Exam of Automatic Controls. January 8^{th}, 2021
Duration: 135 mins

Solve the following problems. Laude is granted if more than 34 points are gained, including the max 3 points from the (optional, upon student's request) oral exam.

1) Calculate the response of the system

$$G(s) = \frac{s+2}{(s+1)^2}$$

to the unitary step. **Pts: 2**

Solution.
The response in the Laplace domain to the unitary step is

$$Y(s) = \frac{s+2}{(s+1)^2} \cdot \frac{1}{s}.$$

Expand $Y(s)$ in partial fractions

$$Y(s) = \frac{s+2}{s(s+1)^2} = \frac{A}{s} + \frac{B}{s+1} + \frac{C}{(s+1)^2},$$

where

$$A = [s\,Y(s)]\,\Big|_{s=0} = 2$$

$$C = \left[(s+1)^2\,Y(s)\right]\Big|_{s=-1} = -1.$$

$$B = \frac{d}{ds}\left[(s+1)^2\,Y(s)\right]\Bigg|_{s=-1} = -A = -2$$

The resulting response in the time domain is

$$y(t) = 2 - 2e^{-t} - te^{-t}.$$

2) Calculate the Laplace transform of the signal

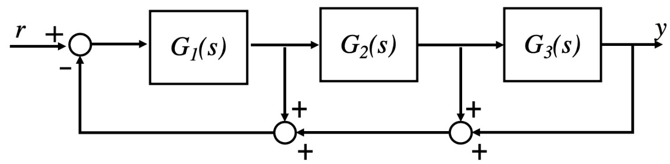
$$y(t) = 4 + 3e^{-3t}\sin 7t + 2(1+t^2)e^{5t}.$$

Pts: 2

Solution.
The Laplace transform $Y(s)$ of $y(t)$ is

$$Y(s) = \frac{4}{s} + \frac{21}{(s+3)^2+49} + \frac{2}{s-5} + \frac{4}{(s-5)^3}.$$

3) Reduce the following block diagram and find the transfer function $G(s) = \frac{Y(s)}{R(s)}$.



Show the intermediate steps. **Pts: 2**

Solution.

$$G(s) = \frac{G_1 G_2 G_3}{1 + G_1 + G_1 G_2 + G_1 G_2 G_3}.$$

4) Solve the problem at the previous point by using the Mason's formula. **Pts: 2**

Solution.
The *transfer function* $G = \dfrac{Y}{R}$ is

$$G = \frac{1}{\Delta} \sum_{i \in \mathcal{P}} P_i \, \Delta_i$$

where

- $\mathcal{P} = \{P_1\} = \{G_1 G_2 G_3\}$,
- the determinant Δ is calculated as

$$\Delta := 1 - \sum_{i \in \mathcal{J}_1} A_i + \sum_{(i,j) \in \mathcal{J}_2} A_i \, A_j - \sum_{(i,j,k) \in \mathcal{J}_3} A_i \, A_j \, A_k + \ldots,$$

  where
  (a) $\mathcal{J}_1 = \{A_1, \ A_2, \ A_3\} = \{-G_1, \ -G_1 G_2, \ -G_1 G_2 G_3\}$,
  (b) $\mathcal{J}_n = \emptyset, \ n > 1$.
- $\Delta_2 = \Delta_1 = 1$.

