Exam of Automatic Controls. February 22nd, 2021
Duration: 135 mins

Solve the following problems. Laude is granted if more than 34 points are gained, including the max 3 points from the (optional, upon student's request) oral exam.

1) Show the region in the complex plane corresponding to poles with
   (a) overshoot not larger than 10%,
   (b) settling time not larger than 1 second,
   (c) period of the pseudo-oscillations less than 2s.

Pts: 2

Solution.
An overshoot less than 10% is obtained with a damping $\delta \geq 0.6$. This corresponds to angles

$$\varphi \leq \arccos \delta = 53.13°.$$

of the line connecting the poles with the origin. Hence, the poles in this sector have damping coefficient less than 0.6.
A settling time less than one second imposes that

$$T_a = \frac{3}{\delta\omega_n} \leq 1.$$

This is a constraint on the real part of the dominant poles

$$\sigma \geq 3.$$

Hence, the poles at the left of the vertical line $\mathcal{R}e \leq -3$ satisfy such constraint. Such constraint is satisfied by $\omega_n \leq 5$, when $\delta \geq 0.6$. The intersection of this region, with the sector from the constraint on the damping coefficient describes the region of the complex plane where the specs (a) and (b) are satisfied.
Finally, the period of the oscillations is kept below 2s for

$$\delta \geq 0.6,\ \omega_n \geq 5,\ \omega_n \geq \frac{\pi}{\sqrt{1-\delta^2}}.$$

2) Sketch the response of the system

$$\ddot{y}(t) + 0.4\dot{y}(t) + 4y(t) = 10u(t-2)$$

to the unitary step.

Pts: 2

Solution.
The response in the Laplace domain to the unitary step is

$$Y(s) = \frac{10e^{-2s}}{s^2+0.4s+4} \cdot \frac{1}{s}.$$

This is the response to a second order system with damping $\delta = 0.1$, natural frequency $\omega_n = 2$, static gain $k = 2.5$ and input delay $\tau = 2s$.

The step response can be sketched with the help of overshoot, settling time and period of the pseudo oscillations

$$
\begin{aligned}
S\% &= 100\, e^{\frac{-\pi\delta}{\sqrt{1-\delta^2}}} = 72.92\%, \\
T_a &= \frac{3}{\delta\omega_n} = 15, \\
T &= \frac{2\pi}{\omega_n\sqrt{1-\delta^2}} = 3.15s
\end{aligned}
$$

3) Calculate the Laplace transform of the signal

$$y(t) = e^{-5t}\sinh(t-2).$$

Pts: 2

Solution.
Recall that

$$\sinh t = \frac{e^t - e^{-t}}{2}.$$

The Laplace transform $Y(s)$ of $y(t)$ is

$$
\begin{aligned}
Y(s) &= \mathcal{L}\left[e^{-5t}\frac{e^{t-2} - e^{-t+2}}{2}\right] = \frac{e^{-2}}{2}\mathcal{L}\left[e^{-4t}\right] - \frac{e^{2}}{2}\mathcal{L}\left[e^{-6t}\right] \\
&= \frac{e^{-2}}{2(s+4)} - \frac{e^{2}}{2(s+6)}
\end{aligned}
$$

4) Consider the system

$$G(z) = \frac{1}{1-0.5z^{-1}}.$$

Calculate the output response at time 2 to a unitary input step, when $y(0) = 10$. Calculate the steady-state output as well. **Pts: 2**

Solution.
The system, in the time domain, is described by the following differences equation

$$y(t) = 0.5y(t-1) + u(t).$$

It follows that

$$y(1) = 0.5 \cdot 10 + 1 = 6,\ \ y(2) = 0.5 \cdot 6 + 1 = 4.$$

In order to calculate the steady-state output, we apply the final value theorem

$$
\begin{aligned}
\lim_{t\to\infty} y(t) &= \lim_{z\to 1}\left[(1-z^{-1})Y(z)\right] \\
&= \lim_{z\to 1}\frac{1-z^{-1}}{1-0.5z^{-1}}\frac{1}{1-z^{-1}} = 2
\end{aligned}
$$

