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In this article we present a procedure for constructing repeating patterns which is based on the concept of contour lines of maps of surfaces generated by periodic functions of two variables. The 3D-graph of a prescribed function is cut by selected horizontal planes, and contour lines are projected onto the plane $z=0$. Then, the corresponding regions are coloured, and the result is a repeating pattern. This procedure is demonstrated on various examples using the command `ContourPlot` of *Mathematica*. By overlaying several such contour maps, we obtain patterns with rich structure that creates impression of depth. We also study how functional properties such as symmetry and parity influence the pattern.

Keywords: pattern; contour line; contour map; wallpaper group

AMS Subject Classifications: 00A66; 52C20; 68U05

1. Introduction

Every repeating pattern is determined by its motif and the order of its repetition. In this article we present a procedure for generating patterns based on the concept of contour lines of maps. The starting point is a periodic function of two variables $f(x, y)$ and a corresponding 3D-graph $z=f(x, y)$. The graph of f is then intersected by k prescribed horizontal planes, and contour lines are projected onto the plane $z=0$. Finally, contour regions are coloured by distinct colours. In this way the motif of the pattern is created. The periodicity of the two variables x and y of the function f assures repetition of the motif in two independent directions, and therefore the final result is a two-dimensional repetitive pattern. This idea has also been considered previously by Lucca [10] and Puc [13,14].

This article is organized in the following way. After we give overviews of patterns and contour lines and maps, our procedure of constructing contour map patterns is presented, and then it is demonstrated using the command `ContourPlot` from Wolfram's *Mathematica*®. Then we study how some particular function properties influence the symmetry of the contour map pattern. We give examples of contour maps of symmetric functions, asymmetric functions and functions that are periodic only in one variable. Finally, we demonstrate the procedure of constructing 'composed' patterns obtained by overlaying several of our contour map patterns.

1.1. *Patterns*

Patterns have been used to decorate surfaces throughout the history of art, craft and design in different cultures across the globe. Nowadays, patterns have a wide range of practical applications in textile and fashion design, architecture, advertising, graphic-, web-, product- and interior design.

In art and design, a pattern is a visual surface formed by an element (or motif or unit) that repeats in selected (mathematical) order. The system of organization generally follows a basic network diagram such as square, drop, brick, diamond, triangle, hexagon, octagon, ogee or circle. The actual network selected forms invisible guidelines and becomes an integral part of the repeating pattern.

Floral and geometrical motifs are frequently used in fashion. Combinations of motifs and repetitions result in various pattern designs. Studio [15] defines four types of patterns – abstraction, graphic, collage and texture. Cole [1] arranged patterns into five categories – conversational, abstract, retro, geometric and organic. Cole [2] grouped patterns by colour rather than by chronology, and so one often finds patterns from the last century juxtaposed with

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contemporary designs. Meller and Elffers [11] divided patterns into floral, geometric, conversational and ethnic and by movements and period styles.

From a mathematical point of view, one often studies the symmetries of (discrete) patterns, and classifies patterns by their symmetry group. The group can only be one of the 17 distinct groups, the so-called *wallpaper groups* (or *plane crystallographic groups* or *plane symmetry groups*). This group characterisation was first done by von Fedorov [17] in 1891 and later, but independently, by Pólya [12] in 1924 (see also [3,6] for a mathematical study of patterns).

Roughly speaking, a symmetry is a mapping that transforms the pattern to look exactly the same, and is an *isometry* (i.e. maps the plane onto itself in such a way that preserves distances). It is well known that only the following type of isometries occur:

T_v : translation along a vector $v \in \mathbb{R}^2$;
$R_{a,\alpha}$: rotation with centre $a \in \mathbb{R}^2$ and angle of rotation α;
F_L : reflection about a line L;
$G_{L,v}$: glide reflection which reflects about a line L then translates by a vector $v \in \mathbb{R}^2$.

Notice that the glide reflection $G_{L,v}$ is nothing but a composition (combination) of a reflection F_L and a translation T_v, i.e. $G_{L,v} = T_v \circ F_L$. Rotation angles can only be 60°, 90°, 120° or 180°.

1.2. *Contour lines and maps*

A *contour line* is a curve that connects points that have the same particular value of a given two variable function. A *contour map* is a map comprised of one or more contour lines. Usually, we obtain such a map by projecting all contour lines onto one plane, say the plane $z = 0$, as shown in Figure 1. In this way, the plane is divided into several regions, which we call *contour regions*. One can see from illustrations in [9] how different shapes and forms are projected onto various selected planes.

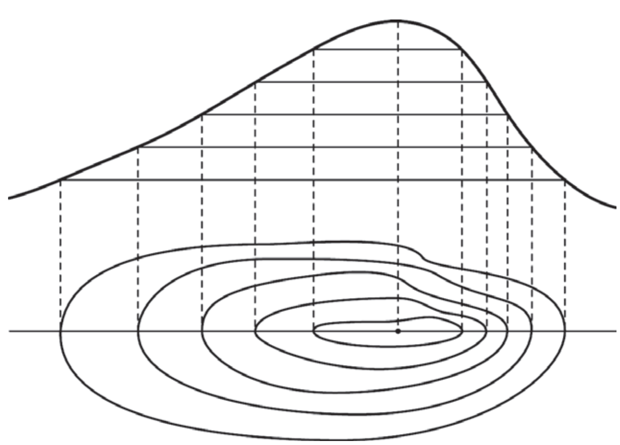

Figure 1. Diagram of contour lines and map.

Widely used examples of this concept are topographic maps, where contour lines join points of equal height, and thus the map visualizes valleys and hills and the steepness of their slopes.

The first use of contour lines is ascribed to Edwin Halley [7,16] for a magnetic chart in 1701. Cruquius [4] drew contour lines for the bed of the river Merwede in 1727. It is noteworthy that Johnson [8] recently observed a much earlier use by Pieter Bruinsz in 1584.

Nowadays, the concept of contour lines is widely used in geography and meteorology. Depending on the discipline, different names which start with 'iso' meaning 'the same' are used. Thus, an *isobar* is a line of equal pressure on a map, an *isotherm* connects points on a map of the same temperature, an *isogon* refers to line of constant wind direction, an *isobath* is a line of equal depth in the water, etc. In contour maps, lines and intervals may be drawn with a distinct width, colour and type to carry some special meaning.

In what follows, we use the following notation for contour lines. Let $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be a function and let $c_1, c_2, \ldots, c_k$ be an increasing sequence of real numbers that all belong to the range of f. For each $i \in \{1, \ldots, k\}$, we define the set $L_i = \{(x, y) : f(x, y) = c_i\}$, so $L_i = f^{-1}(c_i)$. Each L_i represents a 'contour line' for the points of value c_i and the union $L_1 \cup L_2 \cup \cdots \cup L_k$ represents the corresponding contour map. We will denote this map simply by $\mathcal{M}(f;\, c_1, c_2, \ldots, c_k)$.

