, 3.7.12 and 3.7.13. $\square$

3.7.15 Computation of $D(v)$ for an odd vertex v

Suppose $v \in V(T_B)$ is odd. Let $S_0(v), S_1(v), S_2(v)$ denote the partition of $\varphi^{-1}(v)$ constructed in Lemma 3.7.5(c).

Lemma 3.7.16. *Suppose $v \in V(T_B)$ is odd. Let $v'' \in S_0(v), w'' \in N(v''), v' = \varphi_1(v'')$ and $w = \varphi(w'')$.*

(a) *The component $\Gamma_{v'}$ is the strict transform of Γ_v in Y and v' is odd. The image $\psi(\Gamma_{v''})$ is not a point.*

(b) *We have that*

$$\{w'' \in C(v'') \mid m_{w''} = 2\} = S_1(v).$$

We also have that $\#S_1(v) = s_v$.

(c) *If $v'' \in C(w'')$ and $m_{w''} = 2$, then $w = p_v$ and w is odd.*

(d) *If p_v is odd, there exists a unique $u'' \in \varphi^{-1}(p_v)$ such that $v'' \in C(u'')$.*

(e) *The map φ induces a bijection between the sets $\{w'' \in C(v'') \setminus S_2(v) \mid m_{w''} = 1\}$ and $\{w \in C(v) \mid w \text{ is even}\}$.*

(f) *We have that $\Gamma_{v''}.\Gamma_{w''} = 1$.*

Proof.

(a) Since $v'' \in S_0(v)$ and $v' = \varphi_1(v'')$, it follows from Lemma 3.7.5(c)(ii) that $\Gamma_{v'}$ is the strict transform of Γ_v in Y. Since $\varphi_1(v'') = v'$, it follows that $\psi(\Gamma_{v''}) = \psi_2(\Gamma_{v'}) = \Gamma_v$. Therefore $\psi(\Gamma_{v''})$ is not a point. Lemma 3.7.5(c)(ii) also implies that v' is odd.

(b) Suppose $w'' \in C(v'')$ and $m_{w''} = 2$. Let $w' = \varphi_1(w'')$. Since $w'' \in C(v'')$, Lemma 3.5.1(a) implies that $w' \in C(v')$. Since odd components of Y do not intersect and (a) implies that v' is odd, w' is even. Since $m_{w''} = 2$, Lemma 3.7.6 tells us that w is odd and w' is not an even leaf of T_Y. Let T_0, T_1, T_2 be the partition of $\varphi_2^{-1}(w)$ as in Lemma 3.7.5(c).

Since w' is even, Lemma 3.7.4 tells us that $w' \notin T_0$. Since w' is not an even leaf of T_Y, the displayed equation in the proof of Lemma 3.7.6 shows that $w' \in T_1$. Since $w' \in T_1$, Lemma 3.7.5(c)(vii) shows that $p_{w'} \in T_0$. Since $w' \in C(v')$, it follows that $v' = p_{w'} \in T_0$ and therefore $\varphi_2(v') \in \varphi_2(T_0) = \{w\}$, which implies that $v = w$. Finally, $w'' \in \varphi_1^{-1}(w') \subseteq \varphi_1^{-1}(T_1) = S_1(v)$.

Conversely, suppose $w'' \in S_1(v)$. Since $v'' \in S_0(v)$, Lemma 3.7.5(c)(i,vii) show that $w'' \in C(v'')$ and $m_{w''} = 2$. Lemma 3.7.5(c)(iii) implies that $\#S_1(v) = s_v$.

(c) Suppose $v'' \in C(w'')$ and $m_{w''} = 2$. Since $v'' \in C(w'')$ and $\psi(\Gamma_{v''})$ is not a point by (a), Lemma 3.5.1(b) tells us that $v \in C(w)$. Since $m_{w''} = 2$, Lemma 3.7.6 tells us that w is odd.

(d) Suppose p_v is odd. Let $u = p_v$. Let T_0, T_1, T_2 be the partition of $\varphi_2^{-1}(u)$ as in Lemma 3.7.5(c). Let $u' \in T_1$ be the unique vertex such that $\psi_2(\Gamma_{u'}) = \Gamma_u \cap \Gamma_v$. Since (a) implies that $\Gamma_{v'}$ is the strict transform of Γ_v in Y, the proof of Lemma 3.7.5(c)(iii) in the case of the odd vertex u shows that $v' \in C(u')$. Lemma 3.7.5(c) applied to the odd vertex u tells us that φ_1 induces a bijection between $\varphi^{-1}(u)$ and $\varphi_2^{-1}(u)$. This shows that there exists a unique $u'' \in V(T_X)$ such that $\varphi_1(u'') = u'$. Since v' is odd by (a), Lemma 3.7.5(a) and Lemma 3.7.1(a) then imply that $\psi_1^{-1}(\Gamma_{v'}) = \Gamma_{v''}$. Since $\varphi_1^{-1}(u') = \{u''\}$, it follows that $\psi_1^{-1}(\Gamma_{u'}) = \Gamma_{u''}$. Therefore, $\Gamma_{u''} \cap \Gamma_{v''} = \psi_1^{-1}(\Gamma_{u'} \cap \Gamma_{v'}) \neq \emptyset$. This implies that either $u'' \in C(v'')$, or $v'' \in C(u'')$. Since $v' \in C(u')$, Lemma 3.5.1(a) implies that $v'' \in C(u'')$. This proves the existence of u''.

Suppose $u'' \in \varphi^{-1}(u)$ be such that such that $v'' \in C(u'')$. Then, Lemma 3.5.1(a) implies that $\varphi_1(u'') = p_{v'}$. Since v' is odd (by (a)) and $\Gamma_{p_{v'}}$ intersects $\Gamma_{v'}$, Lemma 3.7.1(a) and Lemma 3.7.5(a) imply that $\#\varphi_1^{-1}(p_{v'}) = 1$. This proves uniqueness of $u'' \in \varphi^{-1}(u)$ such that $v'' \in C(u'')$.

(e) Suppose $w'' \in C(v'') \setminus S_2(v)$ and $m_{w''} = 1$. We will first show $\psi(\Gamma_{w''})$ is not a point. Suppose $\psi(\Gamma_{w''})$ is a point. Since $w'' \in C(v'')$, Lemma 3.5.1(b) implies that $w = \varphi(w'') = \varphi(v'') = v$. Since $m_{w''} = 1$, Lemma 3.7.5(c)(i,ii,iii) then imply that $w'' \in S_2(v)$, which is a contradiction. Therefore, $\psi(\Gamma_{w''})$ is not a point. Lemma 3.5.1(a) then implies that

$w \in C(v)$.

Suppose w is odd. Let $w' = \varphi_1(w'')$. Since $\psi(\Gamma_{w''})$ is not a point, $w'' \in S_0(w)$. Part (a) applied to w'' implies that w' is odd. Part (a) implies that v' is odd. Since $w'' \in C(v'')$, Lemma 3.5.1(a) implies that $w' \in C(v')$. This is a contradiction since odd components of Y cannot intersect. Therefore w is even. This shows one inclusion.

Now suppose $u \in C(v)$ is even. Let $u' \in V(T_Y)$ be the vertex corresponding to the strict transform of Γ_u in Y. Part (a) implies that v' is the vertex corresponding to the strict transform of Γ_v and v' is odd. Lemma 3.7.4 implies that u' is even. This in turn implies that ψ_2 is an isomorphism above a neighbourhood of Γ_u, and therefore $u' \in C(v')$. Since v' is odd and $u' \in C(v')$, Lemma 3.7.5(b) applied to u implies that $\#\varphi^{-1}(u) = 1$. Let $\varphi^{-1}(u) = \varphi_1^{-1}(u') = \{u''\}$. Since $\psi_1^{-1}(\Gamma_{v'}) = \Gamma_{v''}$ and $\psi_1^{-1}(\Gamma_{u'}) = \Gamma_{u''}$, it follows that $\Gamma_{v''} \cap \Gamma_{u''} = \psi_1^{-1}(\Gamma_{v'} \cap \Gamma_{u'})$ is not empty. In particular, $u'' \in N(v'')$. Since $\varphi_1(u'') = u' \in C(v') = C(\varphi_1(v''))$, Lemma 3.5.1(a) implies that $u'' \in C(v'')$. This shows the opposite inclusion.

(f) Since $\varphi(v'') = v$ is odd, Lemma 3.7.8 tells us that $\Gamma_{v''}.\Gamma_{w''} < 2$. On the other hand, since $w'' \in N(v'')$, we have that $\Gamma_{v''}.\Gamma_{w''} \geq 1$. □

We will now compute $\sum_{v'' \in S_i(v)} \delta(v'')$ for each $i \in \{0, 1, 2\}$, in terms of l_v, r_v and s_v.

Lemma 3.7.17. *Suppose $v \in V(T_B)$ is odd. Then*

$$\sum_{v'' \in S_0(v)} \delta(v'') = \begin{cases} -2 + l_v + 2s_v & \text{if } p_v \text{ is even} \\ -1 + l_v + 2s_v & \text{if } p_v \text{ is odd.} \end{cases}$$

Proof. Let $S_0 = S_0(v), S_1 = S_1(v)$ and $S_2 = S_2(v)$. Lemma 3.7.5(c)(ii) implies that $\#S_0 = 1$. Let $\tilde{v} \in S_0$. Since S_0 consists of a single vertex $\tilde{v}$,

$$\sum_{v'' \in S_0} \delta(v'') = \delta(\tilde{v}) = (1 - m_{\tilde{v}})\ \chi(\Gamma_{\tilde{v}}) + \sum_{w'' \in N(\tilde{v})} (m_{w''} - 1)\Gamma_{\tilde{v}}.\Gamma_{w''} + \sum_{w'' \in C(\tilde{v})} \Gamma_{\tilde{v}}.\Gamma_{w''}.$$

We will compute each of the three terms in this sum separately.

By Lemma 3.7.5(c)(ii),

$$(1 - m_{\tilde{v}})\ \chi(\Gamma_{\tilde{v}}) = (1 - m_{\tilde{v}})\ \chi(\mathbb{P}^1_k) = (1-2)(2) = -2.$$

Now

$$\begin{aligned}
\sum_{w'' \in N(\tilde{v})} (m_{w''} - 1)\Gamma_{\tilde{v}}.\Gamma_{w''} &= \sum_{w'' \in N(\tilde{v})} (m_{w''} - 1) \quad \text{(by Lemma 3.7.16(f))} \\
&= \sum_{w'' \in S_1} (2-1) + \sum_{w'' \in C(\tilde{v}) \setminus S_1} (1-1) + \sum_{\substack{w'' \in V(T_X) \\ \tilde{v} \in C(w'')}} (m_{w''} - 1) \\
&\hspace{15em} \text{(by Lemma 3.7.16(b))} \\
&= s_v + \sum_{\substack{w'' \in V(T_X) \\ \tilde{v} \in C(w'')}} (m_{w''} - 1) \quad \text{(by Lemma 3.7.16(b))} \\
&= s_v + \sum_{\substack{w'' \in \varphi^{-1}(p_v) \\ \tilde{v} \in C(w'') \\ \varphi(w'') \text{ is odd}}} (m_{w''} - 1) \quad \text{(by Lemma 3.7.16(c))} \\
&= \begin{cases} s_v & \text{if } p_v \text{ is even} \\ s_v + 1 & \text{if } p_v \text{ is odd} \end{cases} \quad \text{(by Lemma 3.7.16(d)).}
\end{aligned}$$

Now

$$\begin{aligned}
\sum_{w'' \in C(\tilde{v})} \Gamma_{\tilde{v}}.\Gamma_{w''} &= \sum_{w'' \in C(\tilde{v})} 1 \quad \text{(by Lemma 3.7.16(f))} \\
&= \sum_{\substack{w'' \in C(\tilde{v}) \\ m_{w''} = 2}} 1 + \sum_{w'' \in S_2} 1 + \sum_{\substack{w'' \in C(\tilde{v}) \setminus S_2 \\ m_{w''} = 1}} 1 \\
&\hspace{15em} \text{(by Lemmas 3.1.5(b), 3.7.5(c)(i,iv,vii))} \\
&= s_v + l'_v + \sum_{\substack{w'' \in C(\tilde{v}) \setminus S_2 \\ m_{w''} = 1}} 1 \quad \text{(by Lemma 3.7.16(b) and Lemma 3.7.5(c)(iv))} \\
&= s_v + l'_v + r_v \quad \text{(by Lemma 3.7.2 since } v \text{ is odd, and by Lemma 3.7.16(e))} \\
&= s_v + l_v.
\end{aligned}$$

Adding the three previous equalities gives us

$$\sum_{v''\in S_0(v)} \delta(v'') = \delta(\tilde{v}) = \begin{cases} -2 + l_v + 2s_v & \text{if } p_v \text{ is even} \\ -1 + l_v + 2s_v & \text{if } p_v \text{ is odd.} \end{cases} \qquad \square$$

Lemma 3.7.18. *Suppose $v \in V(T_B)$ is odd. Then*

$$\sum_{v''\in S_1(v)} \delta(v'') = s_v.$$

Proof. Let $S_1 = S_1(v)$. Let $\tilde{v}$ be the unique element of $S_0(v)$. Suppose $v'' \in S_1$. Lemma 3.7.5(c)(iii,viii) tells us that $\Gamma_{v''} \cong \mathbb{P}^1_k$, $v'' \in C(\tilde{v})$, $m_{v''} = 2$ and $\psi(\Gamma_{v''}) = \Gamma_v \cap \Gamma_u$ for an odd $u \in C(v)$.

