

Exam of Automatic Controls. June 18th, 2021

Duration: 150 mins

Solve the following problems. Laude is granted if more than 35 points are gained, including the max 3 points from the (optional, upon student's request) oral exam and the 3 points from the assignments.

1) Calculate the Laplace transforms of the following signals

- (a) $x(t) = \cos(3t - 9)$,
- (b) $x(t) = 2(1 + t^2)e^{5t}$

Pts: 2

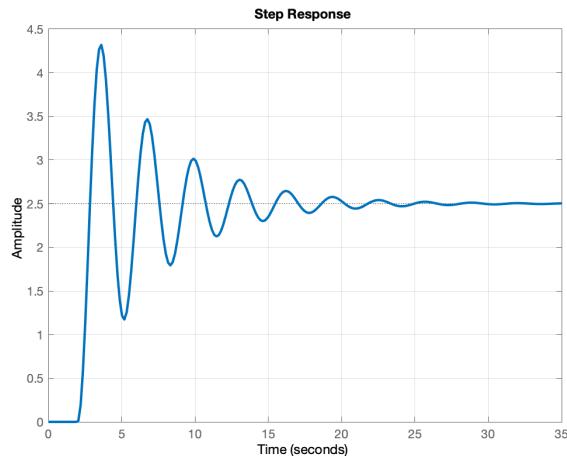
Solution.

$$(a) \mathcal{L}[\cos(3t - 9)] = \mathcal{L}[\cos 3(t - 3)] = e^{-3s} \frac{s}{s^2 + 9},$$

$$(b) \mathcal{L}[2(1 + t^2)e^{5t}] = \frac{2}{s-5} + \frac{4}{(s-5)^3}.$$

2) Derive the transfer function whose response to the unitary step is shown in the figure below.

Pts: 2



Solution.

From the step response the following overshoot, settling time, period of the pseudo oscillations, input delay and steady-state output, respectively, can be measured

$$\begin{aligned}
 S\% &= 72.92\% = 100 e^{\frac{-\pi\delta}{\sqrt{1-\delta^2}}} \Rightarrow \delta = 0.1, \\
 T_a &= 15 = \frac{3}{\delta\omega_n}, \Rightarrow \omega_n = 2, \\
 T &= 3.15s = \frac{2\pi}{\omega_n \sqrt{1-\delta^2}}, \\
 \tau &= 2s, \\
 y_\infty &= 2.5, \Rightarrow k = 2.5.
 \end{aligned}$$

The resulting transfer function is

$$G(s) = \frac{10e^{-2s}}{s^2 + 0.4s + 4}.$$

- 3) The signal $y(t) = A \sin(10t + \varphi)$ is obtained (after the transient) by feeding $u(t) = \sin 10t$ into the filter with TF

$$G(s) = \frac{1}{s + 1}.$$

Calculate A and φ .

Pts: 2

Solution.

At $\omega = 10$ rad/s (one decade after the filter's breaking point $\omega = 1$) the filter G introduces an attenuation $A = -20dB = 0.1$ and a phase shift $\varphi = -90^\circ = -\frac{\pi}{2}$.

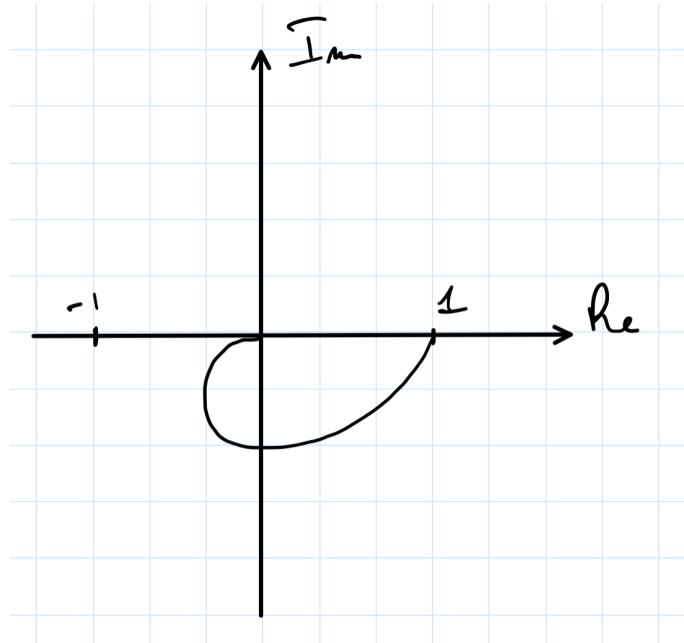
- 4) Show the Nyquist plot of the system

