AN ARITHMETIC COUNT OF THE LINES MEETING FOUR LINES IN $\mathbf{P}^3$

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ABSTRACT. We enrich the classical count that there are two complex lines meeting four lines in space to an equality of isomorphism classes of bilinear forms. For any field k, this enrichment counts the number of lines meeting four lines defined over k in $\mathbf{P}^3_k$, with such lines weighted by their fields of definition together with information about the cross-ratio of the intersection points and spanning planes. We generalize this example to an infinite family of such enrichments, obtained using an Euler number in $\mathbf{A}^1$-homotopy theory. The classical counts are recovered by taking the rank of the bilinear forms.

1. INTRODUCTION

It is a classical result that there are exactly two lines meeting four general lines in $\mathbf{P}^3_{\mathbf{C}}$, and we briefly recall a proof. The lines meeting three of the four are pairwise disjoint and their union is a degree 2 hypersurface. The intersection of the fourth line with the hypersurface is then two points, one on each of the two lines meeting all four. There is a lovely description in, for example, [EH16, 3.4.1]. Over an arbitrary field k, the hypersurface is defined over k, but the two intersection points, and therefore the two lines, may have coefficients in some quadratic extension $k[\sqrt{L}]$ of k. For example, over the real numbers $\mathbf{R}$, there may be two real lines or a Galois-conjugate pair of $\mathbf{C}$-lines.

In this paper, we give a restriction on the field of definition of the two lines combined with other arithmetic-geometric information on the configurations of the lines. More generally, we give an analogous restriction on the lines meeting $2n - 2$ codimension 2 hyperplanes in $\mathbf{P}^n$ with n odd.

These restrictions are equalities in the Grothendieck–Witt group $GW(k)$ of the ground field k, defined to be the group completion of the semi-ring of isomorphism classes of non-degenerate, symmetric bilinear forms on finite dimensional vector spaces valued in k. The Grothendieck–Witt group arises in this context as the target of Morel's degree homomorphism in $\mathbf{A}^1$-homotopy theory. A feature of $\mathbf{A}^1$-homotopy theory is that it produces results over any field. These results can record arithmetic-geometric information about enumerative problems, classically posed over the complex numbers, which admit solutions using algebraic topology. We show this is the case for the two enumerative problems just described and answer the question of what information is being recorded.

This latter question has a fun answer. Given a line L meeting four pairwise non-intersecting lines L_1, L_2, L_3, and L_4, there are four distinguished points $L \cap L_1$, $L \cap L_2$, $L \cap L_3$ and $L \cap L_4$ on L. The line L is isomorphic to $\mathbf{P}^1_{k(L)} \cong L$, where $k(L)$ denotes the field of

definition of L, giving four points on $\mathbf{P}^1_{k(L)}$, which therefore have a cross-ratio, which we denote as λ_L. The planes in $\mathbf{P}^3_k$ containing L are likewise parametrized by a scheme isomorphic to $\mathbf{P}^1_{k(L)}$, and therefore the four planes determined by L and L_1, L and L_2, L and L_3 and finally, L and L_4 also have a cross-ratio, denoted μ_L.

Let $\langle a \rangle$ in $\mathrm{GW}(k)$ denote the isomorphism class of the one-dimensional bilinear form $k \times k \to k$ defined $(x, y) \mapsto axy$, for a in $k^*/(k^*)^2$. For a separable field extension $k \subseteq E$, let $\mathrm{Tr}_{E/K} : \mathrm{GW}(E) \to \mathrm{GW}(k)$ denote the map which takes a bilinear form $\beta : V \times V \to E$ to the composition $\mathrm{Tr}_{V \times V} \circ \beta : V \times V \to k$ of β with the field trace $\mathrm{Tr}_{E/K} : E \to K$. Then the following equality holds in $\mathrm{GW}(k)$.

Theorem 1. *Let k be a field of characteristic not 2, and let L_1, L_2, L_3, and L_4 be general lines in $\mathbf{P}^3_k$ defined over k. Then*

$$\sum_{\substack{\text{lines } L \text{ such that } L \cap L_i \neq \emptyset \\ \text{for } i=1,2,3,4}} \mathrm{Tr}_{k(L)/k} \langle \lambda_L - \mu_L \rangle = \langle 1 \rangle + \langle -1 \rangle.$$

The condition that the lines are general means there is an open set of four-tuples of lines such that the theorem holds. In this case, this open set contains the lines that are pairwise non-intersecting and such that the fourth is not tangent to the quadric of lines meeting the first three.

This result generalizes as follows. Let $\pi_1, \pi_2, \ldots, \pi_{2n-2}$ be general codimension two hyperplanes defined over k in $\mathbf{P}^n_k$ for n odd. Suppose that L is a line in $\mathbf{P}^n$, defined over the field $k(L)$, which intersects all of the π_i. L corresponds to a 2-dimensional subspace W of $k(L)^{n+1}$. The intersection points $\pi_i \cap L$ determine $(2n - 2)$ points of $\mathbf{P}W \cong \mathbf{P}^1_{k(L)}$. The space of hyperplanes containing W corresponds to the projective space $\mathbf{P}(k(L)^{n+1}/W) \cong \mathbf{P}^{n-2}_{k(L)}$. Choosing coordinates on $\mathbf{P}W$ and $\mathbf{P}(k(L)^{n+1}/W)$, let $[c_{i0}, c_{i1}]$ be the coordinates of $\pi_i \cap L = [c_{i0}, c_{i1}]$, and let $[d_{i0}, d_{i1}, \ldots, d_{i,n-2}]$ be the coordinates of the plane spanned by π_i and L. (The choice of these coordinates does not matter, i.e., Theorem 2 below will be true for any choices of coordinates.) We will define a certain normalized lift of these homogeneous coordinates to coordinates (c_{i0}, c_{i1}) and $(d_{i0}, d_{i1}, \ldots, d_{i,n-2})$ respectively in Definition 10. The count of the lines L meeting the π_i will weight each line L by $\mathrm{Tr}_{k(L)/L} \langle i(L) \rangle$ where $i(L)$ is determined by the normalized lifts of the intersection points $\pi_i \cap L$ and the codimension 1 hyperplanes spanned by π_i and L by the formula

$$(1) \qquad i(L) = \det \begin{bmatrix} d_{10}c_{10} & \cdots & d_{i0}c_{i0} & \cdots & d_{2n-2,0}c_{2n-2,0} \\ \vdots & & \vdots & & \vdots \\ d_{1j}c_{10} & \cdots & a_{ij}d_{i0} & \cdots & d_{2n-2,j}c_{2n-2,0} \\ \vdots & & \vdots & & \vdots \\ d_{1,n-2}c_{10} & \cdots & d_{i,n-2}c_{i0} & \cdots & d_{2n-2,n-2}c_{2n-2,0} \\ d_{10}c_{11} & \cdots & d_{i0}c_{i1} & \cdots & d_{2n-2,0}c_{2n-2,1} \\ \vdots & & \vdots & & \vdots \\ d_{1j}c_{11} & \cdots & d_{ij}c_{i1} & \cdots & d_{2n-2,j}c_{2n-2,1} \\ \vdots & & \vdots & & \vdots \\ d_{1,n-2}c_{11} & \cdots & d_{i,n-2}c_{i1} & \cdots & d_{2n-2,n-2}c_{2n-2,1} \end{bmatrix}$$

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Theorem 2. *Let k be a field. Let n be odd and let $\pi_1, \pi_2, \ldots, \pi_{2n-2}$ be general codimension two hyperplanes defined over k in $\mathbf{P}_k^n$. Assume either that k is perfect, or that the extension $k \subseteq k(L)$ is separable for every line L that meets all the planes π_i. Then*

$$\sum_{\substack{\text{lines } L \text{ such that } L \cap \pi_i \neq \emptyset \\ \text{for } i=1\ldots 2n-2}} \mathrm{Tr}_{k(L)/k}\langle i(L) \rangle = \frac{1}{2}\frac{(2n-2)!}{n!(n-1)!}(\langle 1 \rangle + \langle -1 \rangle).$$

There are many tools available for studying $GW(k)$. The Milnor conjecture, proven by Voevodsky, identifies the associated graded ring of the filtration of $GW(k)$ by powers of the fundamental ideal with the étale cohomology group $H^*(k, \mathbb{Z}/2)$, giving rise to invariants valued in $H^*(k, \mathbb{Z}/2)$, or equivalently Milnor K-theory, the first of which are the rank, discriminant, Hasse-Witt, and Arason invariants [Mil69] [Voe03a] [Voe03b]. For many fields, much is understood about $GW(k)$, for example giving algorithms to determine if two given sums of the generators $\langle a \rangle$ for $a \in k^*/(k^*)^2$ of $GW(k)$ are equal, as well as computations of $GW(k)$. See for example [Lam05]. Applying invariants of $GW(k)$ to the equalities of Theorems 1 and 2 produces other equalities, which may be valued in more familiar groups. A selection of such results follows.

For $k = \mathbf{R}$ the real numbers, applying the signature to both sides of Theorem 1, we see that if the two lines are real, the sign of $\lambda_L - \mu_L$ must be reversed for the two lines. More generally, in the situation of Theorem 2 for $k = \mathbf{R}$, half of the real lines will have $i(L)$ negative and half positive. The question of which lines are real in both these situations has been previously studied. Sottile has shown that the lines may all be real [Sot97, Theorem C]. Work on similar questions is also found in [Vak06]. There are also connections to the B. and M. Shapiro conjecture, proven by Eremenko and Gabrielov [EG02] and Mukhin, Tarasov, and Varchenko [MTV09b] [MTV09a]. For example, the positive solution to the Shapiro conjecture gives a large class of examples where the lines meeting four lines in space are real. There is a nice exposition in [Sot11].

For $k = \mathbf{F}_q$ a finite field with q elements, applying the discriminant produces, for example:

Corollary 3. *Let $k = \mathbf{F}_q$ be a finite field with q elements, with q odd. Let L_1, L_2, L_3, L_4 be general lines defined over k in $\mathbf{P}_k^3$. If a line L meeting L_i for $i = 1, \ldots, 4$ is defined over $\mathbf{F}_{q^2}$, then*

$$\lambda_L - \mu_L = \begin{cases} \text{is a non-square for } q \cong 1 \mod 4 \\ \text{is a square for } q \cong 3 \mod 4. \end{cases}$$

The proofs of Theorems 1 and 2 and Corollary 3 can be found in Section 6. For these proofs, we use a strategy based on joint work [KW17] of Jesse Kass and the second named author. Namely, we take a classical enumerative problem over the complex numbers that admits a solution using an Euler class from algebraic topology, and rework it over a field k using an enriched Euler class valued in $GW(k)$. To obtain an enumerative result over k, one then needs a geometric interpretation of certain local indices, which is guessed on a case-by-case basis.