5) Calculate the settling time of the system

$$G(s) = \frac{160(3+0.2s)(s^2+20s+80^2)}{(15s+3)(0.2s+8)(s^2+6s+160)(s^2+8s+100)}.$$

Pts: 2

Solution.
The setting time can be calculated as

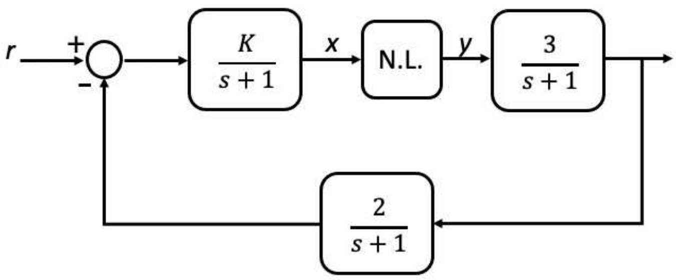
$$T_a = \frac{3}{\sigma_{dom}},$$

where σ_{dom} is the real part of the dominant pole(s) (the closest to the imaginary axis). The real parts of the poles are

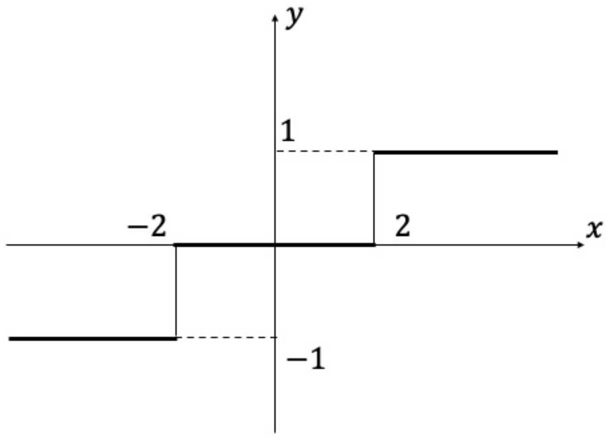
$$G(s) = \frac{160(3+0.2s)(s^2+20s+80^2)}{\underbrace{(15s+3)}_{-0.2}\underbrace{(0.2s+8)}_{-40}\underbrace{(s^2+6s+160)}_{-3}\underbrace{(s^2+8s+100)}_{-4}}.$$

Hence, the settling time is $T_a \approx \frac{3}{0.2} = 15s$

6) Consider the closed-loop system in Figure 1 with $K = 1$. Calculate the reference signals $r_1,\ r_2$



(a) Closed-loop system



(b) Static nonlinearity

Figura 1: Closed-loop system with a static nonlinearity.

corresponding to the equilibrium points $x_1 = 6,\ y_1 = 5$ and $x_2 = 6,\ y_2 = 1$.

Pts: 2

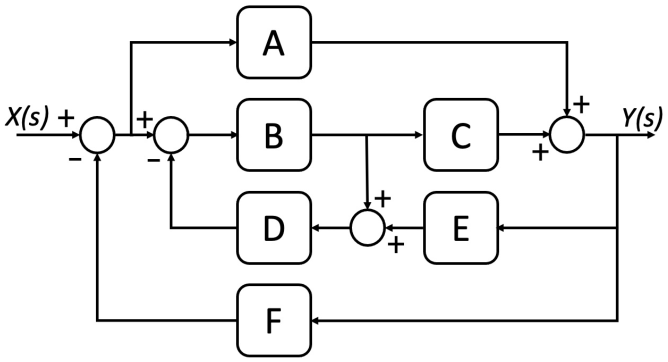
Solution.
We observe that (x_1, y_1) is not an equilibrium point of the closed-loop system because the static nonlinearity imposes that $y = 1,\ \forall\ x \geq 2$.

Denote by $K_1 = 1,\ K_2 = 3,\ K_3 = 2$ the static gains of the three dynamical systems in the control loop. By plugging $x_2 = 6,\ y_2 = 1$ in

$$x = K_1(r - K_2K_3y) \rightarrow r = x + 6y,$$

it follows that $r_2 = 12$

7) Reduce the following block diagram and find the transfer function $G(s) = \frac{Y(s)}{R(s)}$.