$$G(s) = \frac{1}{s^2 + 1.4s + 1}$$

and discuss the stability of the closed-loop system obtained with a proportional controller. **Pts: 2**

Solution.

The Nyquist plot of $G(s)$ is shown in the figure below.

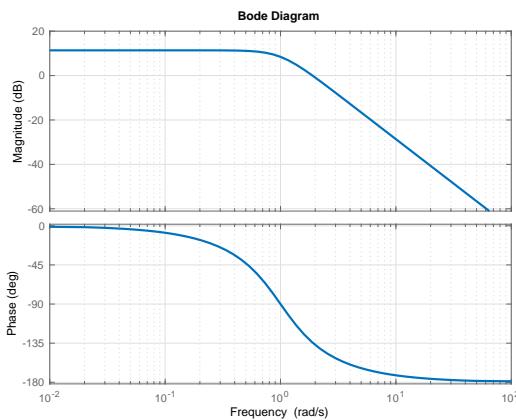


According to the Nyquist criterion, the CL system obtained with a proportional controller is always asymptotically stable.

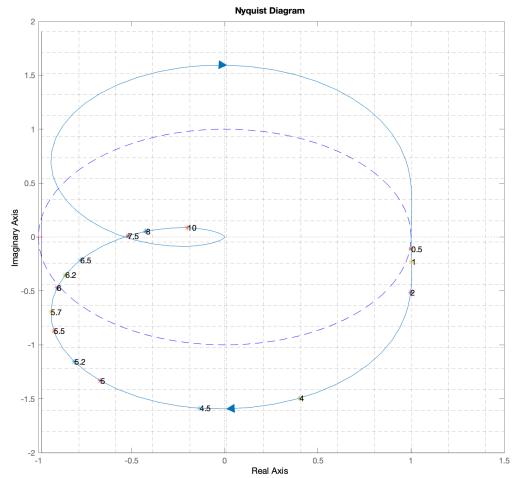
- 5) Calculate the phase and gain margins from the two figures below.

Pts: 2

Solution.



(a) System A



(b) System B

(a) **System A.** $M_\alpha = \infty$, $M_\varphi = 45^\circ$ at $\omega_c = 2$ rad/s

(b) **System B.** $M_\alpha \approx 2$, $M_\varphi \approx 25^\circ$ at $\omega_c \approx 6$ rad/s

6) Analytically calculate the stability margins of the system

$$G(s) = \frac{10}{(s+1)(s+5)(s+10)}$$

Pts: 2

Solution.

Gain margin. The frequency ω_π : $\angle G(j\omega_\pi) = -180^\circ$ is found by solving the equation

$$0 - \angle(1 + j\omega_\pi) - \angle(1 + 5j\omega_\pi) - \angle(1 + 10j\omega_\pi) = -\pi \Rightarrow \omega_\pi = 8.06 \text{ rad/s.}$$

The gain margin is then found as

$$M_\alpha = \frac{1}{|G(j\omega_\pi)|} = 9.88.$$

Phase margin. The frequency ω_c : $|G(j\omega_c)| = 1$ is found by solving the equation

$$\frac{10}{|1 + j\omega_c| |1 + 5j\omega_c| |1 + 10j\omega_c|} = 1 \Rightarrow \omega_c = 1.59 \text{ rad/s.}$$

