The Tiling Patterns of Sebastien Truchet and the Topology of Structural Hierarchy

Author(s): Cyril Stanley Smith and Pauline Boucher

Source: *Leonardo*, 1987, Vol. 20, No. 4, 20th Anniversary Special Issue: Art of the Future: The Future of Art (1987), pp. 373-385

Published by: The MIT Press

Stable URL: https://www.jstor.org/stable/1578535

The Tiling Patterns of Sebastien Truchet and the Topology of Structural Hierarchy

Cyril Stanley Smith
with translation of Truchet's text by
Pauline Boucher

Abstract—A translation is given of Truchet's 1704 paper showing that an infinity of patterns can be generated by the assembly of a single half-colored tile in various orientations. It embodies an early representation of the principles of combinatorial theory and of crystallographic symmetry including color symmetry. Simple rules of the topology of separation and junction are used to extend Truchet's concept of directional choice and, by relaxing symmetry rules, to generate diagrams illustrating field/ground relations, the hierarchy of structural freedom and the origin and nature of structural order and disorder in general.

I. INTRODUCTION

There are few places where the approaches of the artist and the scientist intersect more intimately than in the production and analysis of tiling patterns. In *The Sense of Order: A Study of the Psychology of Decorative Art* [1], Ernst Gombrich reproduces some figures from a book by a Dominican priest named Douat, which was published in Paris in 1722 and is now very rare [2]. Douat's book contains 72 engraved designs, one of which is reproduced in Fig. 1, and nearly 190 pages of tables listing all the possible permutations and combinations of the four letters A, B, C and D (arranged in different sequences to represent the orientation of the unit tiles juxtaposed horizontally, diagonally and vertically, concluding with the 256 possibilities of arrangement of a 4 × 4 array). The book had some influence on European decorative art in the eighteenth century and inspired illustrations in such works as Jeurat's *Traité de perspective* (Paris, 1750) and Diderot's *Encyclopédie*.

Though Douat's book provided a fine source of patterns for the use of craftspersons, the intellectual inspiration behind it was entirely that of a fellow Dominican priest, Sebastien Truchet, an engineer with an interest in mathematics and art; he must have been much like *Leonardo*'s founder, Frank Malina, to whose memory the present paper is dedicated. Truchet had published a short paper "Memoir sur les Combinaisons" in the *Memoires de l'Académie Royale des Sciences* in 1704. A translation of this with reproductions of all the original illustrations forms Section II [3] of the present paper and should be read before the discussion that follows in Section III.

Cyril Stanley Smith (metallurgist, historian of technology, Massachusetts Inst. of Technology), 31 Madison Street, Cambridge, MA 02138, U.S.A.

Pauline Boucher (librarian), Tufts University, Medford, MA 02185, U.S.A.

Received 11 February 1987.

II. TREATISE ON COMBINATIONS BY THE R[EVEREND] F[ATHER] SEBASTIEN TRUCHET

(Translated by Pauline Boucher)

During the last trip that I took to the canal d'Orléans by order of His Royal Highness, in a château called Motte St. Lyé [4] 4 leagues this side of Orléans, I

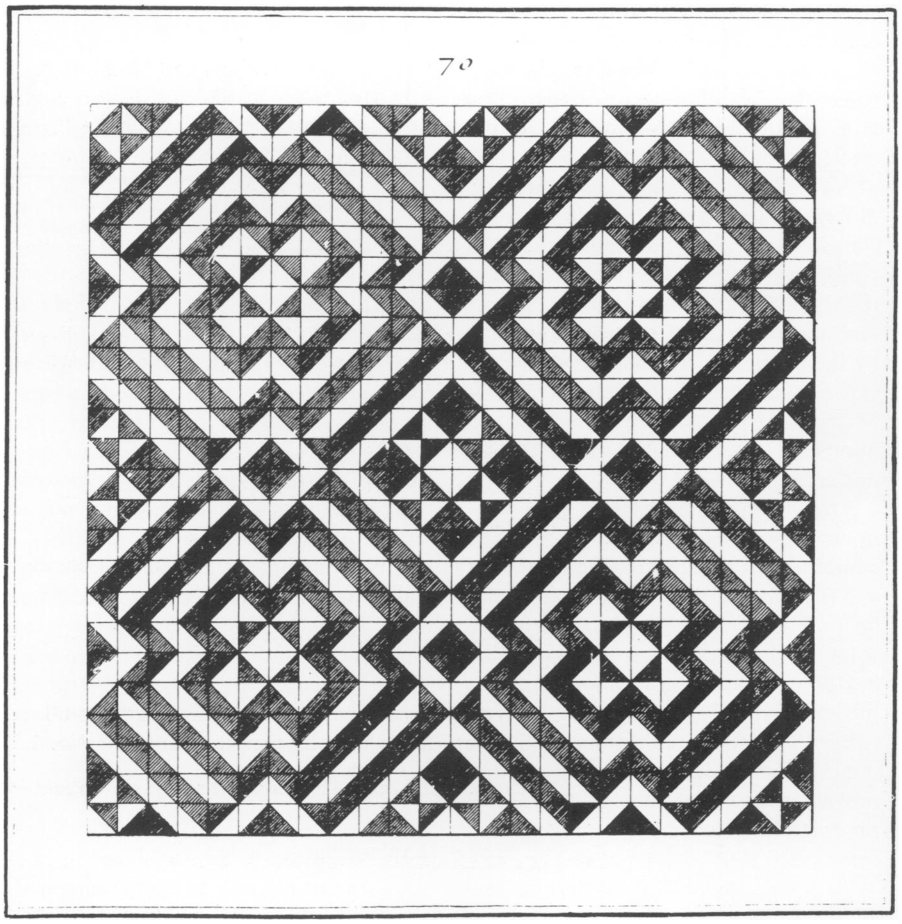

Fig. 1. A pattern of tiles illustrated by Douat [2] in 1722. (Courtesy of the Bibliothèque Nationale, Paris)

Pergamon Journals Ltd.
Printed in Great Britain.
0024-094X/87 $3.00+0.00

found several ceramic tiles that were intended for tiling the floor of a chapel and several other apartments. They were of square shape, divided by a diagonal line into two colored parts. In order to be able to form pleasing designs and patterns by the arrangement of these tiles, I first examined the number of ways in which these tiles could be joined together in pairs, always in checkerboard array.

I found that there were 64 different ways of ranging these tiles in pairs [within a rectangular border], which make 64 combinations: this seems surprising, for two letters or two numbers ordinarily combine only twice, since they change position only by being put one after the other sequentially in a line, the base always remaining the same. However, in the arrangement of two tiles, one of the pair can take four different locations [in relation to the corners of the larger square frame], in each of which the second tile can change 16 times [in position and in orientation] giving the [total of] 64 combinations. These are illustrated and identified by number in table I [Fig. 2].

We then noticed that there were similar patterns within these 64 combinations, and that we could reduce these to 32 different ones because each pattern is repeated twice in the same location and the two patterns differ from one another only by the transposition of the darker tile, as can be seen in table II [Fig. 3, top], where they are all illustrated in pairs and numbered in the same way as in the first table.

We have also found that these 32 different patterns can be reduced to 10 similar ones if we do not take into consideration their location [in relation to the border] and the same point of view, and that the similar patterns differ only by their different location on their four sides, as can be seen in table III [Fig. 3, bottom], where they are illustrated and numbered in the same way as in the first and second table [5].

After having examined the combinations of two tiles, we could put here the combinations that could be made with 3, 4, 5 tiles; but inasmuch as this detail will be long, and we are not yet satisfied with what we have done on that, we shall postpone that article to another paper. [no such paper appeared—Ed.]

Books on civil architecture and those treating combinations were consulted to find if anyone had already made the same observations, but we found nothing that came even close.

Next we sought to form designs and compartments by joining these patterns together, always in checkerboard-like array. We found too great a number to report them all and selected only a hundred of which to make a fair copy, so that everyone can judge with his own eyes the truth of what we have said and the fecundity of these combinations, the origin of which is nevertheless so very simple. Of these designs, only 30 were engraved for this publication so as not to make the volume too thick [see Figs 4–10]. Next to each design is its explanation, describing how it was constructed from table I, which serves as a dictionary for finding the combinations that were used to form them. They were all constructed by the arrangement of the two tiles taken together, placed in the order that is noted for each plate.

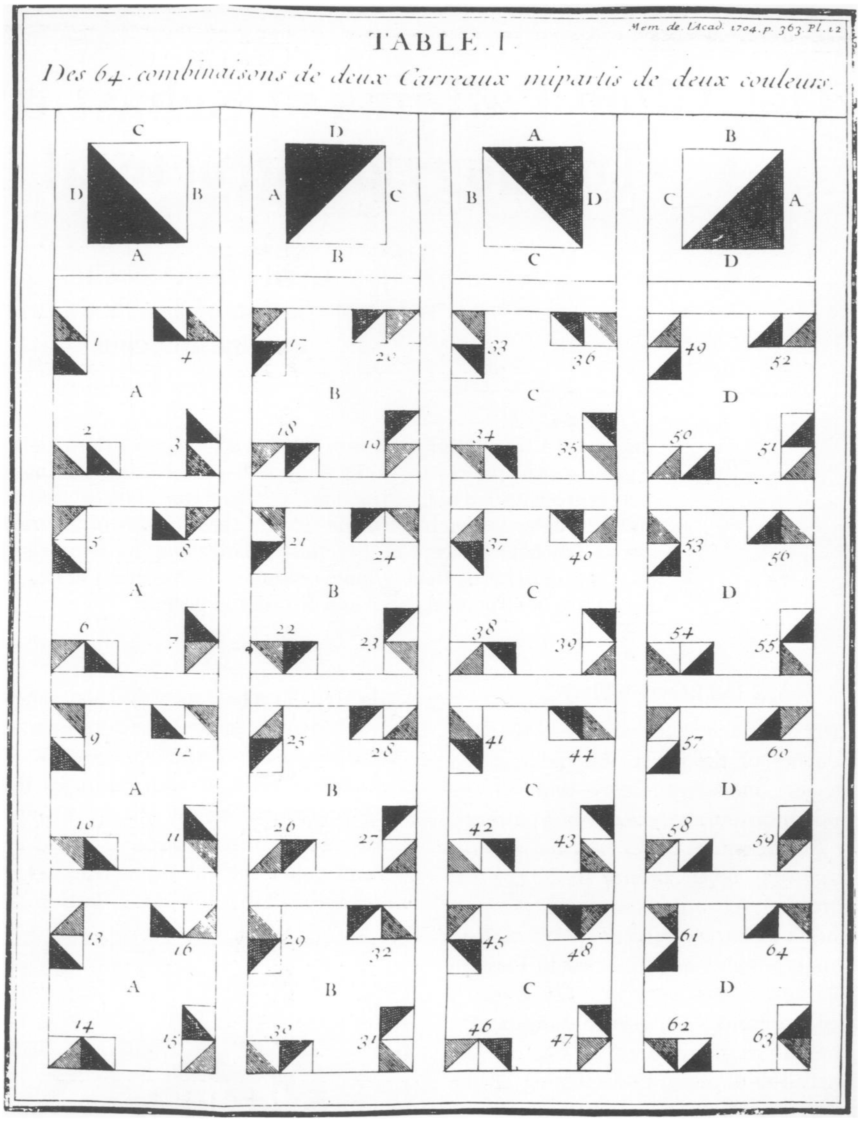

Fig. 2. Truchet's table I shows his bi-colored tiles and their placement in pairs in different orientations. (Courtesy of Burndy Library, Norwalk, Connecticut)

Explanation of Table I

. . . In this plate [Fig. 2] are illustrated the 64 combinations that can be made with two tiles diagonally divided into two colored sections.

