Introduction to Stabilizer Groups and Hamiltonians

%% Add defnitions of stabilizers %%

> [@rp]: Please refer to [stabg#1] for a preliminary introduction to stabilizer states, and/or also the material in [stabg#2]. It will be enough for you to understand the basics of stabilizer formalism, i.e
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> - How to represent an arbitrary quantum state in the formalism,
> - Realize that any quantum arbitrary state can have a stabilized and a non-stabilized (/logical) subspace
> - How to see the analogue of an unitary transformation in the stabilizer formalism
> - The idea of 'Gottesman-Knill' theorem, which claims that its possible to "Classically simulate the time-evolution of Clifford unitaries on a Stabilizer State", this fact is something that will show up in many different forms throughout. It won't be necessary to dive into concepts specific to error-correction such as code-distance, etc. (*unless you find them interesting ;)*
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> Below I will summarise just a few basics notations and concepts that I will be using in rest of the document.

Notation : Given two pauli operators on n-qubits $A, B \in \mathcal{P}^{\otimes n}$ they can either commute or anti-commute and we will use the notation $\odot$ as $A \odot B = \begin{cases} 0\,;[A,B]=0 \\ 1\,;[A,B]\neq 0 \end{cases}$ to indicate their inter-commutation relation.

Isomorphism of Pauli Operators We know that for single qubits there are three non-trivial pauli operators besides the identity namely $\mathcal{P}^{\otimes 1} = \{I, X, Y, Z\}$. However as will be apparent in the later sections it is often convenient to use the fact that $Y = iZX$, to say that $\mathcal{P}^{\otimes 1}$ is generated by the elements X, Z upto an overall phase (i.e if we ignore the i). This fact allows us to say that for a 1-qubit system where the hilbert space is two-dimensional we have two "independent" pauli operators that are X, Z, where the notion of independence corresponds to the fact that the Z cannot be expressed as a scalar-multiple of X or vice versa. However from X and Z, the last one, Y is a multiple of XZ.

Similarly for a n-qubit system we can construct respective pauli operators by taking n-tensor products of the single qubit pauli operators as $\mathcal{P}^{\otimes n} = \bigotimes_{i=1}^{n} \mathcal{P}^{\otimes 1}$, for example $\mathcal{P}^{\otimes 2} = \{I, X_0, Z_0, Y_0\} \otimes \{I, X_1, Z_1, Y_1\}$. Generalizing the result before we can see that an n qubit system must have $2n$ many 'independent' pauli operators, represented as $\{X_0, Z_0, X_1, Z_1 \dots X_n, Z_n\}$. Observe that they must satisfy the commutation relation

$$X_i \odot Z_j = \delta_{ij}$$

,which we will refer to as the 'canonical commutation relation' and is easy to deduce from the fact that the pauli operators only interact non-trivially when they're acting on the same qubit and thus must commute otherwise (remember $X_i = I_1 \otimes \dots \otimes X_i \otimes \dots I_n$).

We can define a canonical transformation $T : \mathcal{P}^{\otimes n} \to \mathcal{P}^{\otimes n}$ which transforms the independent pauli operators to independent pauli-operators as

$$\forall_{i=1}^{n} \ \ X_i \to \tilde{X}_i \ ; \ Z_i \to \tilde{Z}_i$$

where $\tilde{X}_i \otimes \tilde{Z}_j = \delta_{ij}$. Note that $\tilde{X}_i, \tilde{Z}_i$ are not necessarily single qubits operators unlike X_i, Z_i, but could be any element from $\mathcal{P}^{\otimes n}$ as long as they satisfy the commutation relation. For example we can define the following transformation on $\mathcal{P}^{\otimes 2}$ as

$$Z_1 \to X_1 X_2 \ ; \ Z_2 \to Z_1 Z_2$$
$$X_1 \to Z1 \ ; \ X_2 \to X_1$$

, where the the operators Z_1, Z_2 are mapped to non-local pauli-operators. All such pauli groups generated via canonical-transformations are called "isomorphic" and such transformations are called "Clifford" transformations, something we will come back to later.

Stabilizer States For any n-qubit system, a state ψ is called a stabilizer state if we can find n 'independent' and 'commuting' pauli operators $\mathcal{S} = \{S_1, S_2, \dots S_n\} \subset \mathcal{P}^{\otimes n}$ such that $\forall_{S \in \mathcal{S}} \ S\psi = \psi$

For convenience we can use the notation $\mathcal{S} = S_1 S_2 \dots S_n$ to refer to a stabilizer state stabilized by $\mathcal{S}$.

