"BEL"

Isoscalar electric transitions: form factors, energy weighted sum rules and transition rates. A general program to calculate isoscalar transition rates (λ=0,1,2,...) from deformation parameters obtained from DWBA or CC analysis.

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ABSTRACT

An interactive program for the VAX computer was written with which one can calculate isoscalar electric transition rates for multipolarities $\lambda \geq 0$ from deformation parameters obtained from DWBA or CC analysis. Form factors for excitation of giant resonances as well as low-lying excitations of vibrational and deformed nuclei (including small amplitude β-, γ- and octupole vibrations) to be used in the DWBA and CC analyses are derived. Isoscalar energy weighted sum rules and single particle units for $\lambda \geq 0$ transitions are also derived.

I. INTRODUCTION

There are of course a large number of articles [1-7] that have been written on the topics of form factors, transition rates and sum rules for isoscalar electric transition operators. The idea of this report is not just to add one more article to this series. It is the intention of this report to describe the formalism used in the program "BEL" available on the VAX computer. Thus we will discuss here only isoscalar energy weighted sum rules (EWSR) for $\lambda > 0$ transitions, derive the form factors for $\lambda > 0$ transitions assuming full exhaustion of the sum rules and moreover describe a way of obtaining isoscalar electric transition rates from measured inelastic hadron scattering cross sections. The electric transition rates will be calculated from deduced deformation parameters β_λ (i) following Bernstein procedure [1], (ii) assuming a uniform density and/or (iii) assuming a density given by the real or imaginary parts of the optical model potential. This last procedure seems to be most justified [8] for deriving isoscalar electric transition rates especially for comparison with transition rates derived from electromagnetic experiments.

In addition to the form factors derived from the sum rule approach and hence in principle mainly useful in describing giant resonances, also form factors for low-lying excitations of vibrational and deformed nuclei [ref. 9] will be considered. In this respect, form factors for small amplitude vibrations [10,11] (e.g. β-, γ- and octupole vibrations) superposed on static deformations are of interest; and derivation of the electric transition rates to the members of these bands will be discussed.

Although a definition of a single particle unit for $\lambda > 2$ transitions exists, there are no generally accepted definitions for single particle units for $\lambda = 0$ and $\lambda = 1$ transitions. We will attempt to make such definitions and in the process define the various electric transition operators and hopefully clarify some points of confusion in these definitions.

The main usefulness of this report is as mentioned earlier to be referred to as a guide to the formalism used in the interactive program "BEL" which calculates transition rates, EWSR and percentages of EWSR for electric transitions excited by inelastic hadron scattering. However, the program "BEL" itself asks for all the information it needs to calculate the various interesting quantities. It has been written for the VAX computer. Modifications to treat the small amplitude β-, γ- and octupole vibrations superposed on static deformations were already introduced into programs DWUCK and CHUCK [12].

II. ELECTRIC TRANSITION OPERATORS

We will first start by defining the electric transition operators for $\lambda > 0$ transitions. The electric multipole moment of order λ can be written [see e.g. ref. 13]:

$$\mathcal{M}(E\lambda,\mu) = \frac{(2\lambda+1)!!}{q^{\lambda}(\lambda+1)} \int Z\,\rho(\vec{r})\,\frac{\partial}{\partial r}\,[r j_{\lambda}(qr)]\,Y_{\lambda}^{\mu}(\hat{r})\,d\tau$$
$$+ \frac{i(2\lambda+1)!!}{cq^{\lambda-1}(\lambda+1)} \int [\vec{r}\cdot\vec{j}(\vec{r})]\,j_{\lambda}(qr)\,Y_{\lambda}^{\mu}(\hat{r})\,d\tau,$$

where $j_{\lambda}(qr)$ is the Bessel function, $\vec{j}(\vec{r})$ is the current density and $\rho(\vec{r})$ is the charge density normalized to one.

In the long wavelength limit i.e. $qr<<1$, this can be rewritten:

$$\mathcal{M}(E\lambda,\mu) = \frac{(2\lambda+1)!!}{q^{\lambda}(\lambda+1)} \int Z\,\rho(\vec{r})\,\frac{\partial}{\partial r}\,\left[\frac{q^{\lambda} r^{\lambda+1}}{(2\lambda+1)!!}\right]\,Y_{\lambda}^{\mu}(\hat{r})d\tau$$
$$= \int Z\,\rho(\vec{r})\,r^{\lambda}\,Y_{\lambda}^{\mu}(\hat{r})\,d\tau, \qquad (1)$$

using the expansion of the Bessel function $j_{\lambda}(qr)$ in terms of (qr) and taking first order only:

$$j_{\lambda}(qr) = \frac{(qr)^{\lambda}}{(2\lambda+1)!!}\left[1 - \tfrac{1}{2}\,\frac{(qr)^2}{2\lambda+3} + \ldots\right] \qquad (2)$$

If one further substitutes in eq. (1), the charge density given by the distribution of point-like protons

$$\rho(\vec{r}) = \frac{1}{Z} \sum_{\text{protons}} e\,\delta(\vec{r}-\vec{r}_k),$$

one obtains:

$$\mathcal{M}(E\lambda,\mu) = \sum_{\text{protons}} e\,r_k^{\lambda}\,Y_{\lambda}^{\mu}(\hat{r}_k). \qquad (3)$$

Using isospin formalism where $e_p = e(\tfrac{1}{2}-t_z)$, $t_z|p\rangle = -\tfrac{1}{2}$,

and $e_n = e(\tfrac{1}{2}-t_z)$, $t_z|n\rangle = +\tfrac{1}{2}$,

then:

$$\mathcal{M}(E\lambda,\mu) = \tfrac{1}{2}\,e \sum_{\text{nucleons}} r_k^{\lambda}\,Y_{\lambda}^{\mu}(\hat{r}_k) - e \sum_{\text{nucleons}} t_z(k)\,r_k^{\lambda}\,Y_{\lambda}^{\mu}(\hat{r}_k). \qquad (4)$$

Therefore one observes that the electromagnetic operator splits into two parts: The first term independent of isospin and can lead only to isoscalar excitations ($\Delta T=0$) and hence called isoscalar and the second proportional to the z-component of the isospin vector and leads to isovector excitations ($\Delta T=1$) and hence called isovector.

At this point a number of remarks should be made:

(i) since a photon has zero mass and unit spin with spin orientation either parallel or antiparallel to its momentum and since further for a monopole $\lambda=0$ transition all orientations in space should be zero, therefore no photons can exist in such a state. This further implies that in $0^+ \to 0^+$ transitions γ-decay is forbidden; however, such a transition can occur by internal conversion or pair creation. For electron or hadron scattering from a nucleus, monopole transitions can take place and the operator that is responsible for the transition will be similar to eq. (4), but of higher order in expansion of (qr) since the first order leads to a constant which does not induce intrinsic excitations of the nucleus. This will be further discussed shortly.

(ii) The isoscalar term for $\lambda=1$ is proportional to the center of mass coordinate of the nucleus and therefore cannot induce intrinsic isoscalar dipole excitations of the nucleus. This is true in the long wavelength limit ($qr<<1$). Higher order terms in the expansion in terms of qr which become very important if $qr\simeq 1$ (such a condition can be easily satisfied in electron and hadron scattering) can induce such isoscalar dipole excitations.

(iii) Some authors define the multipole operators as:

$$\mathcal{M}(E\lambda) = \tfrac{1}{2}\, e \sum_k r_k^\lambda\, Y_\lambda^0(\hat{r}_k) - e \sum_k t_z(k)\, r_k^\lambda\, Y_\lambda^0(\hat{r}_k),$$

and moreover derive sum rules without summing over μ. This results in a discrepancy of a factor $(2\lambda+1)$. In this report all final results for sum rules will include the summation over μ.

Note: In the following we will drop the unit charge e from all the formulae for the transition operators and matrix elements.

In inelastic electron scattering and also in inelastic hadron scattering, the transition operator responsible for nuclear excitation resembles the electromagnetic multipole operators associated with multipole γ-quanta. This has been discussed in detail by Bernstein [1] for the case of inelastic α-scatter-

ing. One starts from the transition amplitude from an initial to a final state. In DWBA this can be written:

$$T_{fi} = \int \chi_f^{(-)*}(\vec{k}_f,\vec{r}) \langle\psi_f|\sum_k v(|\vec{r}-\vec{r}_k|)|\psi_i\rangle \chi^{(+)}(\vec{k}_i,\vec{r})\, d\tau \qquad (5)$$

where $\chi^{(+)}$ and $\chi^{(-)}$ are distorted incoming and outgoing waves, respectively, ψ_f and ψ_i are final and initial states of the target nucleus, respectively, and $v(|\vec{r}-\vec{r}_k|)$ is the projectile-nucleon interaction. ψ_i and ψ_f are antisymmetric wave functions of A-nucleons:

$$\psi = \frac{1}{(A!)^{\frac{1}{2}}} \det |\phi_k(r_j)|$$

where $\phi_k(r_j)$ are total single particle wavefunctions. Usually in the electromagnetic quanta decay or excitation, the reduced transition matrix element is given by:

$$M(E\lambda) = \langle\psi_f|| \sum_{\text{protons}} r_k^\lambda Y_\lambda(\hat{r}_k)||\psi_i\rangle$$
$$= Z \int \rho_{fi}^{\text{protons}}\, r^{\lambda+2}\, dr. \qquad (6)$$

where

$$\rho_{fi}^{\text{protons}} = \frac{1}{Z}\langle\psi_f|| \sum_{\text{protons}} Y_\lambda(\hat{r}_k)||\psi_i\rangle \qquad (7)$$

If one expands the projectile-nucleon interaction in multipoles, i.e.:

$$v(|\vec{r}-\vec{r}_k|) = \sum_{\lambda,\mu} v_\lambda(r,r_k)\, Y_\lambda^{\mu *}(\hat{r})\, Y_\lambda^\mu(\hat{r}_k),$$