The phase margin is then found as

$$M_\varphi = \angle G(j\omega_c) + \pi = 95.4^\circ.$$

7) Calculate the impulsive response of the system

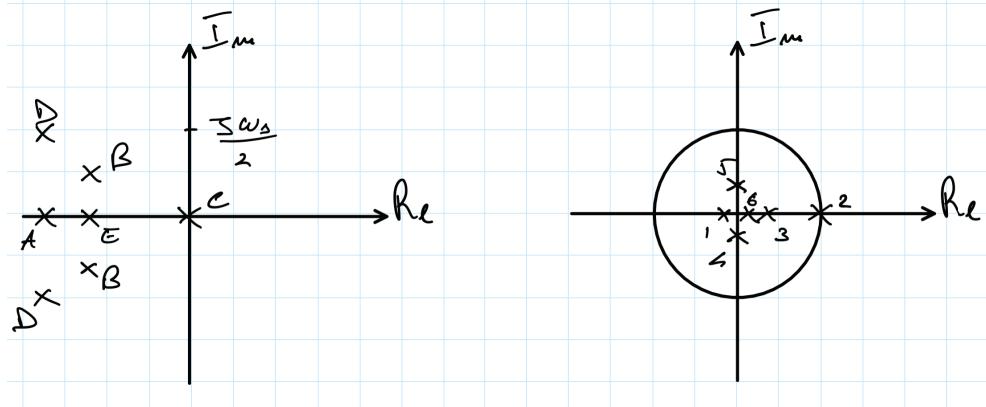
$$G(s) = \frac{1}{s^2(s+1)}$$

Solution.

The impulsive response is the inverse Laplace transform of the TF. Hence,

$$g(t) = \mathcal{L}^{-1}[G(s)] = \mathcal{L}^{-1}\left[\frac{1}{s^2(s+1)}\right] = \mathcal{L}^{-1}\left[-\frac{1}{s} + \frac{1}{s^2} + \frac{1}{s+1}\right] = -1 + t + e^{-t}.$$

8) Match continuous- and discrete-time poles in the figure below.

**Solution.**

- $C \rightarrow 2$,
- $E \rightarrow 3$,
- $A \rightarrow 6$,
- $B \rightarrow 4, 5$,
- $D \rightarrow 1$.

9) Consider the system

$$G(s) = \frac{5(7s-1)}{s(s^2+2s+65)},$$

in closed loop with a proportional controller $C(s) = K$. Use the Routh criterion to find the values of K such that the closed-loop system is asymptotically stable. Pts: 2

Solution.

The characteristic equation is

$$s^3 + 2s^2 + (65 + 35K)s - 5K = 0.$$

From the corresponding Routh table

3	1	$65 + 35K$
2	2	$-5K$
1	$130 + 70K + 5K$	0
0	$-5K$	

it follows that

$$-\frac{130}{75} < K < 0.$$

- 10) Can a limit cycle be established for a system with a pole in the origin and a real pole in closed-loop with a relay? Would such a limit cycle be stable? Motivate your answer. **Pts: 2**

Solution.

In the considered case a limit cycle cannot be established because the Nyquist diagram of the system has phase equal to $-\pi$ and null modulus as $\omega \rightarrow \infty$, while $-\frac{1}{F(X)}$, where $F(X)$ is the describing function of the relay, completely lies on the real negative semi-axis. Hence, they do not intersect.