The plate has four columns from top to bottom, each column being separated into five squares. In the first square of each column, a single large tile is represented, which is placed [and oriented] differently in each, as can be seen by the four letters ABCD which always mark the same sides of each tile (i.e. A and D are the two colored sides, B and C the two white sides), in such a way that in all the squares of the first column the darker tile is always shown as though resting horizontally on side A.

In the second column [the darker tile rests] on side B, in the third, on side C, and in the fourth column, on side D.

In the four squares that make up the first column having the letter A in the center, the 16 combinations are illustrated that can be made with two tiles, one of which, the darker one, remains always with side A horizontal: this tile is colored a darker shade to distinguish it from the one that changes orientation.

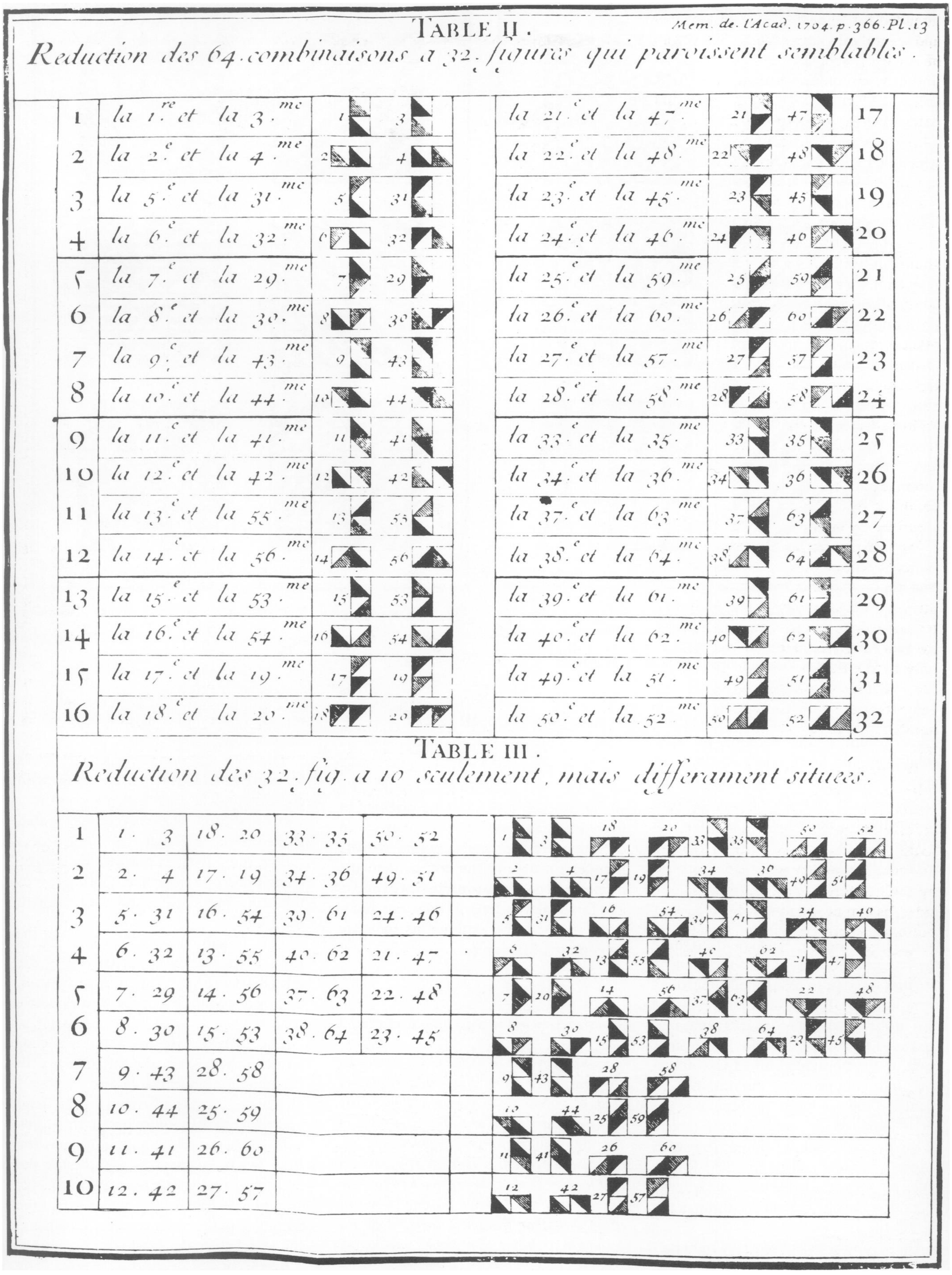

TABLE II. Mem. de l'Acad. 1704. p. 366. Pl. 13

Reduction des 64. combinaisons a 32. figures qui paroissent semblables.

1	la 1.re et la 3.me		la 21.e et la 47.me	17
2	la 2.e et la 4.me		la 22.e et la 48.me	18
3	la 5.e et la 31.me		la 23.e et la 45.me	19
4	la 6.e et la 32.me		la 24.e et la 46.me	20
5	la 7.e et la 29.me		la 25.e et la 59.me	21
6	la 8.e et la 30.me		la 26.e et la 60.me	22
7	la 9.e et la 43.me		la 27.e et la 57.me	23
8	la 10.e et la 44.me		la 28.e et la 58.me	24
9	la 11.e et la 41.me		la 33.e et la 35.me	25
10	la 12.e et la 42.me		la 34.e et la 36.me	26
11	la 13.e et la 55.me		la 37.e et la 63.me	27
12	la 14.e et la 56.me		la 38.e et la 64.me	28
13	la 15.e et la 53.me		la 39.e et la 61.me	29
14	la 16.e et la 54.me		la 40.e et la 62.me	30
15	la 17.e et la 19.me		la 49.e et la 51.me	31
16	la 18.e et la 20.me		la 50.e et la 52.me	32

TABLE III.

Reduction des 32. fig. a 10 seulement, mais differament situées.

1	1 · 3	18 · 20	33 · 35	50 · 52
2	2 · 4	17 · 19	34 · 36	49 · 51
3	5 · 31	16 · 54	39 · 61	24 · 46
4	6 · 32	13 · 55	40 · 62	21 · 47
5	7 · 29	14 · 56	37 · 63	22 · 48
6	8 · 30	15 · 53	38 · 64	23 · 45
7	9 · 43	28 · 58		
8	10 · 44	25 · 59		
9	11 · 41	26 · 60		
10	12 · 42	27 · 57		

Fig. 3. Truchet's tables II and III show the equivalence of many of the arrangements in table I.

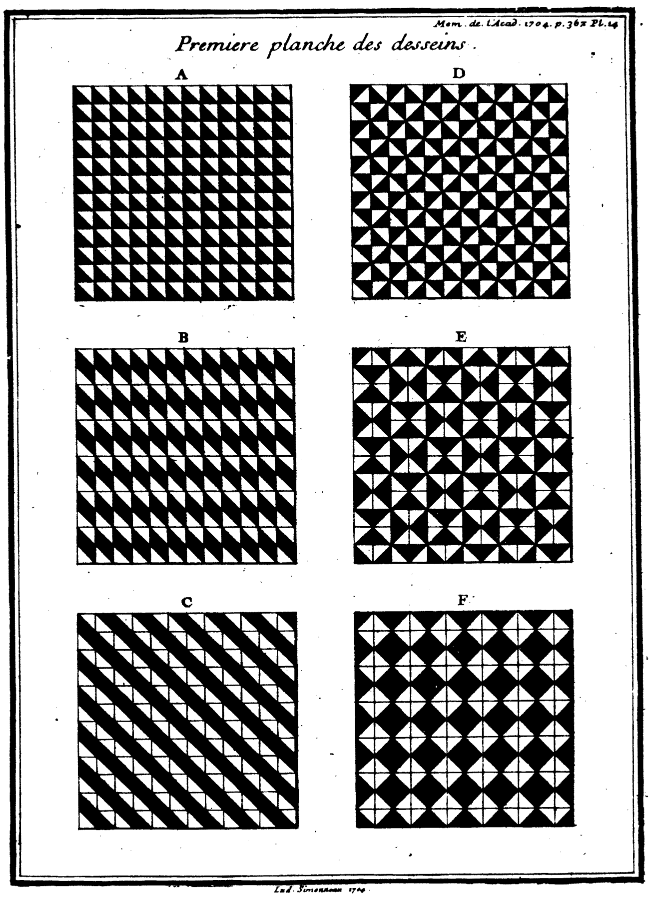

Fig. 4. Truchet's plate 1, showing the simplest patterns to be constructed by the assembly of his tiles in various orientations within square borders. Symmetry is highest in the parallel bands C and the checkerboard D. (Courtesy of Burndy Library, Norwalk, Connecticut)