Then eq. (5) leads, for a definite multipolarity $(\lambda\mu)$, to

$$T_{fi}^{\lambda\mu} = \langle\psi_f| \sum_k O^{\lambda\mu}(\theta_k,r_k)\, Y_\lambda^\mu(\hat{r}_k)|\psi_i\rangle \qquad (8)$$

and

$$O^{\lambda\mu}(\theta_k,r_k) = \int \chi^{(-)*}(\vec{k}_f,\vec{r})\, v_\lambda(r,r_k)\, Y_\lambda^{\mu *}(\hat{r})\, \chi^{(+)}(\vec{k}_i,\vec{r})\, d\tau \qquad (9)$$

where $\theta^{\lambda\mu}$ is the transition operator for scattering to the final state. One

can further easily show that if one takes $\chi^{(-)}$ and $\chi^{(+)}$ to be plane waves (PWBA) and the projectile-nucleon interaction to be a δ-interaction, then eq. (9) reduces exactly to [see e.g. ref. 14]:

$$O^{\lambda\mu}(\theta_k, r_k) \alpha \; j_\lambda(qr_k) \tag{10}$$

where q is the momentum transfer. Therefore eq. (8) reduces to first order in (qr) to a matrix element which is proportional to the electromagnetic matrix element [eq. (6)] for $\lambda > 2$:

$$\begin{aligned} T_{fi}^{\lambda\mu} &\alpha \langle\phi_f| \sum_k j_\lambda(qr_k)\; Y_\lambda^\mu(\hat{r}_k)|\phi_i\rangle \\ &\alpha \langle\phi_f| \sum_k \frac{(qr_k)^\lambda}{(2\lambda+1)!!}\; Y_\lambda^\mu(\hat{r}_k)|\phi_i\rangle \\ &\alpha \frac{q^\lambda}{(2\lambda+1)!!} \langle J_i M_i \lambda\mu | J_f M_f\rangle \langle\phi_f|| \sum_k r_k^\lambda\; Y_\lambda(\hat{r}_k)||\phi_i\rangle \end{aligned} \tag{11}$$

where for hadron scattering this is summed over all nucleons. A similar expression is obtained for the Coulomb matrix element part of high-energy electron scattering, except in this case, the sum would only run over protons. Using the expression for the mass transition density

$$\rho_{fi}^{(\lambda)} = \frac{1}{A} \langle\phi_f|| \sum_{\text{nucleons}} Y_\lambda(\hat{r}_k)\;||\phi_i\rangle \tag{12}$$

then:

$$T_{fi}^{\lambda\mu} \alpha \; A \int \rho_{fi}^{(\lambda)}\; r^{\lambda+2}\; dr \tag{13}$$

Expression (11) was obtained in the limit of PWBA and a short δ-type projectile-nucleon interaction. However, it has been shown by Bernstein [1] for the case of α-scattering that even if the projectile-nucleon interaction is taken to be of the Gaussian type and the scattering waves are distorted by the nuclear and Coulomb fields of the nucleus, the transition operator $O^{\lambda\mu}$ will to a large extent still resemble the electric transition operator as given by eq. (4).

Because the transition operator responsible for inelastic hadron scattering resembles to a large extent the electric transition operator, one can define a general isoscalar mass transition operator to be given by eq.

(10). However, since our aim would finally be to compare electric transition rates obtained from inelastic hadron scattering to those obtained from an electromagnetic experiment, we will define the isoscalar mass transition operator, for $\lambda \geq 2$, following Bernstein [1], to be given by:

$$O^{\lambda\mu} = \frac{Z}{A} \sum_k r_k^\lambda Y_\lambda^\mu(\hat{r}_k), \tag{14}$$

the factor $\frac{Z}{A}$ is used because for truly collective isoscalar states where all nucleons of the nucleus participate in the collective motion (e.g. collective vibrations or rotations), the contribution of the protons to the transition matrix element is $\frac{Z}{A}$ of the total matrix element. Since in electric transitions induced by electromagnetic probes only the protons participate, it is clear that the transition matrix element for an electric collective transition as obtained from electromagnetic measurements [i.e. using operator of eq. (3)] would be the same as the one obtained from an isoscalar probe [i.e. a probe which interacts equally with protons and neutrons such as α-scattering] if the above definition (14) is assumed for the isoscalar mass transition operator. It is also obvious from the above discussion that for single particle transitions or in general non-collective transitions differences may arise between transition rates as obtained from electromagnetic measurements and from inelastic α-scattering for example, because for such transitions the contribution of the proton and neutron densities is in a ratio different from Z/N.

The isoscalar dipole operator and the isoscalar monopole operator are defined, respectively, as:

$$O^{1\mu} = \frac{Z}{2A} \sum_k r_k^3 Y_1^\mu(\hat{r}_k) \tag{15}$$

and

$$O^{00} = \frac{Z}{2A} \sum_k r_k^2 \tag{16}$$

which are second order expansions, of the Bessel function, since the first order terms do not induce intrinsic excitations of the nucleus as discussed earlier. Note further the difference in our definition of the monopole operator of a factor $\frac{1}{2}$ times the monopole operator generally used. This requires a multiplication of the monopole matrix element calculated by "BEL" by a factor of 2 before comparison with other results.

III. TRANSITION RATES AND SINGLE PARTICLE UNITS

The reduced isoscalar transition rate for a transition $J_i \to J_f$ is defined by [see e.g. ref. 13]:

$$B(IS\lambda, J_i \to J_f) = \sum_{\mu M_f} |\langle \psi_f | O^{\lambda\mu} | \psi_i \rangle|^2$$

$$= \sum_{\mu M_f} \langle J_i M_i \lambda\mu | J_f M_f \rangle^2 \; |\langle \psi_f || O^{\lambda} || \psi_i \rangle|^2$$

Note, however, the different definition of the Wigner-Eckart theorem as used here and in refs. 13 and 16. In fact, the definition that we are using makes the reduced matrix element $\langle \psi_f || O^{\lambda} || \psi_i \rangle$ transform similar to a Clebsch-Gordon coefficient under exchange of J_i and J_f. Using the symmetry and completeness relations of the Clebsch-Gordon coefficient we can further write:

$$B(IS\lambda, J_i \to J_f) = \frac{2J_f+1}{2J_i+1} \; |\langle \psi_f || O^{\lambda} || \psi_i \rangle|^2$$

$$= \frac{2J_f+1}{2J_i+1} \; |M(IS\lambda, J_i \to J_f)|^2 \qquad (17)$$

where the reduced transition matrix element $M(IS\lambda)$ for $\lambda \geq 2$ is given by:

$$M(IS\lambda, J_i \to J_f) = \langle \psi_f || O^{\lambda} || \psi_i \rangle$$

$$= \frac{Z}{A} \langle \psi_f || \sum_k r_k^{\lambda} \, Y_{\lambda}(\hat{r}_k) \; || \psi_i \rangle$$

$$= Z \int \rho_{fi}^{(\lambda)} \; r^{\lambda+2} dr, \qquad (18)$$

and the transition density ρ_{fi}, is given by (12):

$$\rho_{fi}^{(\lambda)} = \frac{1}{A} \langle \psi_f || \sum_k Y_{\lambda}(\hat{r}_k) || \psi_i \rangle$$

k runs over all nucleons. We will obtain at a later stage the transition denstiy $\rho_{fi}^{(\lambda)}$, in the limits of various models.

Similar to eq. (17), one can write:

$$B(IS\lambda, J_f \to J_i) = \frac{2J_i+1}{2J_f+1} \; |\langle \psi_i || O^{\lambda} || \psi_f \rangle|^2$$

and using the transformation property of the reduced matrix element, one gets:

$$B(IS\lambda, J_f \to J_i) = |\langle \psi_f || O^{\lambda} || \psi_i \rangle|^2 \tag{19}$$

which then yields the relation:

$$B(IS\lambda, J_i \to J_f) = \frac{2J_f+1}{2J_i+1} B(IS\lambda, J_f \to J_i) \tag{20}$$

In order to have a measure of the collectivity or non-collectivity of an electromagnetic transition, one usually defines for convenience a single particle unit (s.p.u.) of a certain type of electromagnetic transition. Following Weisskopf [15], we define a s.p.u. for an electric transition by starting from the definition of an electric transition rate in the single particle model assuming only one particle makes a transition [see e.g. ref. 13]; for $\lambda \geq 2$:

$$B_{s.p.}(E\lambda, j_i \to j_f) = \frac{2j_f+1}{2j_i+1} |\langle \phi_f || r^{\lambda} Y_{\lambda} || \phi_i \rangle|^2$$

Using the relation

$$\langle \tfrac{1}{2}\ell_f j_f || Y_{\lambda} || \tfrac{1}{2}\ell_i j_i \rangle = (-)^{j_f - \frac{1}{2}} \sqrt{\frac{2\lambda+1}{4\pi}} \langle j_f -\tfrac{1}{2} \lambda 0 | j_i -\tfrac{1}{2} \rangle \tfrac{1}{2}[1+(-)^{\ell_i+\lambda+\ell_f}]$$

one obtains

$$B_{s.p.}(E\lambda, j_i \to j_f) = \frac{2j_f+1}{2j_i+1} \cdot \frac{2\lambda+1}{4\pi} \langle j_f -\tfrac{1}{2} \lambda 0 | j_i -\tfrac{1}{2} \rangle^2 \left| \int u_f^*(r)\, r^{\lambda}\, u_i(r)\, r^2 dr \right|^2$$

$$= \frac{2\lambda+1}{4\pi} \langle j_i \tfrac{1}{2} \lambda 0 | j_f \tfrac{1}{2} \rangle^2 \left| \int u_f^*(r)\, r^{\lambda+2}\, u_i(r)\, dr \right|^2$$