The resulting transfer function is

$$G(s) = \frac{G_1 G_2 G_3}{1 + G_1 + G_1 G_2 + G_1 G_2 G_3}.$$

5) Calculate the steady-state voltage across the resistor R of a RC circuit when the input voltage is $u(t) = U \sin \omega_0 t$. **Pts: 2**

Solution.
The transfer function of a RC circuit is

$$G(s) = \frac{RCs}{1 + RCs}.$$

The steady-state voltage across the resistor v_r is then

$$v_r(t) = \frac{U \omega_0 RC}{\sqrt{1 + \omega_0^2 R^2 C^2}} \cos\left(\omega_0 t - \arctan(\omega_0 RC)\right).$$

6) Consider the system

$$G(s) = \frac{1}{s+3},$$

in closed loop with a PI controller. Calculate the values of $k_p,\ k_i$ leading to a closed-loop stable system. **Pts: 2**

Solution.
With the controller

$$C(s) = \frac{k_p s + k_i}{s},$$

the closed-loop system is

$$G_{cl}(s) = \frac{k_p s + k_i}{s^2 + (k_p + 3)s + k_i}.$$

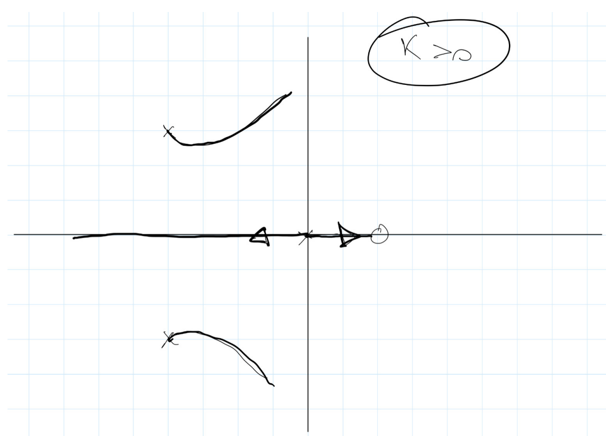
Hence, the closed-loop system is stable for $k_p \geq -3,\ k_i > 0$.

7) Consider the system

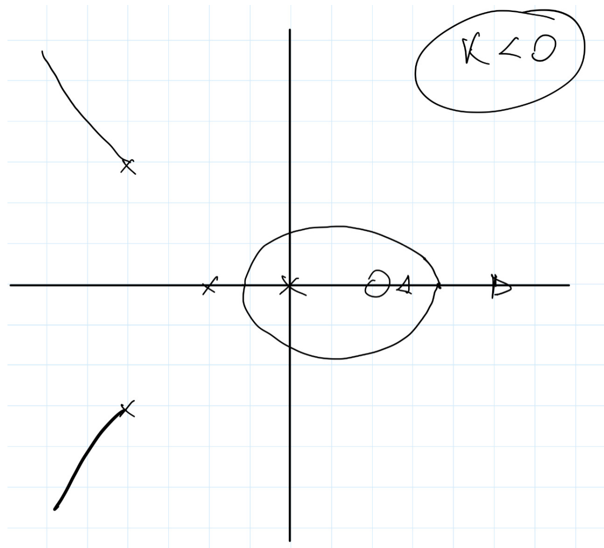
$$G(s) = \frac{10(s-1)}{s(s+1)(s^2+8s+25)},$$

in closed-loop with a proportional controller. Use the root locus to determine whether the closed-loop system is stabilized by a positive or negative feedback gain. **Pts: 2**

Solution.
The direct locus below shows that the system cannot be stabilized with a positive feedback gain.



The inverse locus in the plot below shows that a negative gain stabilizes the system instead.



8) Consider the system

$$G(s) = 10\frac{1+0.5s}{(1+0.2s)(1+s)(1+5s)}.$$

Show the Nyquist diagram of $G(s)$ and explain why the closed-loop system obtained with a static feedback gain K is stable for all $K > 0$. **Pts: 2**

Solution.