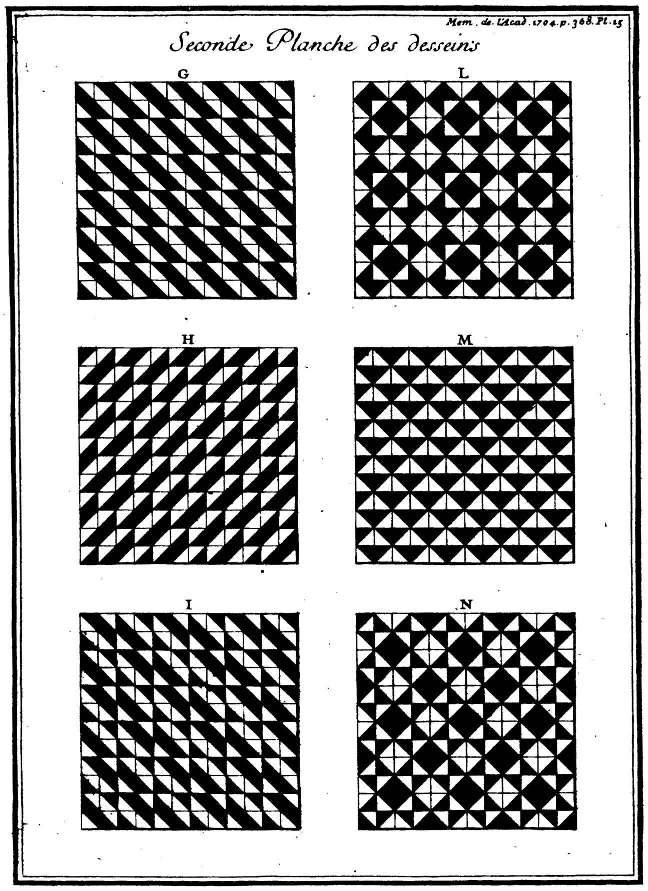

Fig. 5. Truchet's plate 2. (Courtesy of Burndy Library, Norwalk, Connecticut)

The same order is followed in the three other columns: the squares of each column are marked by the same letter, B in the second, C in the third, and D in the fourth column. In each column, to avoid confusion, the combinations of four are separated into four [different orientations in the corners].

Explanation of Table II

. . . This table [Fig. 3] was made to show the reduction of the 64 combinations of the first table [into associated subgroups forming] 32 patterns that appear similar and in the same situation. It is separated first into two large columns of 16 lines each, each subdivided into four others: the first column lists the number to identify the reduced groupings; the second records the numbers from table I of the patterns that are similar [but with transposed color]; the third and fourth columns show the same pairs of patterns that differ only by the transposition of the darker tile.

The second large column [the right half of the table] differs from the first only in that the numbers identifying the groups are placed in the last column.

Explanation of Table III

The third table [Fig. 3] shows that we can further reduce the 32 patterns of the second table to form 10 patterns that are similar to each other, but are oriented in four different ways. This we can see in each line, for each line contains first the number to identify the group, then the numbers of combinations that are similar and finally the patterns of these combinations situated and outlined as they are given in the first table.

Construction of the Six Designs of Plate 1

General Note: In order to construct all the designs that are illustrated in plate 1 [Fig. 4] and the following ones [Figs 5–10], it is necessary to refer to the first table of the 64 combinations in which all the combinations used are indicated by numbers and to take those that are marked to form each row of the designs, putting them one after the other from left to right as the letters in each line of ordinary writing are placed.

The first design, marked A [see Fig. 4], is constructed with combination 2 [Fig. 2] repeated over and over, and begun anew in each row.

The second design, marked B, is formed by making an entire first row with combination 2, and then a second row with number 34. These two rows repeated make the entire design.

The third design, marked C, will be made by forming alternately a first row of the 12th combination, and a second with the 10th.

The fourth design, marked D, is formed alternately by a first row of the 6th combination repeated in succession, and by a second row of the 40th similarly repeated.

The fifth design, marked E, is constructed thus: We make a first row with the two combinations 24 and 14 placed alternately; a second row with the 22nd and the 16th also alternated; a third row with the same two combinations as the first, but by placing the 14th before the 24th; and finally the twenty-fourth [*sic*; should read fourth] row which is like the second with the order reversed by placing the 16th before the 22nd.

The sixth design, marked F, is made by placing alternately in the first row the 24th combination repeated successively,

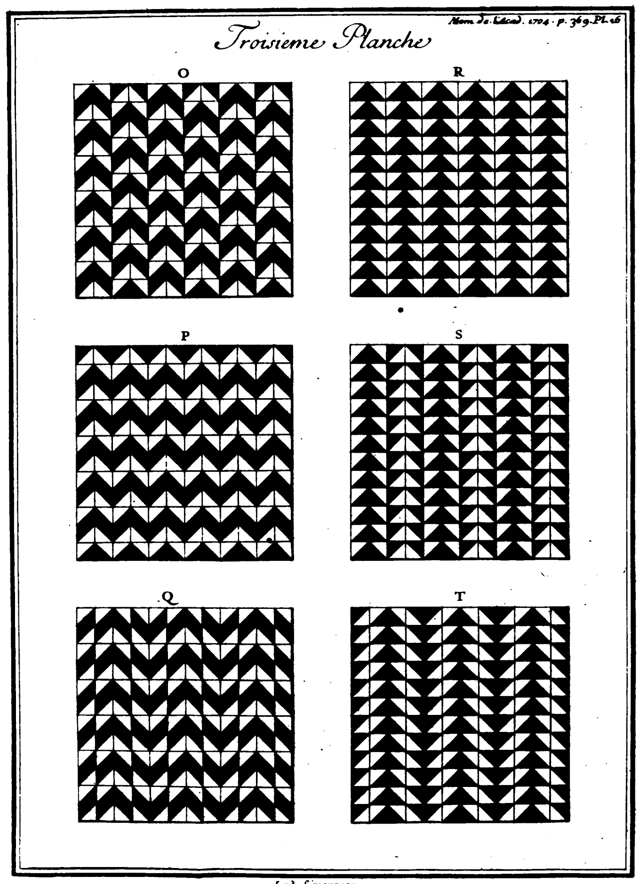

Fig. 6. Truchet's plate 3. (Courtesy of Burndy Library, Norwalk, Connecticut)

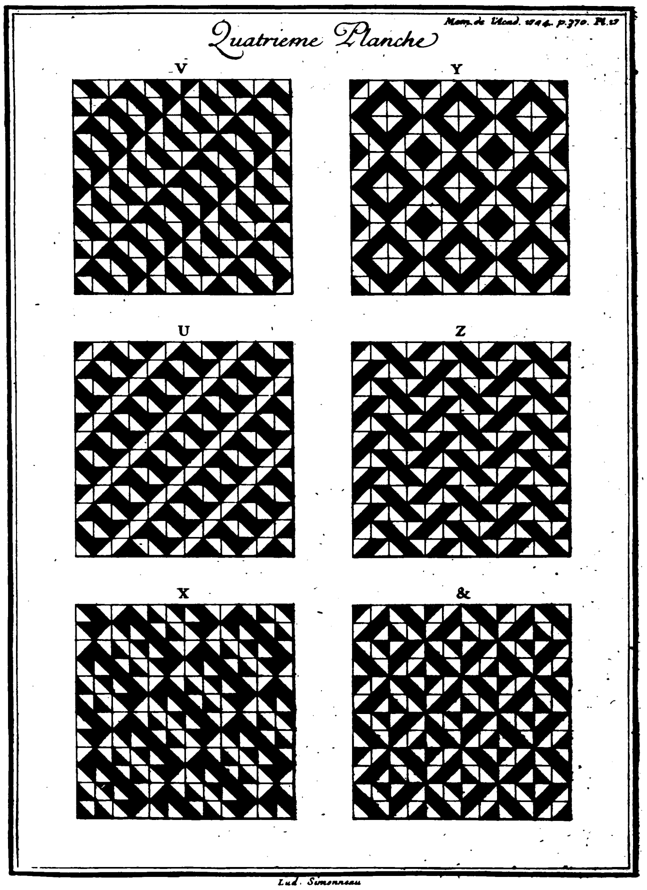

Fig. 7. Truchet's plate 4. (Courtesy of Burndy Library, Norwalk, Connecticut)

and in the second row the 16th similarly repeated.

Construction of the Six Designs of Plate 2

The first design, marked G [Fig. 5], is made with the 42nd combination repeated in succession in the first row, with the 10th similarly repeated in the second. The third row is made like the second; the fourth and fifth rows, like the first.

The second design, marked H, is made with the three combinations 28, 26 and 50 repeated in succession in the first row; then with 26, 50 and 28 similarly in the second; and finally with 50, 28 and 26 in the third row . . . .

[There follow five pages detailing similar numerical descriptions of the composition of each of the remaining patterns illustrated in Figs 6–10. Most of these have not been translated, for the information is immediately evident in the patterns themselves, and we skip directly to the last, which is the most complicated.]

Construction of the Two Designs of Plate 7

. . . The first row in the second design, marked 6 [Fig. 10], is made with 16 and 8 each once, then 22 twice, followed by 40 and 16 each once; the second row is formed with 34, 6, 50, 2, 38 and 18 each once; the third row is made with 12, 8, 26, 10, 40 and 28 in succession, each once; the fourth row by 28, 6, 10, 26, 38 and 12 each once; the fifth row by 50, 8, 34, 18, 40 and 2 in succession; the sixth row by 24 and 32 each once, then 14 twice in succession, 28 and 24 each once; the seventh row with 22 and 40 each once, then 16 twice in succession and 8 and 22 each once; the eighth row is made by 2, 38, 18, 34, 6 and 50 each once; the ninth with 10, 40, 28, 12, 8 and 26 in succession; the tenth row by 26, 38, 12, 28, 6 and 10, also in succession; the eleventh with 18, 40, 2, 50, 8 and 34 each once in succession; finally the twelfth row is made by 14 and 38 each once, 24 twice in succession, 6 and 14 each once. [*End of paper*]

III. DISCUSSION

On Tiling Patterns

Truchet's treatise is of considerable importance for it is in essence a graphical treatment of combinatorics, a subject that, under the influence of Pascal, Fermat and Leibniz, was at the forefront of mathematics at the time. Truchet says that he got the idea when he saw a supply of tiles for paving apartments in a château near Orléans. He had the wit to see that nothing but orientation underlay the almost limitless patterns that the artisan could produce using a simple unit, and his principles can be seen to underlie virtually any pattern or structure. Truchet was particularly fascinated by the hierarchy of closure, and the simple patterns that he shows exploit sub-structural orientational and positional freedoms that relate more to the topology of color symmetry, superlattices and non–periodic lattices than to the conventional symmetries of nineteenth–century crystallographers [6].

Truchet's patterns are superficially similar to those used in the construction of the mosaic tiles so prominent in Islamic architecture, the construction and philosophy of which, based on the intersection of circles of differing radii, has been so well treated by Keith Critchlow [7], but the principles are more fundamental. Beneath the geometry and straight lines, it can be seen that Truchet exploits the qualities of directional interplay between granularities on different scales, of the inversion between positive and negative, of the balance between density and rarity and of the mutual exclusiveness of field and ground.

