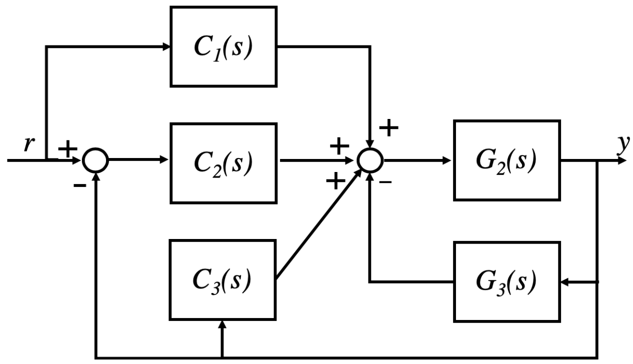
Exam of Automatic Controls. June 8$^{\text{th}}$, 2020

Duration: 1hr

Solve the following problems.

a1) Reduce the following block diagram and find the transfer function $G(s) = \frac{Y(s)}{R(s)}$. Show the intermediate steps. **Pts: 0.1**



Solution. $G(s) = \dfrac{C_1G_2 + C_2G_2}{1+G_2G_3-G_2C_3+C_2G_2}$

a2) Verify your result with the Mason formula. **Pts: 0.05**

Solution.

The *transfer function* $G = \dfrac{Y}{R}$ is

$$G = \frac{1}{\Delta} \sum_{i \in \mathcal{P}} P_i \, \Delta_i$$

where

- $\mathcal{P} = \{P_1,\ P_2\} = \{C_1G_2,\ C_2G_2\}$,
- the determinant Δ is calculated as

$$\Delta := 1 - \sum_{i \in \mathcal{J}_1} A_i + \sum_{(i,j) \in \mathcal{J}_2} A_i \, A_j - \sum_{(i,j,k) \in \mathcal{J}_3} A_i \, A_j \, A_k + \dots,$$

  where

  (a) $\mathcal{J}_1 = \{A_1,\ A_2,\ A_3\} = \{-G_2G_3,\ C_3G_2,\ -C_2G_2\}$,

  (b) $\mathcal{J}_n = \emptyset,\ n > 1$.

- $\Delta_2 = \Delta_1 = 1$.

b) Consider the system

$$G(s) = \frac{K}{s(s+3)}, \quad K = 1. \tag{1}$$

b1) By using the root locus, show that the resulting closed-loop system is asymptotically stable for any proportional feedback controller. Find the maximum controller gain such that the CL poles are real. **Pts: 0.1**

Solution.

The points of the real axis between 0 and -3 belong to the DL. The DL has an asymptote forming angles $\vartheta_{a,0} = \frac{\pi}{2}$, $\vartheta_{a,1} = \frac{3}{2}\pi$ intersecting the negative real semi-axis in $\sigma_a = \frac{3}{2}$. The maximum gain K_1 such that roots are real is found as

$$|K_1| = \frac{\prod_{i=1}^{n} \eta_i}{\prod_{i=1}^{m} \lambda_i} = 1.5^2$$

b2) Use the root locus to design a controller and choose a constant reference such that permanent oscillation are established, with frequency $\omega = 5.19$ rad/s and unitary amplitude. Explain which drawing rules you have used and how. **Pts: 0.2**

Solution.

In order to establish permanent oscillations, the locus asymptotes need to intersect the imaginary axis. Hence, since the asymptotes form with the real negative semi-axis angles

$$\vartheta_{a,\nu} = \begin{cases} \dfrac{(2\,\nu+1)\,\pi}{n-m} & K_1 > 0, \\ & \qquad\qquad \nu = 0,\,1,\,\ldots\,,\,n{-}m{-}1, \\ \dfrac{2\,\nu\,\pi}{n-m} & K_1 > 0. \end{cases}$$

one more pole is needed in order for the asymptotes to form angles $\vartheta_{a,0} = \frac{\pi}{3}$, $\vartheta_{a,1} = \pi, \vartheta_{a,2} = \frac{5}{3}\pi$, thus intersecting the imaginary axis.