Show the intermediate steps. **Pts: 2**

Solution.

$$G(s) = \frac{A(1 + BD) + BC}{1 + AF + BD + BCDE + BCF + AFBD}.$$

8) Solve the problem at the previous point by using the Mason's formula. **Pts: 2**

Solution.
The *transfer function* $G = \dfrac{Y}{R}$ is

$$G = \frac{1}{\mathit{\Delta}} \sum_{i \in \mathcal{P}} P_i \, \mathit{\Delta}_i$$

where

- $\mathcal{P} = \{P_1, P_2\} = \{A, BC\}$,
- the determinant $\mathit{\Delta}$ is calculated as

$$\mathit{\Delta} := 1 - \sum_{i \in \mathcal{J}_1} A_i + \sum_{(i,j) \in \mathcal{J}_2} A_i \, A_j - \sum_{(i,j,k) \in \mathcal{J}_3} A_i \, A_j \, A_k + \dots,$$

  where

  (a) $\mathcal{J}_1 = \{A_1,\ A_2,\ A_3,\ A_4\} = \{-AF, -BD, -BCDE, -BCF\}$,
  (b) $\mathcal{J}_2 = \{(A_1,\ A_2)\} = \{AFBD\}$,
  (c) $\mathcal{J}_n = \emptyset,\ n > 2$,
- $\mathit{\Delta}_1 = 1 + BD$,
- $\mathit{\Delta}_2 = 1$.

The resulting transfer function is

$$G(s) = \frac{A(1 + BD) + BC}{1 + AF + BD + BCDE + BCF + AFBD}.$$

9) Consider the system

$$G(s) = \frac{2(s - 0.2)}{s(s^2 + s + 9)},$$

in closed loop with a proportional controller $C(s) = K$. Use the Routh criterion to find the values of K such that the closed-loop system is asymptotically stable. **Pts: 2**

Solution.
The characteristic equation is

$$s^3 + s^2 + (9 + 2K)s - 0.4K = 0.$$

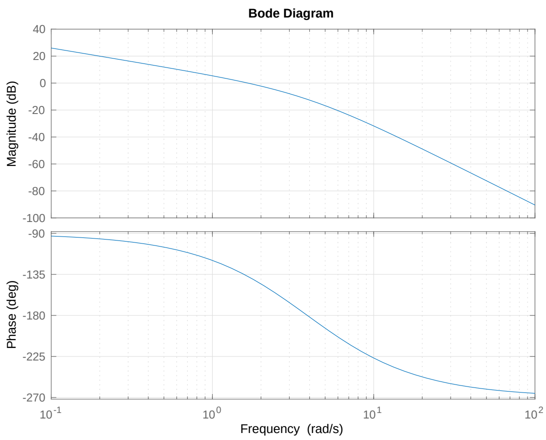
From the corresponding Routh table

$$\begin{array}{c|cc} 3 & 1 & (9+2K) \\ 2 & 1 & -0.4K \\ 1 & 9+2.4K & 0 \\ 0 & -0.4K & \end{array}$$

it follows that

$$-\frac{9}{24} < K < 0.$$

10) Calculate the stability margins and the bandwidth of the closed-loop systems from the following Bode diagrams.



Pts: 2

Solution.
The phase margin is approximatively 44° the gain margin is approximatively $10dB$, while the closed-loop bandwidth is approximatively 1.5 rad/s.

11) Consider the system

$$G(s) = \frac{1}{s^2 + 3s}.$$

Find the positive proportional controller such that the closed-loop system has the minimum settling time. **Pts: 3**

Solution.
As K increases, the two closed-loop poles move toward the vertical asymptote at $\sigma_a = -1.5$ Hence, the minimum settling time is achieved for the maximum gain K^* such that the closed-loop poles are real. K^* is obtained as the gain such that the roots of

$$1 + \frac{K}{s^2 + 3s}$$

are real. That is

$$9 - 4K^* = 0 \rightarrow K^* = 2.25.$$

12) Consider the system

$$G(s) = \frac{-0.33s^2 + s}{s^3 + 6.6s^2 + 72s + 320}.$$

(a) Plot the asymptotic Bode diagrams of modulus and phase (stepped diagram). **Pts: 1**

Solution.