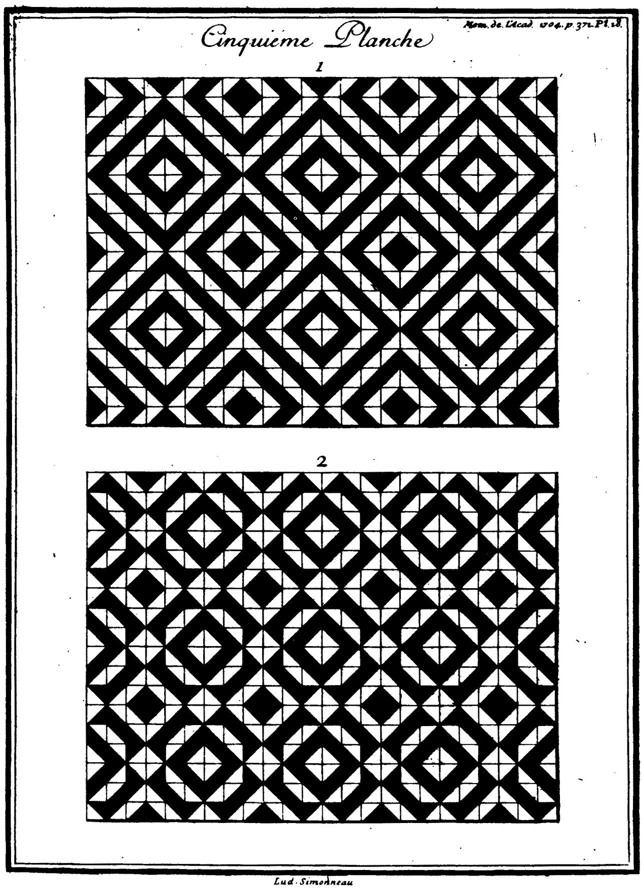

Fig. 8. Truchet's plate 5. (Courtesy of Burndy Library, Norwalk, Connecticut)

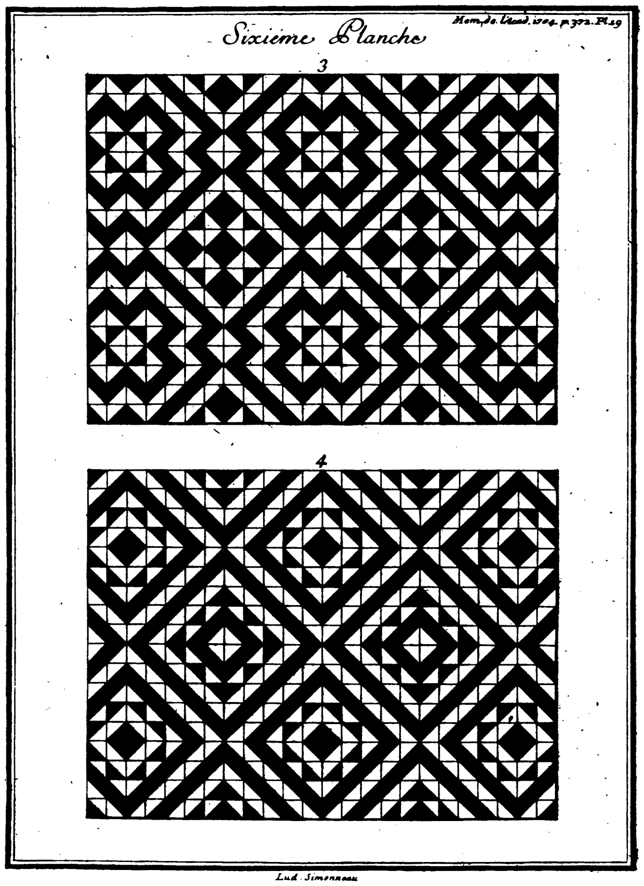

Fig. 9. Truchet's plate 6. (Courtesy of Burndy Library, Norwalk, Connecticut) Note the combination of directional and color symmetry in quincunxial alternation.

His approach was more topological than geometric, and the qualitative aspects of pattern take priority over the metric ones. His principles provide a kind of metaphor for the hierarchy of separation and connection in all things.

Islamic patterns generally exploit the packing of straight-sided tiles of differing shapes (Fig. 11) [8], while Celtic designs, such as those in the Lindisfarne Gospel and the Book of Kells (Fig. 12) [9], are in essence interweavings of continuous closed ribbons, said to be symbolic of eternity. Truchet, however, uses a single polygonal shape—a right-angled isosceles triangle, with and without internal differentiation of color—and obtains diversity by orientation alone. All tilings can generate a vast number of patterns in which the human eye delights in discovering local similarities and regional differences, for their perception exercises both the sensory enjoyment of pattern and the intellectual need for analysis.

Truchet provides a framework within which, solely by change in the scale of resolution and distortion without change in the topological characteristics, the other types can be generated. Any tiling serves to illustrate important mathematical principles, but Truchet's patterns use as the single structural unit the smallest polygon that can be formed of straight lines arranged at different orientations, to join at vertices the valencies of which never exceed 8. But there is the color substructure of a single color that does not affect the grosser array. Despite their ability to produce checkerboard uniformity, the tiles suggest a departure from customary emphasis on either geometry or analysis via trigonometric functions and reciprocals and invite the adoption of the simpler topological approach which is independent of scale and of idealized rule-and-compass symmetry. It involves neither infinitesimals nor negative or irrational numbers—so essential in many calculations—and requires only the balance of simple sums of odd and even integers recording junctions within regions and on the boundaries that define the association of substructural regions.

All tilings combine aspects of lines-of-progress and cyclical closures, but they differ in their relative emphasis on internal or boundary balance: pattern depends less upon the fit of any definable polygons with their immediate neighbors than upon the whole hierarchy of directions. The eye enjoys the alternating perception of boundaries that enclose or exclude or are themselves regions of which the inner structure is remembered and does not need further analysis.

The illustrations given by Truchet and Douat have a high degree of symmetry. Perhaps more interesting, if less pleasing to the eye, are the dissymmetries of the kind illustrated in Figs 13 and 14. The former is an array in which both the orientation and coloring of the tiles are random; in the latter, spiral order has been introduced within this background. (This is reminiscent of the *lei wen* pattern, perhaps the most pleasing ground-filling texture ever devised, used on Chinese ceremonial bronzes, especially of the late Shang dynasty). Figure 15 is an array containing configurations analogous to the lattice imperfections in solid-state physics. The quincunxial balance between the quadrivalent and octavalent vertices will be noticed as well as the positional freedom of the pentavalent vertices representing dislocations.

In Fig. 16 some simple, locally interlocking configurations are shown embedded in a background, within which the tiles have no predetermined orientation and both halves are shown equally gray. Note the configurations of straight, elbowed or hooked terminations within groups of four tiles, and the kiss, missed kiss and sheared kiss of opposing

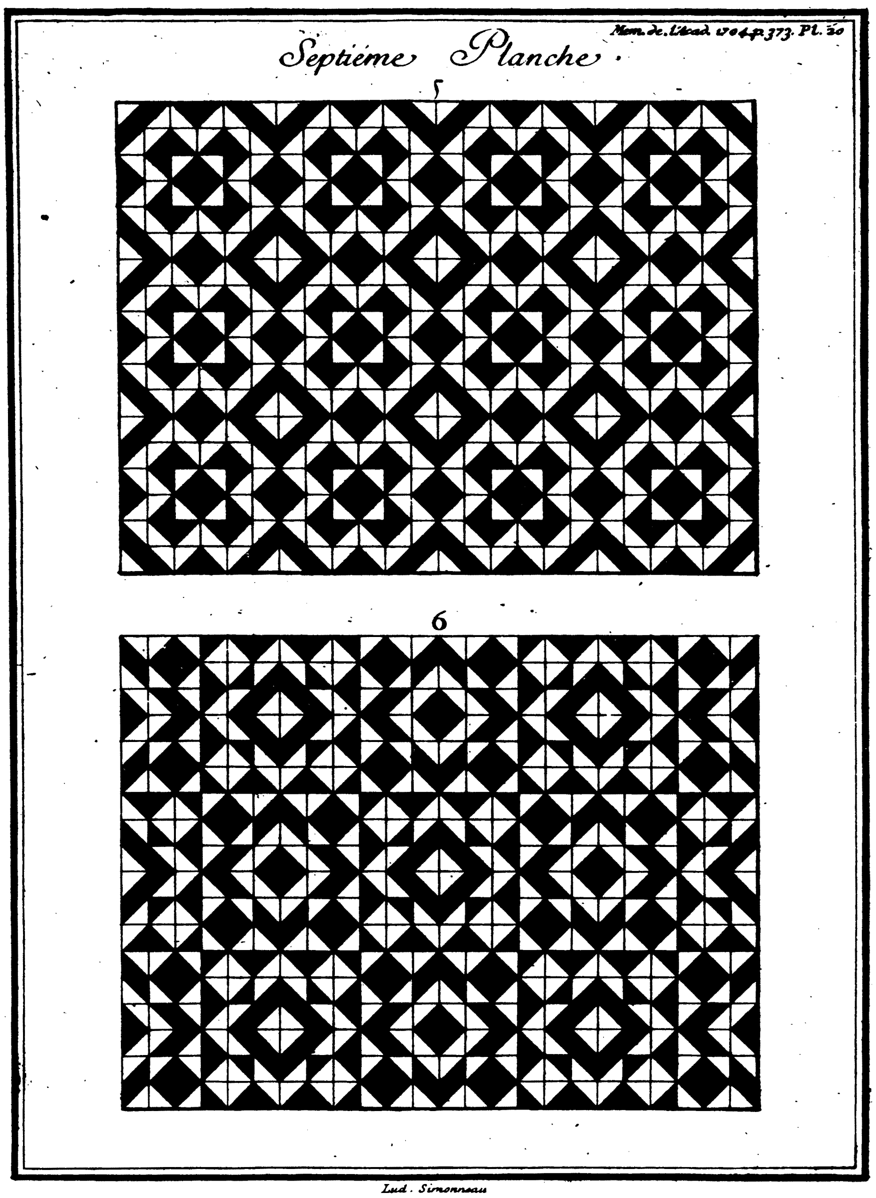

Fig. 10. Truchet's plate 7. (Courtesy of Burndy Library, Norwalk, Connecticut)

diagonals. The shared boundary between black and white can always be extended indefinitely as a straight line in a single direction, even across a lateral interruption as in the parallel lines forming a trigram like those in the Chinese Book of Changes. Physicists might note the resemblance of the diagonal connection between the two square corners to the Feynman diagram. The simplest basis of the relation between convexity and concavity is shown in the diagram at the lower right corner of Fig. 16. Two units, with boundary lengths 6 and 4, combine to give a single one with boundary 6. (Think of the Latin word for it: is it sheer coincidence that the diagrams show the relation between the shapes of the letters X and Y and also chromosome association and division?)