Notice that in general L_i could be a disconnected curve. This depends on the function f and the value c_i. The condition that c_i belongs to the range of f is equivalent to the requirement that the graph of f and the horizontal plane $z = c_i$ intersect.

2. Construction of contour map patterns

Our simple procedure for obtaining a repeating pattern is to consider a (continuous) function of two variables which is periodic in x with period T_1, and periodic in y with period T_2. Next, we consider k planes $z = c_i$ with $i \in \{1, \ldots, k\}$ where each c_i belongs to the range of f; for simplicity, we assume the ordering $c_1 \le c_2 \le \cdots \le c_k$. Our pattern is in fact nothing more than the induced contour map $\mathcal{M}(f;\, c_1, c_2, \ldots, c_k)$.

One natural question that arises is 'What is the motif'? Repetition is assured by the periodicity of the function f. Observe that for every $x, y \in \mathbb{R}$, we have

$$f(x,y) = f(x+T_1, y) = f(x, y+T_2) = f(x+T_1, y+T_2).$$

This shows that the domain $\mathbb{R} \times \mathbb{R}$ of f can be tiled by rectangles of size $T_1 \times T_2$ on which the graph of f looks the same. The region $[0, T_1] \times [0, T_2]$ can be considered as a motif that repeats along two independent translations, one in the direction of the x-axis and the other in the direction of the y-axis. For aesthetic or other reasons, one may choose to shift the motif $[0, T_1] \times [0, T_2]$ to something more appropriate, for example, the rectangle $[-T_1/2, T_1/2] \times [-T_2/2, T_2/2]$.

Figure 2 shows the entire procedure for generating a pattern. Figure 2(a) shows the function $f(x, y) = \sin x \sin y$, where $x, y \in [-2\pi, 2\pi]$, and five horizontal planes $z = -0.75, -0.25, 0.25, 0.5, 0.75$ that intersect the graph of f. Each plane is coloured with distinct but partially transparent colours: olive green for $z = -0.75$, yellow green for $z = -0.25$, black for $z = 0.25$, gray for $z = 0.5$ and light gray for $z = 0.75$. Figure 2(b) illustrates the projection of the contour lines onto the plane $z = 0$. The colour of the curve is deduced from the corresponding plane. For each horizontal plane $z = c_i$ we get the curve L_i that is the solution for the equation $\sin x \sin y = c_i$. In this particular case, when restricted to $[0, 2\pi] \times [0, 2\pi]$, the solution of the equation is comprised of two square-shaped ovals. In Figure 2(c) we have the corresponding contour map pattern.

Since we know the real-valued sine function is a periodic function with period 2π, it follows that for each $x, y \in \mathbb{R}$ we have $f(x, y) = f(x + 2\pi, y) = f(x, y + 2\pi) = f(x + 2\pi, y + 2\pi)$. Thus our motif can be $[0, 2\pi] \times [0, 2\pi]$, which is translated along the x- and y-axis, and so tiles the domain $\mathbb{R} \times \mathbb{R}$.

We may go further by combining (more precisely, by overlaying) several contour maps in order to obtain more complex patterns. For example, layering two copies of the map Figure 2(c) where one has coloured contour regions and one is only slightly translated, results in the pattern in Figure 3. The interrelationship between elements of the composition creates a dynamic sense of movement. At the same time, the colour combinations form impressions of light and depth. In section 6, we present more patterns constructed in this way.

3. Use of Wolfram *Mathematica*

The above procedure for generating patterns can be easily implemented in various programming languages. We find the command `ContourPlot` of Wolfram *Mathematica* [5] both flexible and easy to use, making it unnecessary to write special software to achieve the results we desired. This command is rich with options that enable us to manipulate the image. We found particularly useful the ones enabling us to assign the number of contours and their distribution along the z-axis, assign colours to contours and contour lines, and specify the smoothness of the image.

We now explain the use of this tool for some examples. The easiest way to use `ContourPlot` is to just take a function of two variables, say $f(x, y) = 3(\sin x - \sin^3 x \sin^3 y + \sin y)$ together with some domain, say $D = \{(x, y);\ -3.5\pi \le x \le 2.5\pi,\ -3.5\pi \le y \le 2.5\pi\}$. Figure 4(a) is the result of executing:

```
ContourPlot [3 Sin[x] - 3 Sin[x]^3 Sin[y]^3 + 3
            Sin[y],
            {x, -3.5 Pi, 2.5 Pi}, {y, -3.5 Pi,
            2.5 Pi}].
```

Mathematica automatically selects equally spaced contour levels, whose values minimise the number of digits in their decimal representation. This contributes to the aesthetic of the image. Thus, automatically seven horizontal level planes were assigned using values −8, −6, −4, −2, 0, 2, 4. Regions between contour lines are coloured using the default monochromatic scheme. Higher values are shown lighter and lower values darker. Contour lines are partially transparent.

Figure 4(b) is the result of executing the following command:

```
ContourPlot [3 Sin[x] - 3 Sin[x]^3 Sin[y]^3 + 3
            Sin[y],
            {x, -3.5 Pi, 2.5 Pi}, {y, -3.5 Pi,
            2.5 Pi},
            Contours -> 4,
            ContourShading -> {White, Black,
            RGBColor[0.332113, 0.392157,
            0.392157]},
            Frame -> False].
```

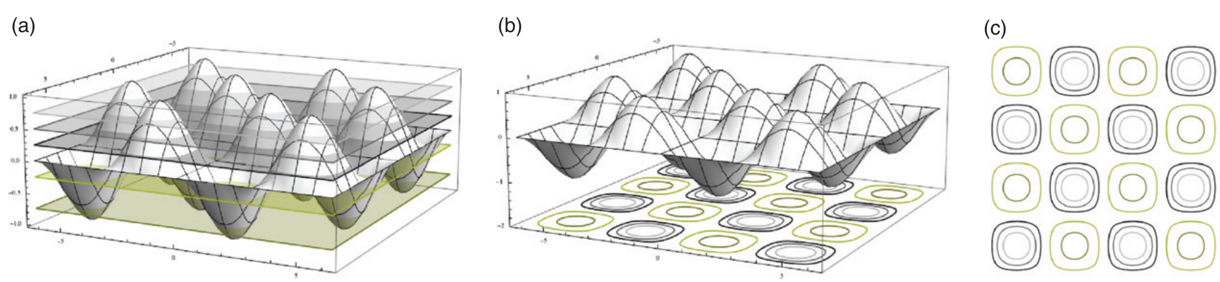

Figure 2. An example of generating a pattern: (a) the graph of the function $f(x, y) = \sin x \sin y$ is intersected by five horizontal planes, (b) projection of contour lines onto the plane $z = 0$, and (c) contour map pattern.