Since $\psi(\Gamma_{v''})$ is a point that belongs to two odd components of $(\mathrm{Bl}_n(\mathbb{P}^1_R))_s$, Lemma 3.7.7(a)(i,ii) tell us that $\#N(v'') = 2$ and $\#C(v'') = 1$. Suppose $w'' \in N(v'')$. Lemma 3.7.7(a)(iii,iv) tell us that $\varphi(w'')$ is odd and $m_{w''} = 2$. Since $\varphi(w'')$ is odd, Lemma 3.7.8 tells us that $\Gamma_{v''}.\Gamma_{w''} < 2$. On the other hand, since $w'' \in N(v'')$, we have that $\Gamma_{v''}.\Gamma_{w''} \geq 1$. This implies that

$$\begin{aligned} \delta(v'') &= (1 - m_{v''})\ \chi(\Gamma_{v''}) + \sum_{w''\in N(v'')} (m_{w''} - 1)\Gamma_{v''}.\Gamma_{w''} + \sum_{w''\in C(v'')} \Gamma_{v''}.\Gamma_{w''} \\ &= (1-2)2 + (2-1)1 + (2-1)1 + 1 \\ &= 1. \end{aligned}$$

Therefore

$$\sum_{v''\in S_1(v)} \delta(v'') = \sum_{v''\in S_1(v)} 1 = s_v \qquad \text{(since Lemma 3.7.5(c)(iii) implies that } \#S_1 = s_v\text{).} \qquad \square$$

Lemma 3.7.19. *Suppose $v \in V(T_B)$ is odd. Then*

$$\sum_{v''\in S_2(v)} \delta(v'') = l_v - r_v.$$

Proof. Let $S_2 = S_2(v)$ and $S_0(v) = \{\tilde{v}\}$. Suppose $v'' \in S_2$. Lemma 3.7.5(c)(iv,viii) tells us that $\Gamma_{v''} \cong \mathbb{P}^1_k$, $v'' \in C(\tilde{v})$, $m_{v''} = 1$ and $\psi(\Gamma_{v''}) = \Gamma_v \cap H$ where H is an irreducible horizontal divisor occuring in (f) on $\mathrm{Bl}_n(\mathbb{P}^1_R)$.

Since $\psi(\Gamma_{v''})$ is a point that belongs to a unique odd component of $(\mathrm{Bl}_n(\mathbb{P}^1_R))_s$, Lemma 3.7.7(a)(i,ii) tell us that $\#N(v'') = 1$ and $\#C(v'') = 0$. Since $v'' \in C(\tilde{v})$, we have that $N(v'') = \{\tilde{v}\}$. Lemma 3.7.5(c)(ii) implies that $m_{\tilde{v}} = 2$. Since $\tilde{v} \in N(v'')$ and $\varphi(\tilde{v})$ $(= v)$ is odd, Lemma 3.7.8 applied to the pair $v'', \tilde{v}$ tells us that $\Gamma_{v''}.\Gamma_{w''} < 2$. On the other hand, since $\tilde{v} \in N(v'')$, we have that $\Gamma_{v''}.\Gamma_{w''} \geq 1$. This implies that

$$\begin{aligned}
\delta(v'') &= (1 - m_{v''})\ \chi(\Gamma_{v''}) + \sum_{w'' \in N(v'')} (m_{w''} - 1)\Gamma_{v''}.\Gamma_{w''} + \sum_{w'' \in C(v'')} \Gamma_{v''}.\Gamma_{w''} \\
&= (1-1)2 + (2-1)1 + 0 \\
&= 1.
\end{aligned}$$

Therefore

$$\sum_{v'' \in S_2(v)} \delta(v'') = \sum_{v'' \in S_2(v)} 1 = l'_v = l_v - r_v \qquad \text{(since Lemma 3.7.5(c)(iv) implies that } \#S_2 = l'_v).$$

□

Lemma 3.7.20. *Suppose $v \in V(T_B)$ is odd (in particular, v is not the root). Then*

$$D(v) = \begin{cases} -2 - r_v + 3s_v + 2l_v & \text{if } v \text{ is odd and } p_v \text{ is even} \\ -1 - r_v + 3s_v + 2l_v & \text{if } v \text{ is odd and } p_v \text{ is odd.} \end{cases}$$

Proof. Combine Lemmas 3.7.17,3.7.18,3.7.19. □

3.7.21 Formula for $D(v)$

Theorem 3.7.22. *Let $v \in V(T_B)$. Then*

$$D(v) = \begin{cases} (l_v \bmod 2) + 2r_v + 2s_v & \text{if } v \text{ is even} \\ -2 - r_v + 3s_v + 2l_v & \text{if } v \text{ is odd and } p_v \text{ is even} \\ -1 - r_v + 3s_v + 2l_v & \text{if } v \text{ is odd and } p_v \text{ is odd.} \end{cases}$$

Proof. This follows directly from Lemma 3.7.14 and Lemma 3.7.20. □

3.8 Comparison of the two discriminants

One might hope that the inequality $D(v) \leq d(v)$ holds for every vertex $v \in V(T_B)$, but this is not true. It is however true after a slight alteration of the function D.

3.8.1 A new break-up of the Deligne discriminant

Define a new function E on $V(T_B)$ as follows:

$$E(v) = \begin{cases} -(l_v \bmod 2) - \sum\limits_{\substack{v' \in C(v) \\ v' \text{ odd}}} (2 - \mathrm{wt}_{v'}(\mathrm{wt}_{v'} - 1)) & \text{if } v \text{ is even} \\ r_v + s_v + 2 - \mathrm{wt}_v(\mathrm{wt}_v - 1) - \sum\limits_{\substack{v' \in C(v) \\ v' \text{ odd}}} (2 - \mathrm{wt}_{v'}(\mathrm{wt}_{v'} - 1)) & \text{if } v \text{ is odd, } p_v \text{ even} \\ r_v + s_v + 1 - \mathrm{wt}_v(\mathrm{wt}_v - 1) - \sum\limits_{\substack{v' \in C(v) \\ v' \text{ odd}}} (2 - \mathrm{wt}_{v'}(\mathrm{wt}_{v'} - 1)) & \text{if } v \text{ and } p_v \text{ are odd.} \end{cases}$$

For $v \in V(T_B)$, set $D'(v) := D(v) + E(v)$.

Using Lemma 3.7.2, we get

$$\sum_{\substack{v' \in C(v) \\ v' \text{ odd}}} 2 = \begin{cases} 2s_v \text{ if } v \text{ is odd} \\ 2r_v \text{ if } v \text{ is even} \end{cases}.$$

We can use this, along with Theorem 3.7.22 to simplify the expression of D'.

$$D'(v) = \begin{cases} 2s_v + \sum_{\substack{v' \in C(v) \\ v' \text{ odd}}} \mathrm{wt}_{v'}(\mathrm{wt}_{v'} - 1) & \text{if } v \text{ is even} \\ 2\big(l_v + s_v\big) - \mathrm{wt}_v(\mathrm{wt}_v - 1) + \sum_{\substack{v' \in C(v) \\ v' \text{ odd}}} \mathrm{wt}_{v'}(\mathrm{wt}_{v'} - 1) & \text{if } v \text{ is odd} \end{cases} \tag{3.1}$$

Lemma 3.8.2. *The following equalities hold.*

$$\sum_{\substack{v \in V(T_B) \\ v \text{ even}}} \sum_{\substack{v' \in C(v) \\ v' \text{ odd}}} -(2 - \mathrm{wt}_{v'}(\mathrm{wt}_{v'} - 1)) + \sum_{\substack{v \in V(T_B) \\ v \text{ odd}}} \left(2 - \mathrm{wt}_v(\mathrm{wt}_v - 1) - \sum_{\substack{v' \in C(v) \\ v' \text{ odd}}} (2 - \mathrm{wt}_{v'}(\mathrm{wt}_{v'} - 1)) \right) = 0.$$

$$\sum_{\substack{v \in V(T_B) \\ v \text{ even}}} -(l_v \bmod 2) + \sum_{\substack{v \in V(T_B) \\ v \text{ odd}}} r_v = 0.$$

$$\sum_{\substack{v \in V(T_B) \\ v \text{ odd} \\ p_v \text{ is odd}}} -1 + \sum_{\substack{v \in V(T_B) \\ v \text{ odd}}} s_v = 0.$$

Proof. The first equality can be rewritten as

$$\sum_{v \in V(T_B)} \sum_{\substack{v' \in C(v) \\ v' \text{ odd}}} -\left(2 - \mathrm{wt}_{v'}(\mathrm{wt}_{v'} - 1)\right) + \sum_{\substack{v \in V(T_B) \\ v \text{ odd}}} \left(2 - \mathrm{wt}_v(\mathrm{wt}_v - 1)\right) = 0.$$

Since the root is an even vertex, every odd vertex has a parent. This implies that

$$\sum_{v \in V(T_B)} \sum_{\substack{v' \in C(v) \\ v' \text{ odd}}} -\left(2 - \mathrm{wt}_{v'}(\mathrm{wt}_{v'} - 1)\right) = - \sum_{\substack{v \in V(T_B) \\ v \text{ odd}}} \left(2 - \mathrm{wt}_v(\mathrm{wt}_v - 1)\right).$$

We have that

$$\begin{aligned} \sum_{\substack{v \in V(T_B) \\ v \text{ even}}} -(l_v \bmod 2) &= \sum_{\substack{v \in V(T_B) \\ v \text{ even} \\ v \text{ has an odd parent}}} -1 \quad \text{(by Lemma 3.7.3)} \\ &= \sum_{\substack{w \in V(T_B) \\ w \text{ odd}}} \sum_{\substack{v \in C(w) \\ w \text{ even}}} -1 \\ &= \sum_{\substack{w \in V(T_B) \\ w \text{ odd}}} -r_w \quad \text{(by Lemma 3.7.2).} \end{aligned}$$

We have that

$$\begin{aligned} \sum_{\substack{v \in V(T_B) \\ v \text{ odd} \\ \mathrm{p_v} \text{ is odd}}} -1 &= \sum_{\substack{w \in V(T_B) \\ w \text{ odd}}} \sum_{\substack{v \in C(w) \\ w \text{ odd}}} -1 \\ &= \sum_{\substack{w \in V(T_B) \\ w \text{ odd}}} -s_w \quad \text{(by Lemma 3.7.2).} \end{aligned}$$