In the present case, there is a classical count of the appropriate number of complex lines as a power of the first Chern class of the line bundle $\mathcal{S}^* \wedge \mathcal{S}^*$ on an appropriate Grassmannian, where $\mathcal{S}^* \wedge \mathcal{S}^*$ denotes the wedge of the dual tautological bundle with itself. This characteristic class is equivalent to the Euler class of $\oplus_{i=1}^{N} \mathcal{S}^* \wedge \mathcal{S}^*$ for an appropriate N. In [KW17], an Euler class is constructed in $GW(k)$ for relatively oriented bundles of rank r on a smooth, compact r-dimensional scheme over k, as a sum over the zeros of a section with only isolated zeros of a local contribution from each zero. This local contribution can be expressed as a local degree in $\mathbf{A}^1$-homotopy theory. As in the classical case, a configuration of codimension 2 hyperplanes (or more precisely the set of equations whose zero loci are the hyperplanes) determines a section of $\oplus_{i=1}^{N} \mathcal{S}^* \wedge \mathcal{S}^*$. We therefore have that a fixed element of $GW(k)$ is a sum over the lines of the local degree in $GW(k)$ of a section, in analogy to the fixed $\mathbf{Z}$-valued Euler class on the complex Grassmiannian from classical algebraic topology expressed as a sum over the lines of the local $\mathbf{Z}$-valued degrees at the zero locus of a section. In the complex case, these latter local degrees happen to be generically all one because complex manifolds and algebraic sections are orientable, giving the number of lines as an Euler class, but over other fields, interesting local degrees or indices arise.

Readers who would like to avoid $\mathbf{A}^1$-homotopy theory may do so, as the construction of the Euler class of [KW17] uses as local indices the classes from the Eisenbud–Khimshiashvili–Levine Signature Formula, which are the local $\mathbf{A}^1$-degrees, but also have a concrete commutative algebra construction that a computer can compute. In the case of Theorem 1, we also find the stronger result that the cross ratios of the points and the planes switch when we switch the two lines over their fields of definition, i.e., $\mu_L = \lambda_L$ and $\mu_L = \lambda_L$ (see Theorem 17 and Example 6.1), which can be verified independently. Note, however, that it is more than just analogy that links our results to algebraic topology; there is a full-fledged theory of $\mathbf{A}^1$-homotopy theory providing a connection [MV99] [Mor12]. Moreover, there is a machine producing enriched results of the form given in Theorems 1 and 2. Once the machine has produced an enrichment, there is guess-work involved in identifying local indices, but once this is accomplished, the end result is by design independent of $\mathbf{A}^1$-homotopy theory and the machine, and one can then, at least in some cases, provide alternate proofs that are also independent. A main tool here is the Euler class of [KW17]. There are older constructions of Euler classes in $\mathbf{A}^1$-homotopy theory in [BM00] [Mor12, 8.2], more recent ones in [DJK18] [LR18], and other constructions of Euler classes for schemes independent of $\mathbf{A}^1$-homotopy theory in [GI80] [MS96] [BS99] [BS00] [BDM06]. See also [Fas08] [AF16] [Lev17b] for results on Euler classes or useful tools for their computation.

Matthias Wendt has a lovely alternate computation of the Euler classes of $\oplus_{i=1}^{N} \mathcal{S}^* \wedge \mathcal{S}^*$, using a Schubert Calculus he has developed [Wen18b]. He also considers the resulting applications to enumerative geometry. His results as well as methods are different from the ones given here. His work [Wen18b] builds on his previous work [Wen18a] and his joint work [HW17] with Hornbostel.

Unique to the present paper are the given computations of the local $GW(k)$-degrees or indices/weights of the lines and their geometric interpretations, and the resulting Theorems 1 and 2 and consequences. We also give computations of the relevant Euler classes and take the opportunity to further the study of the Euler class of [KW17].

This paper fits into a recent program that could be called $\mathbf{A}^1$-enumerative geometry, or enumerative geometry enriched in quadratic forms. See [Hoy14] [KW16a] [KW16b] [Lev17b] [Lev17a] [KW17] [Lev18a] [Lev18b] [Wen18b].

1.1. **Outline.** In Section 2, we give the necessary results and notation to have a well-defined Euler class of $\oplus_{i=1}^N \mathcal{S}^* \wedge \mathcal{S}^*$ in $\mathrm{GW}(k)$ as a sum over lines of a local index or degree. In Section 3, we give formulas for the local index, in particular in terms of the $i(L)$ for Theorem 2 above. In Section 4, we give computations of Euler classes, one using arguments of Fasel/Levine. In Section 5, we prove the connection between the local indices and the cross-ratios appearing in Theorem 1. Section 6 contains the proofs of the stated results in the introduction and an explicit example.

2. Local coordinates on Grassmannians, orientations on vector bundles, and Euler numbers

A vector bundle $V \to X$ is *oriented* (or *weakly oriented* if you prefer) if it is equipped with an isomorphism $\det V \cong \mathcal{L}^{\otimes 2}$ for a line bundle $\mathcal{L}$ on X. A smooth scheme X is oriented if its tangent bundle is. Let U be a Zariski open set. A section in $\mathcal{L}^{\otimes 2}(U)$ is called a *square* if it is of the form $s \otimes s$ for s in $\mathcal{L}(U)$. A trivialization $\psi : V|_U \xrightarrow{\cong} \mathcal{O}|_U^r$ is *compatible* with a given orientation if the composition $\det \mathcal{O}^r|_U \to \det V|_U \to \mathcal{L}^{\otimes 2}$ takes the canonical section in $\det \mathcal{O}^r(U)$ to a square.

There is an Euler class or number in $\mathrm{GW}(k)$ for an oriented vector bundle of rank n on a smooth projective oriented n-dimensional scheme. In fact, the weaker notion of a *relatively oriented* vector bundle $V \to X$ of rank n with X smooth, projective, and dimension n over k suffices to define an Euler class. V relatively oriented means that $\mathrm{Hom}(\det TX, \det V)$ is oriented. While we do not need the notion of relative orientation for this paper, we prove some results in this greater generality for their own interest.

Let n be odd, and let $\mathrm{Gr}(2, n+1)$ denote the Grassmannian parametrizing lines in $\mathbf{P}_k^n$. By a *line* L in $\mathbf{P}_k^n$, we mean a closed point of this Grassmannian. After basechange to $k(L)$, L corresponds to a closed subscheme of $\mathbf{P}_{k(L)}^n$.

Let $\mathcal{S}$ be the tautological bundle on $\mathrm{Gr}(2, n+1)$. Then the line bundle $\mathcal{O}(1)$ corresponding to the Plucker embedding of $\mathrm{Gr}(2, n+1)$ is $\wedge^2 \mathcal{S}^*$. Let $\mathcal{Q}$ denote the quotient bundle on $\mathrm{Gr}(2, n+1)$, defined by

$$0 \to \mathcal{S} \to \mathbf{A}^{n+1} \to \mathcal{Q} \to 0$$

as the cokernel of the inclusion of the tautological bundle into the trivial bundle.

Let $\{e_1, \ldots, e_{n+1}\}$ denote a basis for k^{n+1}, and let $\{\phi_1, \ldots, \phi_{n+1}\}$ denote the dual basis of $(k^{n+1})^*$. The 2-dimensional subspace $ke_n \oplus ke_{n+1}$ spanned by $\{e_n, e_{n+1}\}$ determines a k-point of $\mathrm{Gr}(2, n+1)$. There are local coordinates

$$\mathrm{Spec}\, k[x_1, \ldots, x_{n-1}, y_1, \ldots, y_{n-1}] \to \mathrm{Gr}(2, n+1)$$

of $\mathrm{Gr}(2, n+1)$ around this point such that $(x_1, \ldots, x_{n-1}, y_1, \ldots, y_{n-1})$ corresponds to the span of $\{\tilde{e}_n, \tilde{e}_{n+1}\}$, where $\{\tilde{e}_1, \ldots, \tilde{e}_{n+1}\}$ is the basis of k^{n+1} defined by

$$
\begin{cases}
e_i & \text{for } i = 1, \ldots, n-1 \\
\sum_{i=1}^{n-1} x_i e_i + e_n & \text{for } i = n \\
\sum_{i=1}^{n-1} y_i e_i + e_{n+1} & \text{for } i = n+1.
\end{cases}
$$

These local coordinates determine a local trivialization of $T\,\mathrm{Gr}(2, n+1)$ using the canonical trivialization of the tangent space of $\mathbf{A}^{2(n-1)}$. We will say that coordinates are *compatible* with a given orientation if the corresponding local trivialization of the tangent bundle is. Let $\{\tilde{\phi}_1, \ldots, \tilde{\phi}_{n+1}\}$ be the dual basis for the basis $\{\tilde{e}_1, \ldots, \tilde{e}_{n+1}\}$ of k^{n+1}. The content of the following proof is contained in [KW16a, Lemma 42], but we include the proof in the stated generality for completeness.