One usually estimates a s.p.u. transition for $j_i = \lambda + \frac{1}{2}$ to $j_f = \frac{1}{2}$, this further simplifies the above relation to:

$$B_{s.p.u.}(E\lambda, \lambda+\tfrac{1}{2} \to \tfrac{1}{2}) = \frac{1}{4\pi} \left| \int u_f^*(r)\, u_i(r)\, r^{\lambda+2} dr \right|^2$$

Assuming the radial wavefunctions u_f and u_i to be constant over the nucleus then

$$\int u_f\, u_i\, r^{\lambda+2}\, dr = \frac{C^2}{\lambda+3} r^{\lambda+3} \Big|_0^R = \frac{C^2}{\lambda+3} R^{\lambda+3}$$

but

$$\int C^2 \, r^2 dr = 1 \quad \rightarrow \quad C^2 = \frac{3}{R^3}$$

Therefore

$$\int u_f \, u_i \, r^{\lambda+2} \, dr = \frac{3}{\lambda+3} R^{\lambda}$$

where for a uniform density nucleus $R = 1.2 \, A^{1/3}$. Finally, we have for $\lambda > 2$:

$$B_{s.p.u.}(E\lambda, \lambda+\tfrac{1}{2} \rightarrow \tfrac{1}{2}) = \frac{1}{4\pi} \left(\frac{3}{\lambda+3}\right)^2 (1.2 \, A^{1/3})^{2\lambda}$$

and using eq. (20) one obtains:

$$B_{s.p.u.}(E\lambda, 0 \rightarrow \lambda) = \frac{2\lambda+1}{4\pi} \left(\frac{3}{\lambda+3}\right)^2 (1.2 \, A^{1/3})^{2\lambda} \qquad e^2\text{-fm}^{2\lambda} \qquad (21)$$

For the electric isoscalar dipole operator, we can obtain in a similar manner that:

$$B_{s.p.u.}(E1, 0 \rightarrow 1) = \frac{3}{4\pi} \left[\frac{1}{2} \left(\frac{3}{6}\right)\right]^2 \left[1.2 \, A^{1/3}\right]^6$$

$$= \frac{3}{64\pi} \left[1.2 \, A^{1/3}\right]^6 \qquad e^2\text{-fm}^6 \qquad (22)$$

and for the electric isoscalar monopole operator, we get:

$$B_{s.p.u.}(E0, 0 \rightarrow 0) = \frac{1}{4\pi} \cdot 4\pi \left[\frac{1}{2} \frac{3}{5}\right]^2 \left[1.2 \, A^{1/3}\right]^4$$

$$= \frac{9}{100} \left[1.2 \, A^{1/3}\right]^4 \qquad e^2\text{-fm}^4 \qquad (23)$$

Bohr and Mottelson [see ref. 3, p. 553] define a s.p.u. for a monopole transition in a different way, obtaining thus:

$$B_{s.p.u.}(E0, 0 \rightarrow 0) \simeq 1.0 \, A^{2/3} \qquad e^2\text{-fm}^4$$

However, program "BEL" uses eq. 23 for a monopole s.p.u. transition.

IV. ENERGY WEIGHTED SUM RULES

In the following, we will only consider sum rules for isoscalar tran-

sition operators $O^{\lambda\mu}$ as defined in section II. These are easy to evaluate using simple commutation algebra. Sum rules weighted to any power can be evaluated but are more involved to derive than the simple energy weighted (to power 1) sum rules (EWSR) to which we will limit ourselves here. These sum rules are model independent in as far as the nucleon-nucleon force is considered to be velocity independent. If velocity dependent forces are allowed, corrections to the EWSR have to be included. Moreover, in deriving the EWSR we will assume the nucleon to be elementary with no internal degrees of freedom. This assumption is valid only if one considers excitation energies well below the Δ-resonance energy.

To calculate EWSR of the isoscalar multipole transition operators, it is important to express these operators in terms of intrinsic coordinates referring to the center of mass (c.o.m.) of the nucleus. This is most acute for the isoscalar dipole operator which in first order is proportional to the c.o.m. coordinate and can only induce spurious c.o.m. motion. But even if one takes the second order isoscalar dipole operator as given in eq. (15), which is responsible for intrinsic isoscalar dipole excitations, the c.o.m. should still be taken care of exactly, otherwise gross errors can be committed. An elegant way of obtaining EWSR corrected to c.o.m. motion which we will use here is due to Fallieros and Deal [6,7]. One starts by defining the isoscalar form factor operator:

$$F(\vec{q}) = \frac{1}{A} \sum_k e^{-i\vec{q}\cdot\vec{r}_k}$$

Since a correct treatment requires the use of the transition operator in terms of intrinsic coordinates ($\vec{r}'$) where $\vec{r}'_k = \vec{r}_k - R$, (R being the c.o.m. coordinate: $\vec{R} = \frac{1}{A} \sum_k \vec{r}_k$), we express our form factor in terms of these coordinates. To indicate this we use a tilde (~)

$$\tilde{F}(\vec{q}) = \frac{1}{A} \sum_k e^{-i\vec{q}\cdot\vec{r}'_k} = e^{i\vec{q}\cdot\vec{R}} F(\vec{q})$$

We now assume that this form factor operator commutes with the potential energy part of the Hamiltonian (i.e. no velocity-dependent forces). Using the relation

$$[B,[V^2,D]] = -2(\vec{\nabla}D)\cdot(\vec{\nabla}B)$$

we obtain the following commutator relation:

$$[[\tilde{F}(q_1),H],\tilde{F}^*(q_2)] = \frac{\hbar^2}{mA}\,\vec{q}_1\cdot\vec{q}_2\,\{\tilde{F}(\vec{q}_1-\vec{q}_2) - \tilde{F}(\vec{q}_1)\,\tilde{F}^*(q_2)\}$$

Take the expectation value of this relation with respect to a J=0 ground state and insert a complete set of intermediate states, we get:

$$\sum_n [(E_n-E_o)\tilde{F}_{on}(\vec{q}_1)\,\tilde{F}^*_{no}(\vec{q}_2) + \frac{\hbar^2}{2mA}\,\vec{q}_1\cdot\vec{q}_2\,\tilde{F}_{on}(\vec{q}_1)\,\tilde{F}^*_{no}(\vec{q}_2)]$$
$$= \frac{\hbar^2}{2mA}\,\vec{q}_1\cdot\vec{q}_2\,\langle 0|\tilde{F}(\vec{q}_1-\vec{q}_2)|0\rangle \qquad (24)$$

where $\tilde{F}_{on}(\vec{q}_1) \equiv \langle 0|\tilde{F}(\vec{q}_1)|n\rangle$.

Making a multipole decomposition:

$$\tilde{F}(\vec{q}) = \sqrt{4\pi}\sum_{\lambda=0}^{\infty}(-i)^\lambda\sqrt{2\lambda+1}\,\tilde{F}^{(\lambda)}(q)$$

where $\tilde{F}^{(\lambda)}(q) = \frac{1}{A}\sum_k j_\lambda(qr'_k)\,Y^0_\lambda(\Omega'_k)$

Moreover:

$$A\,\tilde{F}^{(\lambda)}(q) = \frac{q^\lambda}{(2\lambda+1)!!}\,2\tilde{Q}^{(\lambda)} - \tfrac{1}{2}\,\frac{q^{\lambda+2}}{(2\lambda+1)!!(2\lambda+3)}\,2\,\tilde{P}^{(\lambda)} + \ldots.$$

where $\tilde{Q}^{(\lambda)} = \frac{1}{2}\sum_k r'^\lambda_k\,Y^0_\lambda\,(\Omega'_k)$,

is similar to the usual isoscalar electric multipole operator and

$$\tilde{P}^{(\lambda)} = \tfrac{1}{2}\sum_k r'^{\lambda+2}_k\,Y^0_\lambda(\Omega'_k)$$

is similar to the second order isoscalar electric multipole operator which is of importance for isoscalar monopole and dipole excitations.

Moreover, we define the elastic form factor:

$$F_{el}(q) = \langle 0|\frac{1}{A}\sum_k e^{-i\vec{q}\cdot\vec{r}'_k}|0\rangle$$
$$= \langle 0|\frac{1}{A}\sum_k j_0(qr'_k)|0\rangle$$

and also define:

$$\langle r'^2\rangle = \langle 0|\frac{1}{A}\sum_k r_k'^2|0\rangle$$

If we now integrate both sides of eq. (24) over $d(\cos\theta)\ Y_\lambda^0(\theta)$ where θ is the angle between $\vec{q}_1$ and $\vec{q}_2$ and project out a definite multipolarity λ, using the above relations as well as:

$$Y_\lambda^0(\Omega_{12}) = \sqrt{\frac{4\pi}{2\lambda+1}} \sum_{m=-\lambda}^{\lambda} Y_\lambda^m(\Omega_1)\ Y_\lambda^{m*}(\Omega_2),$$

we obtain the relation:

$$\sum_n \{(E_n-E_o)\ \tilde{F}_{on}^{(\lambda)}(q_1)\ \tilde{F}_{no}^{(\lambda)}(q_2) + \frac{\hbar^2}{2mA}\frac{q_1 q_2}{(2\lambda+1)} \cdot$$

$$\cdot\ [(\lambda+1)\ \tilde{F}_{on}^{(\lambda+1)}(q_1)\ \tilde{F}_{no}^{(\lambda+1)}(q_2) + \lambda\tilde{F}_{on}^{(\lambda-1)}(q_1)\ \tilde{F}_{no}^{(\lambda-1)}(q_2)]\} \qquad (25)$$

$$= \frac{\hbar^2}{8m\pi A^2}\frac{q_1 q_2}{(2\lambda+1)}\langle 0|\sum_k[(\lambda+1)j_{\lambda+1}(q_1 r_k')j_{\lambda+1}(q_2 r_k') + \lambda j_{\lambda-1}(q_1 r_k')j_{\lambda-1}(q_2 r_k')]|0\rangle$$