Just as any part of Truchet's symmetrical compositions could be excised and inserted in another orientation or position, so can any random or special array be embedded in an ordered or random environment. Moreover, the black and white regions can be inverted photographically without changing the nature of the local vertex valencies or boundary contrasts and average density. Moreover, no amount of local expansion, contraction, bending or other distortion will change the topological relations and contrasts as long as the vertices are neither dissociated nor combined. Both density and direction are always relative and can be compared only at the points of junction.

The Resolution of Structure

The structure of matter devolves ultimately into the intimate coexistence of something like corpuscles of nothing and atoms of something, segregating through the accidents of history to yield regions differing in density intimately interwoven on different scales. The existence of the world as well as human perception and analysis of any part of it is a matter of the angular scale of resolution and of the time necessary for making *comparison* between the different parts. Both the stability of material structures and human perception of them involve the tracing of directions and terminations, the formation and the finding of closed boundaries surrounding regions of substructure, that may be similar or different in the density or shape and orientation of the parts. Without such variations and without time to compare remembrances of them, nothing can be experienced.

The patterns of Truchet's tiles appear at first glance as variously shaped interlocked regions of black and white, the boundaries between the square tiles being submerged whenever the two regions flanking them have the same color, just as in a real floor the air or cement between the tile edges is not perceived—until one looks closely. The scale of resolution determines what is seen. Though forms like the separate polyhedra, polygons, lines and points of the mathematician can be imposed on material things, including works of art (as the sixteenth-century artist well knew), they mark only boundaries in anisotropic differences between regions of differing density. And the lines themselves, if they are to have conceptual or material significance, must have substructural density different from their surroundings in two or more directions (dimensions?). There is continuity of either density or rarity in one direction and discontinuity in others.

Consider the difference between a fine line made by pen or graver and the more expressive brushstroke of a Chinese calligrapher. Both are polygonal regions definable in terms of the boundary shape and the density of molecules within them, different from those outside. Further structure is ignored because it is seemingly uniform. Either region can be expanded or contracted without change of continuity or discontinuity unless new contacts between regions occur or old ones are broken. One sees only the shape of contrast, but seeing requires movement. As George Kubler so perceptively put it, one sees the Shape of Time.

The substructural nature of regional contrast is seen more easily in engravings than in paintings, for in the latter the significant scale of substructural density is that of the wavelengths of visible light, too small to be resolvable by the cells in the retina but not by the molecules within

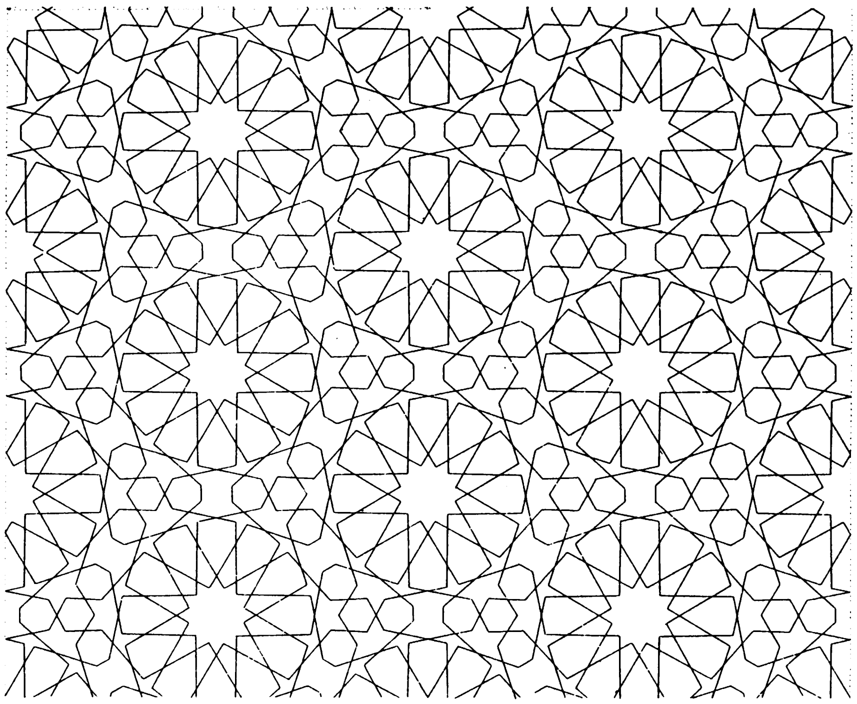

Fig. 11. An Islamic mosaic tile pattern, from Bourgoin [8]. Divalent and quadrivalent vertices are associated with polygons having 4, 6, 7, 8, 10 or 24 sides in accord with equation 2.

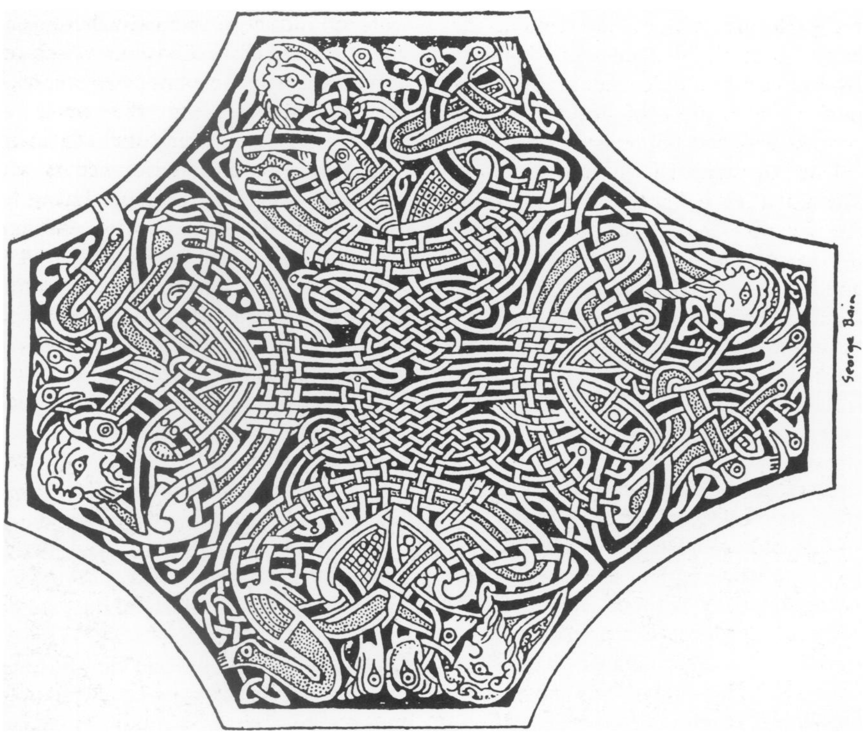

Fig. 12. A pattern from the Book of Kells, redrawn by Bain [9]. With the exception of the zoomorphic figures, the entire design (only 1.2 inches wide in the original) is composed of three continuous ribbons interlaced with themselves and with each other.

them. It is the same in matter itself, where the 'seeing' involves the discovery of patterns of electronic interactions within atoms and the discovery of larger structures that acquire stability as patterns of resonance between mobile photons and phonons with their differing velocities and viscosities.

Truchet's tiles show clearly the hierarchical interdependence of different levels of structure. The color on half the tile surface depends on some unresolved substructure within otherwise identical triangles; their association forms superstructural patterns of any degree of connectivity, closure, symmetry or randomness that depends on the orientation of units in relation to each other—orientation that may result from historical accident or the choice of a human designer. Rotation of a single tile can complete an ordered pattern or destroy it, all without the slightest change in overall density or number of the macro distinctions that are involved. Note particularly the difference between Truchet's designs A and D (Fig. 4). In A the visible vertices are all hexavalent, and the pattern as a whole has a single dominant diagonal direction. In D the directions of the color distinction in each tile find symmetrical balance within their immediate neighbors, and none is globally dominant. The hexavalent vertices have dissociated into pairs of octavalent and quadrivalent ones, without change of the average valence or number of contacts.

The application of a broad wash of color for distinction between the two halves of the tiles is an unnecessary dilution of information, which numerically if not aesthetically requires only a single addition of substructural density. Thus, the distinction between the colored halves of the four tiles in Fig. 17 (a) can be quantitatively represented by the substructures of Fig. 17 (b–d), in which, with no change whatever in the shape of the macro structure or its external relationships, a single line generates an internal monogon or digon or monovalent vertex that has freedom of motion and can be transferred between adjacent triangles—just as the tile maker can put color on either side of the tile or the artisan laying the floor can orient them in whatever direction desired. Nevertheless, the directional balance of direction has the same global combinatorial qualities as does electron spin orientation in quantum theory. If the substructural lines form a digon, as in Fig. 17 (d, e), the structure has a topologically indeterminate quality, for distinction is on curvature alone. Only external bias can determine which of the lines define the boundary between two macro regions and which provide interior 'color' for either. (Incidentally, this definition of color suggests an approach toward a proof of the old four-color map problem, for any four contiguous polygons in a map with trivalent junctions can be divided into groups of four to be distinguished by the assignment of substructural densities of 0, 1, 2 and 3, as in Fig. 17f [10].