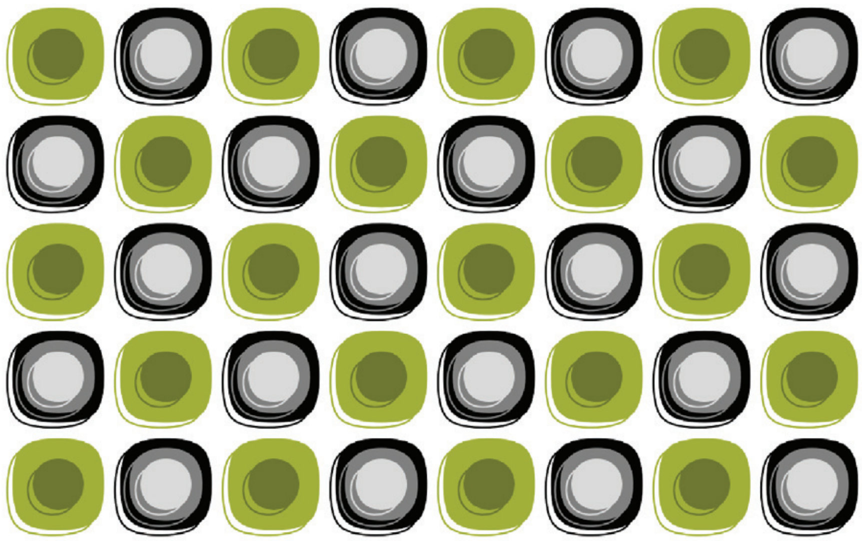

Figure 3. Pattern created by overlaying two contour maps of the function $f(x, y) = \sin x \sin y$ with $(x, y) \in [-4\pi, 4\pi] \times [-3\pi, 2\pi]$.

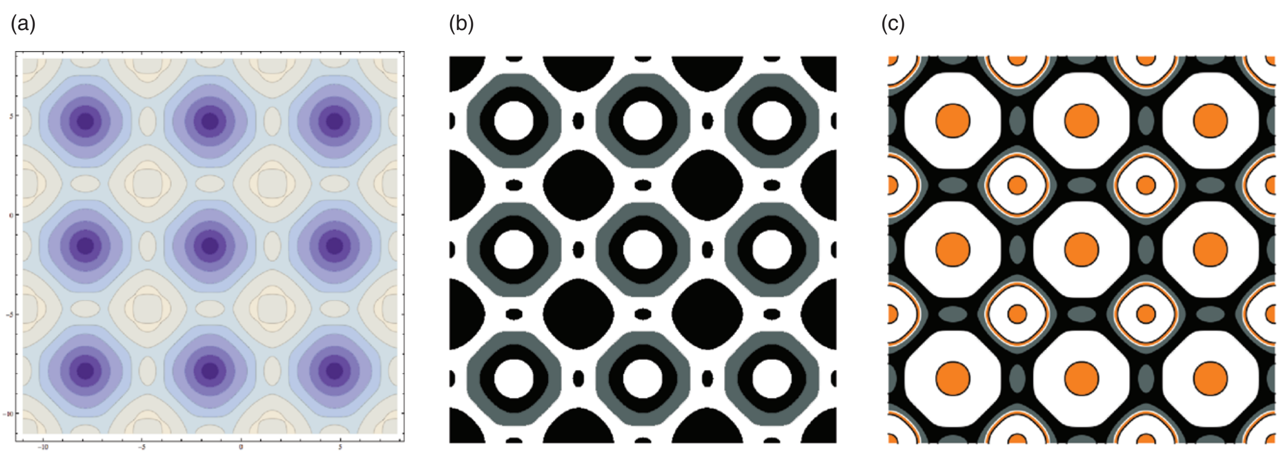

Figure 4. Generating contour map patterns of the function $f(x, y) = 3(\sin x - \sin^3 x \sin^3 y + \sin y)$ using some different options of the command `ContourPlot`.

By invoking the option `Contours`, we are able to specify four contours. Automatically, equally spaced horizontal planes were assigned at levels −5.2, −2.6, 0, 2.6. We selected the corresponding colours for regions between contour lines using the setting `ContourShading`.

The following example gives the pattern shown in Figure 4(c):

```
ContourPlot [3 Sin[x] - 3 Sin[x]^3 Sin[y]^3 + 3
            Sin[y],
            {x, -3.5 Pi, 2.5 Pi}, {y, -3.5 Pi,
            2.5 Pi},
            Contours -> {-6, 0, 2, 3, 3.5},
            ContourShading ->{Orange, White,
            Black, RGBColor[0.332113,
            0.392157, 0.392157]},
            ContourStyle ->{{Black, Thick},
            {White, Thick}}, Frame ->False].
```

In this image, by using the option `Contours`, we specified five horizontal planes at levels −6, 0, 2, 3, 3.5. By the option `ContourStyle`, we assign matching colours and thicker contour lines.

4. Function properties and symmetries of contour maps

Because the contour map $\mathcal{M} = \mathcal{M}(f; c_1, c_2, \ldots, c_k)$ is a projection of the graph of f onto the plane $z = 0$ for certain levels $c_1, \ldots, c_k$, it is natural to expect that symmetries of the graph of f to induce symmetries on the contour map $\mathcal{M}$. For example, any mirror plane

$ax + by + 0z = c$ of the graph of f is projected to a mirror line $y = c/b - ax/b$ in $\mathcal{M}$. Of course, in general, $\mathcal{M}$ may have some additional symmetries that the graph of f does not.

In this section, we consider some functional properties that induce symmetries on the graph of the function, and thus also on the contour map. In the next section, we present some additional examples of simple contour maps and determine the corresponding symmetry groups of their patterns.

Symmetric functions. A function of two variables $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is *symmetric*, if $f(x, y) = f(y, x)$ for every $x, y \in \mathbb{R}$. For example, $\cos x + \cos y$ is symmetric but $\cos x + \sin y$ is not. Notice that if a symmetric function f is periodic in x and also in y, then their periods must have the same value, say T. Symmetric functions possess the following property.

Proposition 4.1: *If $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a (periodic) symmetric function with period T, then for any integer k the line $y = x + kT$ is a reflection in $\mathcal{M}$.*

Proof: It is easy to show that a point with coordinates (a, b) is reflected with respect to the line $y = x + kT$ to the point $(b + kT, a - kT)$. Since $f(b + kT, a - kT) = f(b, a) = f(a, b)$, the proof follows. □

Even and odd functions. The concept of being an even or odd function in one variable can be extended to two variables by restricting attention to the first or second variable. Thus we say that f is *even in the first variable* if $f(-x, y) = f(x, y)$ for every $x, y \in \mathbb{R}$, and similarly we say it is *odd in the first variable*, if $f(-x, y) = -f(x, y)$ for every $x, y \in \mathbb{R}$. As examples, $\cos^3 x + \sin^2 y$ is even in both variables, $\sin x \cos y$ is odd in x and even in y, and $\sin x \cos^2 y$ is odd in both x and y. For functions with such properties, one can observe the following propositions.