For $\lambda\geq 2$ and $q_2\to 0$ the leading transition operator is $Q^{(\lambda)}$, we then get:

$$\sum_n (E_n-E_o)\ \tilde{F}_{on}^{(\lambda)}(q_1)\ \tilde{Q}_{no}^{(\lambda)} = \frac{\hbar^2}{16m\pi A}\langle 0|\lambda q_1 \sum_k j_{\lambda-1}(q_1 r_k')r_k'^{\lambda-1}|0\rangle$$

$$- \sum_n \frac{\hbar^2}{2mA}\lambda\ q_1\ \tilde{F}_{on}^{(\lambda-1)}(q_1)\ \tilde{Q}_{no}^{(\lambda-1)} \qquad (26)$$

To obtain the static energy weighted sum rule for $\lambda\geq 2$ with respect to the operator $Q^{(\lambda)}$, take the limit of $q_1\to 0$, then

$$\sum_n (E_n-E_o)|\tilde{Q}_{no}^{(\lambda)}|^2 = \tfrac{1}{4}\frac{\hbar^2}{8m\pi}\lambda(2\lambda+1)\ A\langle r'^{2\lambda-2}\rangle - \frac{\hbar^2}{2mA}\lambda(\lambda+1)\sum_n |\tilde{Q}_{no}^{(\lambda-1)}|^2$$

Notice that the second term to the right which arises because of the center of mass correction vanishes identically for the case of $\lambda=2$, since in this case $Q^{\lambda-1}\ \alpha\ R$ the c.o.m. coordinate which has vanishing matrix elements for intrinsic states of the nucleus. For $\lambda>2$, the correction term is of the order of A^{-1} which vanishes in the limit of $A\to\infty$. We will further neglect this term. If we further notice that $O^{\lambda 0} = \frac{2Z}{A}Q^{(\lambda)}$, and that summation over μ would introduce another factor of $(2\lambda+1)$, we obtain:

$$\sum_f E_f\, B(IS\lambda, i\to f) = \frac{\hbar^2}{8m\pi}\,\lambda(2\lambda+1)^2\,\frac{Z^2}{A}\,\langle r^{2\lambda-2}\rangle \qquad (27)$$

One can similarly obtain relations for the static EWSR for $\lambda=0$ and $\lambda=1$, except here in the expansions in the limits of $q_2\to 0$ and $q_1\to 0$ one should notice that the lowest order transition operator is $P^{(\lambda)}$. The isoscalar dipole ($\lambda=1$) EWSR case has been discussed extensively in ref. 7, we will only quote the results here. For $\lambda=1$, we get:

$$\sum_n (E_n-E_o)\,\tilde{F}^{(1)}_{on}(q_1)\,\tilde{P}^{(1)}_{no} = -\frac{\hbar^2}{8\pi mA}\,\frac{q_1}{3}\,\langle 0|\sum_k \left[2j_2(q_1 r'_k)r'^2_k - \frac{5}{2}\,j_0(q_1 r'_k)r'^2_k\right]|0\rangle$$
$$+\sum_n \frac{\hbar^2}{2mA}\,\frac{q_1}{3}\left[4\,\tilde{F}^{(2)}_{on}(q_1)\,\tilde{Q}^{(2)}_{no} - 5\,\tilde{F}^{(0)}_{on}(q_1)\,\tilde{P}^{(0)}_{no}\right] \qquad (28)$$

and

$$\sum_f E_f\, B(IS1, i\to f) = \frac{\hbar^2}{8m\pi}\,\frac{3}{4}\,\frac{Z^2}{A}\left\{11\langle r^4\rangle - \frac{25}{3}\langle r^2\rangle^2 - 10\varepsilon\langle r^2\rangle\right\} \qquad (29)$$

where $\varepsilon = \left(\frac{4}{E_{GQR}} + \frac{5}{E_{GMR}}\right)\frac{\hbar^2}{3mA}$ and $E_{GQR} \simeq 65\ A^{-1/3}$ MeV and $E_{GMR} \simeq 80\ A^{-1/3}$ MeV are the excitation energies of the giant quadrupole and giant monopole resonances, respectively. The second and third terms on the r.h.s. of eq. (29) are due to c.o.m. correction. While the second term is large and should always be considered, the third term is proportional to A^{-1} and could in practice be neglected for $A \geq 40$.

For $\lambda=0$ we get:

$$\sum_n (E_n-E_o)\,\tilde{F}^{(0)}_{on}(q_1)\,\tilde{P}^{(0)}_{no} = -\frac{\hbar^2}{8m\pi A}\,q_1\,\langle 0|\sum_k j_1(q_1 r'_k)\,r'_k|0\rangle \qquad (30)$$

and:

$$\sum_f E_f\, B(IS0, i\to f) = \frac{\hbar^2}{2m}\,\frac{Z^2}{A}\,\langle r^2\rangle \qquad (31)$$

In eqs. (27), (29) and (31) we have dropped the prime to indicate coordinates in the intrinsic system. We will do the same in the next sections as well.

V. TRANSITION DENSITIES

i) Sum rule approach

To obtain transition densities for giant resonances which exhaust their respective sum rules eqs. (27), (29) and (31), we follow again the procedure of references 6 and 7. Here we will show how to do this for the case of $\lambda \geq 2$. The case of $\lambda=1$ has been discussed in detail in ref. 7.

We start by using Rayleigh's relation:

$$r^{\lambda} j_{\lambda}(qr) = (-)^{\lambda}\, q^{\lambda}\, (\frac{1}{q}\frac{d}{dq})^{\lambda}\, j_0(qr),$$

in eq. (26) for the case of $\lambda \geq 2$. We drop the center of mass correction term since it is very small, we then obtain:

$$\sum_n (E_n - E_o)\, \tilde{F}^{(\lambda)}_{on}(q)\, \tilde{Q}^{(\lambda)}_{no} = \frac{\hbar^2}{16 m\pi A} \cdot \lambda q < 0 \Big| \sum_k (-)^{\lambda-1} q^{\lambda-1} (\frac{1}{q}\frac{d}{dq})^{\lambda-1} j_0(qr_k) \Big| 0 >$$

which simplifies, upon using the relation for the elastic form factor ($\tilde{F}_{el}$) on page 13, to:

$$\sum_n (E_n - E_o)\, \tilde{F}^{(\lambda)}_{on}(q)\, \tilde{Q}^{(\lambda)}_{no} = (-)^{\lambda-1} \frac{\hbar^2}{16 m\pi}\, \lambda\, q^{\lambda}\, (\frac{1}{q}\frac{d}{dq})^{\lambda-1}\, \tilde{F}_{el}(q) \qquad (32)$$

Similarly we obtain for $\lambda=1$, using eq. (28):

$$\sum_n (E_n - E_o)\, \tilde{F}^{(1)}_{on}(q) \tilde{P}^{(1)}_{no} = -\frac{\hbar^2}{16 m\pi} (3q^2 \frac{d}{dq}\frac{1}{q}\frac{d}{dq} + 5\frac{d}{dq} + \frac{5}{3}\langle r^2 \rangle q + \varepsilon q^2 \frac{d}{dq})\, \tilde{F}_{el}(q) \qquad (33)$$

and for $\lambda=0$, using eq. (30):

$$\sum_n (E_n - E_o)\, \tilde{F}^{(0)}_{on}(q)\, \tilde{P}^{(0)}_{no} = \frac{\hbar^2}{8 m\pi}\, q\, \frac{d}{dq}\, \tilde{F}_{el}(q) \qquad (34)$$

The once integrated sum rules for $\lambda=0$, 1 and ≥ 2 are obtained by Fourier transforming eqs. (34), (33) and (32), respectively. A number of relations are useful here, such as:

orthogonality relation $$\int_0^{\infty} j_{\ell}(qr)\, j_{\ell}(qr')\, q^2 dq = \frac{\pi}{2r^2}\, \delta(r-r'),$$

elastic form factor $$F_{el}(q) = \int j_0(qr)\, \rho_0(r)\, d^3r,$$

ground state density $\rho_0(r) = \frac{1}{2\pi^2} \int F_{el}(q)\, j_0(qr)\, q^2 dq,$

λ-pole transition density $\rho^{(\lambda)}(r) = \frac{2}{\pi} \int F^{(\lambda)}(q)\, j_\lambda(qr)\, q^2 dq,$

the recursion relation $j_{\lambda-1}(qr) = \frac{2\lambda+1}{qr}\, j_\lambda(qr) - j_{\lambda+1}(qr)$

and Rayleigh's relation.

Taking the Fourier transform of both sides of eq. (32), one obtains after tedious mathematical manipulations the once-integrated sum rule for $\lambda \geq 2$,

$$\sum_n (E_n - E_o)\, \rho^{(\lambda)}_{on}(r)\, \tilde{Q}^{(\lambda)}_{no} = -\frac{\hbar^2}{4m} \lambda r^{\lambda-1} \frac{d}{dr} \rho_0(r)\, Y^0_\lambda(\hat{r}) \qquad (35)$$

Similarly we obtain for $\lambda=1$ using eq. (33):

$$\sum_n (E_n - E_o)\, \rho^{(1)}_{on}(r)\, \tilde{P}^{(1)}_{no}$$

$$= -\frac{\hbar^2}{4m} \left[3r^2\frac{d}{dr} + 10r - \frac{5}{3}\langle r^2\rangle\frac{d}{dr} + \varepsilon\left(r\frac{d^2}{dr^2} + 4\frac{d}{dr}\right)\right]\rho_0(r)\, Y^0_1(\hat{r}) \qquad (36)$$

and for $\lambda=0$ using eq. (34):

$$\sum_n (E_n - E_o)\, \rho^{(0)}_{on}(r)\, \tilde{P}^{(0)}_{no} = -\frac{\hbar^2}{2m}\left[3 + r\frac{d}{dr}\right] \rho_0(r)\, Y^0_0(\hat{r}) \qquad (37)$$

