Notation as in the paper. R - complete dvr, algebraically closed residue field k

$K = \mathrm{Frac}(R)$

k = residue field

[To ensure that the Swan term vanishes assume char $k > 2g+1$]

GOAL: Given a hyperelliptic curve X/K with minimal Weierstrass equation $y^2 = f(x)$, compute $-\mathrm{Art}(\mathcal{X}'/S)$ for a suitable proper regular model $\mathcal{X}'/S$ of X & prove $-\mathrm{Art}(\mathcal{X}'/S) \le v(\mathrm{disc}(f))$.

USEFUL FACTS ABOUT $-\mathrm{Art}(\mathcal{X}/S)$

$\mathcal{X}, \mathcal{X}'$ regular models of X/K, then

(1) If $\mathcal{X}/S$ is a simple normal crossings model (snc)

- all (reduced) irreducible components of $\mathcal{X}_s$ are smooth
- $(\mathcal{X}_s)_{\mathrm{red}}$ has at worst nodal singularities

$$-\mathrm{Art}(\mathcal{X}/S) = \sum_{\substack{\Gamma: \text{irr. comp.} \\ \text{of } \mathcal{X}_s}} \Big\{ (1-m_\Gamma)\chi(\Gamma) + \sum_{\Gamma' \neq \Gamma} (m_{\Gamma'} - 1)\, \Gamma \cdot \Gamma' \Big\} + \sum_{\Gamma < \Gamma'} \Gamma \cdot \Gamma'$$

(**Lemma 3.1** in the paper)

(2) If $\mathcal{X} \to \mathcal{X}'$ is a blow-up at a closed point of $\mathcal{X}'_s$, then

$$-\mathrm{Art}(\mathcal{X}/S) = -\mathrm{Art}(\mathcal{X}'/S) + 1$$

NECESSARY INGREDIENTS FOR COMPUTING $-\mathrm{Art}(\mathcal{X}'/S)$ (COMBINATORIAL DATA)

(1) A snc model $\mathcal{X}$ & a map $\mathcal{X} \to \mathcal{X}'$ that is obtained as a sequence of blow-downs $\boxed{\mathcal{X} \to \mathcal{X}'}$ + $\boxed{\text{number of blow-down in } \mathcal{X} \to \mathcal{X}'}$

(2) $\boxed{m_\Gamma}$ for every irreducible component Γ of $\mathcal{X}_s$

(3) $\boxed{\chi(\Gamma)}$ " "

(4) $\boxed{\Gamma \cdot \Gamma'}$ for every pair of components Γ, Γ' of $\mathcal{X}_s$

HOW DO WE CONSTRUCT SNC $\mathcal{X}$?

Construct a suitable model $\mathcal{Y}$ of $\mathbb{P}^1_K$ & take the normalization ($=: \mathcal{X}$) of $\mathcal{Y}$ in $K(\mathcal{Y})(\sqrt{f})$

ALGORITHM FOR COMPUTING $\mathcal{X}'$ & $\mathcal{X}$

Start with $\mathbb{P}^1_S$ & $f \in \mathcal{O}(d)$ [monic even degree polynomial]

$$\operatorname{div}(f) = -d(\infty) + \sum H_i$$

H_i: horizontal irreducible Weil divisors
$\Leftrightarrow$ Irreducible factors of f

STEP 1: Blow-up $\mathbb{P}^1_S$ until the strict transforms of all the H_i are regular [i.e. make horizontal components regular] (of the branch locus)

STEP 2: Make the strict transforms of the H_i disjoint by doing some further blow-ups - Call the result $\mathrm{Bl}_n \mathbb{P}^1_S$ [i.e. separate horizontal components from other horizontal components]

On $\mathrm{Bl}_n \mathbb{P}^1_S$,

$$\operatorname{div}(f) = -d(\infty) + \sum_{\substack{H_i:\\ \text{irreducible}\\ \text{horizontal}\\ \text{divisor}}} H_i + 2\Big(\sum_{\substack{V_i:\\ \text{irr}\\ \text{vertical}\\ \text{divisor}}} m_i V_i\Big) + \sum_{j \in J \subset I} V_j$$

STEP 3: Do some further blow-ups to make the strict transforms of the V_j disjoint. [i.e. separate odd vertical components from other odd vertical components]

STEP 4: Do some further blow-ups to make the strict transforms of the V_j disjoint from the strict transforms of the H_i. Call the resulting model of $\mathbb{P}^1_K$ $\mathcal{Y}'$.
[i.e. separate odd vertical components from the horizontal comp.]

STEP 5: Blow up $\mathcal{Y}'$ further until the intersections of the strict transforms of the H_i with $(\mathcal{Y}'_s)_{red}$ are transverse.
Call the resulting model $\mathcal{Y}$.

STEP 6: $\mathcal{X}'$ = Normalization of $K(\mathcal{Y}')$ in $K(\mathcal{Y}')\sqrt{f}$) – REGULAR
$\mathcal{X}$ = Normalization of $K(\mathcal{Y})$ in $K(\mathcal{Y})\sqrt{f}$) – REGULAR, SNC

$$\begin{array}{ccccccc} \mathcal{X} & \longrightarrow & \mathcal{X}' & & & & \\ \downarrow & & \downarrow & & & & \\ \mathcal{Y} & \longrightarrow & \mathcal{Y}' & \longrightarrow & \mathrm{Bl}_n \mathbb{P}^1_S & \longrightarrow & \mathbb{P}^1_S . \end{array}$$

$\boxed{\text{GOAL: PROVE } -\mathrm{Art}(\mathcal{X}/S) \leq v_s(\mathrm{disc}(f))}$