From the properties of pauli operators discussed above its easy to see the for corresponding to any set of n independent and commuting pauli operators $\mathcal{S}$ we must have a complementary set of n independent and commuting operators $\mathcal{S}^c$ such that for every element of $\mathcal{S}$ there exists exactly one element in $\mathcal{S}^c$ that anti-commutes with it. This relation can be encapsulated by recognizing the paulis in $\mathcal{S}$ and $\mathcal{S}^c$ with $\tilde{\mathcal{Z}}$ and $\tilde{\mathcal{X}}$ respectively, (i.e $S_i \to \tilde{Z}_i$, $S_i^c \to \tilde{X}_i$)

Action of Pauli operators on Stabilizer States Since $\forall_i \ \tilde{Z}_i \tilde{\mathcal{Z}} = \tilde{\mathcal{Z}}$

$$\tilde{Z}_i \tilde{Z}_1 .. \tilde{Z}_j .. - \tilde{Z}_i \dots \tilde{Z}_n \ = \ -\tilde{Z}_1 .. \tilde{Z}_j .. - \tilde{Z}_i \dots \tilde{Z}_n$$
$$\tilde{Z}_j \tilde{Z}_1 .. \tilde{Z}_j .. - \tilde{Z}_i \dots \tilde{Z}_n \ = \ \tilde{Z}_1 .. \tilde{Z}_j .. - \tilde{Z}_i \dots \tilde{Z}_n$$

the only paulis that act non-trivially (i.e not equivalent upto overall phase) on the stabilizer state $\tilde{\mathcal{Z}}$ must be elements of the set $\tilde{\mathcal{X}}$.

The pauli $\tilde{X}_i$ anticommutes exclusively with $\tilde{Z}_i$, thus it should transform the state $\tilde{\mathcal{Z}}$ in manner that measuring the $\tilde{Z}_i$ on this transformed state must yield -1, however measuring any of $\forall_{j \neq i} \ \tilde{Z}_j$ should still yield $+1$, since they commute with $\tilde{X}_i$.

This allows us to represent the transformation induced by $\tilde{X}_i$ as

$$\tilde{X}_i \tilde{Z}_1 .. \tilde{Z}_i \dots \tilde{Z}_n \ = \ \tilde{Z}_1 .. - \tilde{Z}_i \dots \tilde{Z}_n$$

Similarly, transformation induced by any arbitrary operator of the form $\tilde{X} = \tilde{X}_i \tilde{X}_j .. \tilde{X}_k$ can be easily obtained as $\tilde{X}\tilde{\mathcal{Z}} = \forall_i \, (-1)^{\tilde{X} \odot \tilde{Z}_i} \tilde{Z}_i$

Also note that any two stabilizer states on which differ by the action of a non-trivial pauli must by orthogonal to each other, this can be seen from the expectation values of $\tilde{\mathcal{Z}}|\tilde{X}|\tilde{\mathcal{Z}}$ in

$$\begin{aligned} 2\tilde{\mathcal{Z}}|\tilde{X}|\tilde{\mathcal{Z}} &= \tilde{\mathcal{Z}}|\tilde{X}|\tilde{\mathcal{Z}} + \tilde{\mathcal{Z}}|\tilde{X}\tilde{Z}_i|\tilde{\mathcal{Z}} \\ &= \tilde{\mathcal{Z}}|\tilde{X}|\tilde{\mathcal{Z}} - \tilde{\mathcal{Z}}|\tilde{Z}_i\tilde{X}|\tilde{\mathcal{Z}} \\ &= 0 \end{aligned}$$

where $\tilde{Z}_i$ is such that $\tilde{Z}_i \odot \tilde{X} = 1$. (that there will exist a $\tilde{Z}_i$ like this follows from the fact that $\tilde{X}$ is non-trivial) Since there are 2^n such non-trivial operators in $\tilde{\mathcal{X}}$, we can generate 2^n mutually orthogonal stabilizer states by acting $\tilde{\mathcal{Z}}$ with the operators in $\tilde{\mathcal{X}}$ which will be of the form $\{\pm\tilde{Z}_1 .. \pm \tilde{Z}_n\}$. (%%Group theoretically speaking, all the basis vectors are 'cosets' of the stabilizer group $\tilde{\mathcal{Z}}$ under the action of the operators in $\tilde{\mathcal{X}}$ %%) Thus we can use this 'stabilizer-basis' to represent any arbitrary state in the n-qubits hilbert space $\mathcal{H}^{\otimes n}$.