- **Starting point.**

$$G_0(s) \simeq G(s)|_{s\simeq 0} = 10 \;\Rightarrow\; \begin{cases} M_0 = 10, \\ \varphi_0 = 0. \end{cases}$$

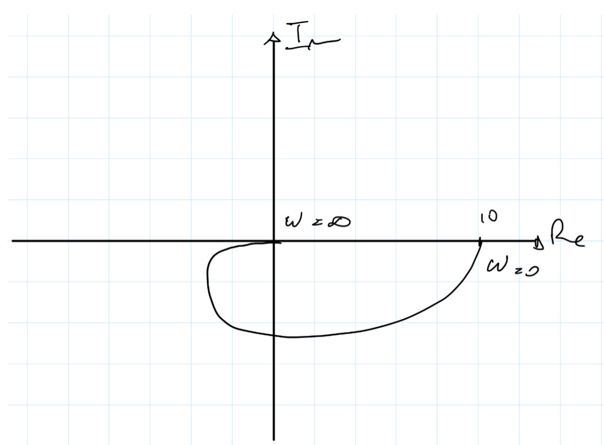
- **Final point.**

$$G_\infty(s) \simeq G(s)|_{s\simeq \infty} = 0 \;\Rightarrow\; \begin{cases} M_\infty = 0, \\ \varphi_\infty = -\pi. \end{cases}$$

  The final phase is approached from above since $\Delta p = -2 + 5 + 1 + 0.2 > 0$.

- **Path direction.** The plot starts clockwise since $\Delta\tau = 0.5 - 0.2 - 1 - 5 < 0$.

The resulting Nyquist plot is



According the Nyquist criterion the system is stabilized by any positive static feedback gain, because the plot cannot encircle the critical point and $G(s)$ does not have any unstable pole.

9) Describe the asymptotic stability criterions for linear continuous-time and discrete-time systems. **Pts: 2**

Solution.
A linear continuous-time system is asymptotically stable if all its poles have negative real part.

A linear discrete-time system is asymptotically stable if all its pole are within the unitary circle.

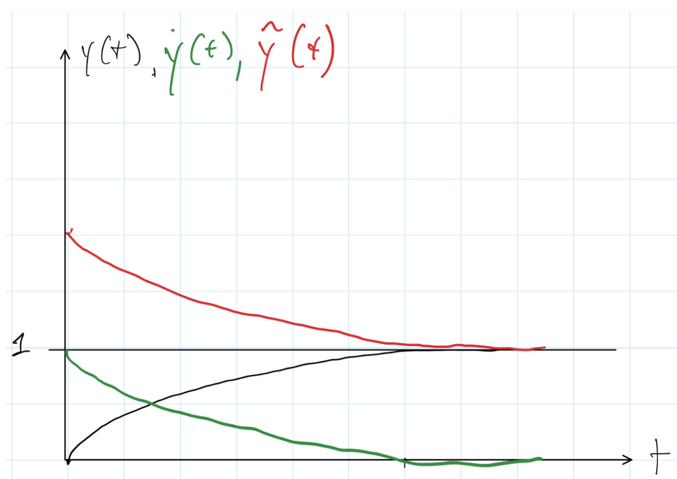
10) Show how the response to a unitary step of the system

$$\tilde{G}(s) = \frac{1+2s}{s+1}$$

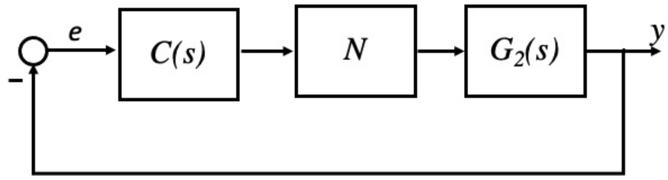
can be graphically obtained from the response of $G(s) = \dfrac{1}{s+1}$. **Pts: 2**

Solution.
We observe that, denoted by $y(t)$ the response to a unitary step of $G(s)$, the response $\tilde{y}(t)$ of $\tilde{G}(s)$ can be written as