Proposition 4.2: *Let $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be an even periodic function in the first variable with period T. Then, for any integer k the line $y = kT/2$ is a reflection in $\mathcal{M}$.*

Proof: We must show $f(kT/2 - x, y) = f(kT/2 + x, y)$ for every $x, y \in \mathbb{R}$. This follows from

$$f\left(\frac{kT}{2} - x, y\right) = f\left(-\frac{kT}{2} + x, y\right)$$
$$= f\left(-\frac{kT}{2} + x + kT, y\right) = f\left(\frac{kT}{2} + x, y\right).$$

□

Proposition 4.3: *Let $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be an odd periodic function in the first variable x with period T. Then, for any integer k the point $(kT/2, 0)$ is the centre for a rotation of $180°$ of $\mathcal{M}$.*

Proof: We must show that $f(kT/2 - x, y) = -f(kT/2 + x, y)$ for every $x, y \in \mathbb{R}$. This follows from

$$f\left(\frac{kT}{2} - x, y\right) = -f\left(-\frac{kT}{2} + x, y\right) = -f\left(\frac{kT}{2} + x, y\right).$$

□

Similarly, one may consider even or odd parity regarding the second variable y and respectively restate the above two propositions to obtain that the lines $x = kT/2$ are reflection lines and the points $(0, kT/2)$ rotation centres.

Proposition 4.4: *If $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a periodic symmetric function with period T and is even in both variables, then for any integer k, $\mathcal{M}$ has reflections in four different directions, namely $y = x + kT$, $y = -x + kT$, $y = kT/2$, and $x = kT/2$.*

Proof: By Proposition 4.1, $y = x + kT$ is a reflection. By Proposition 4.2 and its variant for the second variable y, it follows that $y = kT/2$ and $x = kT/2$ are reflections.

We now show that $y = -x + kT$ is a reflection. Observe that the point with coordinates (a, b) is reflected with respect to the line $y = -x + kT$ to the point $(kT - b, kT - a)$. Since $f(kT - b, kT - a) = f(-b, -a) = f(b, a) = f(a, b)$, the proof is complete. □

The above result tells us that the symmetry group known as `p4m` will be the symmetry group of a contour map of a periodic symmetric function that is even in both variables.

Trigonometric functions. Because the procedure for generating repeating patterns requires periodic functions, it is natural to consider trigonometric functions. After all, the most well-known periodic odd function is $\sin : \mathbb{R} \to [-1, 1]$, and the most well-known even periodic function is its counterpart $\cos : \mathbb{R} \to [-1, 1]$. Both have period 2π. They are related by $\sin(x + \pi/2) = \cos(x)$. Although in this article we mainly restrict our attention to sine and cosine, one could also consider trigonometric functions such as $\tan(x) = \sin(x)/\cos(x)$ and $\cot(x) = \cos(x)/\sin(x)$.

Lucca [10] considered functions that were obtained using functional composition involving trigonometric functions, power functions and rational functions. We chose to consider functions that lie in the ring generated by $x, y, \sin x, \sin y, \cos x, \cos y$. Taking n and m to be positive integers, some examples are:

- $\sin x + \sin^3 x + \sin y + \sin^3 y$,
- $\sin^m x + \sin^n y$,
- $\sin^m (x + y) - \cos^n (x - y)$,
- $(\sin^m x - \sin^m y)(\sin^n x + \cos^n y)$.

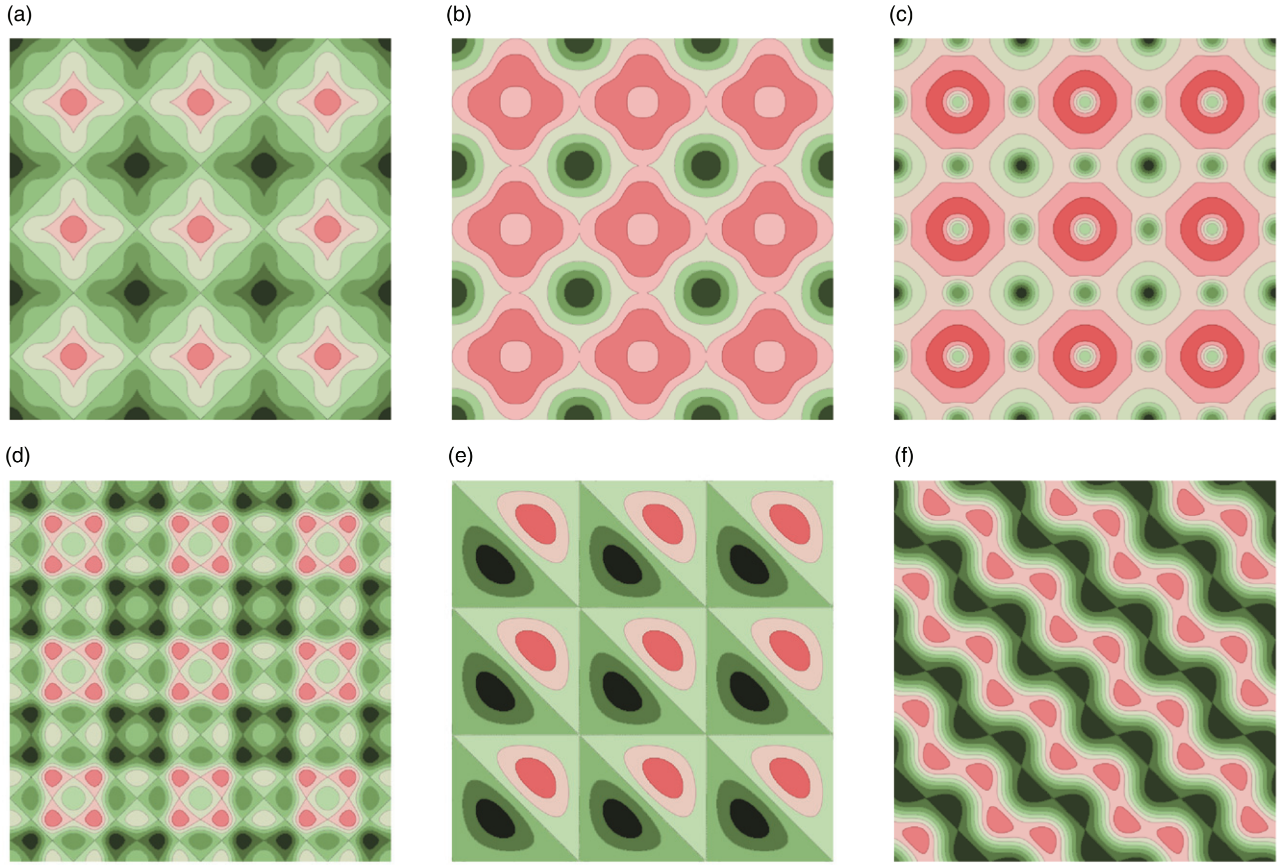

Figure 5. Contour map patterns of some symmetric functions. (a) $\sin^3 x + \sin^3 y$; (b) $-\sin x - \sin y - \sin x \sin y - \sin^2 x \sin^2 y$; (c) $\sin x + \sin y - 5\sin^6 x \sin^6 y$; (d) $\sin(\pi \sin x) + \sin(\pi \sin y)$; (e) $\sin x + \sin y + \sin(x + y)$ and (f) $\sin^3 x + \sin^3 y + \sin^3 (x + y)$.