Taking eq. (35) and assuming that a giant resonant state of multipolarity λ exhausts the respective EWSR [eq. (27)] one can easily show that the transition density for such a λ-pole giant resonance is:

$$\rho^{(\lambda)}(r) = -\frac{\beta_\lambda R}{\sqrt{2\lambda+1}} \cdot \left(\frac{r}{R}\right)^{\lambda-1} \cdot \frac{d}{dr}\rho_0(r) \qquad (38)$$

where $\beta^2_\lambda = \frac{2\pi\hbar^2}{mAE_x} \frac{\lambda R^{2\lambda-4}}{\langle r^{2\lambda-2}\rangle}$ (39)

and β_λ is the collective coupling parameter for the isoscalar λ-pole giant resonance, E_x its excitation energy and R the half-density radius of the Fermi mass distribution. Similarly for the $\lambda=1$ isoscalar giant dipole resonance one obtains [see ref. 7]:

$$\rho^{(1)}(r) = -\frac{\beta_1}{R\sqrt{3}}\left[3r^2\frac{d}{dr} + 10r - \frac{5}{3}\langle r^2\rangle\frac{d}{dr} + \varepsilon\left(r\frac{d^2}{dr^2} + 4\frac{d}{dr}\right)\right]\rho_0(r) \qquad (40)$$

$$\text{where}\quad \beta_1^2 = \frac{2\pi\hbar^2}{mAE_x}\,\frac{3R^2}{\left[11\langle r^4\rangle - \frac{25}{3}\langle r^2\rangle^2 - 10\varepsilon\langle r^2\rangle\right]} \qquad (41)$$

and ε is as given on page 15. For the $\lambda=0$ isoscalar giant monopole resonance one obtains:

$$\rho^{(0)}(r) = -\beta_0\left[3 + r\frac{d}{dr}\right]\rho_0(r) \qquad (42)$$

$$\text{where}\quad \beta_0^2 = \frac{2\pi\hbar^2}{mAE_x}\cdot\frac{1}{\langle r^2\rangle} \qquad (43)$$

Transition densities (40) and (42) satisfy, respectively, the conditions of translational invariance $\int \rho^{(1)}(r)\, r^3 dr \equiv 0$ and the conservation of the number of particles $\int \rho^{(0)}(r)\, r^2 dr \equiv 0$ required by isoscalar dipole and monopole oscillations.

The above transition densities are known as the Tassie transition densities and should in principle be used only for the description of giant resonances which exhaust a large percentage of their respective EWSR. However, they have quite often been used, and successfully, in DWBA for the description of states which constitute only small fragments of the EWSR. Moreover, in hadron scattering it has become the habit to use transition densities for $\lambda>2$ derived from surface oscillations as will be described in the next subsection.

ii) Surface oscillations and static deformations

In general, one can expand the nuclear radius as follows:

$$R = R_0\left[1 + \sum_{\lambda\mu}\alpha_{\lambda\mu}\, Y_\lambda^{\mu *}(\Omega)\right] \qquad (44)$$

In the case of surface oscillations, it is assumed that the shape of the nucleus is not statically deformed but rather oscillates about a spherical mean. In this case the $\alpha_{\lambda\mu}$'s are treated as dynamical variables and could be expanded in terms of phonon creation and destruction operators [17],

$$\alpha_{\lambda\mu} = \left(\frac{\hbar\omega_\lambda}{2C_\lambda}\right)^{\frac{1}{2}}\left[b_{\lambda\mu} + (-)^\mu\, b^*_{\lambda-\mu}\right] = (-)^\mu\,\alpha^*_{\lambda-\mu},$$

of a phonon with angular momentum λ and Z-component μ. The energy of each phonon is $\hbar\omega_\lambda$, and C_λ is the restoring force. One can expand the ground state distribution about R_0 in a Taylor series. We will consider first order terms here which can lead to one-phonon excitation. For derivation of second order terms we refer the reader to the extensive work by Tamura [4]. The ground state denstiy after expansion in a Taylor series about R_0 is:

$$\rho_0(r-R) = \rho_0(r-R_0) + (R-R_0)\frac{d}{dR}\rho_0(r-R)\Big|_{R=R_0}$$

$$= \rho_0(r-R_0) - R_0\frac{d}{dr}\rho_0(r-R_0)\sum_{\lambda\mu}\alpha_{\lambda\mu}\,Y_\lambda^{\mu\,*}$$

Therefore the transition density for a surface oscillation of multipolarity λ is given by:

$$\rho^{(\lambda)}(r) = -\frac{\beta_\lambda R_0}{\sqrt{2\lambda+1}}\frac{d}{dr}\rho_0(r) \qquad (45)$$

where R_0 is the half-density radius of the Fermi mass distribution, and β_λ is the rms deformation due to zero-point oscillations of the ground state of the nucleus,

$$\beta_\lambda^2 = \langle 0|\sum_\mu \alpha_{\lambda\mu}^2|0\rangle = (2\lambda+1)\,(\hbar\omega_\lambda/2C_\lambda).$$

The transition density given by eq. (45) is the one most often used for describing excitation of giant resonances of multipolarity λ in inelastic hadron scattering. Note the difference with the Tassie transition density for multipolarity λ as given by eq. (38) which has a stronger weighting of the derivative of the ground state density at larger radii.

If the ground state is statically deformed, then the deformation parameters $\alpha_{\lambda\mu}$ in eq. (44) are not dynamical variables anymore, but instead are constant in time, giving the nucleus its static deformed shape. We will only consider axially symmetric static deformations here, but allow for surface oscillations to be superposed on these static deformations. Therefore the nuclear radius can now in general be written as:

$$R(\theta',\phi') = R_0\left[1 + \sum_\lambda \beta_\lambda Y_\lambda^0(\Omega') + \sum_{\lambda'\mu'}\alpha_{\lambda'\mu'}\,Y_{\lambda'}^{\mu'}(\Omega')\right], \qquad (46)$$

where the static axially symmetric shape is given by the sum over λ, and the

dynamical axial and non-axial terms are given by a sum over λ' and μ'. Further, $\Omega' = (\theta',\phi')$ is the solid angle in the intrinsic frame of the nucleus.

Since in a statically deformed nucleus K, the Z-component of the angular momentum along the symmetry axis of the nucleus, is a good quantum number, this usually leads to a non-degeneracy of the various bands built from the superposition of the surface oscillation with multipolarity λ' on the axially symmetric statically deformed shape. Therefore for a surface oscillation of multipolarity λ' one expects bands in a deformed nucleus with K-projections $\mu' = 0, 1, \ldots, \lambda'$ and parity $(-)^{\lambda'}$. Moreover, since K is a good quantum number, several $\alpha_{\lambda'\mu'}$ with the same μ' can contribute to the excitation of the members of a band with K-projection μ'. Here we will illustrate in the most general case how to obtain the various multipole transition densities for the excitation of the members of a small amplitude vibrational band with K-projection μ' from the ground state (g.s.) band. We will then focus on three common types of bands in deformed nuclei, i.e. β-bands, γ-bands and octupole bands. At the same time we will obtain the transition densities for excitation within the same rotational band.

We start by expanding ρ_0, the ground state density in terms of the small amplitude dynamic deformation parameters $\alpha_{\lambda'\mu'}$ [see eq. (46)] assuming only terms with the same projection μ' contribute to the excitation of a certain band with K-projection μ', we get:

$$\rho_0(r-R,\Omega') = \rho_0(r-R_\beta,\Omega') + \sum_{\lambda'} [\alpha_{\lambda'\mu'} Y_{\lambda'}^{\mu'} + \alpha_{\lambda'-\mu'} Y_{\lambda'}^{-\mu'}] \rho_1(r-R_\beta,\Omega')$$

where $R_\beta = R_0 \left[1 + \sum_\lambda \beta_\lambda Y_\lambda^0(\Omega')\right]$

and $$\rho_1(r-R_\beta,\Omega') = R_0 \frac{\partial}{\partial R} \rho_0(r-R,\Omega')\Big|_{R=R_\beta} = -R_0 \frac{\partial}{\partial r} \rho_0(r-R_\beta,\Omega') \qquad (47)$$

One can now expand the deformed densities ρ_0 and ρ_1 in Legendre polynomials P_ℓ,

$$\rho_0(r,\Omega') = \sum_\lambda \rho_0^{(\lambda)}(r)\, Y_\lambda^0(\Omega') + \sum_{\lambda\lambda'} \bar{\rho}_1^{(\lambda\lambda')} [\alpha_{\lambda'\mu'} Y_\lambda^{\mu'}(\Omega') + \alpha_{\lambda'-\mu'} Y_\lambda^{-\mu'}(\Omega')]$$

with $$\bar{\rho}_1^{(\lambda\lambda')}(r) = \sum_\ell \rho_1^{(\ell)}(r) \sqrt{\frac{(2\lambda'+1)(2\ell+1)}{4\pi(2\lambda+1)}} \langle \ell 0 \lambda' 0 | \lambda 0 \rangle \langle \ell 0 \lambda' \pm\mu' | \lambda \pm\mu' \rangle \qquad (48)$$

and $$\rho_i^{(\ell)}(r) = \sqrt{4\pi(2\ell+1)} \int_0^1 d(-\cos\theta)\, \rho_i(r,\Omega)\, P_\ell(\cos\theta) \qquad (49)$$

The relation $\int Y_{\ell_1}^{m_1} Y_{\ell_2}^{m_2} Y_{\ell_3}^{m_3 *} d\Omega = \sqrt{\frac{(2\ell_1+1)(2\ell_2+1)}{4\pi(2\ell_3+1)}} \langle \ell_1 m_1 \ell_2 m_2 | \ell_3 m_3 \rangle \langle \ell_1 0 \ell_2 0 | \ell_3 0 \rangle$

is used in obtaining the above result.