Lemma 4. *There is an orientation of $T\,\mathrm{Gr}(2, n+1)$ such that the local coordinates given by the maps $\mathrm{Spec}\, k[x_1, \ldots, x_{n-1}, y_1, \ldots, y_{n-1}] \to \mathrm{Gr}(2, n+1)$ just described are compatible.*

Proof. Let $\{e_1, \ldots, e_{n+1}\}$ and $\{e'_1, \ldots, e'_{n+1}\}$ denote two chosen bases for k^{n+1}. As above, consider the corresponding local coordinates on the open subsets U and U' of $\mathrm{Gr}(2, n+1)$. Under the canonical identification of $T\,\mathrm{Gr}(2, n+1)$ with $\mathrm{Hom}(\mathcal{S}, \mathcal{Q})$ the corresponding trivialization $T\,\mathrm{Gr}(2, n+1)|_U \cong \mathcal{O}_U^{2(n-1)}$ corresponds to the basis of $T\,\mathrm{Gr}(2, n+1)(U)$ given by

$$
\{\tilde{\phi}_n \otimes \tilde{e}_i : i = 1, n-1\} \cup \{\tilde{\phi}_{n+1} \otimes \tilde{e}_i : i = 1, n-1\}
$$

and similarly for U'. (Note the slight abuse of notation when we consider, say $\tilde{e}_1$ at the point p to be an element of $k(p)^{n+1}/(k\tilde{e}_n \oplus k\tilde{e}_{n+1})$.) These trivializations determine clutching functions for $T\,\mathrm{Gr}(2, n+1)$, i.e., isomorphisms

$$(2) \qquad \mathcal{O}^{2(n-1)}|_{U \cap U'} \to T\,\mathrm{Gr}(2, n+1)|_{U \cap U'} \to \mathcal{O}^{2(n-1)}|_{U \cap U'},$$

which are given by the change-of-basis matrix relating

$$
\tilde{\phi}_n \otimes \tilde{e}_1, \tilde{\phi}_n \otimes \tilde{e}_2, \ldots, \tilde{\phi}_n \otimes \tilde{e}_{n-1}, \tilde{\phi}_{n+1} \otimes \tilde{e}_1, \tilde{\phi}_{n+1} \otimes \tilde{e}_2, \ldots, \tilde{\phi}_{n+1} \otimes \tilde{e}_{n-1}
$$

to

$$
\tilde{\phi}'_n \otimes \tilde{e}'_1, \tilde{\phi}'_n \otimes \tilde{e}'_2, \ldots, \tilde{\phi}'_n \otimes \tilde{e}'_{n-1}, \tilde{\phi}'_{n+1} \otimes \tilde{e}'_1, \tilde{\phi}'_{n+1} \otimes \tilde{e}'_2, \ldots, \tilde{\phi}'_{n+1} \otimes \tilde{e}'_{n-1}.
$$

Let $A \in \mathrm{GL}_{n+1}\, \mathcal{O}(U \cap U')$ be the change of basis matrix relating $\{\tilde{e}_1, \ldots, \tilde{e}_{n+1}\}$ to $\{\tilde{e}'_1, \ldots, \tilde{e}'_{n+1}\}$. Since $k\tilde{e}_n \oplus k\tilde{e}_{n+1} = k\tilde{e}'_n \oplus k\tilde{e}'_{n+1}$, we have that A determines sections $B \in \mathrm{GL}_2\, \mathcal{O}(U \cap U')$, given by the change-of-basis matrix relating $\{\tilde{e}_n, \tilde{e}_{n+1}\}$ and $\{\tilde{e}'_n, \tilde{e}'_{n+1}\}$, as well as $C \in \mathrm{GL}_{n-1}\, \mathcal{O}(U \cap U')$, given by the change-of-basis for the quotient bundle relating $\{\tilde{e}_1, \ldots, \tilde{e}_{n-1}\}$ and $\{\tilde{e}'_1, \ldots, \tilde{e}'_{n-1}\}$ (note the same abuse of notation as above). The clutching functions (2) are therefore $(C^{-1})^t \otimes B \in \mathrm{GL}_{2(n-1)}(U \cap U')$, where $(C^{-1})^t$ denotes the inverse transpose of C. Therefore the determinant bundle has clutching functions $(\det C)^{-(n-1)}(\det B)^2 \in \mathcal{O}^*(U \cap U')$. Since n is odd, $(\det C)^{-(n-1)}(\det B)^2$ is a square of an element of $\mathcal{O}^*(U \cap U')$, and the lemma follows. $\qquad \square$

Let $\mathcal{V}$ be the rank $2n - 2$ vector bundle $\oplus_{i=1}^{2n-2} \wedge^2 \mathcal{S}^*$. Given $2n - 2$ codimension two subspaces $\pi_1, \pi_2, \ldots, \pi_{2n-2}$ of k^{n+1}, choose a basis of linear forms α_i and β_i in $(k^{n+1})^*$ vanishing on each π_i. Let σ be the section of $\mathcal{V}$, given by $\sigma = (\alpha_1 \wedge \beta_1, \ldots, \alpha_{2n-2} \wedge \beta_{2n-2})$. (More explicitly, any point L of $\mathrm{Gr}(2, n+1)$ corresponds to a 2-dimensional subspace W

of $k(L)^{\oplus 4}$. The elements α_i and β_i tensored with $k(L)$ then restrict to functionals $\alpha_i|_W$ and $\alpha_i|_W$ on W. The fiber of $\wedge^2 \mathcal{S}^*$ at L is canonically identified with $W^* \wedge W^*$, and $(\alpha_1|_W \wedge \beta_1|_W, \ldots, \alpha_{2n-2}|_W \wedge \beta_{2n-2}|_W)$ determines the section σ.)

The following lemma is standard, but we include it for clarity.

Lemma 5. *Let* L *be a point of* $\mathrm{Gr}(2, n+1)$. *Then* $\sigma(L) = 0$ *if and only if* L *meets all of the hyperplanes* $\pi_1, \pi_2, \ldots, \pi_{2n-2}$.

Proof. If a codimension 2 hyperplane π is cut out by the two linear forms α, β and a dimension 2 subspace L is spanned by the vectors e, f, then $(\alpha \wedge \beta)(e \wedge f) = 0$ if and only if both α and β simultaneously vanish on some linear combination of e and f, or in other words, if $\pi \cap L \neq \emptyset$. $\square$

We use the construction of the Euler number in $\mathrm{GW}(k)$ of [KW17]. Namely, for an oriented vector bundle $\mathcal{E}$ of rank r on a smooth proper oriented k-scheme of dimension $n = r$, equipped with a section σ with only isolated zeros, there is an Euler number $e(\mathcal{E}, \sigma)$ in $\mathrm{GW}(k)$, defined as a sum of local degrees or indices over the zeros of σ,

$$e(\mathcal{E}, \sigma) = \sum_{p:\sigma(p)=0} \mathrm{ind}_p \, \sigma.$$

We apply this construction to the vector bundle $\mathcal{V} = \oplus_{i=1}^{2n-2} \wedge^2 \mathcal{S}^*$ on $\mathrm{Gr}(2, n+1)$. For this, we identify a large subscheme of sections of $\mathcal{V}$ with only isolated zeros. For any such section σ, and L a point of $\mathrm{Gr}(2, n+1)$ such that $\sigma(L) = 0$, the local degree or index $\mathrm{ind}_L \, \sigma$ in $\mathrm{GW}(k)$ (defined in [KW17, Section 4]) is described as follows. Choose oriented local coordinates. Choose an oriented local trivialization of $\mathcal{V}$. This identifies σ with a function $\sigma : \mathbf{A}_k^r \to \mathbf{A}_k^r$. Then $\mathrm{ind}_L \, \sigma$ is the local $\mathbf{A}^1$-degree in the sense of Morel [Mor12]. (This is discussed in more detail in [KW16a].)

If σ has a zero at a point p such that the corresponding function $(f_1, \ldots, f_r) : \mathbf{A}_k^r \to \mathbf{A}_k^r$ satisfies the condition that the Jacobian determinant $J = \det(\frac{\partial f_i}{\partial x_j})_{ij}$ is non-zero in $k(p)$, then we say that σ has a *simple* zero. The local degree at a simple zero p such that $k \subseteq k(p)$ is a separable field extension is computed in [KW17, Proposition 32], and we include the statement for clarity.

Proposition 6. *Suppose* σ *has a simple zero at a point* p. *(In other words, suppose that a corresponding function* $(f_1, \ldots, f_r) : \mathbf{A}_k^r \to \mathbf{A}_k^r$ *satisfies the condition that the Jacobian determinant* $J = \det(\frac{\partial f_i}{\partial x_j})_{ij}$ *is non-zero in* $k(p)$.*) Then the corresponding local degree is* $\mathrm{Tr}_{k(p)/k}\langle J \rangle$ *if* $k \subseteq k(p)$ *is a separable field extension.*

We now identify many sections of $\mathcal{V}$ with isolated zeros. Let $i \colon G := \mathrm{Gr}(2, n+1) \to \mathbf{P}^N$ be the Plücker embedding. Let $\check{\mathbf{P}}^N$ be the dual projective space parametrizing hyperplanes in $\mathbf{P}^N$. Let

$$Z := \{(H_1, H_2, \ldots, H_{2n-2}) \in (\check{\mathbf{P}}^N)^{2n-2} \mid \dim(G \cap H_1 \cap \cdots \cap H_{2n-2}) \neq 0\}.$$

Let $S := \{(\sigma_1, \ldots, \sigma_{2n-2}) \in H^0(G, \wedge^2 \mathcal{S}^*)^{2n-2} \mid \sigma_i \neq 0 \; \forall \; i\}$. We have a natural map

$$p \colon S \to (\check{\mathbf{P}}^N)^{2n-2},$$

which is $\mathbf{G}_{m,k}^{2n-2}$-bundle, and we let $\mathsf{U} := S \setminus p^{-1}(Z)$. The purpose of the following lemma and corollary is to show that the complement of U is codimension at least 2, from which it will follow that $e(\mathcal{V}, \sigma)$ is independent of σ for σ in U.

Lemma 7. Z *is closed and* $\operatorname{codim} Z \geq 2$.

Proof. For any irreducible subvariety $W \subset \mathbf{P}^N$ with $\dim W \geq 1$ and for any hyperplane $H \in \check{\mathbf{P}}^N$, we have $\dim(W \cap H) = \dim W - 1$ if and only if $H \not\supset W$, and $\dim W = \dim(W \cap H)$ otherwise (see, e.g., [Sha94, I 6.2 Theorem 5]). By applying this in turn to the finitely many irreducible components of each of $W = G, G \cap H_1, \ldots, G \cap H_1 \cap H_2 \cap \cdots \cap H_{2n-3}$, we see that $Z = \bigcup_{i=1}^{2n-2} \pi_i^{-1}(Z_i)$, where $Z_i \subset (\check{\mathbf{P}}^N)^i$ is the subset

$$Z_i := \{(H_1, H_2, \ldots, H_i) \in (\check{\mathbf{P}}^N)^i \,|$$
$$H_i \supset A, \text{ for some irreducible component } A \text{ of } G \cap H_1 \cap \cdots \cap H_{i-1}\},$$

and $\pi_i : (\check{\mathbf{P}}^N)^{2n-2} \to (\check{\mathbf{P}}^N)^i$ is the projection on the first i factors. Note that Z_i could equivalently be written

$$Z_i = \{(H_1, H_2, \ldots, H_i) \in (\check{\mathbf{P}}^N)^i \,|\, \dim(G \cap H_1 \cap \cdots \cap H_i) > 2n - 2 - i\}.$$

We will first show Z_i is a closed subset of $(\check{\mathbf{P}}^N)^i$. To see this, consider the incidence variety $I \subset (\check{\mathbf{P}}^N)^i \times \mathbf{P}^N$ defined

$$I := \{(H_1, H_2, \ldots, H_i, x) : H_1, \ldots, H_i \in \check{\mathbf{P}}^N, x \in G \cap H_1 \cap \cdots \cap H_i\}.$$

The projection map restricted to I gives a map $f : I \to (\check{\mathbf{P}}^N)^i$, and Z_i is the locus in $(\check{\mathbf{P}}^N)^i$ of points where the fiber has larger than expected dimension, which is closed by, e.g., [Sha94, I 6.3 Theorem 7].

