Exam of Automatic Controls. July 13th, 2020
Duration: 135 mins

Solve the following problems. Laude is granted if more than 34 points are gained, including the max 3 points from the (optional, upon student's request) oral exam.

1) Provide an asymptotic stability criterion for a discrete time system described by the transfer function
$$G(z) = \frac{b_0\, z^m + b_1\, z^{m-1} + \cdots + b_m}{z^n + a_1\, z^{n-1} + \cdots + a_n}$$
Pts: 2

Solution.
The roots of the polynomials at the denominator must be within the unitary circle.

2) State the Nyquist criterion for OL unstable systems. **Pts: 2**

Solution.
Consider the CL system
$$F(s) = G(s)H(s),\ G_0(s) = \frac{G(s)}{1+F(s)}.$$
If $F(s)$ does not have poles on the imaginary axis, except for one or two poles in the origin, a *necessary and sufficient condition* for the asymptotic stability of the feedback system $G_0(s)$ is that the *complete Nyquist diagram* of $F(s)$ encircles counterclockwise the critical point $-1+j0$ as many times as the number of unstable poles of $F(s)$

3) Show that high loop gain decreases the effect of parametric uncertainties on the output response. Show what happens when the parametric uncertainty is in the sensor TF. **Pts: 2**

Solution.
Consider the TF
$$G(s,\alpha) = G(s,\alpha_0{+}\Delta\alpha) = G(s) + \Delta G(s),$$
where α_0 is a nominal value for the parameter α and $\Delta\alpha$ is its variation wrt α_0.
Express the variation $\Delta G_0(s)$ as function of the variation $\Delta\alpha$.
$$\Delta G_0(s) = \left.\frac{\partial G_0}{\partial \alpha}\right|_{\alpha=\alpha_0} \Delta\alpha = \frac{\partial G_0}{\partial G}\underbrace{\left.\frac{\partial G}{\partial \alpha}\right|_{\alpha=\alpha_0}\Delta\alpha}_{\Delta G(s)} = \frac{\partial G_0}{\partial G}\,\Delta G(s)$$
$$= \frac{\partial}{\partial G}\left(\frac{G}{1{+}G\,H}\right)\Delta G(s) = \frac{1{+}G\,H{-}G\,H}{(1{+}G\,H)^2}\,\Delta G(s)$$
$$= \frac{G}{(1{+}G\,H)^2}\,\frac{\Delta G(s)}{G(s)} = \frac{G_0(s)}{1{+}G(s)\,H(s)}\,\frac{\Delta G(s)}{G(s)}$$
In the frequency range where $|G(j\omega)\,H(j\omega)| \gg 1$, the relative errors of $G_0(s)$ and $G_0(s)$ satisfy the inequality
$$\frac{|\Delta G_0(j\omega)|}{|G_0(j\omega)|} \ll \frac{|\Delta G(j\omega)|}{|G(j\omega)|}$$
For variations $\Delta\beta$ of a parameter β in $H(s,\beta)$ we have:
$$\frac{\Delta G_0(s)}{G_0(s)} = \frac{-G(s)\,H(s)}{1+G(s)\,H(s)}\,\frac{\Delta H(s)}{H(s)}$$

For those frequencies where $|G(j\omega)\, H(j\omega)| \gg 1$,, the relative errors are of the same order of magnitude:

$$\boxed{\frac{|\Delta G_0(j\omega)|}{|G_0(j\omega)|} \simeq \frac{|\Delta H(j\omega)|}{|H(j\omega)|}}$$

4) Calculate the Laplace transform of

$$x(t) = t^2 + e^{-2t}\cos 5(t-1)$$

Solution. **Pts: 2**

$$\mathcal{L}\big[x(t)\big] = \frac{2}{s^3} + e^{-s}\frac{s+2}{(s+2)^2+25}.$$

5) Calculate the response to the unitary step of the system

$$G(s) = \frac{s-1}{s+2}.$$

Check the correctness of your result with the initial and final value theorem. **Pts: 2**