$\square$

Lemma 3.8.3.

$$\sum_{v \in V(T_B)} E(v) = 0.$$

Proof. The sum of the left hand sides of the three equalities in Lemma 3.8.2 equals $\sum_{v \in V(T_B)} E(v)$, which is therefore 0. $\square$

For an odd $v \in V(T_B)$ such that $\mathrm{wt}_v > 2$, let $L_v = \{w \in C(v) \mid \mathrm{wt}_w = 2\}$. Define a new function D'' on $V(T_B)$ as follows:

$$D''(v) = \begin{cases} D'(v) - 2 & \text{if } v \text{ is an odd leaf and } \mathrm{wt}_v = 2 \\ D'(v) & \text{if } v \text{ is odd, not a leaf, and } \mathrm{wt}_v = 2 \\ D'(v) + 2\#L_v & \text{if } v \text{ is odd, and } \mathrm{wt}_v > 2 \\ D'(v) & \text{if } v \text{ is even.} \end{cases}$$

Lemma 3.8.4.

$$\sum_{v \in V(T_B)} D''(v) = \sum_{v \in V(T_B)} D'(v).$$

Proof. For an odd leaf $v \in V(T_B)$ such that $\mathrm{wt}_v = 2$, let q_v denote the least ancestor of v such that $\mathrm{wt}_{q(v)} \geq 3$ (here least ancestor means the ancestor farthest away from the root); such an ancestor exists as the root has weight $2g + 2 \geq 3$. If $v \in V(T_B)$ is odd and $\mathrm{wt}_v = 2$, then p_v must also be odd by Lemma 3.7.2. A repeated application of this fact tells us that if v is an odd leaf such that $\mathrm{wt}_v = 2$, then q_v is odd.

For any vertex $v \in V(T_B)$, let T_v denote the complete subtree of T_B with root v (see section 8 for the definition of complete subtree). Suppose v is an odd vertex such that $\mathrm{wt}_v > 2$. We will now prove the following three claims.

- If $w \in L_v$ and $u \in T_w$, then u is odd and $\mathrm{wt}_u = 2$.

- If $w \in L_v$, then T_w is a chain (that is, every vertex in T_w has at most one child).

- If $v' \in V(T_B)$ is an odd leaf such that $\mathrm{wt}_{v'} = 2$ and $q_{v'} = v$, then there exists a unique $w \in L_v$ such that $v' \in V(T_w)$.

Suppose $w \in L_v$ and $u \in T_w$. Since $u \in T_w$, the definition of the function wt tells us that $\mathrm{wt}_u \leq \mathrm{wt}_w = 2$. On the other hand, Lemma 3.6.2 tells us that $\mathrm{wt}_u \geq 2$. Therefore, $\mathrm{wt}_u = 2$. A repeated application of Lemma 3.7.2 along the path from v to u tells us that u is odd. This proves the first claim.

Suppose $w \in L_v$ and $u \in T_w$. Suppose $u_1, u_2 \in C(u)$ are distinct. The first claim shows $\mathrm{wt}_{u_1} = \mathrm{wt}_{u_2} = 2$. The definition of wt then tells us that $\mathrm{wt}_w \geq \mathrm{wt}_u \geq \mathrm{wt}_{u_1} + \mathrm{wt}_{u_2}$. Since $\mathrm{wt}_w = 2$ and $\mathrm{wt}_{u_1} + \mathrm{wt}_{u_2} = 4$, this is a contradiction. Therefore every vertex in T_v has at most one child, and this proves the second claim.

Suppose $v' \in V(T_B)$ is an odd leaf such that $\mathrm{wt}_{v'} = 2$ and $q_{v'} = v$. Let w be the greatest ancestor of v' such that $\mathrm{wt}_w = 2$ (here greatest ancestor means the ancestor closest to the root). Then, $\mathrm{wt}_{p_w} > 2$. The definition of q then implies $p_w = q_{v'} = v$. This implies that $w \in L_v$. If $w_1, w_2 \in L_v$, then T_{w_1} and T_{w_2} have no vertices in common. This proves that every $v' \in M_v$ can belong to $V(T_w)$ for at most one $w \in L_v$. This finishes the proof of the third claim.

Let $M_v = \{v' \in V(T_B) \mid v' \text{ is an odd leaf, } \mathrm{wt}_{v'} = 2,\ q_{v'} = v\}$. We will now use the claims above to show that there is a bijection $\kappa\colon L_v \to M_v$. Let $w \in L_v$. Let v' be the unique leaf in the chain T_w. Then v' is an odd leaf and $\mathrm{wt}_{v'} = 2$. Furthermore, w is an ancestor of v' such that $\mathrm{wt}_w = 2$ and $\mathrm{wt}_v = \mathrm{wt}_{p_w} > 2$, which shows $q_{v'} = v$. Set $\kappa(w) = v'$. The third claim shows that κ is a bijection. Therefore $\#M_v = \#L_v$.

This implies that

$$\sum_{\substack{v' \text{ is an odd leaf} \\ \mathrm{wt}_{v'}=2}} 2 = \sum_{\substack{v \text{ odd} \\ \mathrm{wt}_v>2}} \sum_{v' \in M_v} 2 = \sum_{\substack{v \text{ odd} \\ \mathrm{wt}_v>2}} 2\ \#M_v = \sum_{\substack{v \text{ odd} \\ \mathrm{wt}_v>2}} 2\#L_v.$$

This tells us that

$$\sum_{v \in V(T_B)} (D''(v) - D'(v)) = \sum_{\substack{v \in V(T_B) \\ v \text{ odd leaf} \\ \mathrm{wt}_v=2}} -2 + \sum_{\substack{v \in V(T_B) \\ v \text{ odd} \\ \mathrm{wt}_v>2}} 2\#L_v = 0. \qquad \square$$

Lemma 3.8.5.

(a) *If* $v \in V(T_B)$, *then*

$$\mathrm{wt}_v \geq l'_v + 3r_v + 2s_v \geq l_v + 2s_v.$$

(b) *If* $r_v = s_v = 0$, *then* $\mathrm{wt}_v = l'_v$.

Proof.

(a) Suppose $u \in C(v)$. Lemma 3.6.2 tells us that $\mathrm{wt}_u \geq 2$. If u is of odd weight, then $\mathrm{wt}_u \geq 3$. Therefore

$$\begin{aligned}
\mathrm{wt}_v &= l'_v + \sum_{u \in C(v)} \mathrm{wt}_u \quad \text{(by the definitions of } l'_v \text{ and wt)} \\
&\geq l'_v + \sum_{\substack{u \in C(v) \\ \mathrm{wt}_u \text{ is odd}}} 3 + \sum_{\substack{u \in C(v) \\ \mathrm{wt}_u \text{ is even}}} 2 \\
&\geq l'_v + 3r_v + 2s_v \\
&= l_v + 2r_v + 2s_v \\
&\geq l_v + 2s_v.
\end{aligned}$$

(b) If $r_v = s_v = 0$, then $C(v) = \emptyset$ and therefore $\mathrm{wt}_v = l'_v + \sum_{u \in C(v)} \mathrm{wt}_u = l'(v)$. □

We are now ready to compare the two discriminants. We first compare the local contributions.

Lemma 3.8.6. *If* $v \in V(T_B)$, *then* $D''(v) \leq d(v)$. *If* v *is even, then* $D''(v) = d(v)$ *if and only if every even child of* v *has weight* 2. *If* v *is odd, then* $D''(v) = d(v)$ *if and only if either* $\mathrm{wt}_v = 2$ *or* $\mathrm{wt}_v = 3$ *and* v *has no even children.*

Proof. If $v \in V(T_B)$ is even, then

$$\begin{aligned}
D''(v) - d(v) &= D'(v) - d(v) \\
&= 2s_v + \sum_{\substack{v' \in C(v) \\ v' \text{ odd}}} \mathrm{wt}_{v'}(\mathrm{wt}_{v'} - 1) - \sum_{v' \in C(v)} \mathrm{wt}_{v'}(\mathrm{wt}_{v'} - 1) \\
&\qquad\qquad \text{(by Lemma 3.6.4 and Equation 3.1)} \\
&= \sum_{\substack{v' \in C(v) \\ v' \text{ even}}} (2 - \mathrm{wt}_{v'}(\mathrm{wt}_{v'} - 1)) \qquad \text{(by Lemma 3.7.2)} \\
&\leq 0 \quad \text{(by Lemma 3.6.2)}.
\end{aligned}$$

From this, it follows that if v is even, then $D''(v) = d(v)$ if and only if the inequality above is actually an equality, that is, if and only if every even child of v has weight 2.

From now on assume $v \in V(T_B)$ is odd. Then

$$\begin{aligned}
D'(v) - d(v) &= 2\big(l_v + s_v\big) - \mathrm{wt}_v(\mathrm{wt}_v - 1) + \sum_{\substack{v' \in C(v) \\ v' \text{ odd}}} \mathrm{wt}_{v'}(\mathrm{wt}_{v'} - 1) - \sum_{v' \in C(v)} \mathrm{wt}_{v'}(\mathrm{wt}_{v'} - 1) \\
&= 2\big(l_v + s_v\big) - \mathrm{wt}_v(\mathrm{wt}_v - 1) - \sum_{\substack{v' \in C(v) \\ v' \text{ even}}} \mathrm{wt}_{v'}(\mathrm{wt}_{v'} - 1),
\end{aligned} \tag{3.2}$$

where the first equality follows from Lemma 3.6.4 and Equation 3.1. Lemma 3.6.2 tells us that $\mathrm{wt}_v \geq 2$. We will handle vertices with $\mathrm{wt}_v = 2$ and with $\mathrm{wt}_v \geq 3$ separately.

Suppose $\mathrm{wt}_v = 2$. Lemma 3.8.5(a) implies that $l'_v + 3r_v + 2s_v \leq \mathrm{wt}_v = 2$. This implies that $r_v = 0$. Lemma 3.8.5(b) implies that either

(i) $l'_v = 2$ and $s_v = 0$, or,

(ii) $l'_v = 0$ and $s_v = 1$.