We will now show $\operatorname{codim} Z_i \geq 2$. Consider the map $\pi_i^0 : (\check{\mathbf{P}}^N)^i \to (\check{\mathbf{P}}^N)^{i-1}$ given by projection onto the first $i-1$ factors. For each $(H_1, \ldots, H_{i-1})$ in $(\check{\mathbf{P}}^N)^{i-1}$ and each irreducible component A of $G \cap H_1 \cap \cdots \cap H_{i-1}$, the dimension of A satisfies $\dim(A) \geq 2n - 2 - (i - 1) \geq 1$, and therefore also $\dim A \geq 1$. Let P, Q be two distinct points on A. Then $Z'_A := \{H \in \check{\mathbf{P}}^N \,|\, H \supset A\} \subset \check{\mathbf{P}}^N$ is contained in the intersection of the two distinct hyperplanes $\{H \in \check{\mathbf{P}}^N \,|\, P \in H\}$ and $\{H \in \check{\mathbf{P}}^N \,|\, Q \in H\}$, and therefore has codimension ≥ 2. Thus the fibers of the restriction of π_i^0 to Z_i have codimension ≥ 2. It follows ([Sha94, I 6.3 Theorem 7]) that $\operatorname{codim} Z_i \geq 2$.

Another application of ([Sha94, I 6.3 Theorem 7]) shows that $\operatorname{codim} \pi_i^{-1}(Z_i) \geq 2$ for every i, and therefore $\operatorname{codim} Z \geq 2$. $\qquad\square$

Let U be the subset of $H^0(G, (\wedge^2 \mathcal{S}^*)^{2n-2})$ defined $\mathsf{U} := S \setminus p^{-1}(Z)$.

Corollary 8. *A section σ of $\mathcal{V} = \oplus_{i=1}^{2n-2} \wedge^2 \mathcal{S}^*$ in U has only isolated zeros and $e(\mathcal{V}, \sigma)$ is independent of the choice of such σ.*

For clarity, we remark that if σ is defined over an extension field E of k, then $e(\mathcal{V}, \sigma)$ is an element of $\mathrm{GW}(E)$ and the claimed independence means that $e(\mathcal{V}, \sigma)$ is the base-change to E of $e(\mathcal{V}, \sigma')$ for some section σ' defined over k.

Proof. A section σ of $\mathcal{V}$ corresponding to a point of U has only isolated zeros because the zeroes of $\sigma = (\sigma_1, \ldots, \sigma_{2n-2})$ are precisely the points in the intersection of G and the hyperplanes corresponding to σ_i.

Since $\mathrm{codim}\, Z \geq 2$, we also have $\mathrm{codim}\, p^{-1}(Z) \geq 2$, so it follows that $H^0(\mathcal{V}) - U$ has codimension ≥ 2. By the proof of [KW17, Lemma 57], the fact that $H^0(\mathcal{V}) - U$ has codimension ≥ 2 implies that any two points of U can be connected by affine lines in U, after possibly passing to an odd degree field extension. (This latter property, which is described in more detail in [KW17, Definition 35 and Corollary 36], is related to $\mathbf{A}^1$-chain connectedness as in [AM11, Section 2.2].) By [KW17, Theorem 3], $e(\mathcal{V}, \sigma)$ is independent of the choice of σ in U. $\qquad\square$

3. A FORMULA FOR THE LOCAL INDEX

Proposition 9. *Let σ be the section $\sigma = \oplus_{i=1}^{2n-2} \alpha_i \wedge \beta_i$ of $\mathcal{V}$. Suppose that σ has a simple zero at the point $L = \mathrm{Span}(e_n, e_{n+1})$ of $\mathrm{Gr}(2, n+1)$. Let $\alpha_i = \sum_j a_{ij}\phi_j$ and $\beta_i = \sum_j b_{ij}\phi_j$ be the expansion of the linear forms α_i and β_i in terms of the chosen k-basis. Then*

$$
\mathrm{ind}_L\,\sigma = \left\langle \det \begin{bmatrix} \cdots & (a_{i1}b_{in+1} - a_{in+1}b_{i1}) & \cdots \\ & \vdots & \\ \cdots & (a_{ij}b_{in+1} - a_{in+1}b_{ij}) & \cdots \\ & \vdots & \\ \cdots & (a_{in-1}b_{in+1} - a_{in+1}b_{in-1}) & \cdots \\ \cdots & (a_{in}b_{i1} - a_{i1}b_{in}) & \cdots \\ & \vdots & \\ \cdots & (a_{in}b_{ij} - a_{ij}b_{in}) & \cdots \\ & \vdots & \\ \cdots & (a_{in}b_{in-1} - a_{in-1}b_{in}) & \cdots \end{bmatrix} \right\rangle
$$

Proof. L corresponds to the origin in the affine patch $\mathrm{Spec}\,k[x_1, \ldots, x_{n-1}, y_1, \ldots, y_{n-1}]$. Trivialize each of the $2n - 2$ summands of $\mathcal{V}$ in a neighbourhood of L using the section $\tilde{\phi}_n \wedge \tilde{\phi}_{n+1}$ of $\mathcal{S}^* \wedge \mathcal{S}^*$. Let $(f_1, f_2, \ldots, f_{2n-2})$ be functions on the affine patch $\mathrm{Spec}\,k[x_1, \ldots, x_{n-1}, y_1, \ldots, y_{n-1}]$ be defined by the relations $\alpha_i \wedge \beta_i = f_i \cdot \tilde{\phi}_n \wedge \tilde{\phi}_{n+1}$.

We now need to compute the matrix of partial derivatives of these functions at the origin and take its determinant. First we have the change of basis formulae

$$
\phi_i = \begin{cases} \tilde{\phi}_i + x_i\tilde{\phi}_n + y_i\tilde{\phi}_{n+1} & \text{for } i = 1, \ldots, n-1 \\ \tilde{\phi}_n & \text{for } i = n \\ \tilde{\phi}_{n+1} & \text{for } i = n+1 \end{cases}.
$$

Now

$$\alpha_i \wedge \beta_i = (\sum_j a_{ij}\phi_j) \wedge (\sum_j b_{ij}\phi_j)$$

$$= [(\sum_{j=1}^{n-1} a_{ij}(\tilde\phi_j + x_j\tilde\phi_n + y_j\tilde\phi_{n+1})) + a_{in}\tilde\phi_n + a_{in+1}\tilde\phi_{n+1}]\wedge$$

$$[(\sum_{j=1}^{n-1} b_{ij}(\tilde\phi_j + x_j\tilde\phi_n + y_j\tilde\phi_{n+1})) + b_{in}\tilde\phi_n + b_{in+1}\tilde\phi_{n+1}]$$

Since we will be evaluating the matrix of partial derivatives at $e_n \wedge e_{n+1}$, we only need to focus on terms that have $\tilde\phi_n \wedge \tilde\phi_{n+1}$ or $\tilde\phi_{n+1} \wedge \tilde\phi_n$. Also, we only need to pick out the linear terms in this expansion, so we may ignore the constant term and higher order terms in the Taylor expansion using the variables x_i and y_i. Therefore

$$\alpha_i \wedge \beta_i = \left[\dots + \sum_j (a_{ij}b_{in+1} - a_{in+1}b_{ij})x_j + \sum_j (a_{in}b_{ij} - a_{ij}b_{in})y_j + \dots\right]\tilde\phi_n \wedge \tilde\phi_{n+1}.$$

Computing partial derivatives and evaluating at $e_n \wedge e_{n+1}$ gives the formula in the statement of the lemma. $\qquad\square$

We wish to express the local index at a simple zero L of σ in terms of the line L and the configuration of the π_i. When $k \subseteq k(L)$ is a separable, we do this in terms of the field of definition $k(L)$, the configuration of the intersection points of $\pi_i \cap L$ on L, and the configuration of hyperplanes spanned by π_i and L in the space of hyperplanes of $\mathbf{P}^n$ containing L, in the following manner.

Let $W \subset k(L)^{n+1}$ denote the dimension 2 vector subspace corresponding to the line L, so L is canonically isomorphic to $\mathbf{P}W$. The space of hyperplanes containing W is canonically identified with the projective space $\mathbf{P}(k(L)^{n+1}/W)$. Let W^* denotes the $k(L)$-linear dual of W, and similarly for $(k(L)^{n+1}/W)^*$.

Although the intersection points $L \cap \pi_i$ and the hyperplanes spanned by the π_i and L only determine points of $\mathbf{P}W$ and $\mathbf{P}(k(L)^{n+1}/W)$, respectively, the section $\sigma = \oplus_{i=1}^{2n-2}\alpha_i\wedge\beta_i$ distinguishes points of W^* and $(k(L)^{n+1}/W)^*$. Namely, since $L\cap\pi_i$ is non-empty, we have that the restrictions of α_i and β_i to W are linearly dependent, i.e., $\alpha_i \wedge \beta_i$ is in

$$\mathrm{Ker} := \mathrm{Ker}((k(L)^{n+1})^* \wedge (k(L)^{n+1})^* \to W^* \wedge W^*).$$

There is a natural map

(3) $$\mathrm{Ker} \to (k(L)^{n+1}/W)^* \otimes W^*.$$

(To see this, note that the map $(k(L)^{n+1})^*\otimes(k(L)^{n+1}/W)^* \to \mathrm{Ker}\,/((k(L)^{n+1}/W)^*\wedge(k(L)^{n+1}/W)^*)$ is surjective with kernel $(k(L)^{n+1}/W)^*\otimes(k(L)^{n+1}/W)^* \subset (k(L)^{n+1})^*\otimes(k(L)^{n+1}/W)^*$, giving rise to a natural isomorphism $\mathrm{Ker}\,/((k(L)^{n+1}/W)^*\wedge(k(L)^{n+1}/W)^*) \cong W^*\otimes(k(L)^{n+1}/W)^*$.)