By changing the scale of resolution, it is always possible to reach a scale at which any pattern can be analyzed as an assembly of regions of low density both separated and joined by denser regions representable as lines—or the converse with the linear region being that of lower density as a sort of three-dimensional photographic negative. In three or more dimensions, both regions can be—

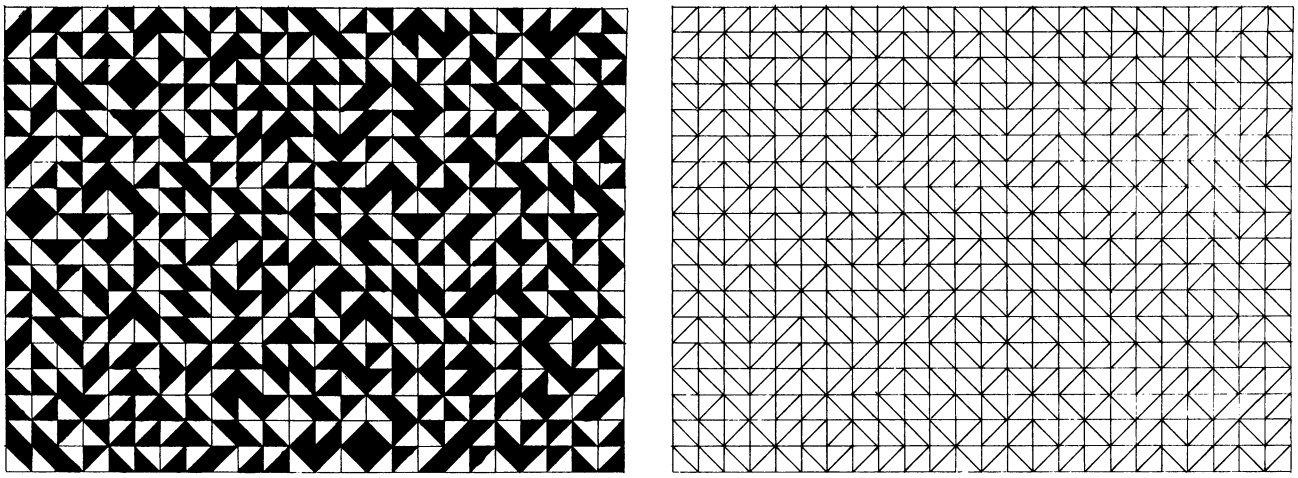

Fig. 13. (a) An array of Truchet tiles oriented at random. (b) The same with color omitted to show the valencies of tile junctions.

indeed, must be—continuous. Lines are the ultimate numerical basis of structure for they both separate and connect, but they are rarely, if ever, the exactly metric or geometric straight lines of Euclidean analysis. Lines can be curved or interwoven in any configuration for only their terminations and junctions matter. A structure is defined numerically by the distribution of the valencies (coordination numbers) of the vertices at which the component lines intersect.

Since each line has two 'ends' and any number of lines may meet at a single vertex, the sum of vertex valencies is exactly equal to twice the number of lines, i.e.

$$2E = \Sigma r V_r \qquad \text{(eq. 1)}$$

where E is the number of lines, and V_r the number of vertices of valence r. It is indifferent to the straightness of the lines or the angles at which they meet. Of course, they are associated with regions of 'empty' space named polygons, polyhedra, Moebius strips, Klein bottles and the like, but these are constrictions, not discontinuities, in the less dense regions.

In a simply connected net in two dimensions, polygons are definable and every line internal to the array is shared by two polygons, while those forming the outer boundary are shared by only one. Equation 1 can then be expanded to:

$$2E = \Sigma r V_r = \Sigma n P_n + E_b + 2E_o \qquad \text{(eq. 2)}$$

where P_n is the number of polygons with n sides, E_b the number of lines constituting the boundary, and E_o the number of lines terminating in a monovalent vertex as in Fig. 17 (c). On the surface of a polyhedron where E_b and E_o are zero, $\Sigma n P_n$ and $\Sigma r V_r$ must be equal [11].

Applying this to Truchet tiles, one can separate the 'color' from the primary shapes. Since all polygons are three-sided and in pairs, $\Sigma n P_n$ is 6T, where T is the number of square tiles. E is $3T + E_b/2$. V is composed of both internal and peripheral ones; in total $V = T + (E_b + 2)/2$.

The distribution of orientations of the tiles therefore affects the distribution of valencies. Despite the internal averaging, every closure is a cellular ribbon the valencies on the two sides of which are adjustable but interdependent. The rotation by 90 degrees of any tile not in contact with the boundary changes the valence of each of its four vertices but does not affect either their sum or that of the 12 vertices in the boundary surrounding the nine tiles with which it shares sides and corners. That boundary, however, has no such freedom: its shape determines the number of lines joining it from inside.

Figure 18 shows all possible permutations of orientation for a single tile surrounded by the eight others with which it shares corners. In each group, there are nine tiles and 18 polygons, formed by 33 lines meeting at 16 vertices. The polygons are all trigons and hence $\Sigma n P_n$ is 54, giving, with E_b 12, the necessary total 66 for 2E.

The lines are separable into those on the unchangeable tile edges (12 on the outer boundary, 12 internally shared) and the 9 diagonal tile divisions which are orientable. The vertex valence sum, 66, is distributable between the outer 12 vertices and the inner four. Maximum segregation to the interior occurs with $2V_8 + 2V_5$ or $2V_7 + 2V_6$, producing the center sum 26 combined with minimum boundary 40, from $8V_3$ and $4V_4$. Conversely, minimum segregation results in the sums of the valencies of the vertices at the center and boundary being 22 and 44, respectively. The three intermediate distributions of valence, of which there

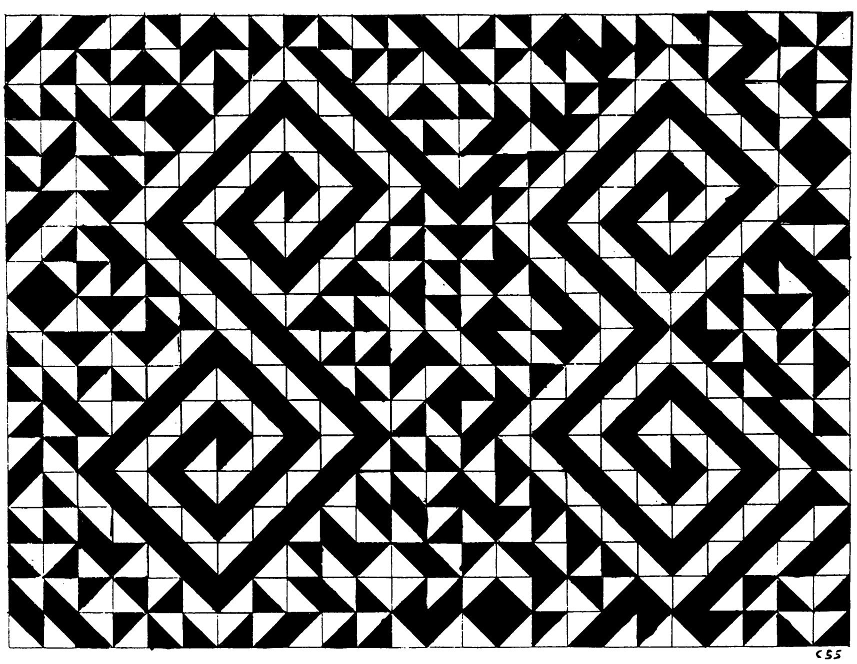

Fig. 14. Spiral patterns produced by pair-wise orientations of tiles within the random field of Fig. 13.

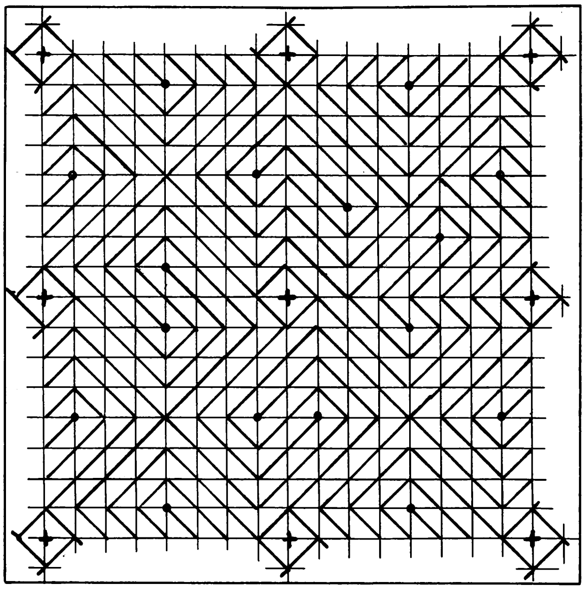

Fig. 15. Truchet tiles arranged to generate structural 'defects' analogous to the lattice dislocations, vacancies and interstitials in solid-state physics. Note the positional freedom of the pentavalent vertices and the quincunxial relations of the quadrivalent and octavalent ones.

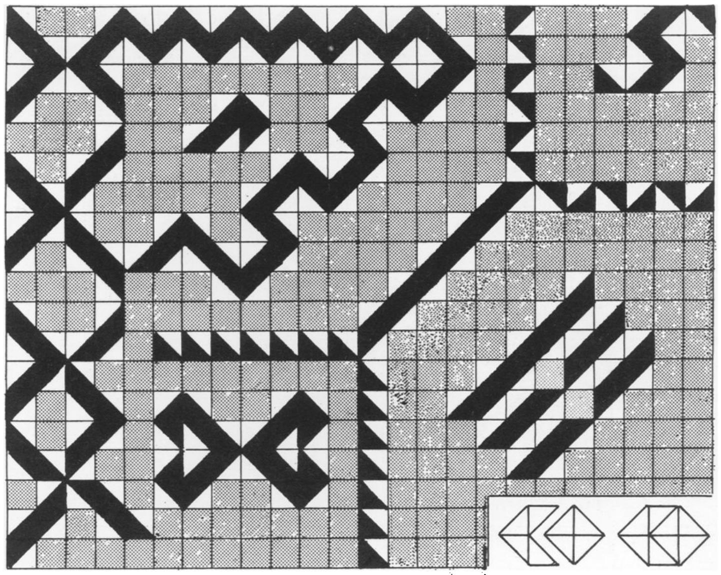

Fig. 16. Various patterns of locally interlocking Truchet tiles embedded in a field within which the tiles have no predetermined orientation and both halves are shown equally gray.

are 34, are based on various combinations of 2, 3, 4 and 5 at the boundary and of 4, 5, 6, 7 and 8 in the center. Some have twofold rotational symmetry; none have four-fold. The 36 patterns shown in the matrix of Fig. 18 were formed by the superposition of the six orientations of the four tiles that share corners with the central tile on the six orientations of those that share sides (i.e. the superposition of A–D on 1–6). Note that the orientation of the central tile has no effect on the division between internal and external valence sums. It is shown as a dotted line: if its two possible orientations were included, the number of combinations would be 72.

It is easy to see how rapidly diversity is developed by the aggregation of the simplest local choices of direction. This is history. Of course, all of these relationships are accurately represented by simple combinatorial mathematics, but to many people a visual display is more meaningful. Artists trying to represent either a scene or an internal vision use the same rules whether or not they are aware of them.