The initial slope of the modulus diagram is 20 dB/dec. The initial branch crosses the line $\omega = 1$ at -50 dB. The slope of the asymptotic diagram is 0 dB/dec between 3 and 5 rad/s, -20 dB/dec between 5 and 8 rad/s and -40 dB/dec for $\omega \geq 8\ rad/s$. The asymptotic diagram is -40 dB at $\omega = 3$, -32 dB at $\omega = 5$ and -28 dB at $\omega = 8$.

The initial and final phases are $\frac{\pi}{2}$ and $-\frac{3}{2}\pi$, respectively.

The zero $z_2 = 3$ introduces a phase shift of $-90°$.

The pole $p_1 = -5$ introduces a phase shift of $-90°$.

The conjugate complex poles with $\omega_n = 8,\ \delta = 0.1$ introduce a phase shift of $-180°$.

(b) Plot the linear approximation of the phase diagram. **Pts: 1**

Solution.

The zero $z_2 = 3$ introduces a slope change between $\omega_a^{z_2} = 0.69$ and $\omega_b^{z_2} = 16.03$.

The pole $p_1 = -5$ introduces a slope change between $\omega_a^{p_1} = 1.04$ and $\omega_b^{p_1} = 24.05$.

The conjugate complex poles with $\omega_n = 8,\ \delta = 0.1$ introduce a slope change between $\omega_a^{\omega_2} = 6.84$ and $\omega_b^{\omega_2} = 9.36$.

(c) Show the deviation of the exact modulus diagram from the asymptotic at the breaking points and sketch the exact modulus diagrams. **Pts: 1**

Solution.

The deviation of the exact modulus diagram from the asymptotic is

- 3-1.7+1=2.3 dB at $\omega = 3$,
- -3++1.5+1≈-0.5 dB at $\omega = 5$,
- 14-1.7+0.5≈12.8 dB at $\omega = 8$.

13) Consider the system

$$G(s) = \frac{1}{s + 3}.$$

Design a PI controller such that its discrete-time version with sampling time $T_s = 0.1s$ provides the CL system with a phase margin of 60° and a bandwidth of 5 rad/s. **Pts: 3**

Solution.

By implementing the PI in discrete-time with sampling time T_s a phase shift $-\frac{T_s}{2}\omega$ is introduced. Hence, the PI should be designed to provide a phase margin

$$M_F = 60° + \frac{T_s}{2}\omega = 60° + \frac{0.1}{2}5 \approx 75°.$$

By setting $L(s) = C(s)G(s)$, where

$$C(s) = K_p + \frac{K_i}{s},$$

the constants $K_p,\ K_i$ have to be chosen such that

$$|L(5j)| = 1,\ \angle L(5j) = 180° - 75° = 105°.$$

Rewrite $L(j\omega)$ in terms of its real and imaginary parts

$$L(j\omega) = \frac{(3K_p - K_i)\omega - (K_p\omega^2 + 3K_i)}{\omega^3 + 9\omega}.$$

By setting the modulus of $L(5j)$ to 1, we obtain

$$850K_p^2 + 34K_i^2 = 28900.$$

By setting $\angle L(5j) = 105°$,

$$K_p = 0.6995K_i$$

follows. By combining the two conditions on $K_p,\ K_i$ we obtain

$$K_p = 5.6063,\ K_i = 8.01.$$

14) Consider the system

$$G(s) = \frac{1}{s+3}. \tag{1}$$

Can a PI controller be designed such that a limit cycle with amplitude $\bar{X} = 2$ and frequency $\bar{\omega} = 5\ rad/s$ is established in presence of a relay with hysteresis with $a = b = 1$? If so, are the oscillations stable?