One can finally rotate to the fixed frame and get:

$$\rho_0(r,\Omega) = \sum_{\lambda\mu} \rho_0^{(\lambda)}(r)\, Y_\lambda^\mu(\Omega)\, D^\lambda_{\mu 0} + \sum_{\lambda'\lambda\mu} \bar{\rho}_1^{(\lambda\lambda')}(r)\, Y_\lambda^\mu(\Omega) \cdot$$
$$\cdot \left[\alpha_{\lambda'\mu'} D^{*\lambda}_{\mu\mu'} + \alpha_{\lambda'-\mu'} D^{*\lambda}_{\mu-\mu'}\right] \qquad (50)$$

The first term in this relation contributes to excitations within the same rotational band. Remembering that the wave functions in the rotational model are given by [3,4,17]

$$\phi^J_{MK} = \sqrt{\frac{2J+1}{16\pi^2}} \left(\chi_K(\zeta)\, D^J_{MK} + (-)^{J+K} \chi_{-K}(\zeta)\, D^J_{M-K}\right) \frac{1}{\sqrt{1+\delta_{K0}}},$$

one can show that the first term in (50) leads to the following transition density between members of the same band:

$$\rho^{(\lambda)}(r) = \rho_0^{(\lambda)}(r) \langle J_f K_f \lambda 0 | J_i K_i\rangle (-)^{J_f - J_i} \delta_{K_f K_i} \qquad (51)$$

where $\rho_0^{(\lambda)}$ is given by (49) and where K is the projection of the angular momentum in the intrinsic frame. If $J \geq \lambda/2$ this transition density has diagonal matrix elements. Such a term that couples a state back to itself is known as the reorientation term.

The second term in (50) treats transitions between the g.s. band and an excited vibrational band of K-projection μ'. We will further assume, that the g.s. band has K-projection 0. The second term in (50) will then lead to transition densities for the excitation of the μ'-band of the type [10]

$$\rho^{(\lambda)}(r) = \sum_{\lambda'} \eta_{\lambda'\mu'} \bar{\rho}_1^{(\lambda\lambda')} (-)^\lambda \langle J_f - \mu' \lambda \mu' | J_i 0\rangle$$
$$= \sum_\ell \rho_1^{(\ell)} (-)^\lambda \langle J_f - \mu' \lambda \mu' | J_i 0\rangle \cdot$$
$$\cdot \sum_{\lambda'} \eta_{\lambda'\mu'} \sqrt{\frac{(2\lambda'+1)(2\ell+1)}{4\pi(2\lambda+1)}} \langle \ell 0 \lambda' 0 | \lambda 0\rangle \langle \ell 0 \lambda' \pm\mu' | \lambda \pm\mu'\rangle \qquad (52)$$

where $\rho_1^{(\ell)}$ is given by (49) and $\eta_{\lambda'\mu'}$ is given by [18]:

$$\eta_{\lambda'\mu'} = \sqrt{\frac{2}{1+\delta_{\mu'0}}} \langle \chi_0(\zeta) | \alpha_{\lambda'\mu'} | \chi_{\mu'}(\zeta)\rangle \qquad (53)$$

For the case of the γ-band ($K^\pi=2^+$) and assuming $\lambda'=2$ only, then (52) and (53) simplify to [10,11,18]:

$$\rho^{(\lambda)}(r) = \sum_{\ell} \rho_1^{(\ell)}(-)^{\lambda} \langle J_f -2\lambda 2 | J_i 0\rangle \cdot \eta_{22}\sqrt{\frac{5(2\ell+1)}{4\pi(2\lambda+1)}} \cdot$$

$$\cdot \langle \ell 020|\lambda 0\rangle \langle \ell 02\pm 2|\lambda \pm 2\rangle \qquad (54)$$

and $$\eta_{22} = \sqrt{2} \cdot \frac{\beta \sin\gamma}{\sqrt{2}} = \beta \sin\gamma = \beta\gamma \qquad \text{for small } \gamma \qquad (55)$$

The octupole bands and the β-bands require a little more careful treatment. This is due to the fact that an octupole vibration induces a spurious center of mass motion and that a β-vibration is not number of particles conserving. The conditions requiring translational invariance for an octupole vibration (i.e. $\int \rho^{(1)} r^3 dr \equiv 0$) and number of particles conservation for a β-vibration (i.e. $\int \rho^{(0)} r^2 dr \equiv 0$) necesitate correction terms. In a pure vibrational model around a spherical shape one can obtain [19] such transition densities for surface dipole and monopole vibrations by assuming diffuseness and radial oscillations such that the coherent addition of these two oscillations satisfy the translational invariance and number of particles conservation conditions, respectively. Here we follow a slightly different approach by adding correction terms in (46) which are number of particles non-conserving or translationally non-invariant to counteract the same type of effects produced by a β-vibration or an octupole vibration, respectively. This leads for the octupole band [$K^\pi=(\mu')^-$] to transition densities of the type:

$$\rho^{(\lambda)}(r) = \sum_{\ell} \rho_1^{(\ell)}(-)^{\lambda} \langle J_f -\mu'\lambda\mu' | J_i 0\rangle \cdot \sqrt{\frac{(2\ell+1)}{4\pi(2\lambda+1)}} \cdot \qquad (56)$$

$$\cdot \{\eta_{3\mu'}\sqrt{7}\langle \ell 030|\lambda 0\rangle\langle \ell 03\pm\mu'|\lambda\pm\mu'\rangle - \xi_{\mu'}\sqrt{3}\,\langle \ell 010|\lambda 0\rangle\langle \ell 01\pm\mu'|\lambda\pm\mu'\rangle\}$$

where $\xi_{\mu'}$, is essentially determined once $\eta_{3\mu'}$ is, by the condition of translational invariance: $\int \rho^{(1)}(r)\, r^3 dr \equiv 0$.

Similarly for the β-band ($K^\pi=0^+$) we obtain:

$$\rho^{(\lambda)}(r) = \sum_{\ell} \rho_1^{(\ell)}(-)^{\lambda} \langle J_f 0\lambda 0| J_i 0\rangle \sqrt{\frac{(2\ell+1)}{4\pi(2\lambda+1)}} \cdot$$

$$\cdot \{\eta_{20}\sqrt{5}\,\langle \ell 020|\lambda 0\rangle^2 - \xi_0\,\langle \ell 000|\lambda 0\rangle^2\} \qquad (57)$$

where again ξ_0 is determined once η_{20} is, by the condition of number of particles conservation: $\int \rho^{(0)}(r)\, r^2 dr \equiv 0$. Moreover η_{20} is: $\eta_{20} \equiv |\beta-\beta_2|$.

VI. FORM FACTORS FOR DWBA AND CC ANALYSES

Taking the transition amplitude T_{fi} [eq. (15)] for scattering from an initial to a final state, one can expand it in multipoles using the multipole expansion of the projectile-nucleon interaction

$$T_{fi}^{\lambda\mu} = \int \chi_f^{(-)*}(\vec{k}_f,\vec{r})\, \langle\phi_f|\sum_k v_\lambda(r,r_k)\, Y_\lambda^\mu(\hat{r}_k)|\phi_i\rangle\, Y_\lambda^{\mu*}(\hat{r})\, \chi_i^{(+)}(\vec{k}_i,\vec{r})\, d\tau$$

$$= \langle J_i M_i \lambda\mu | J_f M_f\rangle \int \chi_f^{(-)*}(\vec{k}_f,\vec{r})\, F_\lambda(r)\, Y_\lambda^{\mu*}(\hat{r})\, \chi_i^{(+)}(\vec{k}_i,\vec{r})\, d\tau$$

where $F_\lambda(r)$ [the form factor in the jargon of hadron scattering] is

$$F_\lambda(r) = \int \rho_{fi}^{(\lambda)}(r')\, v_\lambda(r,r')\, r'^2 dr' \qquad (58)$$

To calculate the above transition amplitudes one usually generates the distorted waves $\chi_f^{(-)}$ and $\chi_i^{(+)}$ in an optical potential which is empirically obtained by requiring a good fit to the elastic scattering data. The form factor $F_\lambda(r)$ can in principle be obtained by folding into a microscopic transition density $\rho_{fi}^{(\lambda)}$ as given by eq. (12) a reasonable projectile-nucleon interaction. One can go a step further by assuming that the transition density need not be obtained from microscopic wave functions, but rather that the transition densities are obtained from a sum rule approach or from a collective (either vibrational or rotational) model approach and expressed as functions of the g.s. density as described in section V. This is the "folding model" procedure for generating form factors for use in DWBA or CC analyses. This folding procedure has also been used [20] to generate optical model potentials from microscopic ground state densities where the optical potential is given as:

$$U_{opt}(r) = \int \rho_0(r')\, v_0(r,r')\, r'^2\, dr' \qquad (59)$$

Since the projectile-nucleon interaction $v(|\vec{r}-\vec{r}'|)$ is complex, one can in principle obtain both the real and imaginary parts of the optical potential by folding. However, the imaginary part of the projectile-nucleon interaction is not well known, and most of the successes were booked in reproducing the real

parts of the phenomelogical optical potentials by using a suitable effective projectile-nucleon interaction. It is further obvious that if one uses the folding procedure to obtain the form factor $F_\lambda(r)$, one has no handle on the imaginary part of the form factor, which is in fact the reason why mostly only real coupling is assumed in the folding model procedure.