Choose bases of W and $k(L)^{n+1}/W$, giving rise to bases of their duals and therefore coordinates of $\mathbf{P}W$ and $\mathbf{P}(k(L)^{n+1}/W)$. Let $[c_{i0}, c_{i1}]$ be the coordinates of $\pi_i \cap L = [c_{i0}, c_{i1}]$, and let $[d_{i0}, d_{i1}, \dots, d_{i,n-2}]$ be the coordinates of the plane spanned by π_i and L.

Definition 10. *A lift of the homogeneous coordinates $[c_{i0}, c_{i1}]$ and $[d_{i0}, d_{i1}, \ldots, d_{i,n-2}]$ to coordinates (c_{i0}, c_{i1}) and $(d_{i0}, d_{i1}, \ldots, d_{i,n-2})$ of vectors in W^* and $(k(L)^{n+1}/W)^*$ are normalized if $(d_{i0}, d_{i1}, \ldots, d_{i,n-2}) \wedge (c_{i0}, c_{i1})$ is the image of $\alpha_i \wedge \beta_i$ under the map (3).*

Let $i(L)$ be defined as in Equation (1) for normalized coordinates (c_{i0}, c_{i1}) and $(d_{i0}, d_{i1}, \ldots, d_{i,n-2})$ of $\pi_i \cap L$ and the plane spanned by π_i and L, respectively, for $i = 1, 2, \ldots, 2n - 1$.

Proposition 11. *Suppose that L is a simple zero of σ such that $k \subseteq k(L)$ is a separable field extension. Then $\mathrm{ind}_L \, \sigma = \mathrm{Tr}_{k(L)/k} \langle i(L) \rangle$.*

Proof. By Proposition 6, it is sufficient to show that the Jacobian determinant J equals $i(L)$, namely $J = i(L)$ in $k(L)^*/(k(L)^*)^2$. In particular, we may assume that $k(L) = k$, and we do this now for notational simplicity. By Lemma 4, we may compute J with local coordinates coming from a basis $\{e_1, \ldots, e_{n+1}\}$ such that L corresponds to $W = \mathrm{Span}(e_n, e_{n+1})$. Since changing the basis of W and k^n/W changes $i(L)$ by a square (n is odd), we may also compute $i(L)$ using the basis $\{e_{n+1}, e_n\}$ of W and the basis of k^n/W determined by the images of $\{e_1, \ldots, e_{n-1}\}$. This choice determines the normalized coordinates of $\pi_i \cap L$ and the plane spanned by π_i and L.

Since $\pi_i = \{\alpha_i = 0, \beta_i = 0\}$ intersects L, there is a unique v_i in W such that $\alpha_i(v_i) = \beta_i(v_i) = 0$. It follows that that the vectors $(a_{in}, a_{i,n+1})$ and $(b_{in}, b_{i,n+1})$ are linearly dependent in the linear dual W^* of W. In particular, by replacing (α_i, β_i) with either $(\alpha_i - c\beta_i, \beta_i)$ for some constant c or $(-\beta_i, \alpha_i)$, we may assume that $(a_{in}, a_{i,n+1}) = (0, 0)$.

By Proposition 9, it follows that

$$
\text{(4)} \qquad \mathrm{ind}_L \, \sigma = \langle \det \begin{bmatrix} \cdots & a_{i1} b_{in+1} & \cdots \\ & \vdots & \\ \cdots & a_{ij} b_{in+1} & \cdots \\ & \vdots & \\ \cdots & a_{in-1} b_{in+1} & \cdots \\ \cdots & -a_{i1} b_{in} & \cdots \\ & \vdots & \\ \cdots & -a_{ij} b_{in} & \cdots \\ & \vdots & \\ \cdots & -a_{in-1} b_{in} & \cdots \end{bmatrix} \rangle.
$$

Note that $[b_{i,n+1}, -b_{in}]$ are coordinates in $\mathbf{P}W$ for the hyperplane $\{\alpha_i = \beta_i = 0\} \cap W$ in W; $[a_{i1}, a_{i2}, \ldots, a_{in-1}]$ are coordinates in $\mathbf{P}(k^n/W)$ for the plane spanned by π_i and L; and that $(b_{i,n+1}, -b_{in})$ and $(a_{i1}, a_{i2}, \ldots, a_{in-1})$ are the normalized coordinates for our chosen bases. $\square$

4. THE EULER CLASS OR NUMBER OF $\mathcal{V}$

Let X be a smooth, proper scheme of dimension n over k. Let $\mathcal{E} \to X$ and $\mathcal{E}' \to X$ be vector bundles of ranks r and r' such that $r + r' = n$ and $\mathcal{E} \oplus \mathcal{E}'$ is relatively orientable.

Suppose that σ and σ' are global sections of $\mathcal{E}$ and $\mathcal{E}'$ respectively that only have isolated zeros admitting Nisnevich local coordinates and such that $e(\mathcal{E} \oplus \mathcal{E}', \sigma \oplus \alpha\sigma')$ is independent of α for all $\alpha \in k^*$. Let $e(\mathcal{E} \oplus \mathcal{E}') := e(\mathcal{E} \oplus \mathcal{E}', \sigma \oplus \sigma')$.

Proposition 12. *If r' is odd, then $e(\mathcal{E} \oplus \mathcal{E}')$ is an integer multiple of the hyperbolic form $h = \langle -1 \rangle + \langle 1 \rangle$ in $\mathrm{GW}(k)$.*

Remark 13. *The hypothesis that $e(\mathcal{E} \oplus \mathcal{E}', \sigma \oplus \alpha\sigma')$ is independent of α for all $\alpha \in k^*$ should be unnecessary, because the Euler number should always be independent of the section. However, since this is not proven at present, we prove the proposition under this hypothesis, which is sufficient for our purposes.*

This proof is extracted from M. Levine's argument that the Euler characteristic of an odd dimensional scheme is a multiple of h [Lev17b, Theorem 7.1]. M. Levine also credits J. Fasel. Since the result and the context of the definitions is different, we give the proposition and proof.

$\mathrm{GW}(k)$ is the zeroth graded summand of the graded ring $K^{MW}(k)$ introduced by Morel, and then refined in joint work with Hopkins, presented in [Mor12, Chapter 3]. $K^{MW}(k)$ has generators $[u]$ of degree 1 for u in k^*, and η of degree -1. The element $\langle u \rangle$ in $\mathrm{GW}(k)$ corresponds to the element $1 + \eta[u]$.

Proof. Note that the set of isolated zeros of $\sigma \oplus \alpha\sigma'$ does not depend on α for any α in k^*. Let x be such a zero of $\sigma \oplus \sigma'$. We first show that

(5) $$\mathrm{ind}_x(\sigma \oplus \alpha\sigma') = \langle \alpha \rangle^{r'} \, \mathrm{ind}_x(\sigma \oplus \sigma')$$

for any $\alpha \in k^*$.

$\mathrm{ind}_x(\sigma \oplus \sigma')$ is computed by the procedure in [KW17]. Heuristically, this computation is accomplished by considering the section $\sigma \oplus \sigma'$ locally to be a function $\mathbf{A}^n \to \mathbf{A}^n$ and the index is the Jacobian determinant. Replacing $\sigma \oplus \sigma'$ by $\sigma \oplus \alpha\sigma'$ has the effect of multiplying the last r' coordinate projections of the associated function $\mathbf{A}^n \to \mathbf{A}^n$ by α, which in turn scales the Jacobian determinant that computes the local index by $\alpha^{r'}$.

Precisely, choose an open neighborhood U of x with Nisnevich local coordinates $\phi : U \to \mathrm{Spec}\, k[x_1, x_2, \ldots, x_n]$ near x. Choose local trivializations $\psi : \mathcal{E}|_U \to \mathcal{O}_U^r$ and $\psi' : \mathcal{E}'|_U \to \mathcal{O}_U^{r'}$ such that $\psi \oplus \psi'$ and ϕ are compatible with the relative orientation. Then $\psi\sigma = (f_1, \ldots, f_r)$ and $\psi\sigma' = (f_1', \ldots, f_r')$ where f_i and f_i' are in $\mathcal{O}(U)$. We choose g_i for $i = 1, \ldots, r$ in $k[x_1, \ldots, x_n]$ and g_i' in $k[x_1, \ldots, x_n]$ for $i = 1, \ldots, r'$ such that $f_i - \phi^* g_i$ and $f_i' - \phi^* g_i'$ are in a sufficiently high power of m_x. For notational convenience, define $(h_1, \ldots h_{r+r'})$ by $(h_1, \ldots h_{r+r'}) = (g_1, \ldots, g_r, g_1', \ldots, g_r')$. Then $(h_1, \ldots h_{r+r'})$ defines the complete intersection

$$\mathcal{O}_{Z,x} \cong k[x_1, \ldots, x_n]_{m_{\phi(x)}} / \langle h_1, \ldots, h_{r+r'} \rangle$$

and $\mathrm{ind}_x(\sigma \oplus \alpha\sigma')$ is represented by the bilinear pairing on $\mathcal{O}_{Z,x}$

$$(a, b) \mapsto \eta(ab),$$

where η is the k-linear map $\eta : \mathcal{O}_{Z,x} \to k$ as constructed in [SS75, p. 182]. For example, when $J = \det(\frac{\partial h_i}{x_j})_{i,j}$ is non-zero in $k(x)$, η can be chosen to be any linear map

with $\eta(J) = \dim_k \mathcal{O}_{Z,x}$ without changing the isomorphism class of the resulting form. For our purposes, it suffices to know that η scales k-linealy with the element of Δ of $\mathcal{O}_{Z,x} \otimes_k \mathcal{O}_{Z,x}$ defined by choosing a_{ij} in $k[x_1, \ldots, x_n] \otimes_k k[x_1, \ldots, x_n]$ such that $h_j \otimes 1 - 1 \otimes h_j = \sum_{ij} a_{ij}(x_i \otimes 1 - 1 \otimes x_i)$, and letting Δ be the image of $\det(a_{ij})$. In particular, both J and Δ are multiplied by α when h_i replaced by αh_i for some fixed i. Changing the section $\sigma \oplus \sigma'$ to $\sigma \oplus \alpha \sigma'$ changes each f_i' to $\alpha f_i'$ and leaves the f_i fixed. We may therefore choose new g_i and g_i' by leaving the g_i fixed and changing g_i' to $\alpha g_i'$. Therefore new a_{ij} can be defined by keeping a_{ij} the same for $j \leq r$ and changing a_{ij} to αa_{ij} for $j = r + 1, \ldots, r + r'$. Therefore Δ is changed to $\alpha^{r'} \Delta$, giving (5).