All of the patterns published by Truchet and Douat play with the possibility of rectangular closures on a diagonal or on vertical–horizontal directions and conform to the straitjacket of the crystallographers symmetry rules. The tiles, however, allow the independent exploitation of the two freedoms, those of position and orientation, and a rich diversity of pattern can be formed if such symmetry is disregarded, as in Figs 13 and 14. The introduction of color is thus to be considered as the introduction of substructure without change of visible macrostructure. It involves a unitary separation of density and rarity and has meaning only in relationship to the hierarchy of boundaries. There is no externally perceivable difference between a point and a circle in Flatland; both are impenetrable and circumnavigable. A closed boundary can be filled or empty, and it disappears if the density on the two sides is identical. But even the existence of the boundary itself depends on the directional scale of resolution. The line defines a difference, but at some scale it itself is the difference—difference in density and in direction, which are inseparable.

Of course, there are many other tile shapes with interesting properties, for example the non-periodic tilings described by Martin Gardner [12]. Then, more Truchet-like, are hexagons divided into two tetragons which assemble to give vertices of average valence 4 if uncolored or 5 if colored with a single internal line, and the square tiles of Fig. 19 with eight vertices and three internal polygons the boundaries of which on assembly in any orientation generate nothing but quadrivalent vertices and form continuous lines extending or closing on any desired scale. As with any net of quadrivalent vertices, the first selection of one of two colors for one polygon determines the pattern of contrast throughout.

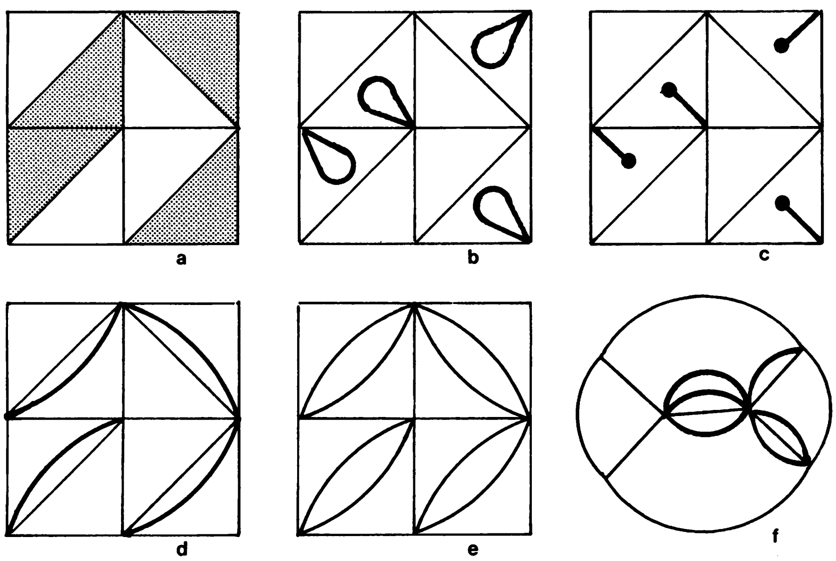

Fig. 17. Diagrams showing how distinctions based on polygon color can be replaced by substructural lines, orientable without change of the external boundaries and their connections.

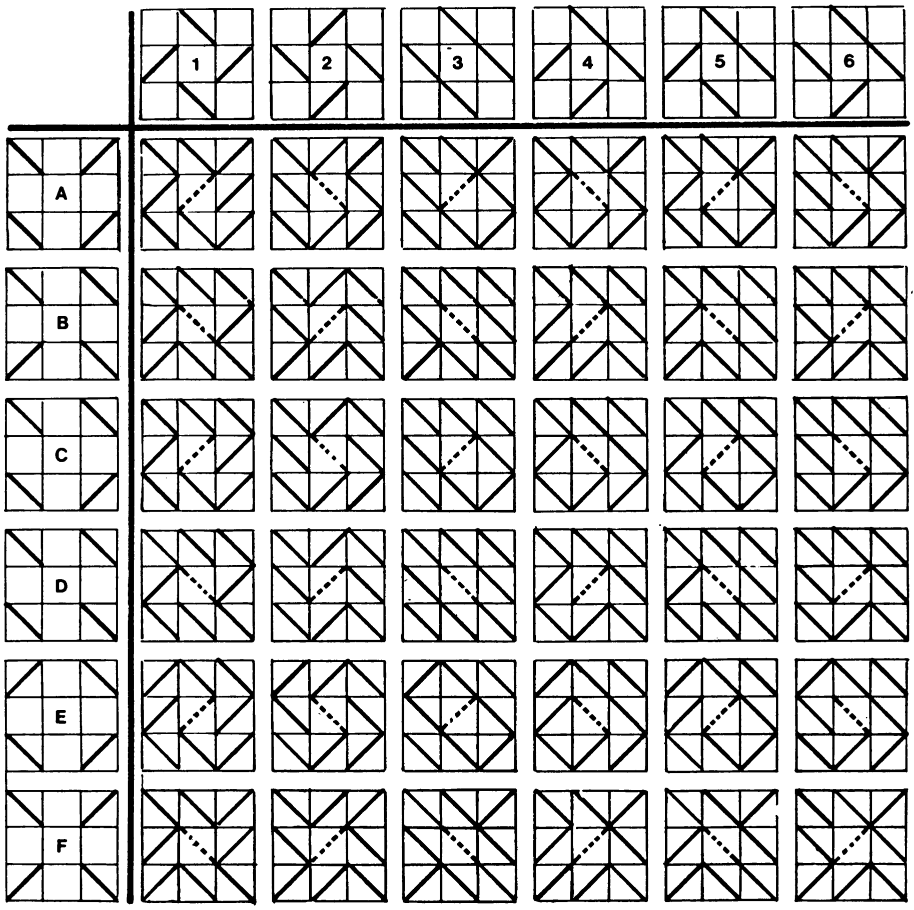

Fig. 18. Groups of nine Truchet tiles, without color, arranged in all possible combinations of their orientations. The orientation of the central tile has no effect on the sum of the valencies of its four corner vertices, but the boundary valencies are strongly influenced by how the tiles are placed.

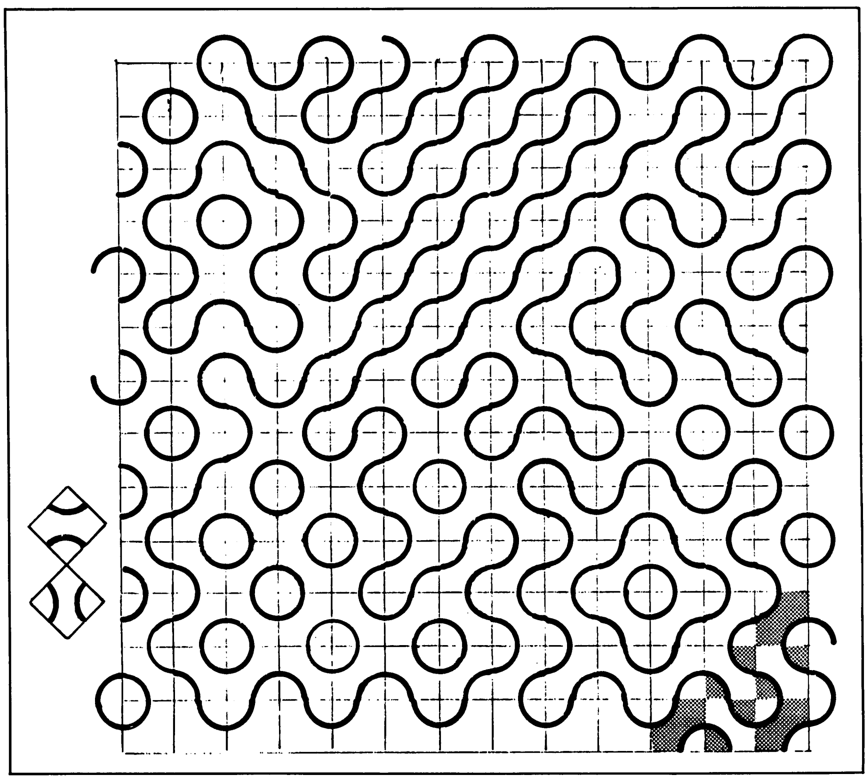

Fig. 19. An orientable square tile with substructural lines that join to produce continuous closures and exclusively quadrivalent internal vertices and to generate a hierarchy of closures and continuity regardless of how they are oriented.

IV. CONCLUSION

Truchet's diagrams, at first glance so symmetrical, illustrate the very basis of asymmetry. Separating three levels of structure, they depend on topology rather than geometry. They have a kind of existential quality in their interior and an almost transcendental relationship to environment. In spirit, they are closer to the corpuscular philosophy of Descartes than to the atomism that replaced it, but they beautifully demonstrate the necessity of both approaches if we are to understand nature's generation of a mimetic hierarchy of boundaries without requiring identity of internal structure.

It must be remembered that the most important factors in relating number to pattern and to the structure of matter itself are topological. As long as the lines representing connections and separations in any diagram retain their identity (even though the interfaces may be fuzzy at the highest resolution) the whole can be deformed in any way without effect on the number and valence of the points of detectable junction. Thus Figs 20 (a) and (b) are topologically identical, despite the fact that the former is a geometrical diagram composed entirely of isosceles right-angle triangles and the latter is an irregular mess of lines of variable curvature and length.

Fig. 21 shows one of the many patterns in which macro-circles can be formed of Truchet tiles. It illustrates the origin of trigonometry in substructural topology and suggests something about the elasticity and plasticity of matter. The basic Truchet triangle can be combined and condensed into nodes of constructed constrictions of space to give three-dimensional nets of any complexity whatsoever.

Note that 'lines' and 'cells' simply mark condensations of fuzziness into one- or three-dimensional features and that such segregation must involve *directional* conflicts and balance: What we term dimensionality is directionality and cannot be detected except as a consequence of the anisotropy of density. The lines and vertices of the mathematician correspond to the limits of resolution of anisotropy; though they ignore material substructure, they depend conceptually upon the reconciliation of directional differences between electron fields as they establish local equilibrium on different scales, with cellular substructural conflicts resolving into cellular macrostructure.