- 11) Consider the system

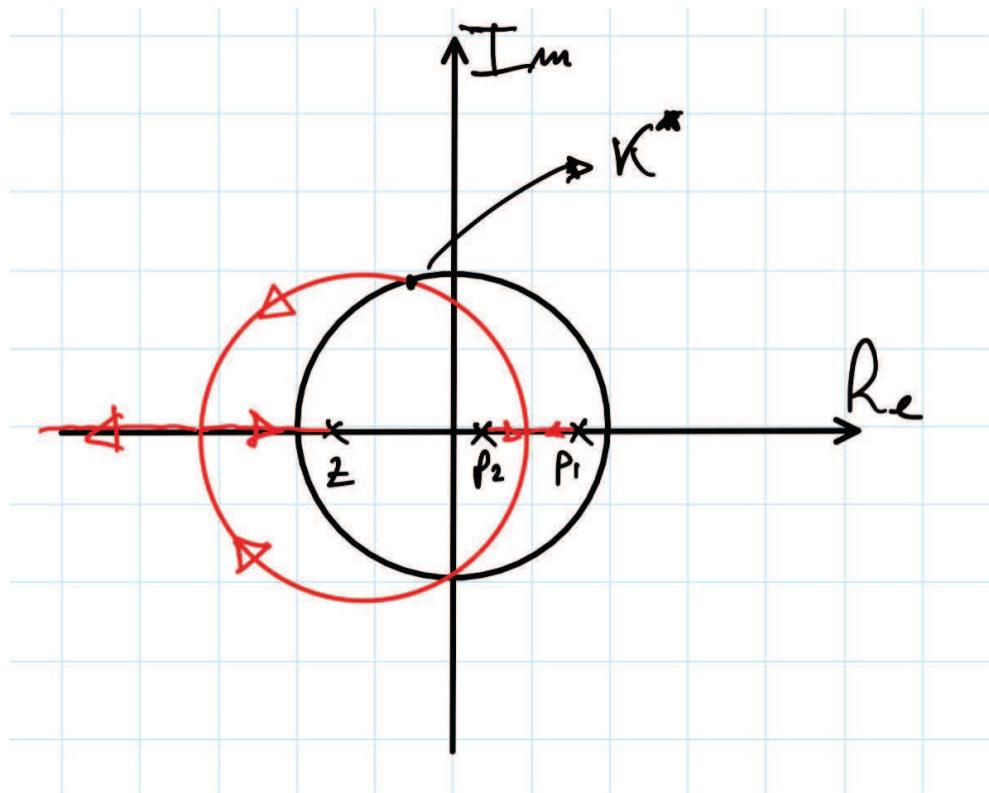
$$G(z) = \frac{z + 0.8}{z^2 - z + 0.1}.$$

Use the root locus to qualitatively study the CL stability.

Pts: 3

Solution.

The system has two real poles $p_1 = 0.89$, $p_2 = 0.11$ and a real zero $z = -0.8$. The resulting (qualitative) root locus is shown in the figure



Hence, the CL system is asymptotically stable for $K < K^*$.

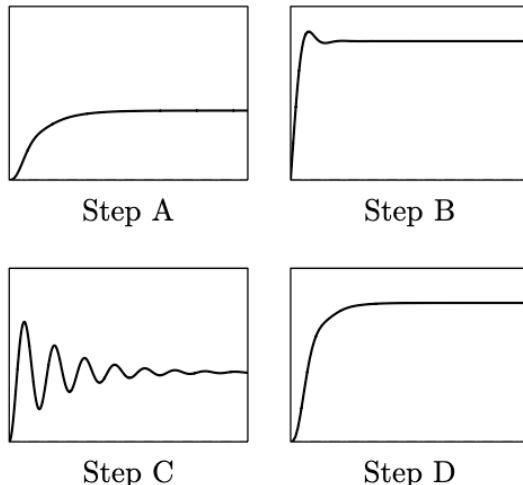
- 12) Match each step response with a transfer function in the table.

Pts: 3

Solution.

Transfer function	Poles	Zeros	$ G(0) $
$G_1(s) = \frac{100}{s^2 + 2s + 100}$	$-1 \pm 10i$		1
$G_2(s) = \frac{1}{s+2}$	-2		$1/2$
$G_3(s) = \frac{10s^2 + 200s + 2000}{(s+10)(s^2 + 10s + 100)}$	$-10, -5 \pm 8.7i$	$-10 \pm 10i$	2
$G_4(s) = \frac{200}{(s^2 + 10s + 100)(s+2)}$	$-2, -5 \pm 8.7i$		1
$G_5(s) = \frac{600}{(s^2 + 10s + 100)(s+3)}$	$-3, -5 \pm 8.7i$		2
$G_6(s) = \frac{400}{(s^2 - 10s + 100)(s+2)}$	$-2, 5 \pm 8.7i$		2

(a) Transfer functions



(b) Step responses. Homologous axes have the same scaling

- $G_1 - C$ because G_1 is poorly damped, which gives and oscillatory behavior,
 - G_2 can be excluded for two reasons. First, this is the only TF with a static gain $\frac{1}{2}$. Second, the only initial slope compatible is the one in Step B. Nevertheless, the presence of an overshoot rules out this possibility.
 - $G_3 - B$ because G_3 has the shortest rise time and some overshoot due to the complex poles. Furthermore the static gain is 2.
 - $G_4 - A$ because the pole in -2 is dominant, which gives a slower response than G_3 and G_5 . Furthermore, the static gain is 1.
 - $G_5 - D$ because the dominant pole is in -3 which is slower than the one in G_3 but faster than the one in G_2 . Furthermore, the static gain is 2.
 - G_6 is excluded because of instability.