For field extensions $k \subseteq L$, let $\mathcal{E}_L \oplus \mathcal{E}_L'$ denote the base change of the vector bundle $\mathcal{E} \oplus \mathcal{E}'$ to L. By functoriality, the Euler number $e(\mathcal{E}_L \oplus \mathcal{E}_L')$ in $\mathrm{GW}(L)$ is the pullback of $e(\mathcal{E} \oplus \mathcal{E}')$ in $\mathrm{GW}(k)$, i.e. $e(\mathcal{E}_L \oplus \mathcal{E}_L') = e(\mathcal{E} \oplus \mathcal{E}') \otimes_k L$.

Furthermore, $e(\mathcal{E}_L \oplus \mathcal{E}_L')$ in $\mathrm{GW}(L)$ can be computed with the section $\sigma + \alpha \sigma'$ for any α in L^*. Thus

$$e(\mathcal{E}_L \oplus \mathcal{E}_L') = e(\mathcal{E}_L \oplus \mathcal{E}_L', \sigma \oplus \alpha \sigma') = \sum_{x : \sigma(x)=0, \sigma'(x)=0} \mathrm{ind}_x(\sigma \oplus \alpha \sigma').$$

Since r' is odd, we have

$$\mathrm{ind}_x(\sigma \oplus \alpha \sigma') = \langle \alpha \rangle^{r'} \mathrm{ind}_x(\sigma \oplus \sigma') = \langle \alpha \rangle \, \mathrm{ind}_x(\sigma \oplus \sigma'),$$

by (5). It follows that

(6) $$e(\mathcal{E}_L \oplus \mathcal{E}_L') = \langle \alpha \rangle e(\mathcal{E}_L \oplus \mathcal{E}_L').$$

for all α in L^*.

To simplify notation, let $e = e(\mathcal{E}_L \oplus \mathcal{E}_L')$ in $\mathrm{GW}(k)$ and let $e_L = e \otimes_k L$ be the pullback to $\mathrm{GW}(L)$. So (6) says that $\langle \alpha \rangle e_L = e_L$ for all α in L^*.

We now show that e is an integer multiple of h. By [Lam05, IX Milnor's Theorem 3.1 p. 306], it suffices to show that $e_{k(t)}$ is an integer multiple of h. There is a map

$$\partial_t : K_0^{\mathrm{MW}}(k(t)) \to K_{-1}^{\mathrm{MW}}(k)$$

corresponding to the local ring $k[t]_{(t)}$ [Mor12, Theorem 3.15]. This map has the following two properties:

(1) For any class f in $\mathrm{GW}(k)$, the class $\langle t \rangle f_{k(t)}$ in $\mathrm{GW}(k(t)) \cong K_0^{\mathrm{MW}}(k(t))$ has image ηf in $K_{-1}^{\mathrm{MW}}(k(t))$ under ∂_t.
(2) For any class f in $\mathrm{GW}(k)$, the class $\partial f_{k(t)} = 0$.

It follows that

$$0 = \partial_t(e_{k(t)}) = \partial_t(\langle t \rangle e_{k(t)}) = \eta(e),$$

where the first equality is (2), the second is (6), and the third is (1).

There is a canonical isomorphism $K_{-1}^{\mathrm{MW}}(k) \cong W(k)$ [Mor12, Lemma 3.10]. Under this isomorphism ηf in $K_{-1}^{\mathrm{MW}}(k)$ corresponds to the image of f under the quotient map $\mathrm{GW}(k) \to \mathrm{GW}(k)/\mathbb{Z}h \cong W(k)$ for any f in $\mathrm{GW}(k)$.

Therefore $e_{k(t)}$ is 0 in $W(k)$, whence a multiple of h and the claim follows. $\square$

Corollary 14.

$$e(\mathcal{V}) = \frac{1}{2}\frac{(2n-2)!}{n!(n-1)!}h.$$

Proof. We know that $e(\mathcal{V})$ is a multiple of h and that its rank is $\frac{(2n-2)!}{n!(n-1)!}$ from [EH16, Proposition 4.12]. $\square$

We now give an alternate, more explicit proof of this calculation when $n = 4$ using the local index calculations from Section 3.

Proposition 15. *Let $\mathcal{S}$ be the tautological bundle of* $\mathrm{Gr}(2,4)$. *Let $\mathcal{V} = \oplus_{i=1}^{4}\Lambda^2(\mathcal{S}^*)$. Then $e(\mathcal{V}) = \langle 1 \rangle + \langle -1 \rangle$.*

Proof. As in Section 3, we will compute this Euler characteristic by adding up local contributions from the explicit section σ of $\mathcal{V}$ coming from four lines in $\mathbf{P}_k^3$. By Corollary 8, we may choose a k-rational section σ with only isolated zeros for this computation. We may moreover assume that the two lines corresponding to the two zeroes of σ are k-rational, by explicit example (see for instance Section 6.1). Classical arguments (see for instance [EH16, Section 3.4.1]) show that these two lines can be taken to be two skew lines in the same ruling of a quadric surface in $\mathbf{P}_k^3$, so after a change of coordinates, we may assume that the two zeroes of σ are the lines corresponding to the subspaces $L' := \mathrm{Span}(e_1, e_2)$ and $L := \mathrm{Span}(e_3, e_4)$ of k^4.

We will now show that a careful choice of α_i, β_i (which in turn determine the section σ), and local coordinates around L' and L lets us show that the matrices computing $\mathrm{ind}_{L'}\sigma$ and $\mathrm{ind}_L\sigma$ are related by row operations and sign swaps.

For any index i, since $L = \mathrm{Span}(e_3, e_4)$ intersects L_i, it follows that $\{\phi_1, \phi_2, \alpha_i, \beta_i\}$ are linearly dependent. In terms of the chosen basis expansions for α_i and β_i, this translates to

$$\det \begin{bmatrix} a_{i3} & a_{i4} \\ b_{i3} & b_{i4} \end{bmatrix} = 0.$$

Since we may replace the pair (α, β_i) by any pair of linearly independent vectors in $\mathrm{Span}(\alpha_i, \beta_i)$, by either adding a multiple of β_i to α_i or by swapping α_i and β_i, we may assume that $a_{i3} = a_{i4} = 0$ without any loss of generality. Similarly from the condition that $L' = \mathrm{Span}(e_1, e_2)$ intersects L_i, it follows that

$$\det \begin{bmatrix} a_{i1} & a_{i2} \\ b_{i1} & b_{i2} \end{bmatrix} = 0,$$

and as before, we can use this relation and further change α_i, β_i to assume that $b_{i1} = b_{i2} = 0$.

With these choices, we get

$$\text{ind}_L \sigma = \det \begin{bmatrix} \cdots & a_{i1}b_{i4} & \cdots \\ \cdots & a_{i2}b_{i4} & \cdots \\ \cdots & -a_{i1}b_{i3} & \cdots \\ \cdots & -a_{i2}b_{i3} & \cdots \end{bmatrix}.$$

To compute $\text{ind}_{L'} \sigma$, we need to choose local coordinates in $\text{Gr}(2,4)$ around this point, and a local trivialization of $\Lambda^2(\mathcal{S})$ compatible with the chosen orientation. By Lemma 4, if we set

$$f_1 = e_3, f_2 = e_4, f_3 = e_1, f_4 = e_2,$$

and define $\{\tilde{f}_1, \tilde{f}_2, \tilde{f}_3, \tilde{f}_4\}$ by

$$\begin{cases} f_i & \text{for } i = 1, 2 \\ x_1 f_1 + x_2 f_2 + f_3 & \text{for } i = 3 \\ y_1 f_1 + y_2 f_2 + f_4 & \text{for } i = 4, \end{cases}$$

then x_1, x_2, y_1, y_2 gives us local coordinates around $L' = \text{Span}(f_3, f_4) = \text{Span}(e_1, e_2)$. Let $\{\tilde{\psi}_1, \tilde{\psi}_2, \tilde{\psi}_3, \tilde{\psi}_4\}$ be the dual basis for the basis $\{\tilde{f}_1, \tilde{f}_2, \tilde{f}_3, \tilde{f}_4\}$ of k^{n+1}. Then as before it follows that $\tilde{\psi}_3 \wedge \tilde{\psi}_4$ gives a trivialization of $\Lambda^2(\mathcal{S})$ around $\text{Span}(e_1, e_2)$ compatible with the chosen orientation and hence also a trivialization of $\mathcal{V}$. With these choices, if we now redo the index calculation in Proposition 9, we obtain the formulae

$$\alpha_i \wedge \beta_i = (a_{i1}\psi_3 + a_{i2}\psi_4) \wedge (b_{i3}\psi_1 + b_{i4}\psi_2)$$

$$(a_{i1}\tilde{\psi}_3 + a_{i2}\tilde{\psi}_4) \wedge (b_{i3}(\tilde{\psi}_1 + x_1\tilde{\psi}_3 + x_2\tilde{\psi}_4) + b_{i4}(\tilde{\psi}_2 + y_1\tilde{\psi}_3 + y_2\tilde{\psi}_4)),$$

and,

$$\alpha_i \wedge \beta_i = \left[\ldots - a_{i2}b_{i3}x_1 - a_{i2}b_{i4}y_1 + a_{i1}b_{i3}x_2 + a_{i1}b_{i4}y_2 + \ldots\right]\tilde{\psi}_3 \wedge \tilde{\psi}_4.$$

This implies that

$$\text{ind}_{L'} \sigma = \det \begin{bmatrix} \cdots & -a_{i2}b_{i3} & \cdots \\ \cdots & -a_{i2}b_{i4} & \cdots \\ \cdots & a_{i1}b_{i3} & \cdots \\ \cdots & a_{i1}b_{i4} & \cdots \end{bmatrix}.$$

From these explicit formulae, we see that the matrices computing $\text{ind}_{L'} \sigma$ and $\text{ind}_L \sigma$ are related by swapping the first and fourth rows, and by multiplying each of the second and third rows by -1. This in turn implies that if $\text{ind}_{L'} \sigma = \langle c \rangle$, then $\text{ind}_L \sigma = \langle -c \rangle$, and therefore $e(\mathcal{V}) = \langle c \rangle + \langle -c \rangle = \langle 1 \rangle + \langle -1 \rangle$. $\qquad\square$

5. THE LOCAL INDEX IN TERMS OF CROSS RATIOS

Definition 16. *Let* L_1, L_2, L_3, L_4 *be four lines in* $\mathbf{P}_k^3$ *and let* L *be another line in* $\mathbf{P}^3$ *that meets all four lines. Assume that the four intersection points of* L *with the* L_i *are distinct. Also assume that the four planes spanned by* L *and the* L_i *are distinct. Let* λ_L *be the cross ratio of the four points* $L \cap L_1, L \cap L_2, L \cap L_3, L \cap L_4$ *on the line* L. *Let* μ_L *be the cross ratio of the four planes containing* L *and each of the four lines* L_1, L_2, L_3, L_4 *in the* $\mathbf{P}_k^1$ *of planes containing* L *in* $\mathbf{P}_k^3$.