A detailed analysis of Truchet's patterns touches upon the most fundamental questions of the relation between mathematical formalism and the structure of the material world. Separations

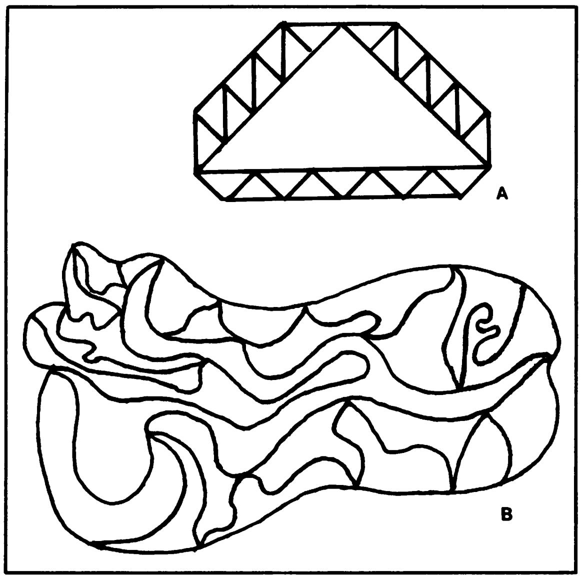

Fig. 20. Two diagrams both composed of 53 lines meeting at 6 trivalent and 22 quadrivalent vertices to form 24 polygons with 3 sides and one polygon with 15 sides, all within a boundary having 16 vertices. Regardless of deformation both diagrams conform, as they must, to equation 2.

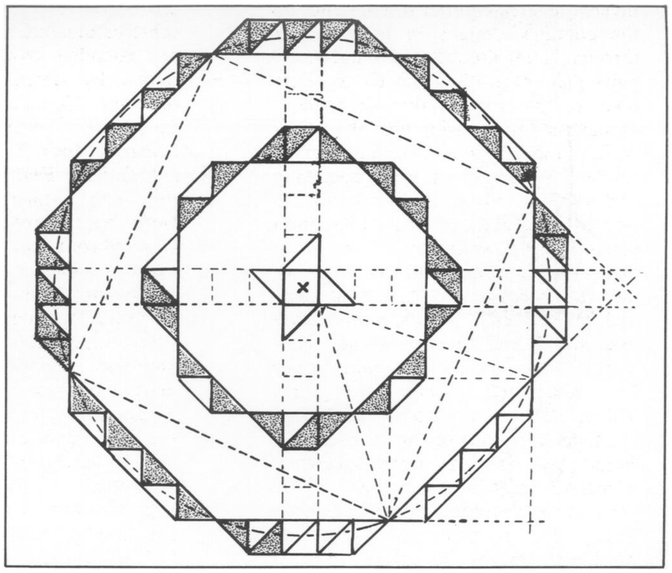

Fig. 21. The transition from a square to a circle within a border composed of Truchet tiles in only two orientations. Many other macroscopically circular combinations are possible, but all have four-fold rotational symmetry of pairs of Truchet tiles in square and rhomboidal juxtaposition or in simple opposition at quadrivalent vertices.

between regions differing in density require that *nothing* be as important as *something* and that large and small cells of both must coexist. The aggregation of unitary choice of directional distinction at interfaces lies at the root of all being and becoming. The success of quantum theory suggests that there is a scale of structural interaction beneath which unitary pulsations of time make directional comparison impossible or at least without external meaning. At intermediate scales in time and space, complex structures form historically and are stabilized by the very impermanence of their components.

There is an interesting difference between human response to the kind of pattern formed by Truchet tiles and to the patterns of musical sounds. Though both are generated in time to explore relations between density and direction, and the appreciation of both involves repeated moiré-like comparison between experienced and remembered structures, the role of human memory is quite different. In viewing visual art, memory can be refreshed and improved by repeated, more detailed examination; time can in a way be reversed. In listening to music, time is unidirectional, and retroactive comparison of detail is achieved only as a feeling of the entire experience. The composer of music directs the attention of the audience to parts in a definite sequence and determines the time allowed for appreciation of each boundary of closure, while the visual artist invites attention to many scales of structure but does not indicate the order in which their relationships are to be appreciated. Short–term and long–term memory capacity is involved quite differently in the different arts.

REFERENCES AND NOTES

1. Ernst Gombrich, *The Sense of Order: A Study of the Psychology of Decorative Art* (Ithaca, NY: Cornell University Press, 1979).
2. Douat, *Methode pour faire une infinité de desseins differents avec des carreaux mipartis de deux couleurs par une ligne diagonale: ou observations du Pere Dominique Douat Religieux Carme de la Province de Toulouse sur un memoire inseré dans l'Histoire de l'Académie Royale des Sciences de Paris l'année 1704, présenté par le Reverend Pere SEBASTIEN TRUCHET religieux du même ordre, Académicien honoraire* (Paris, 1722). Copies can be found in the Bibliothèque Nationale (Paris) and in the Smithsonian Institution Library (Washington, D.C.). The latter copy has a second engraved title page spelling the author's name Doüat and bearing the name of the printer: Jacques Quillau, *Imprimeur Juré de l'Université*. Mark Wilson, P.O. Box 23, West Cornwall, CT 067976, U.S.A., can supply a computer program based on Douat's permutations which dynamically generates spectacular patterns, with and without color, in innumerable combinations.
3. Sebastien Truchet, "Memoir sur les Combinaisons", *Memoires de l'Académie Royale des Sciences*, 363–372 (1704). Plates 12–20 are reproduced as Figs 2–10. This translation has been prepared by Pauline Boucher from the copy in the collection of the Burndy Library, Norwalk, Connecticut.
4. A place named La Motte aux Taurins is on modern maps, just south of the village Coulmiers, 17 km west of Orléans.
5. In the original engravings, the one tile in each group of four pairs that keeps its orientation and whose letter designation defines the column is shaded darker than the other: this is sometimes invisible in the reproduction. The distinction between the orientation of each of the two tiles in the pairs and the horizontal/vertical orientation of the pair itself must, of course, be considered in the layout; however, only the former is of consequence in the complete design. Truchet, in effect, begins his analysis using three colors rather than two (thus obtaining the 64 combinations in table I) and then proceeds to eliminate one of them. The arrangements within the 16 framing squares could have been condensed to form a 3 × 3 matrix around a central tile, the orientation of which has no effect on the boundary.

   In table II, account is taken of this second level of color, reducing the number of pairs by half, though Truchet seems not to have noticed that the distinction in color can duplicate that in pair orientation and that there are actually only 16 distinguishable pairs of bi-colored tiles. Similarly, in table III there are eight, not the stated 10, distinct pairs, for row number 7 is repeated as 10, and 8 as 9. Rows 1 and 2, 3 and 5, 4 and 6, and 7 and 8 are identical except for color reversal. If color is retained but the horizontal/vertical orientation of the pair is disregarded, there are only four forms, those with the two diagonal black/white division lines being either parallel or convergent and oriented either up or down. If all but linear substructure is ignored, there are only two possibilities, those in which the diagonals are either parallel or convergent.

   In the main series of plates, the engraver uses, together with white, only a single density and a single direction of shading throughout. Thus, only eight horizontal pairs and their black/white inversions are required to construct all the complex designs in his plates 1 through 7, but Truchet had to use the 32 pairs of numbers in table III to relate them to the configurations in table I. Douat's notation, using only the letters A, B, C and D to identify orientations of single tiles in any order, is less confusing and leads directly to the 4, 64 and 256 possible permutations of tile orientation assembled in rows having 2, 3 or 4 units.

6. On crystallographic and general symmetry, see particularly A.V. Shubnikov and V.A. Koptsik, *Symmetry in Science and Art*, David Harker, trans. (New York: Plenum Press, 1974) and Arthur Loeb, *Color and Symmetry* (New York: Wiley, 1971). The latter provides an excellent introduction to two-dimensional color symmetry using the concept of rotocenters, which is analogous to the vertex-valencies in equation 2. Loeb gives nearly 100 rectilinear diagrams reproduced in up to six colors. Some of these show Truchet-like patterns in four colors, but the possibilities of cellular substructure/boundary interaction are not extended. Non-periodic tilings are treated by Martin Gardner in "Extraordinary Non-Periodic Tilings That Enrich the Understanding of Tiles" *Scientific American* **236**, No. 1, 110–121 (1977). Peter Pearce and Susan Pearce's *Polyhedra Primer* (New York: Van Nostrand, 1978) contains useful drawings of polyhedra and their stacking; Peter Pearce's *Structure in Nature is a Strategy for Design* (Cambridge, MA: MIT Press, 1978) develops many intriguing structures based on a system of lines meeting at 'universal nodes' as a basis for the division of space by continuous surfaces. Two unusually important books appeared too late to be of use in the preparation of the present article. They are Istvan Hargittai, ed., *Symmetry: Unifying Human Understanding* (New York: Pergamon Press, 1986) and Branko Grünbaum and G.C. Shephard, *Tilings and Patterns* (New York: Freeman, 1986). The former is a fine collection of essays on both the aesthetic and the mathematical aspects of the subject.
7. Keith Critchlow, *Islamic Patterns: An Analytical and Cosmological Approach* (London: Thames and Hudson, 1976).
8. J. Bourgoin, *Arabic Geometrical Pattern and Design* (New York: Dover, 1983).
9. George Bain, *Celtic Art: The Methods of Construction* (Glasgow, 1951; reprint, New York: Dover, 1973).
10. Bourgoin [8].
11. The derivation of these relations and some general discussion of symmetry in art and science are given in C.S. Smith, *A Search for Structure* (Cambridge, MA: MIT Press, 1960), chapters 1 and 14. The possible application of these equations to the map-color problem and to some problems in solid-state physics and thermodynamics is suggested in a paper by C.S. Smith in Nicholas Metropolis, ed., *New Directions in Physics* (Boston: Academic Press, in press, 1987).
12. Gardner [6].

Call for Papers

Art/Science Forum

The Editors of *Leonardo* announce a new regular section of the journal, beginning the first issue of 1988. The Art/Science Forum will feature conference and meeting reports, reviews of group art and technology shows, course descriptions of interdisciplinary university teaching programs, and descriptions of organizations dedicated to the interaction of art, science and technology.

Manuscripts for the Art/Science Forum should be 500 to 1000 words, typed double-spaced, and may include one illustration. The editors reserve the right to edit manuscripts for publication. Only manuscripts written in English and in the journal's basic style and format can be considered.

Please direct inquiries and manuscript submittals to the main editorial office: *Leonardo*, 2020 Milvia, Berkeley, CA 94704, U.S.A.

Leonardo 20-Year Cumulative Subject and Author Indexes

Leonardo announces publication in 1988 of complete subject and author indexes for Volumes 1 through 20 of the journal. The cumulative 20-year indexes will be a significant reference and teaching tool for artists, teachers, librarians, historians and researchers in interdisciplinary fields.

The cumulative indexes will be published separately from the regular journal issues. Further information on availability of the indexes and price (including back issue prices) will be issued shortly to all subscribers.