Solution.
The Laplace transform of the output signal is

$$Y(s) = \frac{s-1}{s+2}\cdot\frac{1}{s} = \frac{A}{s+2} + \frac{1}{s} \Rightarrow \begin{cases} A = \frac{3}{2}, \\ B = -\frac{1}{2} \end{cases}.$$

The inverse transform is

$$y(t) = \mathcal{L}^{-1}\left[\frac{3}{2(s+2)} - \frac{1}{2s}\right] = \frac{3}{2}e^{-2t} - \frac{1}{2}.$$

The initial and final values of the output provided by the initial and final value theorem are

$$y(0) = \lim_{s\to\infty} s\cdot\frac{s-1}{s+2}\cdot\frac{1}{s} = 1,$$

$$y(\infty) = \lim_{s\to 0} s\cdot\frac{s-1}{s+2}\cdot\frac{1}{s} = -\frac{1}{2}.$$

6) Write the transfer function of a 2$^{\text{nd}}$ order system whose output response $y(t)$ to a unitary step has

- $\lim_{t\to\infty} y(t) = 2$,
- has a max overshoot $S = 30\%$,
- has pseudo-oscillations with period $T = 0.1s$

Pts: 2

Solution.
The TF is of the type

$$G(s) = \frac{k}{1 + 2\,\delta\,\frac{s}{\omega_n} + \frac{s^2}{\omega_n^2}},$$

with

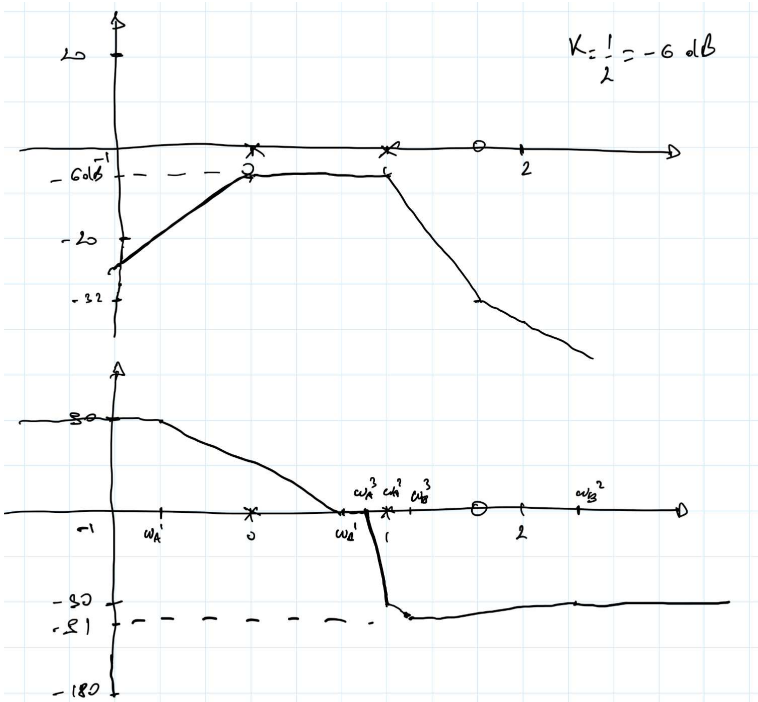
- $K = 2$, determined from $y(\infty)$,
- $S = 0.3 \;\rightarrow\; \delta = 0.35$,
- $\omega_n = \frac{2\pi}{T\sqrt{1-\delta^2}} = 65.86$ rad/s.

7) Plot the asymptotic Bode diagrams of the transfer function

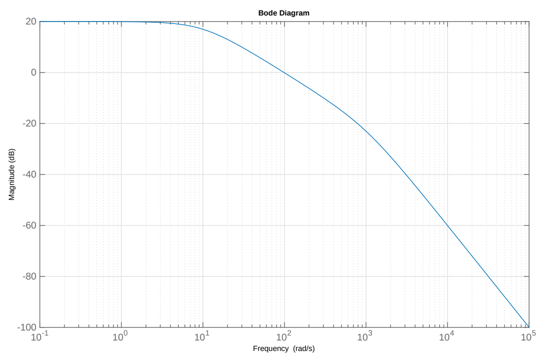
$$G(s) = \frac{s(s+50)}{(s+1)(s^2+4s+100)}.$$

Pts: 2

Solution.