Hint. The relay with hysteresis has the following describing function

$$F(X) = \frac{4b}{\pi X}\left[\sqrt{1 - \left(\frac{a}{X}\right)^2} - j\frac{a}{X}\right], \quad X > a.$$

Pts: 3

Solution.
The TF of a PI is

$$C(s) = K_p + \frac{K_i}{s}.$$

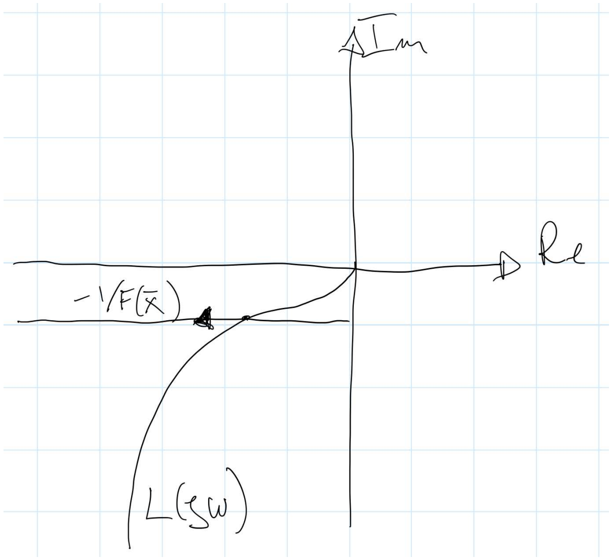
The resulting loop TF is

$$L(s) = C(s)G(s) = \frac{K_p s + K_i}{s(s+3)}.$$

A limit cycle is established with $\bar{X} = 2,\ \bar{\omega} = 5$ if $K_p,\ K_i$ satisfy the pseudo-characteristic equation

$$L(j\bar{\omega}) = -\frac{1}{F(\bar{X})},$$

where $F(X)$ is the DF of the relay with hysteresis.
The qualitative sketch of the Nyquist diagram of $L(s)$, along with the plot of $-\frac{1}{F(X)}$



show that a ***stable*** limit cycle can be established.

By equating the real and imaginary parts of

$$L(j\bar{\omega}) = -\frac{K_i + jK_p\bar{\omega}}{-\bar{\omega}^2 + 3j\bar{\omega}}, \quad -\frac{1}{F(\bar{X})} = -\frac{\pi\bar{X}}{4}\left(\sqrt{1-\left(\frac{1}{\bar{X}}\right)^2} + j\frac{1}{\bar{X}}\right),$$

it follows that

$$K_p = 45.79, \ K_i = -0.15.$$

Simulation results confirm that a limit cycle with $\bar{X} = 2$ and $\bar{\omega} = 5 \ rad/s$ is established.

15) Consider the system

$$G(s) = \frac{1}{s(s+3)(s+5)}.$$

Design a controller such that

(a) the steady-state tracking error is zero for a constant reference and less than 0.1 for a unitary ramp,

(b) the closed-loop bandwidth is not smaller than 5 rad/s,

(c) the damping of the closed-loop poles is not smaller than 0.6.

Pts: 5

Solution.

We start from the static specs. Apply the final value theorem

$$e_\infty = \lim_{s\to 0} s \frac{1}{1 + \dfrac{C(s)}{s(s+3)(s+5)}} Y_{\text{ref}}(s) = \begin{cases} \dfrac{s(s+3)(s+5)}{s(s+3)(s+5)+C(s)}_{|s=0} Y_{\text{ref}} & \text{constant output reference,} \\[2ex] \dfrac{s(s+3)(s+5)}{s(s+3)(s+5)+C(s)}\dfrac{1}{s}_{|s=0} & \text{ramp output reference.} \end{cases}.$$

In order to bound the tracking error for a ramp reference with unitary slope,

$$C(0) \geq \frac{15}{0.1} \Rightarrow C(0) \geq 150$$

must hold. Consider the loop TF

$$G_1(s) = C_1(s)\frac{150}{s(s+3)(s+5)},$$

with $C(s) = 150C_1(s)$. $C_1(s)$ has to be designed to meet the specs on bandwidth and damping of the CL system.