Because of the lack of a good folding procedure for obtaining the imaginary part of the form factor $F_\lambda(r)$, and because in general folding is rather cumbersome, in the sense that one has to find a good transition density and a suitable effective projectile-nucleon interaction to be able to perform the folding, one usually seeks an easy way-out by finding a practical and simple method to generate the form factor $F_\lambda(r)$. This is arrived at by observing that in (58) and (59), in the limit of a zero range interaction, one can write:

$$F_\lambda(r) = \int \rho_{fi}^{(\lambda)}(r')\, v_\lambda(r,r') r'^2\, dr' = \rho_{fi}^{(\lambda)}(r) = f[\rho_0(r)]$$

$$= f\,[U_{opt}(r)]$$

That is $F_\lambda(r)$ will have the same functional form in terms of $U_{opt}(r)$ as the transition density $\rho_{fi}^{(\lambda)}(r)$ has in terms of $\rho_0(r)$. Moreover, because of the short range character of the effective nucleon-nucleon interaction, one can for the case of the large class of transition densities that can be expressed as derivatives of the g.s. density [see section V] approximate [20] the form factor as

$$F_\lambda(r) \alpha - \int \frac{\partial}{\partial r} \rho_0(r')\, v_\lambda(r,r')\, r'^2\, dr'$$

$$= \frac{\partial}{\partial R} \int \rho_0(r')\, v_\lambda(r,r')\, r'^2\, dr'$$

$$\simeq \frac{\partial}{\partial R_{opt}} U_{opt}(r)$$

where R_{opt} is the radius of the optical potential U, and hence
$F_\lambda(r) \alpha - \frac{\partial}{\partial r} U_{opt}(r)$

In general, we will assume that both the real and imaginary parts of the optical potential are deformed in the same way as the ground state density. We will further assume that the form factors $F_\lambda(r)$ are obtained from the phenomelogical optical model potential $U_{opt}(r)$, which is empirically obtained by fitting to the elastic scattering data at the corresponding energy, in the same

way as the transition density $\rho_{fi}^{(\lambda)}(r)$ is obtained from the g.s. density $\rho_0(r)$. This is implicitly assuming that the folding works well for both the real and imaginary parts of both the form factors and the optical potentials.

A number of practical matters arise in performing the DWBA and CC calculations. Since usually the imaginary and real parts of the optical potential have different geometries, the question arises as how to relate the real and imaginary coupling parameters, β_R and β_I, of the form factor. For reasons that will become clear in the next section, I would prefer the relation between β_R and β_I (for $\lambda > 2$) to be such that:

$$\frac{\int \mathrm{Re}F_\lambda(r) r^{\lambda+2} dr}{\int \mathrm{Re}\, U_{opt}(r) r^2\, dr} = \frac{\int \mathrm{Im}\, F_\lambda(r)\, r^{\lambda+2}\, dr}{\int \mathrm{Im}\, U_{opt}(r)\, r^2\, dr} \qquad (60)$$

However, many people use quite often the prescription that $\beta_R R_R = \beta_I R_I$ which rests on the idea of the constancy of the deformation length βR for a certain transition. Yet, there are many people who do not make any distinction between β_R and β_I and just for simplicity take $\beta_R = \beta_I$. The next question that arises then is how to calculate the transition rate for the isoscalar transitions after determining the deformation coupling parameters β_R and β_I from comparisons of experimental cross sections and theoretical ones obtained from DWBA or CC analysis. We will address this question in the next section.

Another practical point is what to do for the case of the form factor for isoscalar dipole transitions using the functional dependence of eq. (40). For the case of the transition density, one takes $\langle r^2 \rangle$ with respect to the g.s. density. However, if the form factor is assumed to be derived from the optical potential then for $\mathrm{Re}F_\lambda$, $\langle r^2 \rangle_R$ should be taken with respect to the real geometry of U_{opt} and for $\mathrm{Im}F_\lambda$, $\langle r^2 \rangle_I$ should be taken with respect to the imaginary geometry of U_{opt}. This is indeed necessary for a proper treatment of the spurious center of mass motion [see also ref. 7 for a discussion on this point].

A similar situation occurs for the dipole transition density (56) and the monopole transition density (57). It is again usual to determine ξ_1 and ξ_0 using the g.s. density so that transition densities $\rho^{(1)}$ and $\rho^{(2)}$ satisfy the conditions of translational invariance and number of particles conservation, respectively. However, for the case of the form factor which is derived from the optical potential there will be two coupling terms ξ_1 which should be apartly determined for the real and imaginary geometries of the optical poten-

tial so that both the real and imaginary parts of the form factor $F_1(r)$ satisfy the condition of translational invariance. Similarly, the real and imaginary parts of the form factor $F_0(r)$ should satisfy the condition of number of particles conservation separately.

VII. ISOSCALAR TRANSITION RATES FROM DWBA AND CC ANALYSIS

The isoscalar transition rates for $\lambda \geq 2$ can be calculated using eqs. (17) and (18):

$$B(IS\lambda, J_i \to J_f) = \frac{2J_f+1}{2J_i+1} \left| M(IS\lambda, J_i \to J_f) \right|^2 \tag{17}$$

and

$$M(IS\lambda, J_i \to J_f) = Z \int \rho_{fi}^{(\lambda)} \, r^{\lambda+2} \, dr \tag{18}$$

For $\lambda < 1$, λ in eq. (18) should be replaced by $\lambda+2$ and an extra factor of $\frac{1}{2}$ should appear in front of the integral as expected from the definitions of the isoscalar multipole operators [eqs. (15) and (16)] for $\lambda \leq 1$.

It is obvious that if one knows the functional form of the transition density then one can always obtain the reduced transition matrix element $M(IS\lambda)$ by numerical integration. In general, however, the above integral can be expressed in terms of the radial moments of the g.s. density. Let us take, for example, the most commonly used case and that is the transition density for $\lambda \geq 2$ as given by eq. (45). One can then write [8]:

$$\begin{aligned} M(IS\lambda, J_i \to J_f) &= -\frac{Z \beta_\lambda R_0}{\sqrt{2\lambda+1}} \int \frac{d\rho_0}{dr} \, r^{\lambda+2} \, dr \\ &= \frac{Z \beta_\lambda R_0 \, (\lambda+2)}{4\pi \sqrt{2\lambda+1}} \langle r^{\lambda-1} \rangle \end{aligned} \tag{62}$$

since $\int \frac{d\rho_0}{dr} \, r^{\lambda+2} \, dr = -\frac{\lambda+2}{4\pi} \langle r^{\lambda-1} \rangle$ by integration by parts and moreover

$\int \rho_0 \, d\tau \equiv 1$ and $\int \rho_0 \, r^{\lambda+2} \, dr \equiv \langle r^\lambda \rangle / 4\pi$.

Similarly one can show that the isoscalar reduced transition matrix element corresponding to the Tassie density (38) for $\lambda \geq 2$ giant multipole resonances is given by:

$$M(IS\lambda, J_i \to J_f) = \frac{Z \beta_\lambda}{4\pi} \sqrt{2\lambda+1} \, \frac{\langle r^{2\lambda-2} \rangle}{R^{\lambda-2}}, \tag{63}$$

the isoscalar dipole reduced transition matrix element corresponding to the isoscalar dipole transition density (40) is given by:

$$M(IS1,J_i\rightarrow J_f) = \frac{Z\,\beta_1}{8\pi\sqrt{3}\,R}\left[11\langle r^4\rangle - \frac{25}{3}\langle r^2\rangle^2 - 10\varepsilon\langle r^2\rangle\right], \qquad (64)$$

and the isoscalar monopole reduced transition matrix element corresponding to the isoscalar monopole transition density (42) is given by:

$$M(IS0,J_i\rightarrow J_f) = \frac{Z\,\beta_0}{\sqrt{4\pi}}\langle r^2\rangle \qquad (65)$$

It is perhaps appropriate at this point to warn that, one should use the same type of transition density to calculate the reduced transition matrix element as was used in the DWBA or CC analysis. If a combination of transition densities was used to perform the DWBA or CC analysis, the same combination with the same phases and coupling amplitudes as used in the DWBA and CC analysis should be used to calculate the reduced transition matrix elements.

The program BEL calculates the reduced transition rates using the above formulae (62), ..., (65) making various assumptions on the g.s. density to obtain the radial moments $\langle r^\lambda\rangle$. For a statically deformed nucleus with transition densities given by (51), ..., (57), the program BEL performs a numerical integration over the transition densities, to obtain the reduced transition rates. In only two cases, i.e. the 2^+ and 4^+ of the g.s. band, does the program have the possibility to calculate the M(ISλ) from approximate formulae given by Bohr and Mottelson [ref. 3, p. 139-140]:

$$M(IS2) = \frac{3\,Z\,R_0^2}{4\pi\sqrt{5}}\left[\frac{4}{3}\frac{\langle r\rangle}{R_0}\beta_2 - \left(\frac{10}{7\pi}\right)^{\frac{1}{2}}\langle 2020|20\rangle\beta_2^2\left\{1+\frac{3}{4}\left(1-\frac{2}{3}\langle r^{-1}\rangle R_0\right)\right\}\right]$$

and

$$M(IS4) = \frac{3\,Z\,R_0^4}{4\pi\sqrt{9}}\left[2\frac{\langle r^3\rangle}{R_0^3}\beta_4 + \frac{5}{3}\frac{\langle r^2\rangle}{R_0^2}\left(\frac{45}{14\pi}\right)^{\frac{1}{2}}\langle 2020|40\rangle\beta_2^2\left\{1-\frac{2}{3}\left(1-\frac{4}{5}\frac{\langle r\rangle R_0}{\langle r^2\rangle}\right)\right\}\right].$$

VIII. DISCUSSION AND CONCLUSION

In deriving isoscalar multipole reduced transition matrix elements from deformation parameters obtained from CC or DWBA analysis, the question arises as to what is the best method of doing so? In most of the common cases for

studying inelastic excitation of multipole strength, the inelastic cross section can be explained as mainly due to real coupling, i.e. the real part of the form factor has the largest contribution to the inelastic cross section. The imaginary part of the form factor usually has a negligible contribution to the inelastic cross section except for medium to high energy proton scattering where in fact for 800 MeV protons [see e.g. ref. 13] the coupling is mainly imaginary. At any rate, one usually takes the coupling which has the largest contribution to the cross sections obtained from inelastic scattering and assumes its deformation length ($\beta_\lambda R$) as a constant to which all other deformation lengths are set equal.