8) Consider the following Bode diagrams.



Find a 1st order approximation. **Pts: 2**

Solution.
The system has clearly two poles in 10 and 1000. By considering the dominant pole only, the resulting transfer function is

$$G(s) = \frac{10}{1+0.1s}.$$

9) Draw the Nyquist plot of the system

$$G(s) = \frac{s+1}{(s+10)(s^2+4s+100)},$$

and mention the plotting rules you used. **Pts: 2**

Solution.

- **Starting point.**

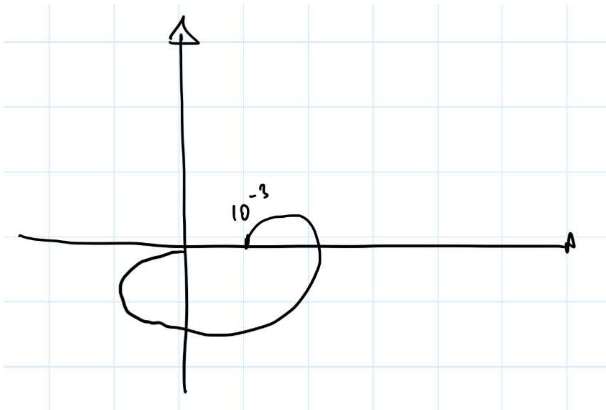
$$G_0(s) \simeq G(s)|_{s \simeq 0} = \frac{1}{10^3} \Rightarrow \begin{cases} M_0 = \frac{1}{1000}, \\ \varphi_0 = 0. \end{cases}$$

- **Final point.**

$$G_\infty(s) \simeq G(s)|_{s \simeq \infty} = 0 \Rightarrow \begin{cases} M_\infty = 0, \\ \varphi_\infty = -\pi. \end{cases}$$

  The final phase is approached from above since $\Delta p = -1 + 10 + 4 > 0$.
- **Path direction.** The plot starts counterclockwise since $\Delta\tau = 1 - \left(0.1 + \frac{4}{100}\right) = 0.74 > 0$.

The resulting Nyquist plot is



10) Consider the system

$$G(s) = -\frac{s+1}{(s+10)(s^2+4s+100)}.$$

Use the Routh criterion to find the max value of a proportional controller such that the closed-loop system is asymptotically stable. **Pts: 2**

Solution.
The characteristic equation is

$$s^3 + 14s^2 + (140-K)s + (1000-K) = 0.$$

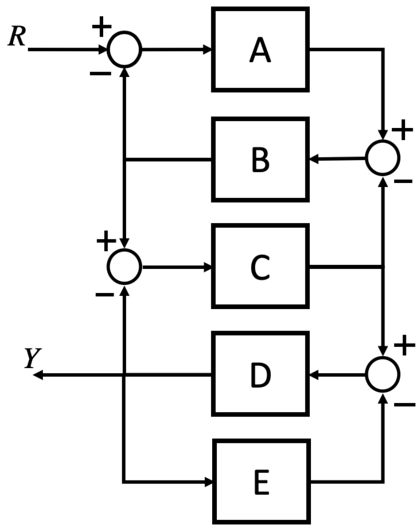
From the corresponding Routh table

$$\begin{array}{c|ccc} 3 & 1 & 140-K & 0 \\ 2 & 14 & 1000-K & 0 \\ 1 & \frac{960-13K}{14} & 0 & \\ 0 & 1000-K & & \end{array}$$