Moreover, a welcome property of sine and cosine is that they are antiperiodic functions with period π. That is,

$$\sin(x + \pi) = -\sin(x) \quad \text{and} \quad \cos(x + \pi) = -\cos(x).$$

As the command `ContourPlot` automatically assigns lighter colours to the higher values of the function, and darker colours to the lower values, this provides us with the perception of colour contrast to go with our geometric symmetry. This becomes even more prominent if the selected colour palette for `ContourPlot` is duotone.

5. Simple contour map examples

Here we give some examples of contour maps that are simple in the sense that we use just one trigonometric function. In the following section, we give ones obtained by overlaying contour maps of several functions. We group the symmetric patterns in Figure 5, asymmetric ones in Figure 6 and finally we give some contour maps of functions that are periodic only in one variable in Figure 7. These patterns were selected subjectively by the authors after experimenting with various functions, usually having some of the properties stated in the previous section. In addition, we intentionally keep the number of contours small to preserve the simplicity of the patterns. In these figures we chose to use the built-in Watermelon palette available in *Mathematica* which assigns red colours to higher values of the function and green colours to lower values of the function.

In Figure 5, we restricted ourselves to symmetric functions. Patterns Figure 5(a)–(d) have symmetry group `p4m`. One can determine this from the figures by observing that each of these patterns has points of rotation for 90° and lines of reflections in four different directions. This also follows from Proposition 4.4. Consider, for example, the function $f(x, y) = \sin^3 x + \sin^3 y$ of Figure 5(a). Note that the function $g(x, y) = \cos^3 x + \cos^3 y$ has the same pattern just translated by $\pi/2$ in both directions since $g(x, y) = f(x + \pi/2, y + \pi/2)$. Obviously, $g(x, y)$ is a symmetric function that is even in both variables, and so Proposition 4.4 assures that it has reflections in four distinct directions.

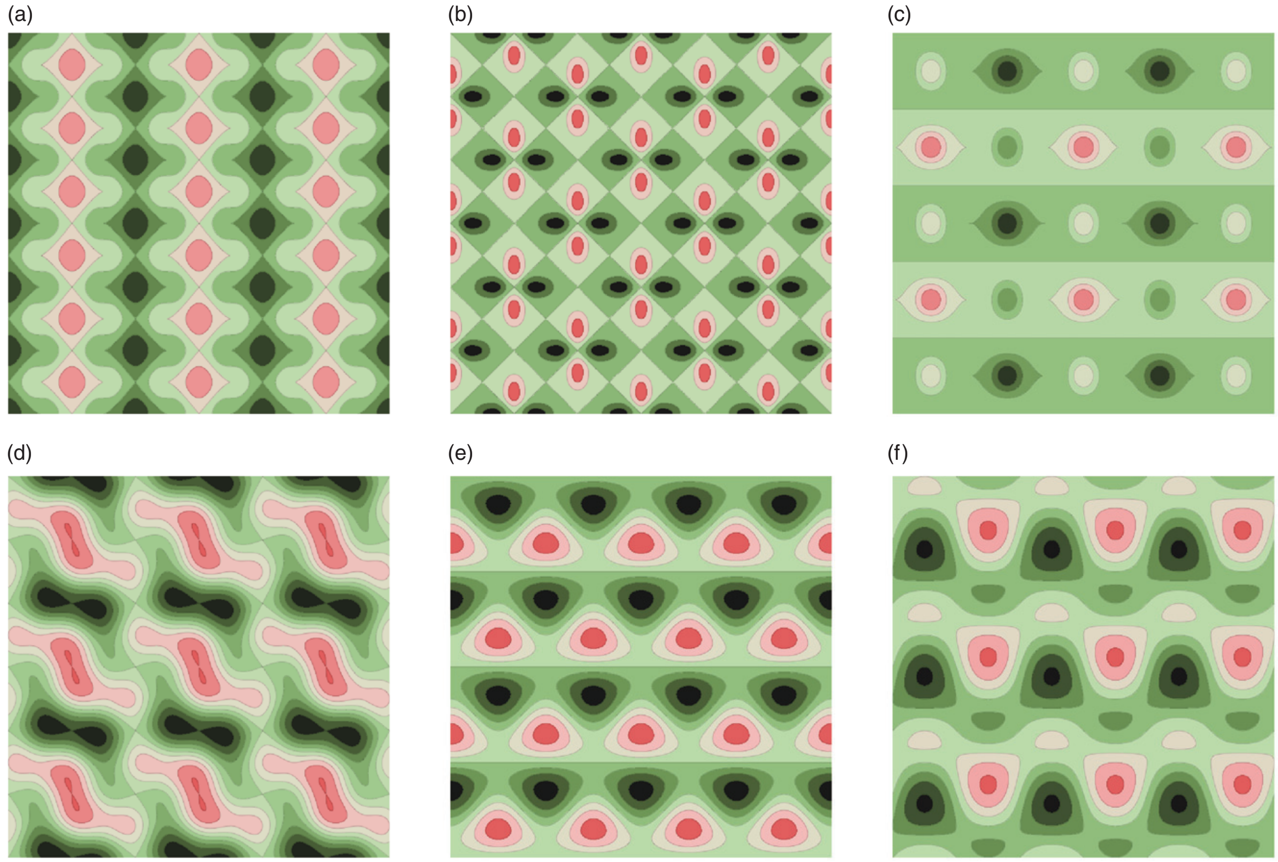

Figure 6. Contour map patterns of some asymmetric functions. (a) $\sin^3 x+\sin^2 y$; (b) $(\sin x-\sin y)(\sin^2 x-\sin^2 y)(\sin^3 x-\sin^3 y)$; (c) $\sin y+3\,\sin^5 x\,\sin^4 y$; (d) $-\sin^2 x+\sin y-\sin^2 y+\sin(x+y)$; (e) $\sin 2x\,\sin y+\sin 2y$ and (f) $\sin x+2\sin x\,\cos y+\sin y$.