13) A magnetic floater, modeled by the TF

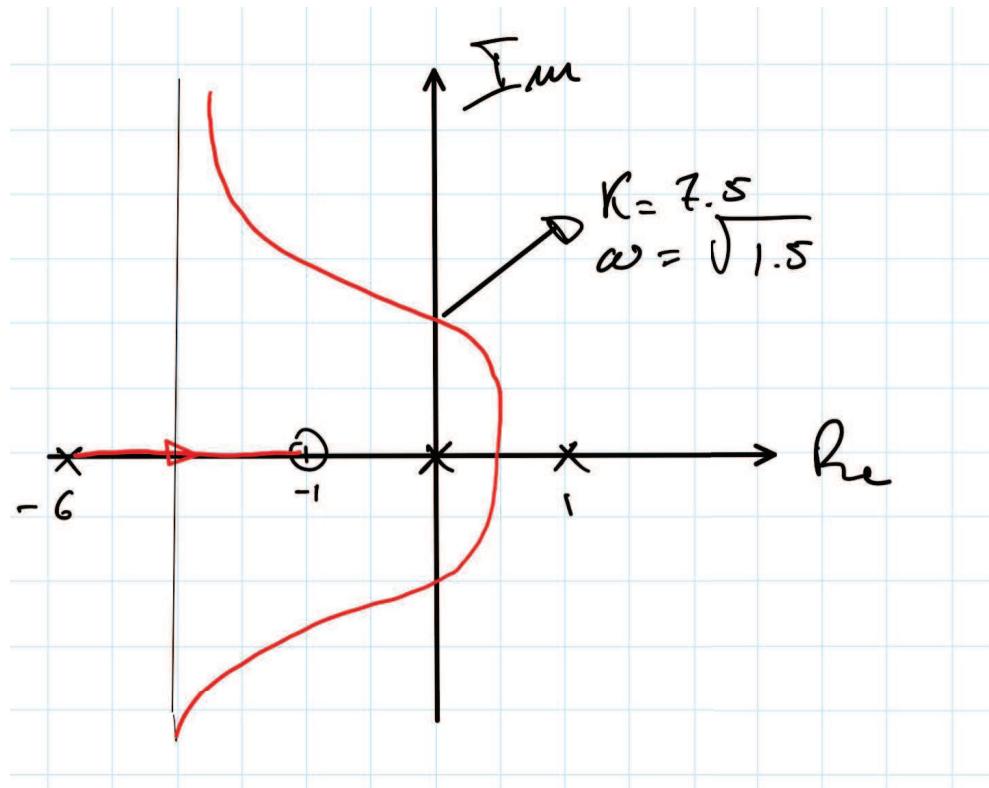
$$G(s) = \frac{s+1}{s(s-1)(s+6)},$$

has to be controlled with a proportional controller K . Use the root locus to discuss the stability and the behavior of the closed-loop system. **Pts: 3**

Solution.