it follows that

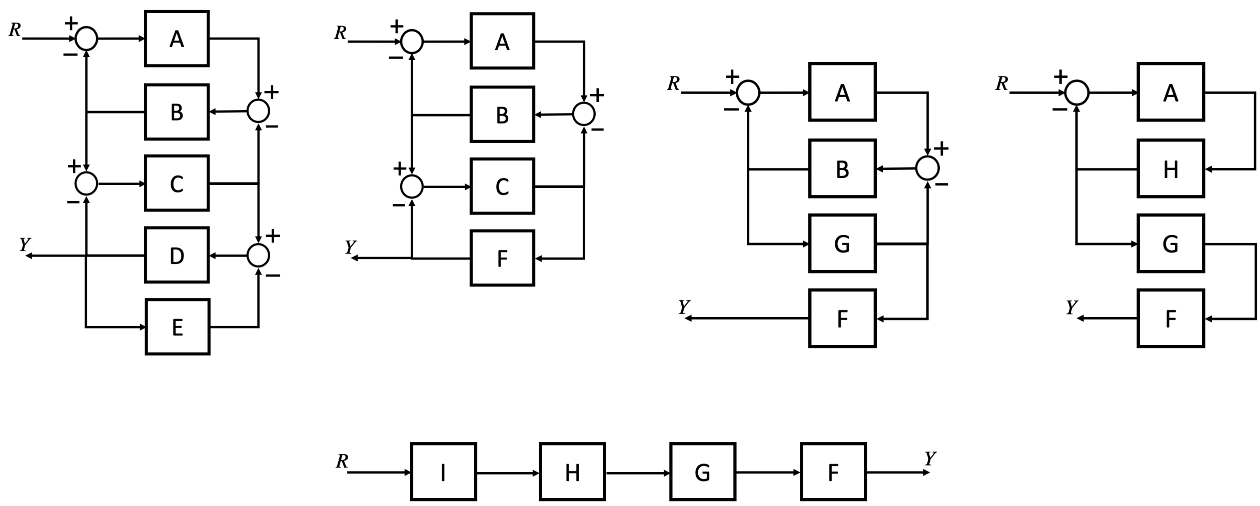
$$K < 1000, \ \underline{K < 73.84}.$$

11) Reduce the following block diagram and find the transfer function $G = \dfrac{Y}{R}$.



Check the correctness of the result with the Mason's formula. **Pts: 3**

Solution.



where

$$F = \frac{D}{1+DE}, \quad G = \frac{C}{1+CF}, \quad H = \frac{B}{1+BG}, \quad I = \frac{A}{1+AH}.$$

The *transfer function* $G = \dfrac{Y}{R}$ is

$$G = \frac{1}{\varDelta} \sum_{i \in \mathcal{P}} P_i \, \varDelta_i$$

where

- $\mathcal{P} = \{ABCD\}$,
- the determinant $\varDelta$ is calculated as

$$\varDelta := 1 - \sum_{i \in \mathcal{J}_1} A_i + \sum_{(i,j) \in \mathcal{J}_2} A_i \, A_j - \sum_{(i,j,k) \in \mathcal{J}_3} A_i \, A_j \, A_k + \ldots,$$

  where

  (a) $\mathcal{J}_1 = \{A_1,\ A_2,\ A_3,\ A_4\} = \{-AB, -BC, -CD, -DE\}$,
  (b) $\mathcal{J}_2 = \{(-AB, -CD), (-AB, -DE), (-BC, -DE)\}$

(c) $\mathcal{J}_n = \emptyset,\ n > 2.$

- $\Delta_1 = 1.$

The resulting TF is

$$G = \frac{ABCD}{1 + AB + BC + CD + DE + ABCD + ABDE + BCDE}.$$

12) Consider a mass-spring-damper system. Calculate the speed $v(0^+)$ when the a constant force F is applied and the initial position and speed are $p(0)$ and $v(0)$, respectively. **Pts: 3**

Solution.
The mathematical model of the mass-spring-damper system is

$$M\ddot{x} = F - kx - b\dot{x},\ p = x,\ v = \dot{x}.$$

In the Laplace domain

$$Ms^2P(s) + bsP(s) + kP(s) = F + Msp(0^-) + bp(0^-) + Mv(0^-),$$

$$P(s) = \frac{F}{Ms^2 + bs + K} + \frac{Msp(0^-) + bp(0^-) + Mv(0^-)}{Ms^2 + bs + K},$$

$$V(s) = \frac{sF}{Ms^2 + bs + K} + s\frac{Msp(0^-) + bp(0^-) + Mv(0^-)}{Ms^2 + bs + K} - p(0^-)$$