Realize that the mostly commonly used computational basis state $\{\bigotimes_{i=1}^{n} z_i \forall_i z_i \in \{0,1\}$ } is also a 'stabilizer-basis' where we have restricted all the stabilizer operators $\forall_i \tilde{Z}_i$ to be acting on the physical ith qubit imposing $\forall_i \tilde{Z}_i = Z_i$ to yield the basis $\{\pm Z_1 .. \pm Z_n\}$. For an example, the state $\psi = \alpha 0 + \beta 1$ can equivalently represented as $\psi = \alpha{+}Z + \beta{-}Z$. A key thing to note is that in general an arbitrary linear composition of different stabilizer states is not a stabilizer state, in the previous example the state ψ is not stabilized by either $\pm Z$ for arbitrary choices of α, β.

Example: Bell State The bell state or more generally the bell basis is characterized by the stabilizer generators $\mathcal{S}_{\beta_{00}} = \{X_0X_1, Z_0Z_1\}$, the corresponding stabilizer-state is $\beta_{00} = 00 + 11$. It is easy to see that both the stabilizers generators yield an eigenvalue $+1$ on the state β_{00}.

Question is how can we generate the entire basis for the two-qubit hilbert space $\mathbb{C}^2$? We can try investigating this by redefining a set of four stabilizer generators as

$$\mathcal{S}_{\beta_{ij}} = \{-1^i X_0X_1, -1^j Z_0Z_1\}$$

. Where for any i, j the structure of the generators $\mathcal{S}_{\beta_{ij}}$ indicate the measurement outcome we should get upon measuring each individual generator, for instance a state β_{01} stabilized by $\mathcal{S}_{\beta_{01}}$ should satisfy the conditions

$$X_0X_1\beta_{01} = \beta_{01} \; ; \; Z_0Z_1\beta_{01} = -\beta_{01}$$

. But how do we guess that state β_{01} (or any β_{ij})? Remember from the previous example that if we define $\tilde{Z}_0 = X_0X_1, \tilde{Z}_1 = Z_0Z_1$ we can have $\tilde{X}_0 = Z_1, \tilde{X}_1 = X_1$ such that $\forall_{i=0,1} \, \tilde{Z}_i \odot \tilde{X}_j = \delta_{ij}$, we can use this property to write

$$\forall_{ij \in \{0,1\}^{\otimes 2}} \;:\; \beta_{ij} \;=\; \tilde{X}_0^i \tilde{X}_1^j \, \beta_{00}$$

thus obtaining an orthogonal basis for the 2-qubit hilbert space $\mathbb{C}^2$.

Partially Stabilized States Up until now we discuss stabilizer states which were fully stabilized in the sense that we had n independent generators for an n qubit system. However for most practical circumstances we will be interested in states which are only partially stabilized i.e will have $k \,:\, k \leq n$ independent generators instead. On a n qubit system with any state ψ with k independent stabilizer-generators $\mathcal{S} = \{\tilde{Z}_1 \ldots \tilde{Z}_k\} \,:\, |\mathcal{S}| = k \leq n$ will be spanned by 2^{n-k} basis vectors of the form $\{\pm\tilde{Z}_{k+1} \cdots \pm \tilde{Z}_n\}$.

For instance, consider a 3 qubit system, on which the stabilizer generators are defined as $\mathcal{S} = \{X_0Z_1, Z_0X_1Z_2\}$, just like before we can recognize the elements in $\mathcal{S}$ as $\tilde{Z}_0 = X_0Z_1$, $\tilde{Z}_1 = Z_0X_1Z_2$. We can complete the $\tilde{Z}$ pauli-operators by picking $\tilde{Z}_2 = Z_1X_2$ and then pick the $\tilde{X}$ operators as $\forall_{i\in\{0,1,2\}}\, \tilde{X}_i = Z_i$ such that the canonical commutation relations ($\forall_{i,j}\, \tilde{X}_i \odot \tilde{Z}_j = \delta_{ij}$) are satisfied. Any state ψ that is 'partially-stabilized' by $\mathcal{S}$ must be spanned by the linear combination of $\{X_0Z_1, Z_0X_1Z_2, Z_1X_2, X_0Z_1, Z_0X_1Z_2, -Z_1X_2\}$ i.e

$$\psi = \alpha X_0Z_1, Z_0X_1Z_2, Z_1X_2 + \beta X_0Z_1, Z_0X_1Z_2, -Z_1X_2$$

for arbitrary α, β. Realize that even though the system has 3 physical qubits, in essence the state ψ is spanned by just two basis vectors and thus is an element of a hilbert-space $\mathbb{C}^2$ and thus is equivalent to single qubit. In the refs. this fact is referred to as ψ encoding just one 'logical qubit'.