Furthermore, in order for the points $P_{1,2} = \pm 5.19j$ to belong to the DL, the following must hold for some integer ν,

$$(2\nu+1)\pi = -\frac{\pi}{2} - \arctan\frac{5.19}{3} - \beta,$$

where β is the angle formed by the pole to be added with $P_{1,2}$. It follows that

$$\beta = \frac{\pi}{6}.$$

By simple geometric arguments it follows that the controller pole should be in $p = -9$.

In order to determine the controller gain $\bar{K}$, the centroid theorem can be used:

$$\underbrace{-9-3+0}_{K=0} = \underbrace{-x}_{K=\bar{K}}.$$

$x = 12$ follows. Hence,

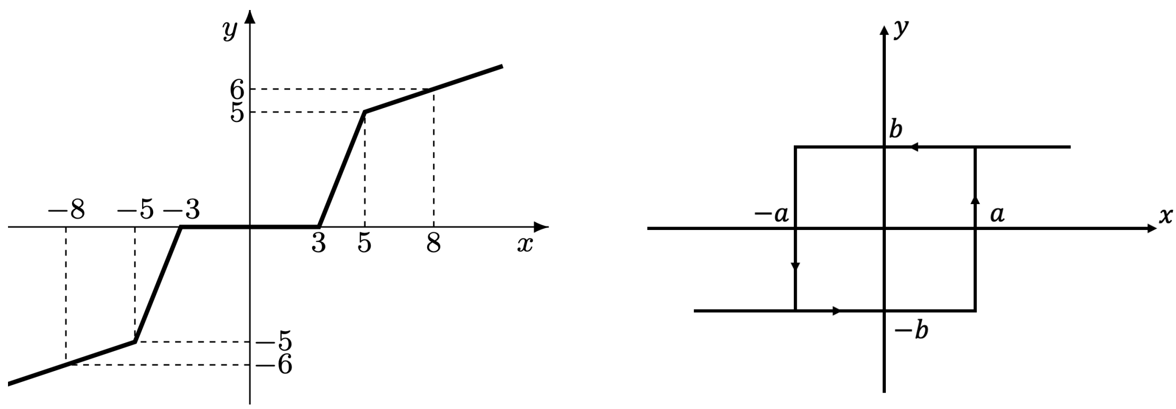
$$\bar{K} = (12\cdot 9)\cdot(12\cdot 3) = 324.$$

The controller

$$C(s) = \frac{324}{s+9},$$

together with a unitary reference signal, enables permanent oscillations with amplitude $X = 1$ and frequency $\omega = 5.19$ rad/s

b3) Consider the following two nonlinearities.



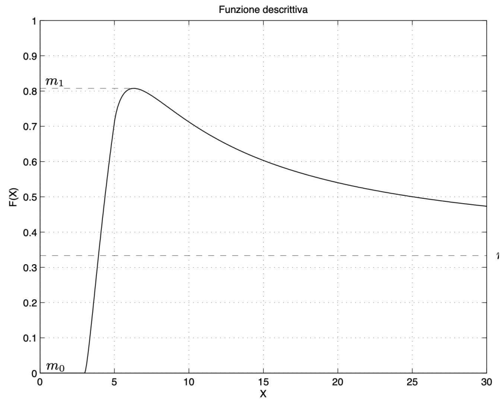
(a) Threshold with partial saturation

(b) Relay with hysteresis

Figura 1: Nonlinearities

Show that the threshold with the partial saturation in closed loop with $G(s)$ cannot enable a limit cycle, while the relay with hysteresis can. Rigorously motivate your answer with plots. **Pts: 0.1**

Hint The qualitative plot of the describing function of the threshold with partial saturation in Figure 1a is



while the relay with hysteresis in Figure 1b has the following describing function

$$F(X) = \frac{4b}{\pi X}\left[\sqrt{1-\left(\frac{a}{X}\right)^2} - j\frac{a}{X}\right], \quad X > a.$$