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The goal of this section is to prove the following theorem.

Theorem 17. *Let $\mathcal{V} = \oplus_{i=1}^4 \Lambda^2 \mathcal{S}^*$. Let L_1, L_2, L_3, L_4 be four lines in $\mathbf{P}_k^3$ such that*

- *the corresponding section $\oplus_{i=1}^4 \alpha_i \wedge \beta_i$ has only simple zeroes,*
- *for any line L meeting all four lines, the four intersection points $L \cap L_i$ are pairwise distinct, and,*
- *for any line L meeting all four lines, the four planes spanned by L and each of the L_i are pairwise distinct.*

(The locus of such lines is an open dense subset of $\mathrm{Gr}(2,4)^4$.) Let λ_L, μ_L be the associated cross ratios as in Definition 16. There exists a section σ of $\mathcal{V}$ such that for all L meeting L_1, L_2, L_3, L_4, we have

$$\mathrm{ind}_L \, \sigma = \mathrm{Tr}_{k(L)/k} \langle \lambda_L - \mu_L \rangle$$

Proof. By Proposition 6 (originally from [KW17]), it is enough to show that $\mathrm{ind}_L \, \sigma$ computed over the field of definition of the line $k(L)$ equals $\langle \lambda_L - \mu_L \rangle$.

Recall that the line L_i is cut out by the two hyperplanes $\alpha_i = \sum_j a_{ij} \phi_j$ and $\beta_i = \sum_j b_{ij} \phi_j$. As in the proof of Proposition 15, we may assume that the two lines meeting these four lines are $L = \mathrm{Span}(e_3, e_4)$ and $L' = \mathrm{Span}(e_1, e_2)$ and that $a_{i3} = a_{i4} = b_{i1} = b_{i2} = 0$ without any loss of generality. We have explicit projective coordinates $[0 : 0 : z_3 : z_4]$ on $L = \mathrm{Span}(e_3, e_4)$ induced from the projective coordinates $[z_1 : z_2 : z_3 : z_4]$ on $\mathbf{P}_k^3$. In these coordinates, the point $L \cap L_i$ is $[0 : 0 : -b_{i4} : b_{i3}]$. Since cross ratios are invariant under automorphisms of L and since $[0 : 0 : z_3 : z_4] \mapsto [0 : 0 : -z_4 : z_3]$ is an automorphism of L, it follows that λ_L is the cross ratio of the four points $[b_{i3} : b_{i4}]$ for $i = 1, 2, 3, 4$. Similarly, with $L' = \mathrm{Span}(e_1, e_2)$, we have that $\lambda_{L'}$ is the cross ratio of the four points $[a_{i1} : a_{i2}]$.

We will now pick explicit coordinates $[w_1 : w_2]$ on the $\mathbf{P}_k^1$ of 2-planes containing $L = \mathrm{Span}(e_3, e_4)$ in $\mathbf{P}_k^3$ to compute μ_L. The isomorphism is given by mapping $[w_1 : w_2]$ to the 2-plane $\mathrm{Span}(e_3, e_4, -w_2 e_1 + w_1 e_2)$ in $\mathbf{P}^3$. In these coordinates, the plane containing L and L_i, namely $\mathrm{Span}(e_3, e_4, -a_{i2} e_1 + a_{i1} e_2)$, is $[a_{i1} : a_{i2}]$. Therefore μ_L is the cross ratio of the four points $[a_{i1} : a_{i2}]$ for $i = 1, 2, 3, 4$. Similarly $\mu_{L'}$ is the cross ratio of the four points $[b_{i3} : b_{i4}]$ for $i = 1, 2, 3, 4$.

Pick $A, B \in \mathrm{GL}_2(k)$ such that their respective images $\overline{A}, \overline{B} \in \mathrm{PGL}(k)$ satisfy

$$\overline{B}[b_{13} : b_{14}] = [1 : 0] \qquad\qquad \overline{A}[a_{11} : a_{12}] = [1 : 0]$$
$$\overline{B}[b_{23} : b_{24}] = [1 : 1] \qquad\qquad \overline{A}[a_{21} : a_{22}] = [1 : 1]$$
$$\overline{B}[b_{33} : b_{34}] = [0 : 1] \qquad\qquad \overline{A}[a_{31} : a_{32}] = [0 : 1]$$
$$\overline{B}[b_{43} : b_{44}] = [1 : \lambda_L] \qquad\qquad \overline{A}[a_{41} : a_{42}] = [1 : \mu_L].$$

Now change coordinates on the underlying $\mathbf{P}^3$ using the block matrix $\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}$ in $\mathrm{GL}_4(k)$. In these new coordinates, applying Proposition 9 we get

$$\mathrm{ind}_L \, \sigma = c \det \begin{bmatrix} 0 & 1 & 0 & \lambda_L \\ 0 & 1 & 1 & \lambda_L \mu_L \\ -1 & -1 & 0 & -1 \\ 0 & -1 & 0 & -\mu_L \end{bmatrix} = c(\lambda_L - \mu_L),$$

where $c \in k$ is an overall nonzero constant coming from the fact that we have picked lifts A and B of the elements $\overline{A}, \overline{B}$ in $\mathrm{PGL}_2(k)$. We can now eliminate c by scaling the section σ by $1/c$, since this scales the local index by the same factor.

Fix a section σ such that $\mathrm{ind}_L \, \sigma = \langle \lambda_L - \mu_L \rangle$. We will show that we also have $\mathrm{ind}_{L'} \, \sigma = \langle \lambda_{L'} - \mu_{L'} \rangle$. This follows from the following two facts.

- As justified in the previous paragraphs, replacing L by L' switches the two cross ratios, so we have $(\lambda_{L'} - \mu_{L'}) = (\mu_L - \lambda_L)$.
- The proof of Proposition 15 shows that $\mathrm{ind}_{L'} \, \sigma = \langle -1 \rangle \, \mathrm{ind}_L \, \sigma$. $\qquad\square$

6. Proofs of Main Theorems and an example

Proof of Theorem 1. Corollary 8 tells us that we may compute $e(\mathcal{V})$ by making any auxiliary choice of section $\sigma \in H^0(\mathrm{Gr}(2, n+1), \mathcal{V})(k) \setminus Z(k)$. We would like to use the explicit section $\sigma = (\alpha_1 \wedge \beta_1, \alpha_2 \wedge \beta_2), \alpha_3 \wedge \beta_3, \alpha_4 \wedge \beta_4)$ arising from four general lines L_1, L_2, L_3, L_4 in $\mathbf{P}^3_k$ for our computations. To be able to apply Proposition 15, which in turn relies on Proposition 6, we need the section σ to have only simple zeroes. Further, to be able to interpret the local index in terms of cross-ratios as in Theorem 17, we need the cross-ratios to be well-defined, which in turn needs the points of intersection of the simple zero L with the four lines L_i to be pairwise distinct, and the planes spanned by the simple zero L and each of the four lines L_i to also be pairwise distinct. All of these conditions are satisfied by an open subset of lines $(L_1, L_2, L_3, L_4) \in \mathrm{Gr}(2, 4)^4$. This open locus for instance contains the locus of four lines that are pairwise non intersecting, and such that the fourth line is not tangent to the unique quadric containing the first 3. The proof of the theorem is now a direct application of Proposition 15 and Theorem 17. $\qquad\square$

The proof of Theorem 2 is similar to the proof of Theorem 1.

Proof of Theorem 2. For an open subset of the product of Grassmannians parametrizing $2n - 2$-tuples of codimension 2 planes, the corresponding sections $\sigma = \oplus_{i=1}^{2n-2} \alpha_i \wedge \beta_i$ satisfies the condition that its zeroes are all isolated and simple (see Corollary 8 and the definition just before Proposition 6). If we assume that either that k is perfect or that $k(L)/k$ is separable for all zeroes L of σ, then we may compute the local index over the field of definition $k(L)$ of the line and then apply $\mathrm{Tr}_{k(L)/k}$ to obtain the local index over k. The theorem now follows from Corollary 8, Proposition 11 and Corollary 14. $\qquad\square$

Proof of Corollary 3. By Theorem 1, we have that $\mathrm{Tr}_{\mathbf{F}_{q^2}/\mathbf{F}_q}\langle\lambda_L - \mu_L\rangle$ is a square for $q \equiv 1$ mod 4 and is a non-square for $q \equiv 3 \mod 4$. Moreover, $\mathrm{Disc}(\mathrm{Tr}_{\mathbf{F}_{q^2}/\mathbf{F}_q}\langle a\rangle)$ is a non-square if a is a square and a square if a is a non-square by, for example, [CP84, II.2.3]. $\square$

6.1. **Example.** We conclude the paper with an explicit example of Theorem 1. Let

$$L_1\colon [t:s] \mapsto [t:0:0:s]$$
$$L_2\colon [t:s] \mapsto [t:s:t:s]$$
$$L_3\colon [t:s] \mapsto [s+t:2s:2s+2t:s]$$
$$L_4\colon [t:s] \mapsto [3t:t:s:3s]$$

be the parametric equations of 4 lines in $\mathbf{P}^3 = \mathrm{Proj}\, k[x, y, z, w]$. When the characteristic is not 2, 3 or 5, these lines pairwise do not intersect and the first three lines lie on the quadric $xy - zw$. (In characteristic 3 the lines L_2 and L_3 intersect, and we can instead work with the example where we replace L_3 alone by $[-(s+t) : -s : s+t : s]$. Similarly, in characteristic 5 the lines L_3 and L_4 intersect, and we can replace L_4 alone by $[-2t : t : 2s : -s]$.) The parametric equations of the lines that meet all four lines are $L\colon [t:s] \mapsto [s:t:t:s]$ and $L'\colon [t:s] \mapsto [-s:t:-t:s]$, with

$$L_1 \cap L = [1:0:0:1] \leftrightarrow [0:1] \qquad L_1 \cap L' = [-1:0:0:1] \leftrightarrow [0:1]$$
$$L_2 \cap L = [1:1:1:1] \leftrightarrow [1:1] \qquad L_2 \cap L' = [-1:1:-1:1] \leftrightarrow [1:1]$$
$$L_3 \cap L = [1:2:2:1] \leftrightarrow [2:1] \qquad L_3 \cap L' = [-1:2:-2:1] \leftrightarrow [2:1]$$
$$L_4 \cap L = [3:1:1:3] \leftrightarrow [1:3] \qquad L_4 \cap L' = [-3:-1:1:3] \leftrightarrow [-1:3].$$

The formula for the cross ratio of four points z_1, z_2, z_3, z_4 in k is $\frac{(z_4-z_1)(z_2-z_3)}{(z_2-z_1)(z_4-z_3)}$. This leads to $\lambda_L = 1/3$ and $\lambda_{L'} = -1/5$.