In both cases, since $r_v = 0$ and v is odd, Lemma 3.7.2 tells us that

$$\sum_{\substack{v' \in C(v) \\ v' \text{ even}}} \mathrm{wt}_{v'}(\mathrm{wt}_{v'} - 1) = 0.$$

In case (i), we have that v is an odd leaf of weight 2 and

$$\begin{aligned}
D''(v) - d(v) &= D'(v) - d(v) - 2 \\
&= 2\big(l_v + s_v\big) - \mathrm{wt}_v(\mathrm{wt}_v - 1) - \sum_{\substack{v' \in C(v) \\ v' \text{ even}}} \mathrm{wt}_{v'}(\mathrm{wt}_{v'} - 1) - 2 \\
&= 2(2+0) - 2(2-1) + 0 - 2 \\
&= 0.
\end{aligned}$$

In case (ii), we have that v is not a leaf and $\mathrm{wt}_v = 2$ and

$$\begin{aligned}
D''(v) - d(v) &= D'(v) - d(v) \\
&= 2\big(l_v + s_v\big) - \mathrm{wt}_v(\mathrm{wt}_v - 1) - \sum_{\substack{v' \in C(v) \\ v' \text{ even}}} \mathrm{wt}_{v'}(\mathrm{wt}_{v'} - 1) \\
&= 2(0+1) - 2(2-1) - 0 \\
&= 0.
\end{aligned}$$

Now suppose $\mathrm{wt}_v \geq 3$. By definition, $\#L_v \leq s_v$.

$$\begin{aligned}
2\#L_v + 2\big(l_v + s_v\big) - \mathrm{wt}_v(\mathrm{wt}_v - 1) &\leq 2\big(l_v + 2s_v\big) - \mathrm{wt}_v(\mathrm{wt}_v - 1) \qquad (3.3)\\
&\leq 2\,\mathrm{wt}_v - \mathrm{wt}_v(\mathrm{wt}_v - 1) \qquad \text{(by Lemma 3.8.5(a))} \\
&= \mathrm{wt}_v(3 - \mathrm{wt}_v) \\
&\leq 0.
\end{aligned}$$

This implies that

$$\begin{aligned}
D''(v) - d(v) &= D'(v) - d(v) + 2\#L_v \\
&= 2\big(l_v + s_v\big) - \mathrm{wt}_v(\mathrm{wt}_v - 1) - \left(\sum_{\substack{v' \in C(v) \\ v' \text{ even}}} \mathrm{wt}_{v'}(\mathrm{wt}_{v'} - 1) \right) + 2\#L_v \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{(by Equation 3.2)} \\
&\leq - \sum_{\substack{v' \in C(v) \\ v' \text{ even}}} \mathrm{wt}_{v'}(\mathrm{wt}_{v'} - 1) \quad \text{(by Equation 3.3)} \\
&\leq 0 \quad \text{(by Lemma 3.6.2)}.
\end{aligned}$$

If v is odd and $D''(v) = d(v)$, then either $\mathrm{wt}_v = 2$ or $\mathrm{wt}_v = 3$ and $r_v = 0$ and $\#L_v = s_v$. By Lemma 3.7.2, $r_v = 0$ if and only if v has no even children. Since every child of v has weight aleast 2 and has weight bounded above by $\mathrm{wt}_v = 3$, Lemma 3.7.2 tells us that $\#L_v = s_v$. $\square$

We are now ready to prove the main theorem.

Proof of Theorem 1.0.1. Construct the proper regular model X as above. Let $n(X)$ denote the number of irreducible components of the special fiber of X and let n be the number of components of the special fiber of the minimal proper regular model $\mathcal{X}$ of C .

To prove $-\operatorname{Art}(X/S) \leq \nu(\Delta)$, sum the inequality of Lemma 3.8.6 over all vertices of T_B and use Lemmas 3.8.3 3.8.4.

We have the equalities

$$-\operatorname{Art}(X/S) = n(X) - 1 + \tilde{f}$$

$$-\operatorname{Art}(\mathcal{X}/S) = n - 1 + \tilde{f}$$

where $\tilde{f}$ is the conductor of the ℓ-adic representation $\mathrm{Gal}(\overline{K}/K) \to \mathrm{Aut}_{\mathbb{Q}_\ell}(H^1_{\mathrm{et}}(X_{\overline{\eta}}, \mathbb{Q}_\ell))$ [Liu94, p.53, Proposition 1]. The minimal proper regular model can be obtained by blowing down some subset (possibly empty) of irreducible components of the special fiber of X_s, so $n \leq n(X)$.

Putting everything together, we get

$$-\operatorname{Art}(\mathcal{X}/S) \leq -\operatorname{Art}(X/S) \leq \nu(\Delta).$$

$\square$

Remark 3.8.7. Lemma 3.8.6 and the proof of Theorem 1.0.1 tell us that $-\operatorname{Art}(\mathcal{X}/S) = \nu(\Delta)$ if and only if the model X is already minimal and the tree T_B satisfies certain strict conditions. Call a subset S of vertices of T_B a connecting chain if

- for any $v \in V(T_B)$, if v lies in the path between two vertices of S, then $v \in S$, and,
- every vertex in S has exactly two neighbours in T_B.

If $-\operatorname{Art}(X/S) = \nu(\Delta)$, then the conditions on the tree T_B tell us that if we replace every connecting chain of 3 or more vertices with a chain of 2 vertices (or equivalently, disregard the length of the chains in T_B and just consider the underlying topological space of T_B), then the tree T_B has height at most 2 (that is, the path from any vertex to the root has at most one other vertex), and all children of the root have at most 3 neighbours. The model X is not minimal if and only if it has contractible -1 curves, and this happens if and only if the tree T_B has an odd vertex v such that $l'_v = 0$, v has an even parent, and v has exactly one child, and that child is even.

Corollary 3.8.8. *Let n be the number of components of the special fiber of the minimal proper regular model of C over R. Then,*

$$n \leq \nu(\Delta) + 1.$$

Proof. Since the conductor $\tilde{f}$ is a nonnegative integer, $n - 1 \leq n - 1 + \tilde{f} \leq \nu(\Delta)$. $\square$

Remark 3.8.9. The equality $n = \nu(\Delta) + 1$ holds if and only if $\tilde{f} = 0$ in addition to all the conditions for $-\operatorname{Art}(\mathcal{X}/S) = \nu(\Delta)$ to hold. By the Néron-Ogg-Shafarevich criterion, $\tilde{f} = 0$ if and only if the Jacobian of C has good reduction.

Chapter 4

Computing sizes of component groups and Tamagawa numbers of Jacobians

4.1 Notation and Definitions

4.1.1 Models of curves

Let R be a Henselian discrete valuation ring with algebraically closed residue field k. Let K be the fraction field of R. A K-variety is a separated scheme of finite type over K. A nice curve C over a field K is a smooth, projective, geometrically integral K-variety of dimension 1. Let $S = \operatorname{Spec} R$. A regular model for a nice curve C is a proper, flat, regular S-scheme X, whose generic fiber is isomorphic to C. A regular S-curve X is an S-scheme that is a regular model for its generic fiber. For a regular S-curve X, let X_s denote its special fiber, let X_η denote its generic fiber and let $X_{\overline{\eta}}$ denote its geometric generic fiber. For a scheme X, let X_{red} denote the associated reduced subscheme. A regular model X is a simple normal crossings (snc) model if the irreducible components of $(X_s)_{\mathrm{red}}$ are smooth, and $(X_s)_{\mathrm{red}}$ has at worst nodal singularities.

In Section 4.3 we assume that R is a Henselian discrete valuation ring with perfect residue field (not necessarily algebraically closed). In this case, let $\overline{k}$ denote the algebraic closure of k and let R^{st} denote the strict Henselization of the Henselian ring R.

4.1.2 Graph theory

A directed weighted multigraph G is a triple $(V(G), E(G), w)$, where

- $V(G)$ is a set called the vertices of G,
- $E(G)$ is a multiset called the directed edges of G, where each element of $E(G)$ is an ordered pair of elements of $V(G)$, and,
- $w\colon E(G) \to \mathbb{N}$ is a non-negative integer valued function called the weight function.

If $e \in E(G)$ is of the form $e = (u, v)$, then the head of e, denoted e^+ equals v and the tail of e, denoted e^-, equals u. The endpoints of e, denoted $D(e)$ is the set $\{e^+, e^-\}$. Let $u, v \in V(G)$. A directed path in G from u to v is an ordered set of directed edges $e_1, \ldots, e_i, \ldots, e_k$ such that $e_1^- = u, e_k^+ = v$ and $e_i^+ = e_{i+1}^-$ for all $i \in [1, k-1]$. If $v \in V(G)$, we will also use v to denote the element $(\delta_{vu})_{u \in V(G)} \in \mathbb{Z}^{V(G)}$. Let $v \in V(G)$. A spanning tree directed into v is a directed weighted multigraph $T = (V(T), E(T), w')$ such that

- $V(T) = V(G)$,
- $E(T) \subset E(G)$ and for every $e \in E(T), w'(e) = w(e)$,
- for every $u \in V(T)$, there is a unique directed path in T from u to v, and,
- $e^- \neq v$ for every $e \in E(T)$.

The weight of a spanning tree equals the product of the weights of all the directed edges in the spanning tree. The vertex v is called a sink if there is a directed path in G from u to v for every vertex $u \in V(G) \setminus \{v\}$. Assume that v is a sink, let $W = V(G) \setminus \{v\}$ and let $\Delta\colon \mathbb{Z}^W \to \mathbb{Z}^W$ be the linear map defined by $u \mapsto (\sum_{e \in E(G), e^- = u} w(e))u - \sum_{t \in W}(\sum_{e \in E(G), e^+ = t, e^- = u} w(e))t$ for every $u \in W$. The map Δ is called the reduced Laplacian of G with respect to the sink v.

We recall the statement of the Matrix-Tree theorem for directed weighted multigraphs.

Theorem 4.1.3. [PPW13, p.5, Th.2.5] *The determinant of the reduced Laplacian of G is the sum of the weights of all its directed spanning trees into the sink.*

A graph G is a directed weighted multigraph such that $E(G)$ is symmetric, that is, there is a bijection $e \mapsto e_{\text{rev}}$ of $E(G)$ such that $e^+ = e^-_{\text{rev}}, e^- = e^+_{\text{rev}}$ and $w(e) = w(e_{\text{rev}})$ for every $e \in E(G)$. In this case, we replace $E(G)$ by $E(G)/\sim$ (the edges of the graph G) where $\sim$ is the equivalence relation that identifies e with e_{rev}. The weight function descends to equivalence classes. A subgraph of a graph $G = (V(G), E(G), w_G)$ is a graph $H = (V(H), E(H), w_H)$ such that $V(H) \subset V(G), E(H) \subset E(G)$ and $w_G|_{E(H)} = w_H$. A spanning tree of a graph G is an equivalence class of directed spanning trees of the underlying directed weighted multigraph, where we declare any two directed spanning trees with the same set of edges as being equivalent. For a graph G, let $S(G)$ denote the set of spanning trees of G. For a vertex v in a graph G, let $N_G(v)$ denote the set of neighbours of v in G, that is, the set of vertices v' such that there is an edge between v and v'. Let $\Delta\colon \mathbb{Z}^{V(G)} \to \mathbb{Z}^{V(G)}$ be the linear map defined by $u \mapsto (\sum_{e \in E(G), u \in D(e)} w(e))u - \sum_{t \in W}(\sum_{e \in E(G), D(e) = \{t,u\}} w(e))t$ for every $u \in V(G)$. Let L denote the matrix of Δ with respect to the standard basis of $\mathbb{Z}^{V(G)}$. The map Δ is called the Laplacian of G. For a vertex $v \in V(G)$, let L_v denote the absolute value of the minor of the element L_{vv} of L. We recall the statement of the Matrix-Tree theorem for graphs.

Theorem 4.1.4. [CS97, p.3, Th.1] *For any vertex v, the number L_v equals the sum of the weights of all the spanning trees of the graph G.*

A cycle C in a graph G is an ordered set of vertices $V(C) := \{v_1, v_2, \ldots, v_k\}$ and an ordered set of edges $E(C) := \{e_1, e_2, \ldots, e_k\}$ such that $D(e_i) = \{v_i, v_{i+1}\}$ for all i, where $v_{k+1} = v_1$. A vertex v belongs to a cycle C if $v \in V(C)$; similarly, an edge e belongs to a cycle C if $e \in E(C)$. A chain C in a graph G is an ordered set of vertices $V(C) := \{v_1, v_2, \ldots, v_k\}$ and an ordered set of edges $E(C) := \{e_1, e_2, \ldots, e_{k-1}\}$ such that $D(e_i) = \{v_i, v_{i+1}\}$ for all i; the length of the chain is k. A connecting chain in a graph G is a chain $\{v_1, v_2, \ldots, v_k\}, \{e_1, e_2, \ldots, e_{k-1}\}$ such that for every integer $i \in [1, k-1]$, there exists a partition $\{V_1, V_2\}$ of the set $V(G)$ (depending on i), such that the only edge with one endpoint in V_1 and another endpoint in V_2 is e_i. Contracting a connecting chain C in a graph G gives rise to another graph G', where $V(G') = V(G)/\sim$, where $\sim$ is the equivalence relation that identifies all the vertices in $V(C)$ and $E(G') = E(G) \setminus E(C)$; an edge $e \in E(G) \setminus E(C)$ between the endpoints $D(e) = \{u, v\}$

in $V(G)$ is interpreted as an edge between the corresponding equivalence classes in $V(G')$.