For any system of k stabilizer-generators on an n dimensional system, we can have a logical space of $n-k$ qubits (i.e of dimension 2^{n-k}) , correspondingly there will be $2(n-k)$ independent pauli-operators that commute with paulis in $\mathcal{S}$ we will indicate as $\mathcal{L}(\mathcal{S})$. In the above example $\mathcal{L}(\mathcal{S}) = \{Z_2, Z_1X_2\}$.

Given Stabilizer Group (i.e a commuting set of pauli) we can always construct a stabilizer hamiltonian of the form

$$H_{\mathcal{S}} = -\sum_{S\in\mathcal{S}} S$$

. We can probe into the eigen spectrum of the hamiltonian by using few simple properties. Any state ψ we have $S\psi = \psi$ or $S\psi = -\psi$ since $S \in \mathbb{P}^{\otimes n}$, thus

$$H_{\mathcal{S}}\psi = -\sum_{S\in\mathcal{S}} S\psi = -\sum_{S\in\mathcal{S}} (-1)^{\psi_S}\,\psi$$

where $S\psi = -1^{\psi_S}\psi$ and $\forall_{S\in\mathcal{S}} \,:\, \psi_S \in \{0,1\}$. If we have k generators (i.e $|\mathcal{S}| = k$) , then $H_{\mathcal{S}}$ has $\frac{(k+1)!}{k!1!} = k$ distinct eigenvalues of the form $\epsilon(H_{\mathcal{S}}) = \{k, k-2, \ldots, -k\}$. We can enumerate the eigenvalues simply by choosing the number of stabilizers $S \in \mathcal{S}$ that are in their $+1$ eigenspace i.e we can partition the stabilizer generators as $\mathcal{S} = \mathcal{S}^+ \cup \mathcal{S}^-$ where the set $\mathcal{S}^+$ and $\mathcal{S}^-$ correspond to the set of stabilizers with +1 and -1 eigenspace respectively, the energy configuration can be represented compactly as $\psi_{\mathcal{S}} = (\psi_{S_1}, \psi_{S_2}, \ldots, \psi_{S_k}) \in \{0,1\}^k$. So if $\epsilon(\psi_{\mathcal{S}}) = E$ we know that $|\mathcal{S}^+| = \frac{E+k}{2}$ and $|\mathcal{S}^-| = \frac{-E+k}{2}$ where there are $\frac{k!}{|\mathcal{S}^+!||\mathcal{S}^-|}$ many configurations corresponding to the energy level E. Further since an energy configuration $\psi_{\mathcal{S}}$ only specifies the state of k stabilizer-measurements the corresponding eigenspace has a degeneracy equivalent to the dimension of $(\mathbb{C}^2)^{\otimes n-k}$ dimensional logical space.

From the section #Action of Pauli operators on Stabilizer States we can see that any eigenstate of $H_{\mathcal{S}}$ corresponding to the energy configuration $\psi_{\mathcal{S}}$ would transform under the action of pauli operator P would transform to another state that has the energy configuration is $\tilde{\psi}_{\mathcal{S}} = (\forall_{S_i \in \mathcal{S}}\ (P \odot S_i) \oplus \psi_{S_i})$ where $\oplus$ is addition modulo 2 (aka XOR addition).

Example : For Bell states, $H_\beta = -(X_1 X_2 + Z_1 Z_2)$, realize that the state β_{ij} corresponds to the energy configuration $\psi_\beta = (i, j)$. And has the following eigen-decomposition. $\epsilon(\beta_{00}) = -2$, $\epsilon(\beta_{01}) = \epsilon(\beta_{10}) = 0$,$\epsilon(\beta_{11}) = 2$.

Effect of Perturbation on Stabilizer Hamiltonians

Now that we understand a bit about the structure of the eigenspectrum of stabilizer hamiltonians we can look into how their eigenspaces are modified under the influence of external perturbations. Any generic perturbations represented as the weighted sum of pauli operators as $H_\delta = \lambda_1 E_1 + \lambda_2 E_2 + \ldots$ where the weights λ indicate the strength of the perturbations and are usually of small magnitude. For most physically inspired problems of interest the error operators $E_1, E_2 \ldots$ are single qubit paulis, (for instance the problems in [VMLdel21] and [VMLdel23])

MBQC Ansatz Construction using PGA