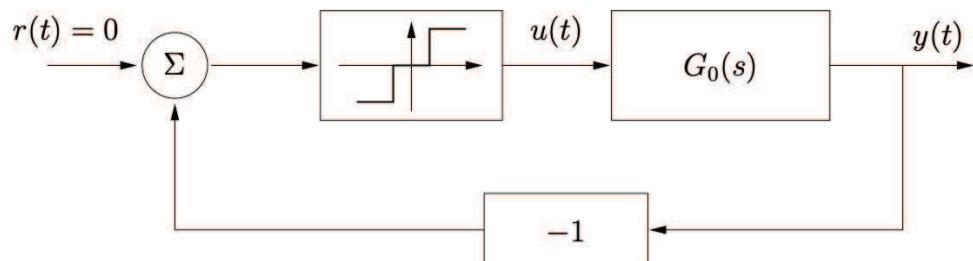
- **Start/end points.** The locus starts from 0, 1 and -6 and ends in -1 and goes to infinity along $3 - 1 = 2$ asymptotes.
- **Asymptotes.** The two asymptotes forms angles of $\pm\frac{\pi}{2}$ with the real axis. They intersect the real axis in $\frac{1}{2}[0 + 1 + (-6) - (-1)] = -2$
- the intervals $[-6, -1]$ and $[0, 1]$ of the real axis belong to the locus.
- **Intersections with the real axis** can be found by setting $s = j\omega$ in the characteristic equation and solving it. This leads to $K = 0$, $\omega = 0$ and $K = 7.5$, $\omega = \pm\sqrt{\frac{3}{2}}$.

The resulting locus is reported in the figure below.



From the root locus, we observe that the unstable open-loop system needs to be stabilized. This happens for $K > 7.5$. For $K > 7.5$ the CL system is stable and oscillates. For increasing values of K the CL system becomes faster and more damped. The dominant time constant is always larger than $\frac{1}{2}s$. For increasing values of K the damping starts decreasing.

- 14) The temperature control system in the figure



has to be designed with $G_0(s) = \frac{1}{s(s+1)^2}$, such that the amplitude of the oscillations is 2.5. Calculate the width D of the dead zone, the output level H of the relay and the frequency of the oscillations.

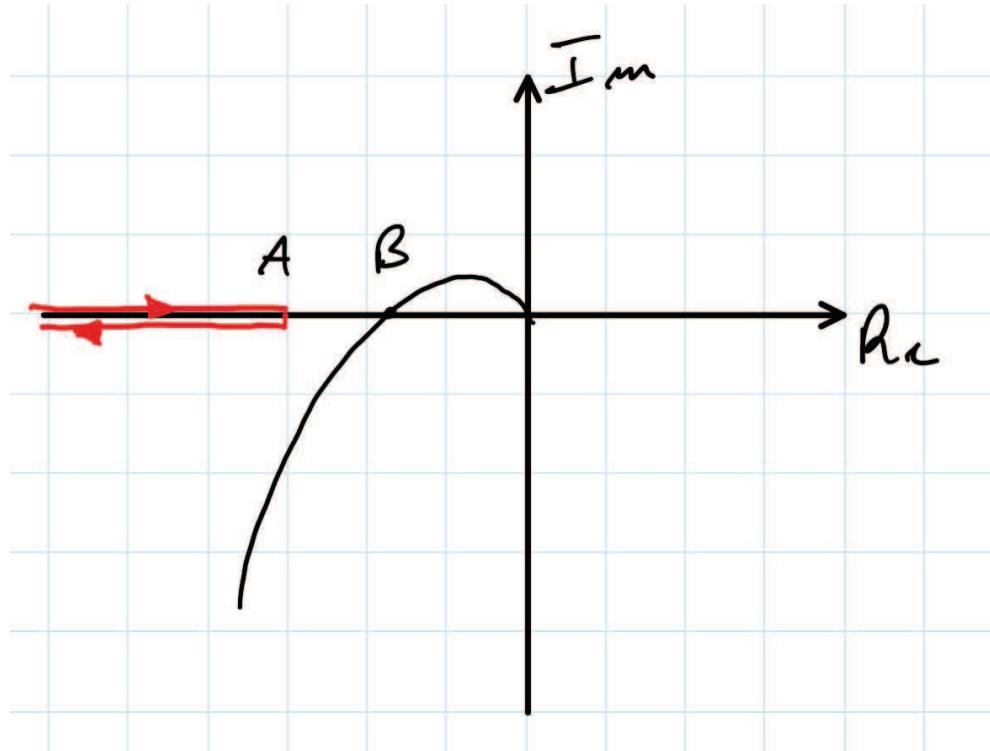
The describing function for a relay with dead zone is

$$F(X) = \frac{4H}{\pi X} \sqrt{1 - \frac{D^2}{X^2}}.$$

Pts: 3

Solution.