4.2 Explicit computation of the sizes of component groups

Let C be a nice K-curve and let X be a regular model of the curve C over R. Let G be the graph defined as follows. The vertices of G are the irreducible components of X_s. For each vertex v of G, let Γ_v denote the corresponding irreducible component of X_s and let m_v denote the multiplicity of Γ_v in X_s. The number of edges between two distinct vertices v and w equals $\Gamma_v.\Gamma_w$, and the weight of each such edge e (denoted $\mathrm{wt}(e)$) equals $m_v m_w$. The graph G is called the weighted dual graph of X_s. In this section, we will compute the size of the component group of the Néron model of the Jacobian of C over R, in terms of the graph G. Special cases of this formula already appear in [Lor93], see for instance Proposition 4.19 therein. A version of this formula also appears embedded in the proof of [Lor89, p.489, Cor. 3.5].

4.2.1 The formula

Let m be the greatest common divisor of the m_v for $v \in V(G)$. Let $n = \#V(G)$. Let $\alpha \colon \mathbb{Z}^{V(G)} \to \mathbb{Z}$ be the map $(a_v)_{v\in V(G)} \mapsto \sum_{v\in V(G)} m_v a_v$. Let $(M_{vv'})$ be the $n \times n$ matrix defined as follows. If $v, v' \in V(G)$, then $M_{vv'} = \Gamma_v.\Gamma_{v'}$. Let $\beta \colon \mathbb{Z}^{V(G)} \to \mathbb{Z}^{V(G)}$ be the linear operator corresponding to the matrix M. Since $(\sum_{v'\in V(G)} m_{v'}\Gamma_{v'}).\Gamma_v = 0$, we have that $\operatorname{im}\beta \subset \ker\alpha$. [BLR90, p.274, 9.6, Th.1] tells us that the component group of the special fiber of the Néron model of the Jacobian is equal to $\ker\alpha/\operatorname{im}\beta$. Call this group Φ.

Theorem 4.2.2.

$$\#\Phi = m^2 \left(\sum_{T\in S(G)} \prod_{v\in V(T)} m_v^{\#N_T(v)-2} \right). \tag{4.1}$$

Proof. Let K_v denote the absolute value of the $(n-1)\times(n-1)$ minor that we get from M by deleting the row and column corresponding to the vertex v. [BLR90, p. , 9.6,Cor. 4] shows that if $v \in V(G)$, then the size of the component group is equal to $(\frac{m}{m_v})^2 K_v$. Let $(N_{vv'})$ be the $n \times n$ matrix defined by $N_{vv'} = m_v m_{v'} M_{vv'}$ for every $v, v' \in V(G)$. For any vertex v of G, let L_v denote the absolute value of the $(n-1)\times(n-1)$ minor that we get from N by deleting the row and column corresponding to the vertex v. Fix a vertex $\tilde{v}$ of G. Then

$$\begin{aligned}
\#\Phi &= \left(\frac{m}{m_{\tilde{v}}}\right)^2 K_{\tilde{v}} \\
&= \frac{m^2}{(\prod_{v\in V(G)} m_v)^2} L_{\tilde{v}} \\
&= \frac{m^2}{(\prod_{v\in V(G)} m_v)^2} \sum_{T\in S(G)} \prod_{e\in E(T)} \mathrm{wt}(e) \qquad \text{(using Theorem 4.1.4)} \\
&= \frac{m^2}{(\prod_{v\in V(G)} m_v)^2} \sum_{T\in S(G)} \prod_{v\in V(T)} m_v^{\#N_T(v)} \\
&= m^2 \left(\sum_{T\in S(G)} \prod_{v\in V(T)} m_v^{\#N_T(v)-2} \right) \qquad \text{(since } V(T)=V(G) \text{ for any spanning tree } T \text{ of } G\text{).}
\end{aligned}$$

□

Remark 4.2.3. The formula also holds under the assumption that R is merely strictly Henselian, provided we also assume that X admits a section after a finite étale extension.

Remark 4.2.4. If the dual graph is a tree, then the formula simplifies to $\#\Phi = m^2 \prod_{v\in V(G)} m_v^{\#N_G(v)-2}$.

Remark 4.2.5. Specializing to the case where C is an elliptic curve, we recover the correct sizes of the component groups for each of the Kodaira types.

4.2.6 Applications of the formula

Criterion for uniform bounds on the sizes of component groups

In the case of elliptic curves, the size of the component group is bounded above by 4 if we omit curves of reduction type I_n. In the theorem below, we provide a generalization of this fact for higher genus curves. We begin by proving a lemma in graph theory.

Lemma 4.2.7. *A vertex v of a graph G belongs to some cycle of G if and only if there exists a spanning tree T of G and an edge $e \in E(G)$ such that $v \in D(e)$ and $e \notin E(T)$.*

Sketch of proof. The 'if' direction follows from the fact that adding any edge of $E(G) \setminus E(T)$ to a spanning tree produces a graph that contains a cycle. The 'only if' direction follows from the fact that any spanning tree of $(V(G), E(G) \setminus \{e\}, w)$ is a spanning tree of G, where e is one of the two edges in a cycle having v as an endpoint. $\square$

Theorem 4.2.8. *Assume that X/S is the minimal proper regular model of the curve C/K and that the genus g of C satisfies $g \geq 2$. Let G be the dual graph of X_s. Assume further that if $v \in V(G)$ corresponds to a (-2)-curve (that is, a $\mathbb{P}^1_k$ of self-intersection -2), then v does not belong to any cycle of G. Then there exists an integer $n(g)$, depending only on the genus g of C, such that the size of the component group of the special fiber of the Néron model of the Jacobian of C is bounded above by $n(g)$.*

Proof. Lemma 4.2.7 and the assumption on (-2)-curves in the theorem imply that the edges in any chain of (-2)-curves belong to every spanning tree. In other words, every chain of (-2)-curves is a connecting chain. Contracting all chains of (-2)-curves produces a graph G'. [Win74, p.224, Cor. 4.3] implies that G' is the dual graph of the special fiber of a regular S-curve X'. Since a vertex v that is part of a connecting chain has exactly two neighbours in every spanning tree, the exponent $N_T(v) - 2 = 0$ for every spanning tree T for every such vertex. It follows that such a vertex does not contribute to the size of the component group by formula (4.1). This implies that the size of the component group of G' equals the size of the component group of G. Since we also assumed that X is minimal and $g \geq 2$, [AW71, p.375, Th. 1.6] implies that there are only finitely many possibilities for the weighted dual graph of X'_s, if we fix the genus g of X'_η. For a fixed g, we can compute the size of the component group of the special fiber of the Néron model of the Jacobian for each of these finitely many weighted dual graphs and take the maximum of these numbers to be $n(g)$. This finishes the proof. $\square$

Remark 4.2.9. A naïve bound for $n(g)$ that one gets by examining the Artin-Winters argument is $n(g) \sim O((2g-2)^{8(2g-2)^2})$. It might be possible to improve this bound by analyzing the combinatorics of these graphs more carefully.

Structure of the component group: periodicity

We will now prove an analogue of [BN07, p.782, Cor. 4.7]. The key step in the proof of Theorem 4.2.8 is the fact that contracting connecting chains in the dual graph does not affect the size of the component group. The structure of the group, however, does depend on the length of the chain. For example, the Kodaira types I_n^* all have component groups of size 4, but the component group is either $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ or $\mathbb{Z}/4\mathbb{Z}$ depending on whether n is even or odd. In this example, the structure of the component group only depends on the length of the connecting chain modulo 2. This suggests that if we have a family of dual graphs which only differ in the length of a single connecting chain, and if all the vertices in the connecting chain have multiplicity m, then the structure of the component group should be m-periodic, i.e., it should only depend on the length of the chain modulo m. More precisely,

Theorem 4.2.10. *Let G be the dual graph of the special fiber of a regular S-curve. Let L be a connecting chain in G of length ℓ. Let the multiplicity of the vertices in the chain L be m. Assume $\ell > m$. Let G' be the graph corresponding to a partial contraction of the chain L, i.e., where we replace the chain L in G by a chain L' of length $\ell - m$. Then G' is the dual graph of the special fiber of another regular S-curve and the component group of G' is isomorphic to the component group of G.*

Proof. [Win74, p.224, Cor. 4.3] shows that G' is the dual graph of the special fiber of a regular S-curve. Let $\Phi(G) = \ker\alpha/\operatorname{im}\beta$ and let $\Phi(G') = \ker\alpha'/\operatorname{im}\beta'$ as in section 4.2.1. Let $V(L) = \{v_1, v_2, \ldots, v_\ell\}$ such that for any $i \in [2, \ell-1]$, the vertex v_i is adjacent to v_{i-1} and v_{i+1} and let $V(L') = \{v'_1, v'_2, \ldots, v'_{\ell-m}\}$ (again v'_i is adjacent to v'_{i-1} and v'_{i+1}). Extend the natural map $f\colon V(G)\setminus V(L) \to V(G')\setminus V(L')$ to $V(G)$ by setting $f(v_i) = v'_i$ if $i \leq \ell - m$ and $f(v_i) = 0$ if $i > \ell - m$. Dualizing this map gives us a map $f^*\colon \mathbb{Z}^{V(G')} \to \mathbb{Z}^{V(G)}$.

Since $\alpha f^* = \alpha'$, it follows that f^* restricts to a map from $\ker\alpha'$ to $\ker\alpha$. We now claim that it maps $\operatorname{im}\beta'$ into $\operatorname{im}\beta$. By linearity, it suffices to check that $f^*(\beta'(v')) \in \operatorname{im}\beta$ for every $v' \in V(G')$. We divide this calculation into three cases.

- If $v' \in N_{G'}(v'_{\ell-m})) \setminus \{v'_{\ell-m-1}\}$, then $f^*(\beta'(v')) = \beta(f^*(v')) + \Gamma_{v'}.\Gamma_{v'_{\ell-m}}(v_{\ell-m} - v_\ell)$.
- $f^*(\beta'(v'_{\ell-m})) = \beta(v_{\ell-m} + v_\ell) - v_{\ell-m+1} - v_{\ell-1} + 2v_\ell$.

- In all other cases, $f^*(\beta'(v)) = \beta(f(v))$.

To prove the inclusion $f^*(\operatorname{im}\beta') \subset \operatorname{im}\beta$, it suffices to prove that $v_{\ell-m} - v_\ell \in \operatorname{im}\beta$ and that $-v_{\ell-m+1} - v_{\ell-1} + 2v_\ell \in \operatorname{im}\beta$. The proof of these facts is most conveniently stated using the language of chip-firing games; we will now recall some of the basic terminology that we will use.

An element $A \in \mathbb{Z}^{V(G)}$ will be referred to as a configuration of chips. Firing a single vertex $v \in V(G)$ replaces the configuration A by $A + \beta(v)$. Given $S \subset V(G)$ and a function $n\colon S \to \mathbb{Z}$, firing the pair (S, n) replaces A by $A + \sum_{v \in S} n(v)\beta(v)$ (the integer $n(v)$ represents the number of times the vertex v is fired). If we talk about firing a set S without specifying a function n, then we take n to be the multiplicity function, i.e., $n(v) = m_v$.

Since L is a connecting chain, the complement $G \setminus L$ breaks up into two connected components D and D'. Assume that D is adjacent to v_1 and D' is adjacent to v_ℓ. For $1 \leq i \leq \ell - 1$, let D_i be the connected graph that you get by adding the vertices $\{v_1, v_2, \ldots, v_i\}$ (and the corresponding edges) to D. Let $D_0 = D$. Note that firing D_i has the effect of replacing a configuration A by $A - mv_i + mv_{i+1}$; this follows easily from the relation $\sum_w m_w \Gamma_v.\Gamma_w = 0$ for any $v \in V(G)$.