Solution.
Since $\underline{/G(j\omega)} = -\pi$, $\omega \to \infty$, while $-\frac{1}{F_1(X)}$, with $F_1(X)$ the DF of the threshold with partial saturation, completely lies on the negative semi-axis, a limit cycle cannot be established with such nonlinearity.
$-\frac{1}{F_2(X)}$, with $F_2(X)$ the DF of the relay with hysteresis, is an horizontal line with negative y-coordinate, there will be a limit cycle.

b4) Design the parameters of the relay with hysteresis (i.e., a and b), in order to obtain a limit cycle with the same amplitude and frequency as in problem b2). **Pts: 0.1**
Solution.
The relay with hysteresis enable a limit cycle if $\bar{X} = 1$, $\bar{\omega} = 5.19$ solve the pseudo-characteristic equation

$$G(j\bar{\omega}) = -\frac{1}{F_2(\bar{X})},$$

where $F_2(X)$ is the DF of the relay with hysteresis. By equating the real and imaginary parts of

$$G(j\bar{\omega}) = -\frac{1+3j}{\bar{\omega}^3+9\bar{\omega}}, \quad -\frac{1}{F_2(\bar{X})} = -\frac{\pi\bar{X}}{4b}\left(\sqrt{1-\left(\frac{a}{\bar{X}}\right)^2} + j\frac{a}{\bar{X}}\right),$$

it follows that

$$a = \frac{3}{\sqrt{10}} = 0.95, \ b = \frac{\pi a}{4}\frac{\bar{\omega}^3+9\bar{\omega}}{3} = 46.32.$$

b5) So far you have shown two ways (i.e., solutions of problems b2) and b4)) to obtain permanent oscillations for the system $G(s)$.
Assume $K \in \left[1, \bar{K}\right]$ in (1). Which of the two methods does still lead to permanent oscillations? Motivate your answer. **Pts: 0.1**

Solution.

If the system gain K is larger than 1, then the two poles of the CL system move to the rhp and the oscillations diverge.

The limit cycle established with the relay with hysteresis, instead, is stable. Hence, if the plant gain increases, there will still a solution of the pseudo-characteristic equation.

c) Consider the system

$$G(s) = \frac{s(s-5)}{(s+1)(s^2+s+25)}.$$

c1) Plot the asymptotic Bode diagrams of modulus and phase (stepped diagram). **Pts: 0.15**

Solution.

The initial slope of the modulus diagram is 20 dB/dec. The initial branch cross the line $\omega = 1$ at -14 dB.

The initial phase is $-\frac{\pi}{2}$. The final phase is $-\frac{5}{2}\pi$.

The zero $z_2 = 5$ introduces a phase shift of $-90°$.

The pole $p_1 = -1$ introduces a phase shift of $-90°$.

The conjugate complex poles with $\omega_n = 5,\ \delta = 0.1$ introduce a phase shift of $-180°$.

c2) Plot the linear approximation of the phase diagram. **Pts: 0.05**

Solution.

The zero $z_2 = 5$ introduces a slope change between $\omega_a^{z_2} = \frac{1}{4.81 \cdot 0.2} = 1.03$ and $\omega_b^{z_2} = \frac{4.81}{0.2} = 24.05$.

The pole $p_1 = -1$ introduces a slope change between $\omega_a^{p_1} = \frac{1}{4.81} = 0.21$ and $\omega_b^{p_1} = 4.81$.

The conjugate complex poles with $\omega_n = 5,\ \delta = 0.1$ introduce a slope change between $\omega_a^{\omega_2} = \frac{5}{4.81^{0.1}} = 4.27$ and $\omega_b^{\omega_2} = 5 \cdot 4.81^{0.1} = 5.85$.

c3) Show the deviation of the exact modulus diagram from the asymptotic at the breaking points and sketch the exact modulus diagrams. **Pts: 0.05**

Solution.

The deviation of the exact modulus diagram from the asymptotic is

- -3+0.2+0.5=-2.3 dB at $\omega = 1$,
- 3+14-0.1≈17 dB at $\omega = 5$.