The Nyquist diagram of $G_0(s)$ and $-\frac{1}{F(X)}$ are qualitatively shown in the figure below.

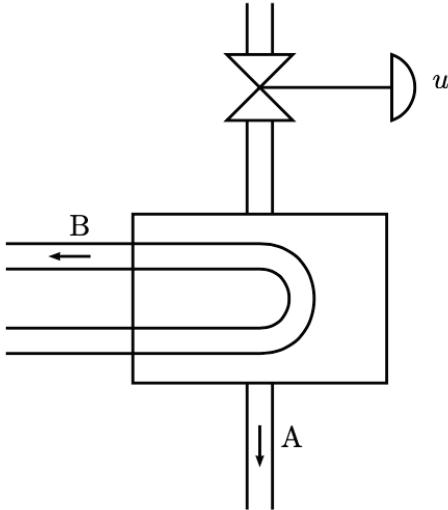


$-\frac{1}{F(X)}$ changes direction in A, which corresponds to the point X where $F(X)$ attains its maximum. By differentiating F and setting to zero, we obtain $X_A = D\sqrt{2}$ and $A = -\frac{\pi D}{2H}$.

Since $\angle G_0(j\omega) = -\pi$ for $\omega = 1$ and $|G_0(1j)| = \frac{1}{2}$, then $B = -\frac{1}{2}$.

The limit condition for the existence of a limit cycle is that $B = A$. Since the amplitude of the oscillations must be 2.5, then $\sqrt{2}D = 2.5 \Rightarrow D = \frac{5\sqrt{2}}{4}$ and $H = \frac{5\pi\sqrt{2}}{8}$. The frequency of the oscillations is $\omega = 1$.

- 15) The outflow temperature T of the liquid B can be controlled in a heat exchanger by controlling the flow of the liquid A with a valve commanded by the signal u . Measurements have been made using a sinusoidal input u and the modulus and phase shift measured at different frequencies are reported in the Table
- Make a Bode diagram for the heat exchanger.
 - What is the largest critical frequency ω_c that can be achieved with a proportional controller and a phase margin of at least 50° ?



(a) The heat exchanger

Frequency [rad/s]	Gain	Phase shift
0.05	1.37	-67°
0.1	0.80	-106°
0.2	0.34	-153°
0.3	0.18	-185°
0.4	0.11	-210°

(b) Measured modulus and phase shift

- (c) Design a controller that doubles the CL system bandwidth compared to the previous point and keeps the same phase margin.

Pts: 5

Solution.

- (a) From the approximate Bode diagrams $\omega_c \approx 0.08$, $M_\varphi \approx 90^\circ$, $M_\alpha \approx 5$.
 (b) From the approximate Bode diagrams, the phase diagrams crosses -130° at approximately 0.15 rad/s. This is the maximum critical frequency that can be obtained with a proportional gain. Such a gain can be computed as

$$K = \frac{1}{|G(0.15j)|} \approx \frac{1}{0.525} = 1.9.$$

- (c) We observe that at the critical frequency $\omega_c = 0.30$ rad/s (twice as larger as the critical frequency at the previous point) the phase margin is almost 0. Hence, a lead network

$$C(s) = K \frac{1 + s\tau}{1 + s\alpha\tau}$$

should be designed providing a phase shift of 50° . Hence,

$$M_\varphi = \arcsin \frac{1 - \alpha}{1 + \alpha} = 50^\circ \Rightarrow \alpha = 0.13.$$

Such a phase lead is attained at $\omega_c = \frac{1}{\tau\sqrt{\alpha}} \Rightarrow \tau = \frac{1}{0.30\sqrt{0.13}} = 9.47$.

Finally, the controller gain K is chosen by imposing $\omega_c = 0.30$ rad/s. That is,

$$1 = K|C(j\omega_c)||G(j\omega_c)| \approx K \frac{1}{\sqrt{\alpha}} \frac{1}{M_\alpha}, \Rightarrow K \approx 1.83.$$

The resulting controller is

$$C(s) = 1.83 \frac{1 + 9.47s}{1 + 1.23s}.$$