We will now start with the configuration $v_{\ell-m} - v_\ell$ and perform a sequence of chip-firing moves to reach the configuration 0. For $\ell - m + 1 \leq i \leq \ell - 1$, let $S_i = V(D_i)$, and let $n_i \in \mathbb{Z}^{S_i}$ be the function which when restricted to $V(D_{i-1})$ is the multiplicity function and on v_i takes the value $(i - \ell + m)(m - 1)$. One can then check that firing the pair (S_i, n_i) (which is the same as firing D_{i-1} followed by $(v_i, (i - \ell + m)(m - 1)v_i)$) has the following effect on the configuration A:

$$\begin{aligned} A &\mapsto (A - mv_{i-1} + mv_i) + (i - \ell + m)(m - 1)(v_{i-1} - 2v_i + v_{i+1}) \\ &= A + ((i - \ell + m - 1)(m - 1) - 1)v_{i-1} - ((2i - 2\ell + 2m - 1)(m - 1) - 1)v_i \\ &\quad + (i - \ell + m)(m - 1)v_{i+1}. \end{aligned} \tag{4.2}$$

Let $\tilde{S}_i = S_{\ell-m+i}$ and let $\tilde{n}_i = n_{\ell-m+i}$ for every i satisfying $1 \leq i \leq m - 1$. We now claim that the sequence of chip-firing moves $(\tilde{S}_1, \tilde{n}_1), (\tilde{S}_2, \tilde{n}_2), \ldots, (\tilde{S}_j, \tilde{n}_j)$ for some $j \leq m - 1$ has the effect of relpacing the configuration $v_{\ell-m} - v_\ell$ by $(-(j-\ell+m)m + j - \ell + m + 1)v_j + (j-\ell+m)(m-1)v_{j+1} - v_\ell$. We will prove this by induction on j. For $j = \ell - m + 1$, this follows from the formula 4.2. Now assume the statement for j and we will prove this for $j+1$. Formula 4.2 together with the induction hypothesis tells us that the effect of the sequence of chip-firing moves $(\tilde{S}_1, \tilde{n}_1), (\tilde{S}_2, \tilde{n}_2), \ldots, (\tilde{S}_{j+1}, \tilde{n}_{j+1})$ is

$$\begin{aligned}
v_{\ell-m} - v_\ell &\mapsto (-m+2)v_{\ell-m+1} + (m-1)v_{\ell-m+2} \\
&\mapsto (-2m+3)v_{\ell-m+2} + 2(m-1)v_{\ell-m+3} \\
&\mapsto \cdots \\
&\mapsto (-(j-\ell+m)m + j - \ell + m + 1)v_j + (j-\ell+m)(m-1)v_{j+1} - v_\ell \\
&\mapsto (-(j-\ell+m)m + j - \ell + m + 1)v_j + (j-\ell+m)(m-1)v_{j+1} - v_\ell \\
&\quad + ((j-\ell+m)(m-1)-1)v_j - ((2j-2\ell+2m+1)(m-1)-1)v_{j+1} \\
&\quad + (j-\ell+m+1)(m-1)v_{j+2} \\
&= (-(j-\ell+m+1)(m-1)+1)v_{j+1} + (j-\ell+m+1)(m-1)v_{j+2} - v_\ell \\
&= (-(j-\ell+m+1)m + j - \ell + m + 2)v_{j+1} + (j-\ell+m+1)(m-1)v_{j+2} - v_\ell.
\end{aligned}$$

This tells us that the sequence of chip-firing moves $(\tilde{S}_1, \tilde{n}_1), (\tilde{S}_2, \tilde{n}_2), \ldots, (\tilde{S}_{\ell-1}, \tilde{n}_{\ell-1})$ takes $v_{\ell-m} - v_\ell$ to $m(m-2)(v_\ell - v_{\ell-1})$. Let $S' = V(D') \cup \{v_\ell\}$ and let $n' \in \mathbb{Z}^{S'}$ be the function which sends a vertex v to $m_v(m-2)$. One can then check that firing the pair (S', n') sends $m(m-2)(v_\ell - v_{\ell-1})$ to 0.

The proof of $-v_{\ell-m+2} - v_{\ell-1} + 2v_\ell \in \operatorname{im}\beta$ is similar. For $1 \le i \le m-1$, let $S_i = \{v_{\ell-m+i}\}$ and $n_i = (i-1)v_i$. Then one can check that the sequence of chip-firing moves $(S_2, n_2), (S_3, n_3), \ldots, (S_{m-1}, n_{m-1})$ takes $-v_{\ell-m+2} - v_{\ell-1} + 2v_\ell$ to $m(v_\ell - v_{\ell-1})$. Let $n'' \in \mathbb{Z}^{S'}$ be the function $v \mapsto mm_v$. One can then check that firing the pair (S', n'') sends $m(v_\ell - v_{\ell-1})$ to 0.

So f^* induces a well-defined map from $\Phi(G')$ to $\Phi(G)$. Since $|\Phi(G')| = |\Phi(G)|$ (by the proof of Theorem 4.2.8), in order to prove that this map is an isomorphism, it suffices to prove that it is a surjection. Let $A \in \ker\alpha$. For $0 \le i \le m$, let $w_i = v_{\ell-i}$. We will inductively construct a sequence of elements $A_0, A_1, \ldots, A_m$ such that

- $A_0 = A$ and $A_m \in \operatorname{im} f^*$,
- For every i satisfying $1 \leq i \leq m$, the configuration A_{i+1} is obtained from A_i by firing the vertex w_{i+1} some number of times.
- For every i satisfying $1 \leq i \leq m$, the coefficient of w_j for $j \leq i-1$ in A_i is 0.

For the inductive step, suppose $A_i = \sum -a_v v$. Let A_{i+1} be the result of firing the pair $(\{w_{i+1}\}, a_{w_i} w_{i+1})$ in the configuration A_i. Then,

$$A_{i+1} = A_i + a_{w_i}(w_i + w_{i+2}) - 2a_{w_i} w_{i+1}.$$

This tells us that the coefficient of w_j for $j \leq i-1$ in A_{i+1} is the same as that in A_i, and that the coefficient of w_i in A_{i+1} is $-a_{w_i} + a_{w_i} = 0$. So to finish the construction, it now suffices to prove that $A_m \in \operatorname{im} f^*$.

The natural identification of $V(G')$ with $V(D_{\ell-m}) \cup V(D')$ tells us that f^* restricts to an isomorphism of $\mathbb{Z}^{V(G')}$ with the subset of elements of $\mathbb{Z}^{V(G)}$ supported on $V(D_{\ell-m}) \cup V(D')$. Since f is multiplicity preserving, this isomorphism maps elements of $\ker \alpha'$ to elements of $\ker \alpha$ supported on $V(D_{\ell-m}) \cup V(D')$. Since $A_m \in \ker \alpha$ and is supported on $V(D_{\ell-m}) \cup V(D')$, this shows that $A_m \in \operatorname{im} f^*$. Since A_m is obtained from A_0 by a sequence of chip-firing moves, $A_m - A_0 \in \operatorname{im} \beta$. This completes the proof of surjectivity of f^*. □

The Néron component series for Jacobians

Let $\mathbb{N}'$ be the set of positive integers prime to the residue characteristic of k. For $d \in \mathbb{N}'$, let $K(d)$ denote the unique tamely ramified extension of K of degree d and let $R(d)$ denote its ring of integers. Let $\mathcal{A}$ denote the Néron model of the Jacobian of X and let $\mathcal{A}(d)$ denote the Néron model of the Jacobian of $X \times_K K(d)$. Since the construction of the Néron model does not commute with base extension, it is natural to ask how much $\mathcal{A} \times_R R(d)$ and $\mathcal{A}(d)$ differ. The behaviour of the size of the component group under tame extensions is recorded in a precise fashion by the Néron component series

$$\sum_{d \in \mathbb{N}'} |\Phi(\mathcal{A}(d))| T^d.$$

One of the key results in [HN] is a proof of the rationality of the Néron component series[HN, p. 25, Ch. 3, Th. 3.1.5]. The main ingredient in the proof of this result is [HN, p.24, Ch.3, Prop.3.1.1], which involves a reduction to the mixed characteristic case. We provide an alternate proof of this proposition below, that works in both the equal characteristic and mixed characteristic cases, using the explicit formula 4.1 and the description of the special fiber of suitable models of $X \times_K K(d)$ as described in [HN, Ch.3, p.17, Prop. 1.3.2, p.23, Lemma 2.3.2].

We begin with a lemma about continued fractions. Let n and r be positive integers, such that $0 < r < n$ and $\gcd(r, n) = 1$. Let m_1, m_2 be positive integers such that $\gcd(m_1, n) = \gcd(m_2, n) = 1$ and $rm_2 + m_1 = 0 \bmod n$. The Hirzebruch-Jung continued fraction expansion of n/r is given by

$$\frac{n}{r} = b_1 - \cfrac{1}{b_2 - \cfrac{1}{\cdots - \cfrac{1}{b_\lambda}}}.$$

If $\lambda = 1$, let $\mu_1 = (m_1 + m_2)/n$. Otherwise, let μ_i be the unique solution to the system of equations

$$\begin{bmatrix} b_1 & -1 & & & & & \\ -1 & b_2 & -1 & & & & \\ & -1 & b_3 & -1 & & & \\ & & & \ddots & & & \\ & & & & -1 & b_{\lambda-1} & -1 \\ & & & & & -1 & b_\lambda \end{bmatrix} \begin{bmatrix} \mu_1 \\ \mu_2 \\ \vdots \\ \mu_{\lambda-1} \\ \mu_\lambda \end{bmatrix} = \begin{bmatrix} m_2 \\ 0 \\ \vdots \\ 0 \\ m_1 \end{bmatrix}$$

For the proof of uniqueness, see [CES03, p.355, Cor. 2.4.3].

Lemma 4.2.11.

$$\frac{n}{m_1 m_2} = \frac{1}{m_2\mu_1} + \frac{1}{\mu_1\mu_2} + \cdots + \frac{1}{\mu_{\lambda-1}\mu_\lambda} + \frac{1}{\mu_\lambda m_1}.$$

Proof. We will prove this by induction on the length λ of the continued fraction expansion of n/r. Since $\gcd(n, r) = 1$, if $\lambda = 1$, it follows that $r = 1, b_1 = n$ and $\mu_1 = (m_1 + m_2)/n$.

$$\frac{1}{m_2\mu_1} + \frac{1}{\mu_1 m_1} = \frac{m_1 + m_2}{\mu_1 m_1 m_2} = \frac{n}{m_1 m_2}.$$

Now assume $\lambda > 1$. Let $n = b_1 r - r'$. We have $rm_2 + m_1 = n\mu_1$ by [CES03, p.355, Cor. 2.4.3]. One can then check that the continued fraction expansion of r/r' is

$$\frac{r}{r'} = b_2 - \cfrac{1}{b_3 - \cfrac{1}{\cdots - \cfrac{1}{b_\lambda}}}.$$

We also have

$$\begin{aligned} m_1 + r'\mu_1 &= m_1 + (b_1 r - n)\mu_1 \\ &= m_1 + b_1 r\mu_1 - n\mu_1 \\ &= m_1 + b_1 r\mu_1 - m_1 - rm_2 \\ &= r(b_1\mu_1 - m_2) \\ &= r\mu_2. \end{aligned}$$

The induction hypothesis applied to the tuple (r, r', m_1, μ_1) (in place of the original (n, r, m_1, m_2)) then tells us that

$$\frac{r}{m_1\mu_1} = \frac{1}{\mu_1\mu_2} + \ldots + \frac{1}{\mu_{\lambda-1}\mu_\lambda} + \frac{1}{\mu_\lambda m_1}.$$

Now,

$$\begin{aligned} \frac{n}{m_1 m_2} &= \frac{m_1 + rm_2}{\mu_1 m_1 m_2} \\ &= \frac{1}{m_2\mu_1} + \frac{r}{m_1\mu_1} \\ &= \frac{1}{m_2\mu_1} + \frac{1}{\mu_1\mu_2} + \cdots + \frac{1}{\mu_{\lambda-1}\mu_\lambda} + \frac{1}{\mu_\lambda m_1}. \end{aligned}$$ □

Since we assumed that the genus of X_η is non-zero, it follows that there exists a minimal snc model $X_{\min}$ of X_η [Liu02, p.426, Ch. 9, Prop. 3.36]. For the rest of this section we set $X = X_{\min}$.

