

Finiteness theorems for reductions of Hecke orbits

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JMM 2021
January 8th, 2021

- 1 Lifting isogenies and endomorphisms of abelian varieties
- 2 A Galois-theoretic criterion for finiteness of Hecke orbits
- 3 Verifying Galois-theoretic criterion for supersingular \bar{A}
- 4 Applications to CM-lifting theorems

Lifting p -isogenies from characteristic p to characteristic 0

K : finite extension of \mathbb{Q}_p

A/K : Abelian variety over K with good reduction

$\overline{A}/\mathbb{F}_q$: Reduction of A

$I_p(A)$: $\{B/K' \mid [K' : K] < \infty, B \text{ is } p\text{-power isogenous to } A\}$

$I_p(\overline{A})$: $\{\overline{B}/\mathbb{F}_{q'} \mid [\mathbb{F}_{q'} : \mathbb{F}_q] < \infty, \overline{B} \text{ is } p\text{-power isogenous to } \overline{A}\}$

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Note: All abelian varieties in $I_p(A)$ also have good reduction.

$\overline{I_p(A)} :=$ Reductions of all the abelian varieties in $I_p(A)$.

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Main Question 1 (Lifting p -isogenies):

How large is the subset $\overline{I_p(A)}$ of $I_p(\bar{A})$?

Lifting endomorphisms from char. p to char. 0

Definition

Let A be a g -dimensional abelian variety over a characteristic 0 local field K . We say that A is a **CM-abelian variety** if there is an embedding

$$F \hookrightarrow \operatorname{End}(A) \otimes_{\mathbb{Z}} \mathbb{Q}$$

of a commutative, semisimple \mathbb{Q} -algebra F of dimension $2g$.

Main Question 2 (Existence of CM-lifts):

For which $\overline{A}/\overline{F}_p$ does there exist a CM-abelian variety A over a characteristic 0 local field with reduction \overline{A} ?

- Honda-Tate (Lifting up to isogeny)

Every $\overline{A}/\overline{\mathbb{F}}_p$ is isogenous to a $\overline{B}/\overline{\mathbb{F}}_p$ with a CM-lift.

- Serre-Tate (Canonical lifts for *ordinary* abelian varieties)

Every *ordinary* abelian variety $\overline{A}/\overline{\mathbb{F}}_p$ admits a CM-lift A .

All *isogenies* of such \overline{A} lift to isogenies of the canonical lift A .

- Oort/Conrad-Chai-Oort (Non-existence of CM lifts)

There are *supersingular* abelian varieties $\overline{A}/\overline{\mathbb{F}}_p$ with *no CM lifts*.

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Applications to CM-lifting theorems

Theorem (Kisin, Lam, Shankar, S.)

Fix a lift A/K of $\overline{A}/\overline{\mathbb{F}}_p$ to a characteristic 0 local field.

Assume that \overline{A} is *supersingular*

Then, the subset $\overline{I_p(A)}$ of $I_p(\overline{A})$ is finite.

We prove an analogous result for p -divisible groups over \mathcal{O}_K where the p -adic Galois representation has *reductive monodromy*.

Theorem (Kisin, Lam, Shankar, S.)

- ① Only finitely many *supersingular*^a abelian varieties $\overline{A}/\overline{\mathbb{F}}_p$ of a given dimension admit CM-lifts.
- ② Only finitely many *supersingular* K3 surfaces $\overline{X}/\overline{\mathbb{F}}_p$ admit CM-lifts when $p \geq 5$.

^aWe also prove a common generalization of the results for ordinary/supersingular strata to other Newton strata.

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- K : finite extension of \mathbb{Q}_p
- G_K : absolute Galois group of K
- I_K : inertia subgroup of G_K

- A : abelian variety over K with good reduction
- \mathcal{G} : p -divisible group over K with good reduction

- V : rational p -adic Tate module of A or \mathcal{G}
- ρ : p -adic Galois representation $G_K \rightarrow \mathrm{GL}(V)$

A Galois-theoretic criterion for finiteness

$$\rho: G_K \rightarrow \mathrm{GL}(V) \cong \mathrm{GL}_{2g}(\mathbb{Q}_p).$$

Proposition (*“Totally ramified up to finite index” criterion*)

If $\rho(I_K)$ has finite index in $\rho(G_K)$, then the reduction of the p -Hecke orbit of the corresponding A or \mathcal{G} is finite.

