Automatic Controls exam aids

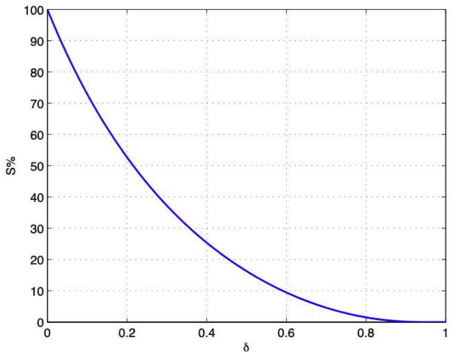
- **Second order systems**
  - Step response

$$y(t) \quad = 1 - \frac{e^{-\delta\omega_n t}}{\sqrt{1-\delta^2}} \; \sin(\omega\, t + \varphi)$$

$$\begin{aligned} \omega &:= \omega_n \sqrt{1-\delta^2} \\ \sigma &:= \delta\omega_n \\ \varphi &:= \arccos\delta = \arctan\frac{\sqrt{1-\delta^2}}{\delta} \end{aligned}$$

  - Overshoot

$$S\% = 100\,(y_{\max} - 1) = 100\; e^{\frac{-\pi\delta}{\sqrt{1-\delta^2}}}$$



  - 5% settling time

$$T_a = \frac{3}{\delta\omega_n}$$

  - Phase margin and damping

$$\delta \approx \frac{M_F}{100}.$$

- **Mason's formula**. Given a block scheme, with input X and output Y, the *transfer function* $G = \dfrac{Y}{X}$ is

$$G = \frac{1}{\mathit{\Delta}} \sum_{i\in\mathcal{P}} P_i\, \mathit{\Delta}_i$$

where the determinant is calculated as

$$\mathit{\Delta} := 1 - \sum_{i\in\mathcal{J}_1} A_i + \sum_{(i,j)\in\mathcal{J}_2} A_i\, A_j - \sum_{(i,j,k)\in\mathcal{J}_3} A_i\, A_j\, A_k + \dots ,$$

- **Root locus.**
  - Asymptotes. Intersection with the real axis at

$$\sigma_a = \frac{1}{n-m}\left(\sum_{i=1}^{m} p_i - \sum_{i=1}^{n} z_i\right)$$

  Angles with the real axis:

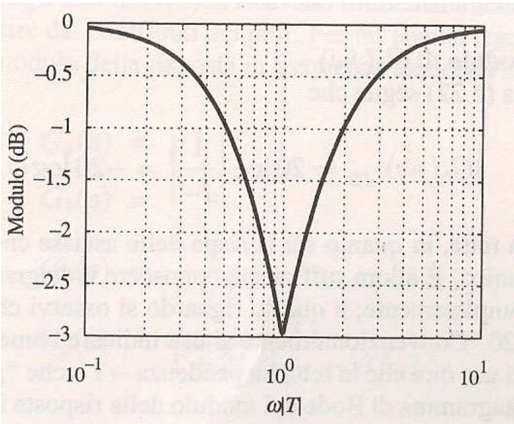
$$\vartheta_{a,\nu} = \begin{cases} \dfrac{(2\,\nu+1)\,\pi}{n-m} & K_1 > 0, \\[2ex] \dfrac{2\,\nu\,\pi}{n-m} & K_1 > 0. \end{cases} \qquad \nu = 0,\, 1,\, \dots,\, n{-}m{-}1$$

  - Points of the locus. A point s belongs to the locus if

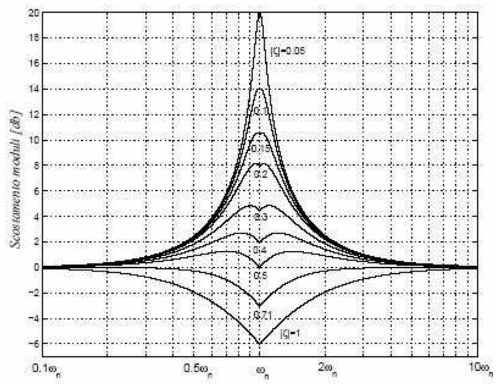
$$(2\,\nu+1)\,\pi + \sum_{j=1}^{m} \arg(s - z_j) - \sum_{j=i}^{n} \arg(s - p_j)\ ,$$

  for some integer ν. This point corresponds to a value of the gain K_1 given by $|K_1| = \dfrac{\prod_{i=1}^{n} \eta_i}{\prod_{i=1}^{m} \lambda_i}$,

- **Bode diagrams.**
  - Phases asymptotic Bode diagrams.
    - First order terms: $\pm 90^o$ phase shift. Slope change between $\omega_a = \frac{1}{4.81\tau}$, $\omega_b = \frac{4.81}{\tau}$.
    - Second order terms: $\pm 180^o$ phase shift. Slope change between $\omega_a = \frac{\omega_n}{4.81^\delta}$, $\omega_b = 4.81^\delta \omega_n$.
  - Deviations of the exact modulus Bode diagrams from the asymptotic ones.



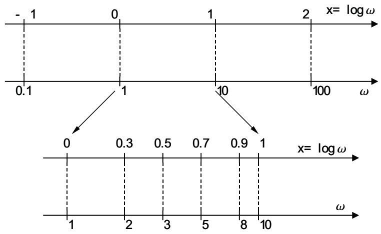
(a) First order terms.



(b) Second order terms.

Figure 1: Error between exact and asymptotic module Bode diagrams.

  - 10-base logarithmic scale



- **Discrete-time systems.**
  - Discretization methods
    - Backward differences

$$D(z) = D(s)|_{s=\frac{1-z^{-1}}{T}} \, ,$$

    - Forward differences

$$D(z) = D(s)|_{s=\frac{z-1}{T}} \, ,$$

    - Bilinear transformation

$$D(z) = D(s)|_{s=\frac{2}{T}\frac{1-z^{-1}}{1+z^{-1}}}$$

    - Poles/zeros matching

$$(s+a) \quad \rightarrow \quad (1-e^{-aT}z^{-1})$$

$$(s+a \pm jb) \quad \rightarrow \quad (1-2e^{-aT}\cos bTz^{-1} + e^{-2aT}z^{-2}).$$

- **Lead/lag networks**. Inversion formulas

$$\tau_1 = \frac{M-\cos\varphi}{\omega \sin\varphi}, \qquad \tau_2 = \frac{\cos\varphi - \frac{1}{M}}{\omega\sin\varphi}$$