Definition 4.2.12. [HN, p.20, Ch. 3, Def. 2.2.2] A component Γ of X_s is principal if

- either the genus of Γ is non-zero, or,
- $\Gamma \setminus \Gamma^\circ$ contains at least 3 points.

(Here Γ° denotes the intersection of Γ with the complement of the singular locus of $(X_s)_{\mathrm{red}}$.)

Definition 4.2.13. [HN, p.20, Ch. 3, Def. 2.2.3] Let I denote the set of principal components of X. The stabilization index of X_η, denoted $e(X_\eta)$, is given by

$$e(X_\eta) = \operatorname*{lcm}_{\Gamma \in I} m_\Gamma.$$

Proposition 4.2.14. [HN, p.24, Ch. 3, Prop. 3.1.1] *Let $d \in \mathbb{N}'$ be an integer prime to $e(X_\eta)$ and let t denote the toric rank of $\mathcal{A}$. Then,*

$$|\Phi(\mathcal{A}(d))| = d^t |\Phi(\mathcal{A})|.$$

Proof. Let X' be the minimal desingularization of the normalization of $X \times_R R(d)$. Let Γ be the dual graph of X_s and let Γ' be the dual graph of X'_s. It follows from [HN, p.17, Ch.3, Prop. 1.3.2] and [HN, p.23, Ch.3, Lem. 2.3.2] that X' is a snc model and that Γ' is a subdivision of Γ. This subdivision gives rise to a natural surjective map σ from $E(\Gamma')$ to $E(\Gamma)$, where we map an edge of Γ' to the unique edge in Γ that it is a subdivision of.

First assume that X_s is a loop of rational curves, where the curves in the loop have multiplicity strictly greater than 1; this implies that X_η is a genus 1 curve without a rational point whose Jacobian has multiplicative reduction. Finish this argument.

Now assume that X_s is not a loop of rational curves, where the curves in the loop have multiplicity strictly greater than 1. Let v and w are neighbouring vertices of Γ. In this case, the proof of [HN, p.23, Lemma 2.3.2] tells us that $\gcd(d, m_v, m_w) = 1$. Let

- $h_v = \gcd(d, m_v)$,

- $h_w = \gcd(d, m_w)$,
- $m_v = h_v m'_v$,
- $m_w = h_w m'_w$, and,
- $d = h_v h_w d'$.

Then $dm'_v m'_w = d' m_v m_w$. Let v' and w' be the corresponding vertices of Γ' and let $u_1, u_2, \ldots, u_\lambda$ be the intermediate vertices. Then [HN, p.23, Lemma 2.3.2] tells us that $m_{v'} = m'_v$ and $m_{w'} = m'_w$. [HN, p.28, Ch.3, Prop. 4.2.5] tells us how to calculate $m_{u_1}, m_{u_2}, \ldots, m_{u_\lambda}$. Let $\mu_1, \mu_2, \ldots, \mu_\lambda$ be as in Lemma 4.2.11, with $n = d', m_2 = m'_v$ and $m_1 = m'_w$. Then $m_{u_i} = \mu_i$. Lemma 4.2.11 then tells us that

$$\frac{d}{m_v m_w} = \frac{d'}{m'_v m'_w} = \frac{1}{m'_v \mu_1} + \frac{1}{\mu_1 \mu_2} + \cdots + \frac{1}{\mu_{\lambda-1}\mu_\lambda} + \frac{1}{\mu_\lambda m'_w}. \tag{4.3}$$

For a graph G, let $\mathcal{C}(G)$ be a subset of the power set of $E(G)$ defined as follows. A subset B of $E(G)$ belongs to $\mathcal{C}(G)$ if and only if deleting the edges in B gives rise to a spanning tree of G. The first Betti number $t(G)$ of a connected graph G equals $|V(G)| - |E(G)| + 1$. Since $|V(G)| = |V(T)|$ for any connected graph G and for any spanning tree T of G, and the first Betti number of a tree equals 2, it follows that any element of $\mathcal{C}(G)$ for a connected graph G corresponds to a subset of $E(G)$ of size $t(G)$. For a spanning tree T of a dual graph G, let $\varphi(T) = \prod_{v \in V(G)} m_v^{N_T(v)-2}$. The function φ can equivalently be viewed as a function on the set $\mathcal{C}(G)$, as we now explain. For any element $B \in \mathcal{C}(G)$, set $\varphi(B)$ equal to the value of φ at the spanning tree that we get by deleting the edges corresponding to the subset B.

We claim that the map $\sigma \colon E(\Gamma') \to E(\Gamma)$ naturally extends to a surjective map $\mathcal{C}(\Gamma') \to \mathcal{C}(\Gamma)$ given by $B' \mapsto \sigma(B')$. To prove this claim, it suffices to check that $\sigma(B') \in \mathcal{C}(\Gamma)$ for every $B' \in \mathcal{C}(\Gamma')$. This follows from the fact that the graph Γ' is a subdivision of Γ. Let $B' \in \mathcal{C}(\Gamma')$ and let $\sigma(B') = B$. Let T and T' be the spanning trees corresponding to B and B'.

- Since Γ' is a subdivision of Γ and t equals the first Betti number of Γ([BLR90, 9.2.5, 9.2.8]), it follows that $t(\Gamma') = t(\Gamma) = t$.

- There is a bijective, multiplicity-preserving correspondence between the vertices of T' of degree greater than or equal to 3 to those of T of degree greater than or equal to 3.

- Since $|B| = |B'| = t$, it follows that for every $b \in B$, there is a unique element in $\sigma^{-1}(b) \cap B'$.

- Conversely, given $B \in \mathcal{C}(\Gamma)$, and an element $b' \in \sigma^{-1}(b)$ for every $b \in B$, then $\{b' \mid b \in B\} \in \mathcal{C}(\Gamma')$.

Putting the last four facts together with Equation 4.3, we get that for any $B \in \mathcal{C}(\Gamma)$,

$$\sum_{B' \in \sigma^{-1}(B)} \varphi(B') = d^t \varphi(B).$$

Formula 4.1 tells us that

$$|\Phi(\mathcal{A})| = \sum_{T \in S(\Gamma)} \varphi(T) = \sum_{B \in \mathcal{C}(\Gamma)} \varphi(B).$$

Similarly,

$$|\Phi(\mathcal{A}(d))| = \sum_{T' \in S(\Gamma')} \varphi(T') = \sum_{B' \in \mathcal{C}(\Gamma')} \varphi(B').$$

Putting the last three equations together, we get that

$$|\Phi(\mathcal{A}(d))| = \sum_{B' \in \mathcal{C}(\Gamma')} \varphi(B') = \sum_{B \in \mathcal{C}(\Gamma)} \sum_{B' \in \sigma^{-1}(B)} \varphi(B') = d^t |\Phi(\mathcal{A})|. \qquad \square$$

4.3 Explicit computation of Tamagawa numbers

In this section, we do not assume that the residue field k is algebraically closed and we let $\overline{k}$ denote the algebraic closure of k. Let C be a nice curve over K and let X be a regular model for C over R. The component group Φ which was defined in Section 4.2 is in fact the set of $\overline{k}$ points of a finite étale k-group scheme φ, called the component group scheme. The size of the set of k points of this scheme is the Tamagawa number associated to the curve C. In this section, we will show that we can use a version of the matrix-tree theorem for directed weighted multigraphs to compute Tamagawa numbers.

4.3.1 The quotient graph $\widetilde{G}$

Let $X^{\mathrm{st}} = X \times_R R^{\mathrm{st}}$. Let G denote the dual graph of X_s^{st}, and let $V = V(G)$. We now define a directed weighted multigraph $\widetilde{G}$, which we call the quotient graph. The set of vertices of $\widetilde{G}$, which we denote $\widetilde{V}$, is the space of orbits of V under the natural action of $\mathrm{Gal}(\overline{k}/k)$. The set $\widetilde{V}$ can also naturally be identified with the set of irreducible components of X_s. The natural map $X_s^{\mathrm{st}} \to X_s$ maps irreducible components to irreducible components, and this gives rise to a quotient map $\pi\colon V \to \widetilde{V}$ where we map a vertex v to the vertex corresponding to the image of Γ_v under this map. For any two vertices $\tilde{w}$ and $\tilde{v}$ in $\widetilde{V}$, the number of directed edges from $\tilde{w}$ to $\tilde{v}$, which we denote $\alpha_{\tilde{w}\tilde{v}}$, is defined as follows. Fix any $w \in \pi^{-1}(\tilde{w})$. Then, $\alpha_{\ tildew\tilde{v}} = \sum_{v \in \pi^{-1}(\tilde{v})} \Gamma_v.\Gamma_w$. This can be checked to be well-defined, independent of the choice of $w \in \pi^{-1}(\tilde{w})$. This defines the vertices and directed edges of $\widetilde{G}$.

We now define the weight function on $\widetilde{G}$. For every $\tilde{v} \in \widetilde{V}$, let $|\tilde{v}|$ denote the size of the orbit corresponding to $\tilde{v}$. For any $\tilde{v} \in \widetilde{V}$, let $\Gamma_{\tilde{v}}$ denote the corresponding irreducible component of X_s, and let $m_{\tilde{v}}$ denote the multiplicity of $\Gamma_{\tilde{v}}$ in the special fiber X_s. For any directed edge e of the form $(\tilde{w}, \tilde{v})$, the weight of the edge e, denoted $\mathrm{wt}(e)$, equals $m_{\tilde{v}} m_{\tilde{w}} |\tilde{w}|$.

4.3.2 Preliminaries

We first recall the statement of [BL99, p.282, Th.1.17]. We restate it in our notation, for the convenience of the reader. The statement of the theorem involves two maps β and α, whose definitions we now recall.

$$\begin{aligned} \beta\colon \mathbb{Z}^{\widetilde{V}} &\to \mathbb{Z} \\ (b_{\tilde{v}})_{\tilde{v} \in \widetilde{V}} &\mapsto \sum_{\tilde{v} \in \widetilde{V}} b_{\tilde{v}} m_{\tilde{v}} |\tilde{v}| \end{aligned}$$

Fix some $w \in \pi^{-1}(\tilde{w})$ for every $\tilde{w} \in \widetilde{V}$. The map below is well-defined, independent of this choice.

$$\begin{aligned}
\alpha \colon \mathbb{Z}^{\widetilde{V}} &\to \mathbb{Z}^{\widetilde{V}} \\
(b_{\tilde{v}})_{\tilde{v} \in \widetilde{V}} &\mapsto \left(\sum_{\tilde{v} \in \widetilde{V}} b_{\tilde{v}} \sum_{v \in \pi^{-1}(\tilde{v})} \Gamma_v . \Gamma_w \right)_{\tilde{w} \in \widetilde{V}}
\end{aligned}$$

In the formula above, $\Gamma_v.\Gamma_w$ denotes the intersection number of the components Γ_v and Γ_w of X_s^{st}. By using the fact that the intersection number of any vertical divisor of X_s^{st} with the special fiber X_s^{st} is 0, one can check that $\mathrm{im}(\alpha) \subset \ker(\beta)$.