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Proof sketch:

- All abelian varieties in the p -Hecke orbit of A are defined over the fixed field K_ρ of $\ker(\rho)$.
- Assumption \Rightarrow The residue field of K_ρ is a finite field.
- The existence of the moduli space \mathcal{A}_g + Zarhin's trick \Rightarrow there are only finitely many isomorphism classes of abelian varieties of a given dimension defined over a fixed finite field.

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$$\rho: G_K \rightarrow \mathrm{GL}(V).$$

Consider the exact sequence of algebraic groups by taking Zariski closures in $\mathrm{GL}(V)$.

$$1 \rightarrow \overline{\rho(I_K)} \rightarrow \overline{\rho(G_K)} \rightarrow T \rightarrow 1.$$

$\text{Sen} + \epsilon \Rightarrow$ If T is finite, then $\rho(I_K)$ has finite index in $\rho(G_K)$.

The unramified quotient T

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Sen + ϵ \Rightarrow If T is finite, then $\rho(I_K)$ has finite index in $\rho(G_K)$.

Goal: Show T is finite if \bar{A} is supersingular.

V_T : a faithful \mathbb{Q}_p -representation of T

σ : image of a Frobenius element in $\mathrm{GL}(V_T)$,
a generator for the image of T in $\mathrm{GL}(V_T)$.

We will show σ is semisimple and its eigenvalues are roots of unity.

T is finite if \bar{A} is supersingular

$$1 \rightarrow \overline{\rho(I_K)} \rightarrow \overline{\rho(G_K)} \rightarrow T \rightarrow 1,$$

V_T a faithful repn. of T and $\langle \sigma \rangle = T \subset \mathrm{GL}(V_T)$ (Frobenius).

Proof sketch:

- V_T is in the Tannakian category generated by $V (= V_p(A))$.
- V_T is unramified by the definition of T and crystalline, so the eigenvalues of σ are *p -adic units*.
- Since \bar{A} is *supersingular*, Frobenius acts semisimply on $\mathbb{D}(V) \otimes \mathbb{Q}_p$ with eigenvalues rational powers of p up to roots of unity.
- The only *power of p* that is a p -adic unit is 1.

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Applications to CM-lifting theorems

Theorem (Kisin, Lam, Shankar, S.)

Fix a lift A/K of $\overline{A}/\overline{\mathbb{F}}_p$ to a characteristic 0 local field.

Assume that \overline{A} is *supersingular* or that the p -adic Galois representation $\rho: G_K \rightarrow \mathrm{GL}(T_p(A))$ has *reductive monodromy*.

Then, the subset $\overline{I_p(A)}$ of $I_p(\overline{A})$ is finite.

Theorem (Kisin, Lam, Shankar, S.)

- ① Only finitely many *supersingular*^a abelian varieties $\overline{A}/\overline{\mathbb{F}}_p$ of a given dimension admit CM-lifts.
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^aWe also prove a common generalization of the results for ordinary/supersingular strata to other Newton strata.

Only finitely many supersingular abelian varieties of a given dimension have CM-lifts

Proof strategy:

- 1 There are *only finitely many supersingular* \bar{A} with a given p -divisible group $\mathcal{G} := \bar{A}[p^\infty]$. (Oort)
- 2 For fixed dimension, *finitely many choices* for the CM-subalgebra F of $\text{End}(\mathcal{G}) \otimes \mathbb{Q}_p$.
For fixed F , *only finitely many possibilities* for the p -adic CM type $\Phi: F \rightarrow \prod_{i=1}^g \overline{\mathbb{Q}_p} = \text{End}_F(\text{Lie } \mathcal{G}_{\overline{\mathbb{Q}_p}})$.
- 3 Upto unramified twists, there is *only one isogeny class* \mathcal{G}_Φ/K of p -divisible group over local field with CM type Φ . (Conrad-Chai-Oort)
- 4 Since \mathcal{G}_Φ has CM, the reduction of its p -Hecke orbit is *finite* by our reductive monodromy theorem.