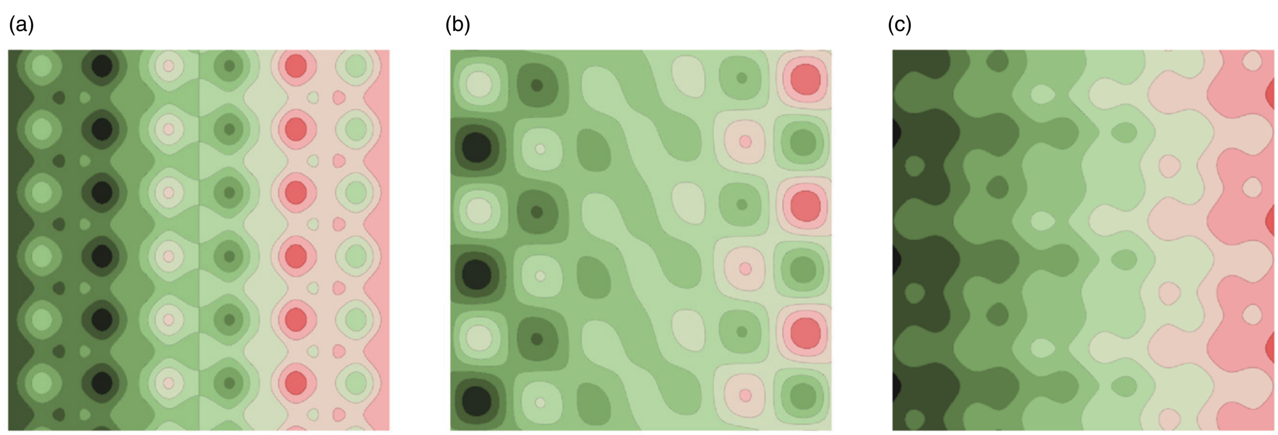

Figure 7. Contour map patterns of some functions periodic in only one variable. (a) $x-4\sin^3 x+3\,\sin x\,\sin 2y$; (b) $\frac{x}{2}+x\,\sin x\,\sin y-\sin(x+y)$ and (c) $x+\sin 2x+\sin y+\sin 2y$.

A similar reasoning applies also to other patterns of this family. Functions for the patterns from Figure 5(e) and (f) are symmetric but they are not even in the first or in the second variable, and each of them has $y=x$ as a line of reflection. The pattern (e) has no point of rotation, but the pattern (f) has one. Thus, the pattern (e) has the symmetry group pm and the pattern (f) has the group pmg.

In Figure 6, we consider asymmetric functions, i.e. functions for which $f(x,y)=f(y,\ x)$ does not

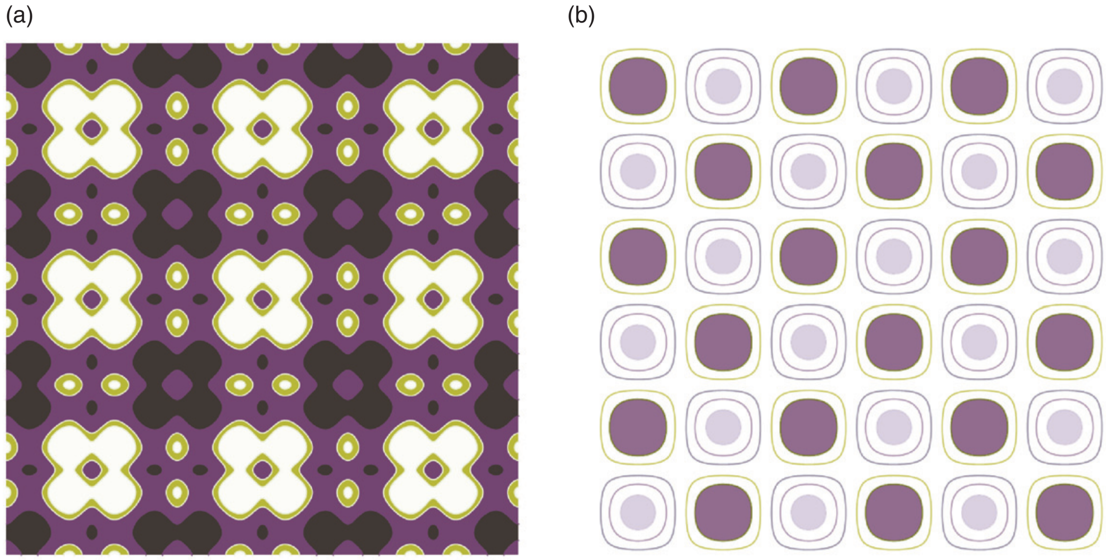

Figure 8. Contour map patterns of the functions $f(x, y) = \sin x - \sin^3 x + \sin^5 x - \sin^7 x + \sin y - \sin^3 y + \sin^5 y - \sin^7 y$ and $g(x, y) = \cos x \cos y$.

always hold. Since these functions are asymmetric, we do not obtain rotations for 90°, but the first four of them do admit rotations by 180°. The first three patterns (a)–(c) have reflections in two directions, and every rotational point lies on some reflection line, hence they have the symmetry group pmm. The pattern (d) has no reflection or glide-reflection and so it has the symmetry group p2. Finally, the last two patterns (e) and (f) have reflections in one direction, so they have symmetry group pm.

Finally, in Figure 7 we give examples of functions that are non-periodic in x but periodic in y. Because of these properties repetition appears only in the vertical direction, and so we cannot speak about wallpaper group symmetry.

6. Composed contour map patterns

Overlaying two or more contour maps may provide us with new, unexpected and exciting patterns. We simply call them *composed* patterns. This coincides with image layering in graphic programs such as Adobe's Photoshop®. In what follows, we present several composed patterns obtained by using the command `Show` from *Mathematica*.

By layering the two contour maps presented in Figure 8, we create the composed pattern of Figure 9. We start with the symmetric function $f(x, y) = \sin x - \sin^3 x + \sin^5 x - \sin^7 x + \sin y - \sin^3 y + \sin^5 y - \sin^7 y$ that is used as the background of the pattern. We specify three horizontal planes at levels −0.4, −0.2, 0.35 to provide better definition of the floral shapes after projection onto the plane $z = 0$. The colour combinations are based on a strong effect of light–dark contrast with yellow–green accents. Next, we generate a transparent upper layer for the pattern. We use the function $g(x, y) = \cos x \cos y$, five horizontal planes $z = -0.5, -0.2, 0.2, 0.5, 0.8$ and a colour palette similar to the background. This layer is comprised of concentric circles with transparent inner regions. Circular shapes are defined with outlines in contrasting or harmonious colours. When both layers were combined in the final composition, the resulting pattern produced a visually interesting interaction of geometrical shapes that imitate natural forms.

A wide range of patterns can be created in a similar manner. Figure 10 shows an example where we used the symmetric function $f(x, y) = a \sin x - 6 \sin^3 x + a \sin y - 6 \sin^3 y$. The pattern is composed of two layers that have different values for a. For the background, we specify $a = 6$ and six horizontal planes $z = -4, -3, -2, 2, 3, 4$, while for the upper layer $a = 3$ and four horizontal planes at levels −4, −2, 2, 4. Changing the value a causes horizontal stretching of the graph. The motif is again floral, but in this example it is on a white background. Shapes are filled in with a colour gradient in a contrasting palette.

A composed pattern obtained by using the asymmetric function $f(x, y) = 3 \sin x + \sin^5 x - \sin y + \sin^3 y$ and the symmetric function $g(x, y) = \sin x \sin y$ is shown in Figure 11. A stripes and polka dots pattern creates the illusion of depth and rhythm of motion. We selected seven horizontal planes at levels −3, −2, −1, 0, 1, 2, 3 for the stripes background and three horizontal planes $z = -0.85, -0.75, 0.98$ for the dots.