Recall that the bandwidth if the CL system upper bounds the bandwidth of the OL system ($G_1(s)$ in our case). Furthermore, since (*see exam aid sheet*)

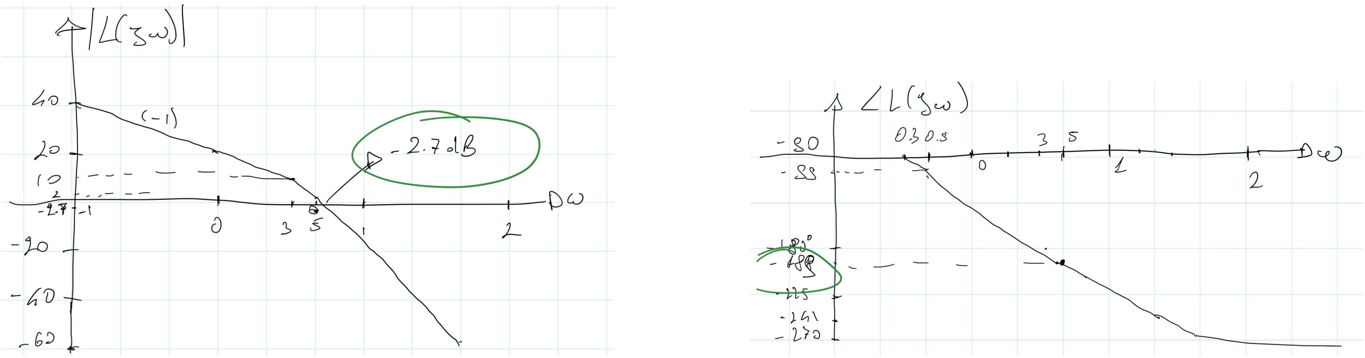
$$\delta \approx \frac{M_F}{100},$$

it follows that $C_1(s)$ must be designed such that

(a) $G_1(s)$ crosses the 0 dB at a frequency not smaller than 5 rad/s,

(b) $G_1(s)$ has a phase margin not smaller than 60°.

Next the Bode diagrams are reported, which can be conveniently used to determined the necessary type (lead/lag) of network.



(a) Modulus (b) Phase

Figura 2: Bode diagrams of $G_1(j\omega)$.

In order to determine the necessary corrections of modulus and phase, the Bode diagrams should be plotted as accurately as possible around $\omega = 5\ rad/s$. We next apply the corrections of the real modulus diagram and plot the linearly approximated phase diagram.

- **Modulus**. The initial slope is $-20\ dB/dec$ and the modulus is 20dB at $\omega = 1$. The modulus at 3 rad/s is then 10 dB. The diagram changes slope to $-40\ dB/dec$ between 3 and 5 rad/s and then to $-60\ dB/dec$ beyond 5 rad/s. Hence, the modulus decreases with slope $-40\ dB/dec$ from 10 dB from 3 to 5 rad/s. By recalling that 3 rad/s is approximately at half decade and 5 rad/s is at 0.7 decade, the asymptotic diagram at 5 rad/s is

$$10\ dB - (0.7 - 0.5) * 40\ dB/decade = 2dB.$$

The exact diagram deviates of $-3\ dB$ from the asymptotic because of the pole in 5 rad/s. Furthermore, the pole in 3 rad/s contributes with an additional deviation that is between -1.5 and -2 dB. We choose -1.7dB. Hence, the exact modulus is -2.7dB at $\omega = 5 rad/s$.

- **Phase** The initial phase is $-90°$. The phase decreases with a slope of approximately $-45°/dec$ between 0.3 and 0.5 (one decade before the poles) and $-90°/dec$ between 0.5 and 5 rad/s. Thus the phase at $\omega = 0.3$ is approximately $-99°$ and at $\omega = 5$ is approximately $-189°$.

From the Bode diagrams, we observe that at $\omega = 5\ rad/s$ $C_1(s)$ has to introduce a gain of $M = 2.7\ dB$ and a phase of $\phi = 69°$. By applying the inversion formulas we obtain

$$\tau_1 = \frac{M - \cos\varphi}{\omega \sin\varphi} = 0.21, \qquad \tau_2 = \frac{\cos\varphi - \frac{1}{M}}{\omega \sin\varphi} = -0.08.$$

Hence the resulting controller is

$$C(s) = 150\frac{1 + 0.21s}{1 - 0.08s}.$$