Now we choose explicit projective coordinates on the $\mathbf{P}^1_k$ of planes containing L and the $\mathbf{P}^1_k$ of planes containing L' in order to compute the cross ratios μ_L and $\mu_{L'}$. For the former, let $[t:s]$ be the coordinates of the 2-plane $\mathrm{Span}((1,0,0,1),(0,1,1,0),(t,s,0,0))$ in $\mathbf{P}^3$, and for the latter let it be the coordinates of the 2-plane $\mathrm{Span}((-1,0,0,1),(0,-1,1,0),(t,s,0,0))$. In these coordinates, we have

$$\mathrm{Span}(L, L_1) = [1:0] \qquad\qquad \mathrm{Span}(L', L_1) = [1:0]$$
$$\mathrm{Span}(L, L_2) = [1:-1] \qquad\qquad \mathrm{Span}(L', L_2) = [1:1]$$
$$\mathrm{Span}(L, L_3) = [1:-2] \qquad\qquad \mathrm{Span}(L', L_3) = [1:2]$$
$$\mathrm{Span}(L, L_4) = [3:1] \qquad\qquad \mathrm{Span}(L', L_4) = [3:1],\ .$$

which leads to $\mu_L = -1/5 = \lambda_{L'}$ and $\mu_{L'} = 1/3 = \lambda_L$, and $\lambda_L - \mu_L = 8/15 = -(\lambda_{L'} - \mu_{L'})$. Finally we have $\langle 8/15\rangle + \langle -8/15\rangle = \langle 1\rangle + \langle -1\rangle$ in $\mathrm{GW}(k)$.

REFERENCES

[AF16]   Aravind Asok and Jean Fasel, *Comparing euler classes*, 2016. *Preprint to appear in Q.J. Math*, available at `https://arxiv.org/abs/1306.5250`. ↑1

[AM11]   Aravind Asok and Fabien Morel, *Smooth varieties up to $\mathbb{A}^1$-homotopy and algebraic h-cobordisms*, Adv. Math. **227** (2011), no. 5, 1990–2058. MR2803793 ↑2

[BDM06]   S. M. Bhatwadekar, Mrinal Kanti Das, and Satya Mandal, *Projective modules over smooth real affine varieties*, Invent. Math. **166** (2006), no. 1, 151–184. MR2242636 ↑1

[BM00]   Jean Barge and Fabien Morel, *Groupe de Chow des cycles orientés et classe d'Euler des fibrés vectoriels*, C. R. Acad. Sci. Paris Sér. I Math. **330** (2000), no. 4, 287–290. MR1753295 ↑1

[BS00]   S. M. Bhatwadekar and Raja Sridharan, *The Euler class group of a Noetherian ring*, Compositio Math. **122** (2000), no. 2, 183–222. MR1775418 ↑1

[BS99]   _____, *Zero cycles and the Euler class groups of smooth real affine varieties*, Invent. Math. **136** (1999), no. 2, 287–322. MR1688449 ↑1

[CP84]   P. E. Conner and R. Perlis, *A survey of trace forms of algebraic number fields*, Series in Pure Mathematics, vol. 2, World Scientific Publishing Co., Singapore, 1984. MR761569 ↑6

[DJK18]   Frédéric Déglise, Fangzhou Jin, and Adheel Khan, *Fundamental classes in motivic homotopy theory*, 2018. *Preprint*, available at `https://arxiv.org/abs/1805.05920`. ↑1

[EG02]   A. Eremenko and A. Gabrielov, *Rational functions with real critical points and the B. and M. Shapiro conjecture in real enumerative geometry*, Ann. of Math. (2) **155** (2002), no. 1, 105–129. MR1888795 ↑1

[EH16]   David Eisenbud and Joe Harris, *3264 and all that—a second course in algebraic geometry*, Cambridge University Press, Cambridge, 2016. MR3617981 ↑1, 4, 4

[Fas08]   Jean Fasel, *Groupes de Chow-Witt*, Mém. Soc. Math. Fr. (N.S.) **113** (2008), viii+197. MR2542148 ↑1

[GI80]   D. Ju. Grigor'ev and N. V. Ivanov, *On the Eisenbud-Levine formula over a perfect field*, Dokl. Akad. Nauk SSSR **252** (1980), no. 1, 24–27. MR572114 ↑1

[Hoy14]   Marc Hoyois, *A quadratic refinement of the Grothendieck-Lefschetz-Verdier trace formula*, Algebr. Geom. Topol. **14** (2014), no. 6, 3603–3658. MR3302973 ↑1

[HW17]   Jens Hornbostel and Matthias Wendt, *Chow-Witt rings of classifying spaces of symplectic and special linear groups*, 2017. *Preprint*, available at `https://arxiv.org/abs/1703.05362`. ↑1

[KW16a]   Jesse Kass and Kirsten Wickelgren, *The class of Eisenbud–Khimshiashvili–Levine is the local $\mathbb{A}^1$-brouwer degree*, 2016. *Preprint*, available at `https://arxiv.org/abs/1608.05669`. ↑1, 2, 2

[KW16b]   _____, *A classical proof that the algebraic homotopy class of a rational function is the residue pairing*, 2016. *Preprint*, available at `https://arxiv.org/abs/1602.08129`. ↑1

[KW17]   _____, *An arithmetic count of the lines on a smooth cubic surface*, 2017. *Preprint*, available at `https://arxiv.org/abs/1708.01175`. ↑1, 2, 2, 4, 5, 6.2

[Lam05]   T. Y. Lam, *Introduction to quadratic forms over fields*, Graduate Studies in Mathematics, vol. 67, American Mathematical Society, Providence, RI, 2005. MR2104929 ↑1, 4

[Lev17a]   Marc Levine, *The intrinsic stable normal cone*, 2017. *Preprint*, available at `https://arxiv.org/abs/1703.03056`. ↑1

[Lev17b]   _____, *Toward a new enumerative geometry*, 2017. *Preprint*, available at `https://arxiv.org/abs/1703.03049`. ↑1, 4

[Lev18a]   _____, *Motivic Euler characteristics and Witt-valued characteristic classes*, 2018. *Preprint*, available at `https://arxiv.org/abs/1806.10108`. ↑1

[Lev18b]   _____, *Toward an algebraic theory of Welschinger invariants*, 2018. *Preprint*, available at `https://arxiv.org/abs/1808.02238`. ↑1

[LR18]   Marc Levine and Arpon Raksit, *Motivic Gauss–Bonnet formulas*, 2018. *Preprint*, available at `https://arxiv.org/abs/1808.08385`. ↑1

[Mil69]   John Milnor, *Algebraic K-theory and quadratic forms*, Invent. Math. **9** (1969/1970), 318–344. MR0260844 ↑1

[Mor12] Fabien Morel, $\mathbb{A}^1$-*algebraic topology over a field*, Lecture Notes in Mathematics, vol. 2052, Springer, Heidelberg, 2012. MR2934577 ↑1, 2, 4, 4, 4

[MS96] Satya Mandal and Raja Sridharan, *Euler classes and complete intersections*, J. Math. Kyoto Univ. **36** (1996), no. 3, 453–470. MR1417820 ↑1

[MTV09a] E. Mukhin, V. Tarasov, and A. Varchenko, *Schubert calculus and representations of the general linear group*, J. Amer. Math. Soc. **22** (2009), no. 4, 909–940. MR2525775 ↑1

[MTV09b] Evgeny Mukhin, Vitaly Tarasov, and Alexander Varchenko, *The B. and M. Shapiro conjecture in real algebraic geometry and the Bethe ansatz*, Ann. of Math. (2) **170** (2009), no. 2, 863–881. MR2552110 ↑1

[MV99] F. Morel and V. Voevodsky, $\mathbb{A}^1$-*homotopy theory of schemes*, Inst. Hautes Études Sci. Publ. Math. **90** (1999), 45–143 (2001). MR1813224 (2002f:14029) ↑1

[Sha94] Igor R. Shafarevich, *Basic algebraic geometry. 1*, Second, Springer-Verlag, Berlin, 1994. Varieties in projective space, Translated from the 1988 Russian edition and with notes by Miles Reid. MR1328833 ↑2

[Sot11] Frank Sottile, *Real solutions to equations from geometry*, University Lecture Series, vol. 57, American Mathematical Society, Providence, RI, 2011. MR2830310 ↑1

[Sot97] ———, *Enumerative geometry for the real Grassmannian of lines in projective space*, Duke Math. J. **87** (1997), no. 1, 59–85. MR1440063 ↑1

[SS75] Günter Scheja and Uwe Storch, *Über Spurfunktionen bei vollständigen Durchschnitten*, J. Reine Angew. Math. **278/279** (1975), 174–190. MR0393056 (52 #13867) ↑4

[Vak06] Ravi Vakil, *Schubert induction*, Ann. of Math. (2) **164** (2006), no. 2, 489–512. MR2247966 ↑1

[Voe03a] Vladimir Voevodsky, *Motivic cohomology with* **Z**/2-*coefficients*, Publ. Math. Inst. Hautes Études Sci. **98** (2003), 59–104. MR2031199 ↑1

[Voe03b] ———, *Reduced power operations in motivic cohomology*, Publ. Math. Inst. Hautes Études Sci. **98** (2003), 1–57. MR2031198 ↑1

[Wen18a] Matthias Wendt, *Chow-Witt rings of Grassmannians*, 2018. *Preprint* available at https://arxiv.org/abs/1805.06142. ↑1

[Wen18b] ———, *Oriented Schubert calculus in Chow–Witt rings of Grassmannians*, 2018. *Preprint* available at https://arxiv.org/abs/1808.07296. ↑1

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