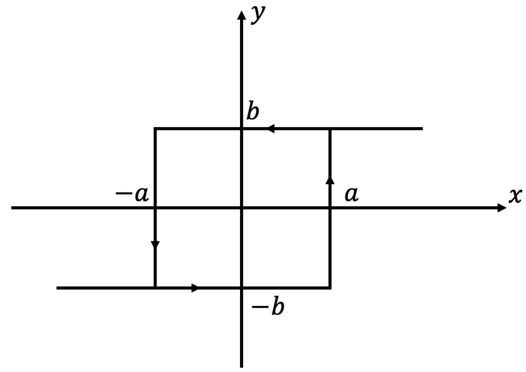
$$\tilde{y}(t) = y(t) + 2\dot{y}(t).$$



11) Consider the system



where the nonlinearity N is the relay with hysteresis



with $a = b = 1$, and the plant $G(s)$ has transfer function

$$G(s) = \frac{5}{(1+s)^2}.$$

Design the regulator $C(s)$ such that permanent oscillations are established with

$$e(t) = 2\cos 0.6t.$$

Pts: 3

Hint. The describing function of the relay with hysteresis is

$$F(X) = \frac{4b}{\pi X}\left[\sqrt{1-\left(\frac{a}{X}\right)^2} - j\frac{a}{X}\right], \quad X > a.$$

Solution.
The following condition must be satisfied

$$C(0.6j)G(0.6j) = -\frac{1}{F(2)}.$$

It follows that

$$C(0.6j) \approx 0.43e^{-j1.54}.$$

Such condition can be satisfied with

$$C(s) = \frac{k}{1+s\tau},$$

provided that

$$|C(0.6j)| = \frac{k}{\sqrt{1+(0.6\tau)^2}} = 0.43, \ \arg C(0.6j) = -\arctan 0.6\tau = -1.54.$$

From the second condition, $\tau = 54.1$ follows and $k = 14$. By plotting the Nyquist diagram of $C(s)G(s)$ and $-\frac{1}{F(X)}$ it can be verified that the oscillations are also stable.

12) Consider the system

$$G(s) = \frac{s(s-5)}{(s+1)(s^2+s+25)}.$$

(a) Plot the asymptotic Bode diagrams of modulus and phase (stepped diagram). **Pts: 1**

Solution.

The initial slope of the modulus diagram is 20 dB/dec. The initial branch cross the line $\omega = 1$ at -14 dB.

The initial phase is $-\frac{\pi}{2}$. The final phase is $-\frac{5}{2}\pi$.

The zero $z_2 = 5$ introduces a phase shift of $-90°$.

The pole $p_1 = -1$ introduces a phase shift of $-90°$.

The conjugate complex poles with $\omega_n = 5,\ \delta = 0.1$ introduce a phase shift of $-180°$.

(b) Plot the linear approximation of the phase diagram. **Pts: 1**

Solution.

The zero $z_2 = 5$ introduces a slope change between $\omega_a^{z_2} = \frac{1}{4.81 \cdot 0.2} = 1.03$ and $\omega_b^{z_2} = \frac{4.81}{0.2} = 24.05$.

The pole $p_1 = -1$ introduces a slope change between $\omega_a^{p_1} = \frac{1}{4.81} = 0.21$ and $\omega_b^{p_1} = 4.81$.

The conjugate complex poles with $\omega_n = 5,\ \delta = 0.1$ introduce a slope change between $\omega_a^{\omega_2} = \frac{5}{4.81^{0.1}} = 4.27$ and $\omega_b^{\omega_2} = 5 \cdot 4.81^{0.1} = 5.85$.

(c) Show the deviation of the exact modulus diagram from the asymptotic at the breaking points and sketch the exact modulus diagrams. **Pts: 1**

Solution.

The deviation of the exact modulus diagram from the asymptotic is

- -3+0.2+0.5=-2.3 dB at $\omega = 1$,
- 3+14-0.1≈17 dB at $\omega = 5$.

13) Consider the system

$$G(s) = \frac{K}{s(s+5)}. \tag{1}$$

By using the root locus, show that the resulting closed-loop system is asymptotically stable for any proportional feedback controller $K > 0$. Find the maximum controller gain such that the CL poles are real. **Pts: 3**

Solution.