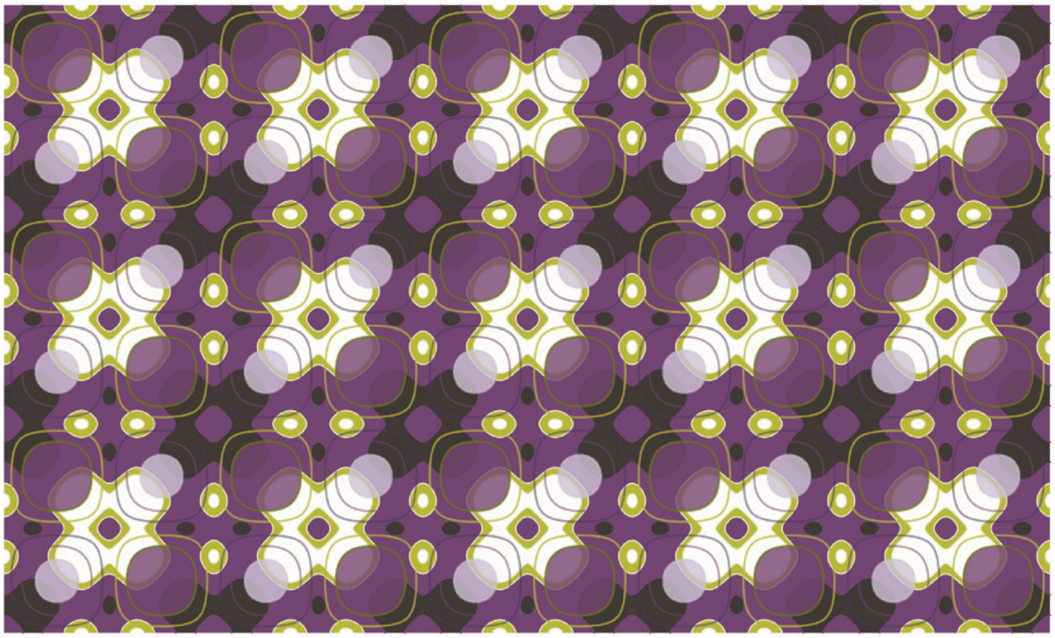

Figure 9. Pattern created by overlaying the contour maps from Figure 8.

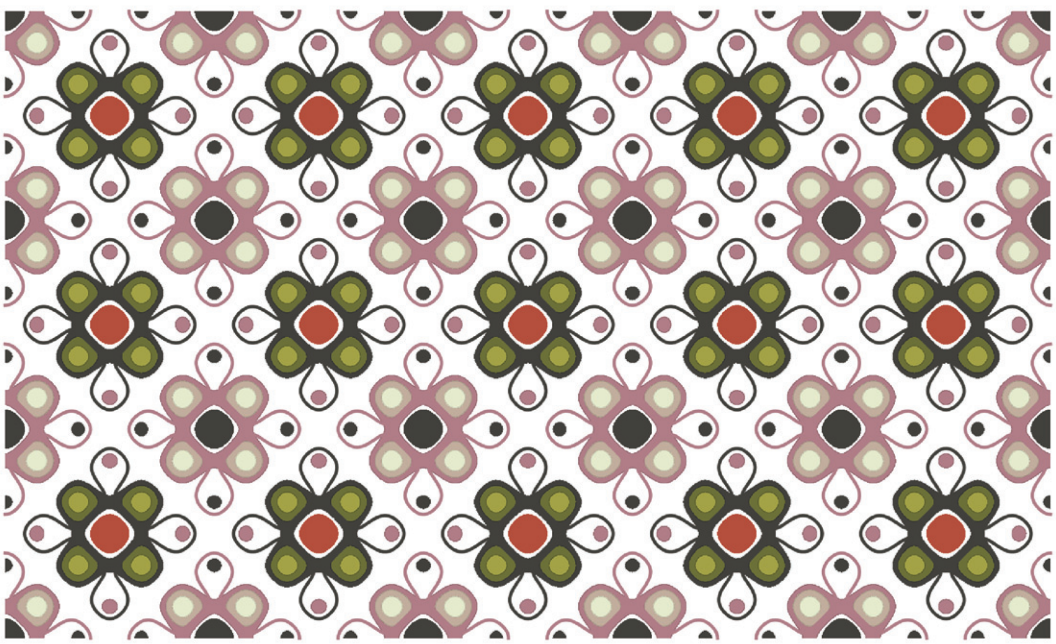

Figure 10. Pattern created by two distinct contour maps of the function $f(x, y) = a\ \sin x - 6\ \sin^3\ x + a\ \sin y - 6\ \sin^3\ y$.

Figure 12 shows a composed pattern obtained by overlaying six contour maps. For the inner texture and structure in white–gray–blue colours we used the function $f(x, y) = \sin(bx + c)\sin(ey + f)$ and combined it with functions $g(x, y) = \sin^{\sin y}\ 25x$ (orange–gray, layer two) and $h(x, y) = 3\sin 2x - 6\ \sin^3 2x + 3\ \sin 2y - 6\ \sin^3 2y$ (white–red, layer four). The values are $b, e = 8$ and $c, f = 0$ for the gray background, and $b = 1$, $c = \frac{5}{2}$, $e = 20$ and $f = 0$ for the blue third layer. The beige layer is generated by the function $i(x, y) = \sin 2x + \sin^3 2x + \sin\frac{y}{4} + \sin^3\frac{y}{4}$ and six horizontal planes $z = -3, -2, -1, 1, 2, 3$. The upper most

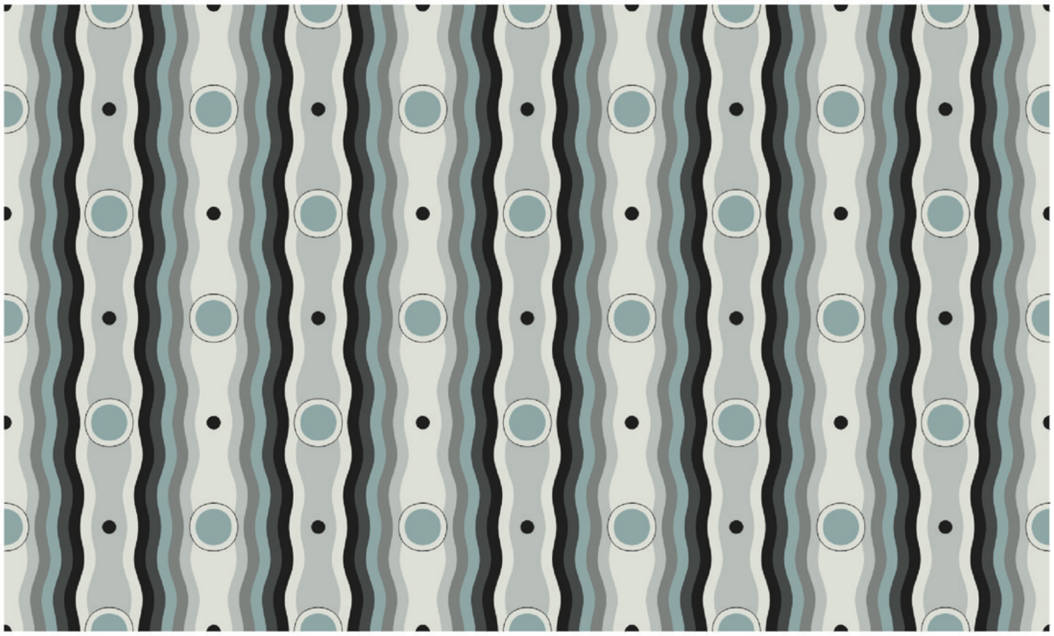

Figure 11. Composed pattern created by overlaying of contour maps of the functions $f(x, y) = 3\sin x + \sin^5 x - \sin y + \sin^3 y$ and $g(x, y) = \sin x \sin y$. See insert for colour version of this figure.