Since $\begin{array}{c}\mathcal{X}\\ \downarrow\\ \mathcal{Y}\end{array}$ is a finite morphism, given an irreducible component Γ of $\mathcal{Y}_s$, ~~for every~~ to compute the combinatorial data of $\mathcal{X} \times_{\mathcal{Y}} \Gamma$, it suffices to know $(m_{\tilde{\Gamma}}, \chi(\tilde{\Gamma}), \tilde{\Gamma}\cdot\tilde{\Gamma}'...)$

- $v_\Gamma(s)$ (valuation of s along Γ; $\mathcal{O}_{\Gamma_{d.v.r.}} \subset K(\mathcal{Y})$)
- parity of $v_\Gamma(f)$
- # of intersection points of Γ with the branch locus (Then Riemann-Hurwitz will help us compute $\chi(\tilde{\Gamma})$) (for every $\tilde{\Gamma}$ mapping down to Γ)
- intersection of Γ with other components of $\mathcal{Y}_s$ & the behaviour of the intersection points in the double cover (i.e., the neighbours of Γ in $\mathcal{Y}_s$)

Since $\mathcal{Y}$ is constructed by an iterative procedure (STEP 1–STEP 5), we will attempt to find iterative formulas for $v_\Gamma(s)$, $v_\Gamma(f)$ for every component Γ of $\mathcal{Y}_s$.

$$\mathcal{Y} \to \cdots \to Bl_m\mathbb{P}^1_S \xrightarrow{\varphi} Bl_{m-1}\mathbb{P}^1_S \to \cdots \to Bl_1\mathbb{P}^1_S \to \mathbb{P}^1_S$$

Each of the morphisms in the above composition is the blow-up at a single closed point of the special fiber.

- Let φ be the blow-up at P & let Γ be the exceptional curve.

Case 1 P lies on a unique component Γ' of $Bl_{m-1}\mathbb{P}^1_S$.

Then,
$$v_\Gamma(\cancel{s}) = v_{\Gamma'}(\cancel{s})$$
$$v_\Gamma(f) = \mathrm{mult}_P(f) = \sum_{\substack{\text{irr. horizontal}\\ \text{comp. } H_i\\ \text{in div}(f) \text{ on } Bl_{m-1}(\mathbb{P}^1_S)}} \mathrm{mult}_P(H_i) + \mathrm{mult}_P(\Gamma')$$

Case 2 P is the intersection point of Γ' & Γ''. Then,
$$v_\Gamma(\cancel{s}) = v_{\Gamma'}(\cancel{s}) + v_{\Gamma''}(\cancel{s})$$
$$v_\Gamma(f) = \sum \mathrm{mult}_P(H_i) + \mathrm{mult}_P(\cancel{\Gamma}') + \mathrm{mult}_P(\Gamma'')$$

* New problem: find iterative formulae for $\mathrm{mult}_p(H_i)$

* Also need to figure out a suitable way to decompose the minimal discriminant, compatible with the decomposition of the Artin conductor, so that we may compare them locally.

HOPE: Resolve both these issues using a description coming from the "Puiseux series" expansion of the roots of the polynomial $f(x)$

Currently even the strategy only seems to make sense for $R=\mathbb{C}((t))$.

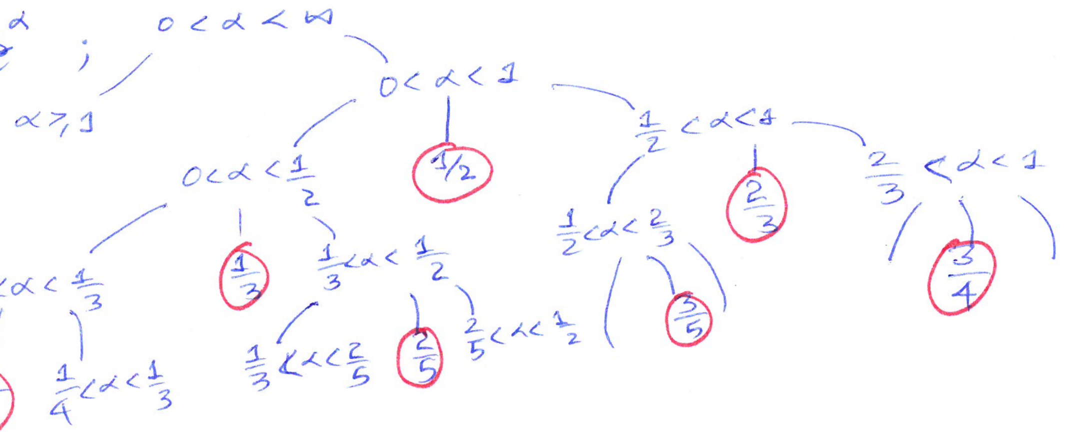
Factor $f(x) = f_1(x)\, f_2(x) \dots f_k(x)$ $\qquad$ $f_i(x)$ irreducible, $f_i(x) \in R[x]$ $\qquad$ $R = \mathbb{C}[[t]]$

$$\begin{array}{ccc} f_1(x) & f_2(x) & f_k(x) \\ \updownarrow & \updownarrow & \updownarrow \\ H_1 & H_2 & H_k \end{array}$$

Roots of f_i: $\quad t^{\alpha_1}+\dots \quad t^{\alpha_2}+\dots \quad t^{\alpha_k}+\dots$

Separating terms (roots of different f_i) by their "leading order" involves repeated subdivisions along one of the branches of the dual graph

Leading term $= t^{\alpha}$;



Farey fractions