The points of the real axis between 0 and -5 belong to the DL. The DL has an asymptote forming angles $\vartheta_{a,0} = \frac{\pi}{2},\ \vartheta_{a,1} = \frac{3}{2}\pi$ intersecting the negative real semi-axis in $\sigma_a = \frac{5}{2}$. The maximum gain K_1 such that roots are real is found as

$$|K_1| = \frac{\prod_{i=1}^{n} \eta_i}{\prod_{i=1}^{m} \lambda_i} = 2.5^2$$

14) Consider the system

$$G(s) = \frac{10}{s^3 + 3s^2 + 3s + 1}.$$

The lag network

$$C(s) = \frac{1 + 2.47s}{1 + 20s}.$$

has been designed to obtain a phase margin of 60° at $\omega = 0.56\ rad/s$. The controller $C(s)$ has to be implemented in discrete time.

(a) Calculate the sampling time of the discrete time version of $C(s)$, such that the resulting phase margin is not smaller than 50°.

(b) Write the equation in the time domain of the discrete-time controller obtained by using the Tustin discretization method with the sampling time calculated at the previous point.

Pts: 3

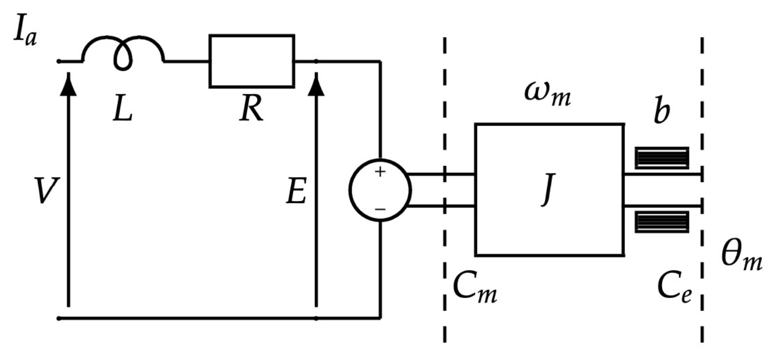
Solution.

(a) Since the ZOH introduces a phase shift of $-\frac{T}{2}\omega$, where T is the sampling time. Then it must be $\frac{T}{2}0.56 < 10\frac{pi}{180}$ It follows $T = 0.62\ s$.

(b) By discretizing $C(S)$ with Tustin and $T = 0.62\ s$, we obtain

$$u(t) = 0.9695u(t-1) + 0.1369e(t) - 0.1064e(t-1).$$

15) Consider the following DC motor



with the following physical parameters

- Armature resistance $R = 0.4\ Ohm$,
- Armature inductance $L = 0.05\ H$,
- Rotor inertia $J = 1\ Kg \cdot m^2$,
- Back EMF constant $K_e = 1.25\ Vs/rad$,
- friction coefficient $b = 0.01\ Nms/rad$.

Design a controller $C(s)$ such that

(a) the tracking error is zero for constant velocity set point and constant load torque, while is not larger than 0.01 for a unitary ramp reference speed. Pts: 2

(b) the friction torque is kept below 1.9 Nm for a step reference speed of 1500 rpm. Pts: 3

Pts: 5

Solution.

(a) The transfer functions from the voltage V and the load torque C_e to the rotating speed ω are

$$G_1(s) = \frac{K_e}{(L+J)s^2 + (RJ + bL)s + bR + K_E^2}, \tag{2}$$

$$G_2(s) = \frac{-(R + Ls)}{(L+J)s^2 + (RJ + bL)s + bR + K_E^2}. \tag{3}$$

By applying the final value theorem

$$e_\infty = \lim_{s\to 0} s \frac{1}{1+CG_1}\frac{1}{s} = \frac{1}{1+C(0)\frac{K_e}{bR+K_e^2}}.$$