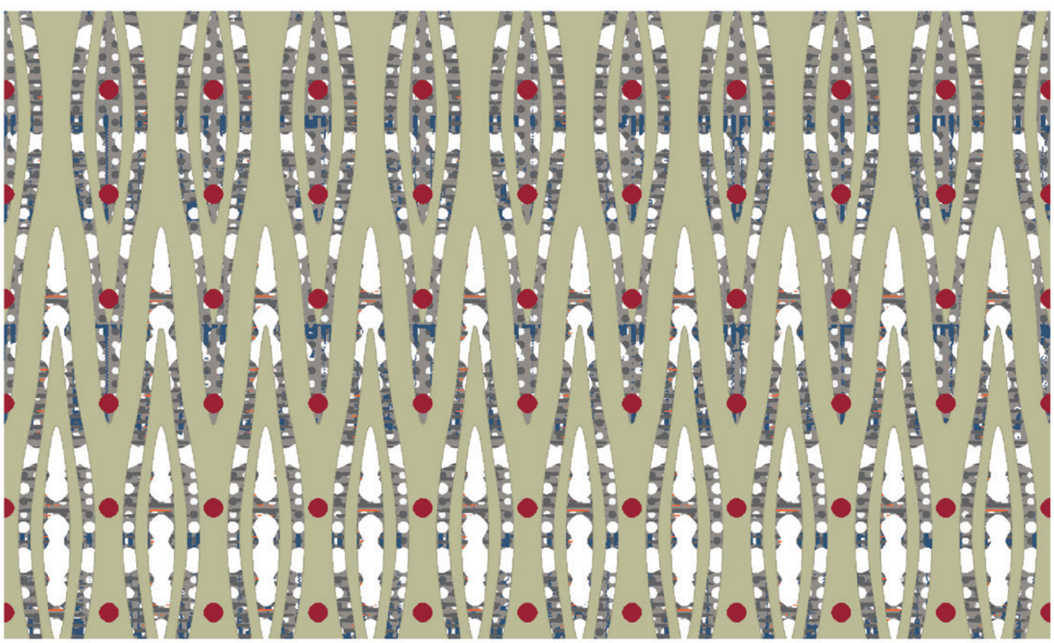

Figure 12. Pattern created by overlaying contour maps of the functions $f(x, y) = \sin(bx + c)\sin(ey + f)$, $g(x, y) = a\sin 2x - b\sin^3 2x + c\sin dy - e\sin^3 fy$, and $h(x, y) = \sin^{\sin y} 25x$. See insert for colour version of this figure.

layer with red dots is the same function as the fourth layer without white shapes.

Using the asymmetric function $f(x, y) = \sin 2x + \sin^3 2x + \sin\frac{y}{4} + \sin^3\frac{y}{4}$ and the symmetric function $g(x, y) = \sin^2 x + \sin^4 x + \sin^6 x + \sin^2 y + \sin^4 y + \sin^6 y$ results in the pattern in Figure 13. For the background, we selected three horizontal planes $z = 1, 3, 5.2$. Next, we generated the upper layer with six horizontal planes at levels -3, -2, -1, 1, 2, 3. Overlaying geometric shapes over dynamic stripes background creates the rhythmic motion of pattern.

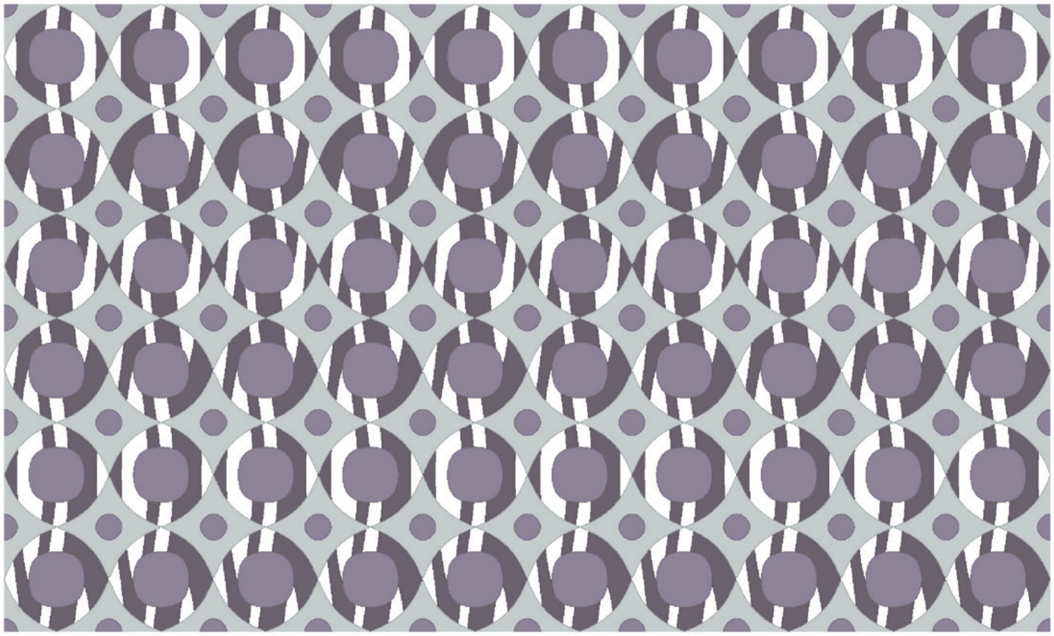

Figure 13. Pattern created by overlaying contour maps of the functions $f(x, y) = \sin^2 x + \sin^4 x + \sin^6 x + \sin^2 y + \sin^4 y + \sin^6 y$ and $g(x, y) = \sin 2x + \sin^3 2x + \sin \frac{y}{4} + \sin^3 \frac{y}{4}$.

7. Conclusions and further work

In this article, we have presented a simple procedure for constructing aesthetic coloured repeating patterns with the use of contour lines and maps.

As the reader may have noticed, we did not provide contour maps for all of the 17 symmetry groups. This was not our goal but we believe that by choosing an appropriate landscape all of these groups can be obtained. However, it may not be easy to find a 'simple' function whose 3D-graph corresponds to such a landscape.

As future work, our main interest is in obtaining a wider range of examples of various patterns like the ones presented in the last section. Moreover, in the overlaying step we wish to involve some patterns which are not contour maps. We would also like to experiment further with transparency and various blending modes.

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