In one method which has been used quite often in the past one assumes that the ground state density is uniform with a radius $R = 1.2\ A^{1/3}$ fm. Here one assumes that this g.s. density gets deformed with a deformation length $[\beta_\lambda\ R]$ being equal to that obtained from DWBA or CC analysis $[\beta_\lambda\ R]_{opt}$. One can then easily calculate the transition rates, the EWSR and hence the percentages of the EWSR, by substituting in each of the formulae for the reduced transition matrix elements and the EWSR's the radial moments of a uniform mass distribution

$$\langle r^\lambda \rangle = \frac{3}{3+\lambda} R^\lambda; \quad R = 1.2\ A^{1/3}\ \text{fm}. \tag{66}$$

The program BEL calculates the reduced transition matrix elements, the EWSR's and the percentages of the EWSR's using this method. However, this method has been shown [1,8] to lead to results which are in disagreement with results obtained from electromagnetic measurements.

Another method was proposed by Bernstein [8] to rectify this disagreement between transition rates obtained from inelastic hadron scattering measurements using the uniform density prescription and those obtained from electromagnetic measurements. The essence of this method lies in the fact that after all the g.s. density is not uniform but has a Fermi shape distribution well described, for most of the nuclei, by a half-density radius $R = 1.115\ A^{1/3} - 0.53\ A^{-1/3}$ fm and a diffuseness a = 2.5/4.4 fm (except for ^{16}O, a = 2.0/4.4 fm). One assumes again that the g.s. density is deformed with a deformation length equal to the scattering deformation length $[\beta_\lambda\ R]_{opt}$. Again the radial moments are calculated for the g.s. Fermi mass distribution (here by integration) and substituted in the formulae for the reduced transition matrix elements and the EWSR. This method, which we will call Bernstein method, was

shown by Bernstein [8] to be in general successful especially for $\lambda<3$ and large A. However, later inelastic α-scattering studies [21,22] showed that even this method leads to transition rates and percentages of EWSR's in disagreement with those obtained from electromagnetic studies especially for light nuclei and high multipolarities. Indeed the discrepancy increased [21,22] as a function of λ. The program BEL also uses the Bernstein method to calculate transition rates from deformation parameters obtained from inelastic hadron scattering.

The third method used by the program BEL to calculate transition rates and which we believe to be the most correct is based on a theorem by Satchler[8] which states that if a spherical distribution h(r) is obtained from another spherical distribution f(r) by folding into with a scalar function g(s), i.e.:

$$h(\vec{r}_1) = \int f(\vec{r}_2)\, g(|\vec{r}_1-\vec{r}_2|)\, d\tau_2$$

then the following statement is true [see eq. 17 of ref. 8]:

$$4\pi \int h(r)\, r^{\lambda+2}\, dr = [4\pi \int f(r)\, r^{\lambda+2}\, dr]\cdot[4\pi \int g(r)\, r^2\, dr]$$

This, of course, certainly holds for example for the optical potential and the form factors which are obtained from the g.s. and transition densities by folding into with a scalar projectile-nucleon interaction; see eqs. (58) and (59). However, as we have discussed earlier, the form factor can be derived from the optical potential in the same way the transition density is derived from the g.s. density under the implicit assumption that the optical potential could be derived from the g.s. density by folding. Therefore one can write the following relations:

$$\int \rho_{fi}^{(\lambda)}\, r^{\lambda+2}\, dr = [\int F_\lambda(r)\, r^{\lambda+2}\, dr] \,/\, [4\pi \int v(r)\, r^2 dr] \qquad (67)$$

$$\int \rho_0(r)\, r^{\lambda+2}\, dr = [\int U_{opt}(r)\, r^{\lambda+2}\, dr] \,/\, [4\pi \int v(r)\, r^2 dr] \qquad (68)$$

and

$$\int \rho_0(r)\, r^2 dr = [\int U_{opt}(r)\, r^2 dr] \,/\, [4\pi \int v(r)\, r^2 dr] \qquad (69)$$

Since $\int \rho_0(r)\, d\tau \equiv 1$, (69) yields:

$$\int U_{opt}(r)\, r^2 dr = \int v(r)\, r^2 dr$$

which by using in (68) leads to:

$$\langle r^\lambda \rangle_{g.s.} = \langle r^\lambda \rangle_{opt} \qquad (70)$$

That is to say, if the g.s. density is deformed, the optical potential should be deformed in such a way to keep the radial moments constant.

Since in inelastic hadron scattering inelastic cross sections are heavily weighted by the radial moments of the optical potential, then one should approximate all the formulae for the isoscalar reduced transition matrix elements such that the radial moments are with respect to the real geometry of the optical potential (for 800 MeV proton scattering the imaginary geometry should be considered instead). This is performed by the program BEL which then compares the sum rule strength $E_x\ B(IS\lambda, J_i \to J_f)$ to the EWSR as calculated from the g.s. Fermi mass distribution using the parameters given by Bernstein [8] [see also above]. Note that the sum rule is a property of the g.s. density. Using the above method one can see that since the optical potential geometry differs, especially for light nuclei, from the g.s. Fermi mass distribution, then the radial moments $\langle r^\lambda \rangle$ with respect to the real geometry will differ appreciably, especially for large λ, from the radial moments $\langle r^\lambda \rangle$ with respect to the g.s. density. This in a way is the reason why the Bernstein method fails for light nuclei and high λ.

The program BEL asks for all the information it needs. The nucleus, the optical potential geometry, the multipolarity of the transition, the types of transition densities used, the deformation and coupling parameters, it then calculates and prints the sum rules, the single particle units and the reduced transition matrix elements, and the percentages of the EWSR according to the formulae and methods described above. In case of dipole and monopole transitions using transition densities (56) and (57), it also prints c.o.m. spurious and volume matrix elements, respectively, so one can easily check whether the transition densities $\rho^{(1)}$ and $\rho^{(0)}$ satisfy the conditions of translational invariance and number of particles conservation, respectively, and if not, ξ_1 and ξ_0 could be modified until the conditions are satisfied.

We do hope that the use of program BEL and this accompanying report will facilitate for many the computation of transition rates using the various

prescriptions as discussed above. We have tried to be consistent in our definitions of the various quantities. If nevertheless some feel uneasy about some of the definitions please contact the author of this report for more discussions on the subject.

Important notes concerning the use of BEL

1. BEL in its present form assumes that the initial state has $J_i^\pi = 0^+$ and hence the final state has $J_f^\pi = \lambda^\pi$. It is, however, not difficult to modify the program for having $J_i^\pi \neq 0^+$. See further the formulae in this report for a simple transformation of $B(IS\lambda, 0 \to \lambda)$ to $B(IS\lambda, J_i \to J_f)$.

2. Although in this report we define $M(IS\lambda) = \sqrt{\frac{2J_i+1}{2J_f+1} B(IS\lambda)}$, the $M(IS\lambda)$ printed by the program BEL is instead $M(IS\lambda) = \sqrt{B(IS\lambda)}$.

3. The monopole matrix element M(E0) is $\frac{1}{2}$ the commonly used definition of M(E0), since our monopole operator is defined as $O^{00} = \frac{1}{2} \sum_k r_k^2$ while in general most people define $O^{00} = \sum_k r_k^2$.

4. The transition strength in s.p.u. printed by BEL is obtained by dividing the transition strength obtained from a uniform density by the s.p.u. strength as defined in section III.

REFERENCES

1. A.M. Bernstein, in Advances in Nuclear Physics, eds. M. Baranger and E. Vogt, vol. 3 (Plenum Press, New York) p. 325; and references therein.
2. G.R. Satchler, in Nuclear Spectroscopy and Nuclear Reactions with Heavy Ions, Enrico Fermi Course 42, ed. H. Faraggi and R.A. Ricci, North Holland, Amsterdam (1976); and references therin.
3. A. Bohr and B.R. Mottelson, Nuclear Structure, Vol. II (Benjamin, Reading, Mass. 1975); and references therein.
4. T. Tamura, Rev. Mod. Phys. 37 (1965) 679; and references therein.
5. G.R. Satchler, Part. and Nuclei 5 (1972) 105; and references therein.
6. T.J. Deal, Nucl. Phys. A217 (1973) 210; T.J. Deal, thesis, Brown University, 1972 (unpublished).
7. M.N. Harakeh and A.E.L. Dieperink, Phys. Rev. C23 (1981) 2329.
8. G.R. Satchler, J. Math. Phys. 13 (1972) 1118; L.W. Owen and G.R. Satchler, Nucl. Phys. 51 (1964) 155.
9. R.H. Bassel, G.R. Satchler, R.M. Drisko and E. Rost, Phys. Rev. 128 (1962) 2693.
10. T. Tamura, Nucl. Phys. 73 (1965) 241; Annu. Rev. Nucl. Sci. 19 (1969) 99.
11. K. van der Borg, M.N. Harakeh and B.S. Nilsson, Nucl. Phys. A325 (1979) 31.
12. P.D. Kunz, programs DWUCK and CHUCK, Univ. of Colorado, unpublished.
13. A. Bohr and B.R. Mottelson, Nuclear Structure, vol. I (Benjamin, New York, N.Y., 1969).
14. W.T. Pinkston and G.R. Satchler, Nucl. Phys. 27 (1961) 270.
15. J.M. Blatt and V.F. Weisskopf, Theoretical Nuclear Physics (John Wiley and Sons, New York, N.Y., 1952).
16. A. de-Shalit and I. Talmi, Nuclear Shell Theory (Academic Press, New York, N.Y., 1963).
17. A. Bohr and B.R. Mottelson, Kgl. Dankse Videnskab. Selskab, Mat.-fys. Medd. 27, No. 16 (1953).
18. L. Ray, G.S. Blanpied and W.R. Coker, Phys. Rev. C20 (1979) 1236.
19. G.R. Satchler, Nucl. Phys. A100 (1967) 481.
20. G.R. Satchler and W.G. Love, Nucl. Phys. A172 (1971) 449; and references therein.

21. M.N. Harakeh, A.R. Arends, M.J.A. de Voigt, A.G. Drentje, S.Y. van der Werf and A. van der Woude, Nucl. Phys. A265 (1976) 189.

22. K. van der Borg, M.N. Harakeh and A. van der Woude, Nucl. Phys. A365 (1981) 243.