It follows that zero steady-state tracking error is achieved in presence of constant speed reference if the controller $C(s)$ has a pole in the origin.
Similarly, in presence of a constant load disturbance,

$$e_\infty = \lim_{s\to 0} s \frac{-G_2}{1+CG_1}\frac{1}{s} = \frac{R}{bR+K_e^2+C(0)K_e}.$$

It follows that a pole in the origin also compensate the effect of a constant load at steady state. Hence the controller can be written as

$$C(s) = \frac{C_1(s)}{s},$$

with $C_1(s)$ to be determined in order to satisfy the specifications on the steady-state error for a ramp reference speed and the specifications at point (b).
Let's choose $C_1(s) = k(1+s\tau)$. The gain k has to be chosen such that, when the speed reference is a unitary ramp, the steady state tracking error is less than 0.01. That is,

$$e_\infty = \lim_{s\to 0} s \frac{1}{1+\frac{C_1(s)}{s}G_1}\frac{1}{s^2} = \frac{bR+K_e^2}{kK_e} \leq 0.01.$$

It follows that

$$k \geq \frac{bR+K_e^2}{0.01K_e} = 125.32.$$

(b) Before designing $C_1(s)$ let's check that the set point is admissible. That's it, that the friction torque C_f at the steady-state speed of 1500 rpm is below the given bound. Since,

$$C_{f|\omega=1500\ rpm} = 0.01 \cdot 157.08 = 1.58\ Nm < 1.9,$$

the reference speed of 1500 rpm generates an admissible friction torque at steady state.
The controller $C_1(s)$ has to be designed such that the speed does not exceed

$$\omega_{max} = \frac{1.9}{0.01} = 190\ rad/s = 1814\ rpm$$

during the transient as well. That is, $C_1(s)$ has to be designed such that the overshoot S is

$$S \leq \frac{1814-1500}{1500} = 21\%.$$

Such maximum overshoot corresponds to a minimum damping coefficient $\delta_{min} = 0.45$.
The time constant τ in $C_1(s)$ should then be chosen such that the closed-loop poles have the prescribed damping $\delta = 0.45$. In order to easily solve such problem, let's observe that, from the DC-motor system equations

$$\begin{cases} L\dot{I}_a &= -RI_a - K_e\,\omega_m + V, \\ J\dot{\omega}_m &= K_eI_a - b\,\omega_m - C_e, \end{cases}$$

it is clear that the time constant $\tau_{el} = \dfrac{R}{L} = 0.125$ of the electrical part is much smaller than the time constant of the mechanical part $\tau_{mech} = \dfrac{J}{b} = 100$. We can then neglect the transient of the electrical part and just consider the model of the mechanical part for the design of $C_1(s)$. By setting $\dot{I}_a = 0$ in

$$L\dot{I}_a = -RI_a - K_e\,\omega_m + V,$$

we obtain in the Laplace domain

$$I_a = \frac{V - K_e\Omega}{sL + R},$$

where Ω is the Laplace transform of the rotating speed ω. It follows that

$$G_3(s) = \frac{\Omega}{V} = \frac{\frac{K_e}{R}}{sJ + \frac{K_e^2}{R} - b}.$$

The time constant τ in $C_1(s) = 125.32(1 + s\tau)$ has to be chosen such that the roots of

$$1 + C(s)G_3(s) = 0$$

are conjugate complex with damping $\delta = 0.45$. Since the roots of

$$1 + 125.32\frac{1 + s\tau}{s}\frac{3.125}{s + 3.896} = 0$$

solve the equation

$$s^2 + (3.896 + 391.64\tau)s + 391.62 = 0,$$

it is enough to impose that

$$\delta = \frac{3.896 + 391.62\tau}{2\omega_n} \geq 0.45,\ \ \omega_n = \sqrt{391.62} = 19.7895.$$

It follows

$$\tau \geq \frac{17.8106 - 3.896}{391.62} = 0.0355.$$