Theorem 4.3.3. *[*[BL99, p.282, Th.1.17]*] Assume that* $\mathrm{Gal}(\overline{k}/k)$ *is procyclic. Let X be a regular model of a nice curve C over K. Let g be the genus of C. Let m be the greatest common divisor of the $(m_{\tilde{v}})_{\tilde{v} \in \widetilde{V}}$ and let $\tilde{m}$ be the greatest common divisor of the $(m_{\tilde{v}}|\tilde{v}|)_{\tilde{v} \in \widetilde{V}}$. Let φ denote the étale k-group scheme corresponding to the component group of the Néron model of the Jacobian of C over S. Then, there exists an exact sequence*

$$0 \to \ker \beta / \mathrm{im}\, \alpha \to \varphi(k) \to (qm\mathbb{Z})/\tilde{m}\mathbb{Z} \to 0 \tag{4.4}$$

where $q = 1$ if $\tilde{m}|g - 1$ and $q = 2$ otherwise.

Remark 4.3.4. In [BL99, p.279, Cor. 1.7], the authors prove that $qm|\tilde{m}$ where q is defined as in the statement of the theorem, and hence the the quotient group on the right in 4.4 is well-defined.

Remark 4.3.5. The theorem above follows from an analysis of the long exact sequence in Galois cohomology associated to the short exact sequence of $\mathrm{Gal}(\overline{k}/k)$ modules

$$0 \to \mathrm{im}\, \overline{\alpha} \to \ker \overline{\beta} \to \varphi(\overline{k}) \to 0,$$

where the maps $\overline{\alpha}$ and $\overline{\beta}$ are the α and β that appear in Section 4.2.

4.3.6 Explicit computation of Tamagawa numbers

In this section, we will show how to combine Theorem 4.3.3 and Theorem 4.1.3 to explicitly compute Tamagawa numbers. In the rest of this section, we will assume that $\mathrm{Gal}(\overline{k}/k)$ is procyclic in order to be able to apply Theorem 4.3.3.

Remark 4.3.7. We can also use Theorem 4.1.3 to compute the size of the component group, which corresponds to the special case where $k = \overline{k}$. The formula that we obtain for the component group using Theorem 4.1.3 can be checked to be equal to the formula obtained in Theorem 4.2.2. We include the proof in Section 4.2 since the graph that appears in Section 4.2 is more closely related to the dual graph that appears in Algebraic Geometry.

We will now show that if we can make the appropriate modifications to the maps α and β, then we can use Theorem 4.1.3 to compute Tamagawa numbers. Define two maps $\delta_1, \delta_2 \colon \mathbb{Z}^{\widetilde{V}} \to \mathbb{Z}^{\widetilde{V}}$ as follows.

$$\begin{aligned} \delta_1 \colon \mathbb{Z}^{\widetilde{V}} &\to \mathbb{Z}^{\widetilde{V}} \\ (b_{\tilde{v}}) &\mapsto (b_{\tilde{v}} m_{\tilde{v}} |\tilde{v}|) \end{aligned}$$

$$\begin{aligned} \delta_2 \colon \mathbb{Z}^{\widetilde{V}} &\to \mathbb{Z}^{\widetilde{V}} \\ (b_{\tilde{v}}) &\mapsto (b_{\tilde{v}} m_{\tilde{v}}) \end{aligned}$$

Let

$$\alpha_1 = \delta_1 \circ \alpha \circ \delta_2.$$

Let

$$\begin{aligned} \beta_1 \colon \mathbb{Z}^{\widetilde{V}} &\to \mathbb{Z} \\ (b_{\tilde{v}}) &\mapsto \sum_{\tilde{v} \in \widetilde{V}} b_{\tilde{v}}, \end{aligned}$$

and let

$$\begin{aligned} \beta_2 \colon \mathbb{Z} &\to \mathbb{Z}^{\widetilde{V}} \\ b &\mapsto (b, b, \ldots, b). \end{aligned}$$

As before, one can easily check that $\beta_1 \circ \delta_1 \circ \alpha = 0$ and $\alpha \circ \delta_2 \circ \beta_2 = 0$. Therefore $\operatorname{im}(\alpha_1) \subset \ker \beta_1$.

We now prove a lemma in linear algebra that allows us to compare the sizes of the finite groups $\ker(\beta_1)/\operatorname{im}(\alpha_1)$ and $\ker(\beta)/\operatorname{im}(\alpha)$. .

Lemma 4.3.8. *Let n be a non-negative integer. Let $D, D' \colon \mathbb{Z}^n \to \mathbb{Z}^n$ be two rank n linear operators whose matrices are diagonal with respect to the standard basis of $\mathbb{Z}^n$. Let $A \colon \mathbb{Z}^n \to \mathbb{Z}^n$ be a rank $n-1$ linear operator. Let $S \colon \mathbb{Z}^n \to \mathbb{Z}$ denote the linear operator which takes a vector to the sum of its coordinates, and let $\Delta \colon \mathbb{Z} \to \mathbb{Z}^n$ denote the linear operator corresponding to the diagonal embedding with respect to the standard basis of $\mathbb{Z}^n$. Assume $SDA = 0$ and $AD'\Delta = 0$. Let d, d' be positive integers such that $(d) = \operatorname{im}(SD)$ and $(d') = \operatorname{im}(SD')$ respectively. Then,*

$$\#\left(\frac{\ker(SD)}{\operatorname{im} A}\right) = \left(\frac{d}{|\det D|}\right) \#\left(\frac{\ker S}{\operatorname{im}(DA)}\right) = \left(\frac{dd'}{|\det D||\det D'|}\right) \#\left(\frac{\ker S}{\operatorname{im}(DAD')}\right).$$

Proof. We can reduce to the case that $d = d' = 1$ by replacing D by $\frac{1}{d}D$ and replacing D' by $\frac{1}{d'}D'$. This is possible since $\operatorname{rank} A = n-1$ and $\operatorname{rank} D = \operatorname{rank} D' = n$ imply that

- $\operatorname{im}(DA)$ has index d^{n-1} in $\operatorname{im}\left(\frac{1}{d}DA\right)$,
- $\operatorname{im}(DAD')$ has index $d^{n-1}d'^{n-1}$ in $\operatorname{im}\left(\frac{1}{dd'}DAD'\right)$,
- $\det D = d^n \det\left(\frac{1}{d}D\right)$, and,
- $\det D' = d'^n \det\left(\frac{1}{d'}D'\right)$.

We now prove the first equality. Since D is an injection, we have $\ker(SD) = D^{-1}(\ker S)$ and

$\operatorname{im} A = D^{-1}(\operatorname{im}(DA))$. This gives us the following short exact sequence,

$$0 \to \frac{\ker(SD)}{\operatorname{im} A} \xrightarrow{D} \frac{\ker S}{\operatorname{im}(DA)} \to \frac{\ker S}{\ker S \cap \operatorname{im} D} \to 0,$$

where the map on the right is the quotient map induced by the inclusion $\operatorname{im}(DA) \subset \ker S \cap \operatorname{im} D$. The inclusion of $\ker S$ into $\mathbb{Z}^n$ induces the following exact sequence:

$$0 \to \frac{\ker S}{\ker S \cap \operatorname{im} D} \to \frac{\mathbb{Z}^n}{\operatorname{im} D} \to \frac{\mathbb{Z}^n}{\ker S + \operatorname{im} D} \to 0.$$

Since $d = 1$, we have $\operatorname{im}(SD) = \mathbb{Z}$ $(= \operatorname{im} S)$ and therefore $\frac{\mathbb{Z}^n}{\ker S + \operatorname{im} D} \cong \frac{\operatorname{im} S}{\operatorname{im}(SD)} = 0$. Putting these two exact sequences together, we get

$$0 \to \frac{\ker(SD)}{\operatorname{im} A} \xrightarrow{D} \frac{\ker S}{\operatorname{im}(DA)} \to \frac{\mathbb{Z}^n}{\operatorname{im} D} \to 0.$$

Since all the groups in the above exact sequence are finite and $\#\left(\frac{\mathbb{Z}^n}{\operatorname{im} D}\right) = |\det D|$, we get

$$\#\left(\frac{\ker(SD)}{\operatorname{im} A}\right)|\det D| = \#\left(\frac{\ker S}{\operatorname{im}(DA)}\right).$$

This finishes the proof of the first equality.

We now prove the other equality. Let $A' = DA$. Since D is injective and A has rank $n-1$, it follows that A' has rank $n-1$. Since $A'D'\Delta = 0$, it follows that $\operatorname{im}(D'\Delta) \subset \ker A'$. Since A' has rank $n-1$ and $d' = 1$, we also have $\ker A' \subset \operatorname{im}(D'\Delta)$ and therefore $\ker A' \subset \operatorname{im} D'$. Therefore,

$$\frac{(A')^{-1}(\operatorname{im}(A'D'))}{\operatorname{im} D'} = \frac{\ker A' + \operatorname{im} D'}{\operatorname{im} D'} \cong \frac{\ker A'}{\ker A' \cap \operatorname{im} D'} = 0.$$

Now the equality we want to show follows directly from the following two exact sequences, just as before. The maps in the first exact sequence are the natural inclusion/quotient maps induced by the inclusions $\operatorname{im}(A') \subset \ker S$ and $\operatorname{im}(A'D') \subset \operatorname{im} A'$.

$$0 \to \frac{\operatorname{im} A'}{\operatorname{im}(A'D')} \to \frac{\ker S}{\operatorname{im}(A'D')} \to \frac{\ker S}{\operatorname{im} A'} \to 0.$$

$$0 \to \frac{(A')^{-1}(\operatorname{im}(A'D'))}{\operatorname{im} D'} \to \frac{\mathbb{Z}^n}{\operatorname{im} D'} \xrightarrow{A'} \frac{\operatorname{im} A'}{\operatorname{im}(A'D')} \to 0. \qquad \square$$

Let m and $\tilde{m}$ be defined as in Theorem 4.3.3.

Lemma 4.3.9.

$$\#\left(\frac{\ker \beta}{\operatorname{im} \alpha}\right) = \frac{m\tilde{m}}{\prod_{\tilde{v}\in\widetilde{V}} m_{\tilde{v}}^2 |\tilde{v}|} \; \#\left(\frac{\ker \beta_1}{\operatorname{im} \alpha_1}\right).$$

Proof. This follows directly from Lemma 4.3.8 with $A = \alpha, D = \delta_1$ and $D' = \delta_2$.

$\square$

We will now show how we can use Theorem 4.1.3 to compute $\#(\ker \beta_1/ \operatorname{im} \alpha_1)$. Recall the definition of the graph $\tilde{G}$ as defined in the beginning of this section.

Lemma 4.3.10. *Fix a vertex $\tilde{v}$ of the directed weighted multigraph $\widetilde{G}$. Let $\widetilde{S}_{\tilde{v}}(\widetilde{G})$ be the set of directed spanning trees into the vertex $\tilde{v}$. Then,*

$$\#\left(\frac{\ker \beta_1}{\operatorname{im} \alpha_1}\right) = \sum_{T\in\widetilde{S}_{\tilde{v}}(\widetilde{G})} \prod_{e\in E(T)} \operatorname{wt}(e).$$

Proof. Since X_s is connected, every vertex of $\widetilde{G}$ is a sink. The proof then reduces to a direct application of Theorem 4.1.3, after we observe that $\#\left(\frac{\ker \beta_1}{\operatorname{im} \alpha_1}\right)$ is equal to the absolute value of the determinant of the reduced Laplacian operator on $\widetilde{G}$, with the fixed vertex $\tilde{v}$ as the sink. $\square$

Let q, m and $\tilde{m}$ be defined as in Theorem 4.3.3.

Theorem 4.3.11.

$$\#\varphi(k) = q \; \frac{m^2}{\prod_{\tilde{w}\in\widetilde{V}} m_{\tilde{w}}^2 |\tilde{w}|} \; \left(\sum_{T\in\widetilde{S}_{\tilde{v}}(\widetilde{G})} \prod_{e\in E(T)} \operatorname{wt}(e)\right).$$

Proof. Combine Theorem 4.3.3, Lemma 4.3.9 and Lemma 4.3.10. $\square$

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