By applying the initial value theorem

$$\begin{aligned}
V(0) &= \lim_{s\to\infty} sV(s) = \lim_{s\to\infty} s\frac{\bar{F}}{s}\frac{1}{Ms^2 + bs + K} + s^2\frac{Msp(0^-) + bp(0^-) + Mv(0^-)}{Ms^2 + bs + K} - sp(0^-) \\
&= \lim_{s\to\infty} \frac{Ms^3p(0^-)}{Ms^2 + bs + K} + \frac{bs^2p(0^-)}{Ms^2 + bs + K} + \frac{Ms^2v(0^-)}{Ms^2 + bs + K} - sp(0-) \\
&= \frac{b}{M}p(0^-) + v(0^-)
\end{aligned}$$

13) Write the differences equation of a 1st order digital filter that attenuates the signal

$$u(t) = A\sin\omega t,\ \omega \in [10, 50]\text{rad/s},$$

of at least a factor 10, without introducing a delay larger than $\tau_{\max} = 0.2s$ in $[10, 50]$rad/s. For convenience, discretize the filter with the bilinear discretization method. **Pts: 3**

Solution.
The modulus of the TF of the filter must be at least -20 dB at $\omega = 10$rad/s. Set

$$G(s) = \frac{1}{s+1}.$$

It follows that

$$\underline{/G}j(\bar{\omega}) = -90° = \underline{/G}(j\underline{\omega}).$$

Such a phase shift introduces a delay in the considered frequency range between

$$\underline{\tau} = \frac{\pi}{2}\cdot\frac{1}{\underline{\omega}},\ \text{and}\ \bar{\tau} = \frac{\pi}{2}\cdot\frac{1}{\bar{\omega}}.$$

The ZOH introduces a phase shift of $-\frac{T}{2}\omega$, where T is the sampling time. The it must be $\max\left\{\underline{\tau} + \frac{T}{2}, \bar{\tau} + \frac{T}{2}\right\} \le \tau_{\max}$. It follows $T = 0.1$.
By discretizing the filter with Tustin, we obtain

$$y(t) = \frac{1}{21}\left[19y(t - 1 + u(t) + u(t-1))\right].$$

14) Consider the system

$$G(s) = \frac{10}{s^3 + 3s^2 + 3s + 1}.$$

Design a lag network such that the phase margin is 60°. **Pts: 3**

Solution.
The qualitative plot of the Nyquist diagram clearly shows that the intersection of the diagram with the imaginary axis is within the feasibility region of the lag network and can be conveniently chosen as the point A. We have to look for the $\omega_A : \mathcal{R}e[G(j\omega_A)] = 0$.

$$\begin{aligned} G(j\omega) &= \frac{10}{(1+j\omega)^3} = \frac{10}{(1-3\omega^2) + j(3\omega - \omega^3)} \\ &= \frac{10(1-3\omega^2) - 10j(3\omega - \omega^3)}{(1-3\omega^2)^2 + (3\omega - \omega^3)^2}, \\ \mathcal{R}e[G(j\omega)] &= \frac{10(1-3\omega^2)}{(1-3\omega^2)^2 + (3\omega - \omega^3)^2}. \\ \mathcal{R}e[G(j\omega)] &= 0, \ \rightarrow \ \omega_A = \frac{1}{\sqrt{3}}. \end{aligned}$$

Modulus and phase at ω_A are, respectively,

$$\varphi_A = 270^\circ, \ M_A = \frac{10(3\omega_A - \omega_A^3)}{(1-3\omega_A^2)^2 + (3\omega_A - \omega_A^3)^2} = 6.49.$$

Point A must be moved to a point B with $\varphi_B = 240^\circ, \ M_B = 1$. Hence,

$$M = \frac{M_B}{M_A} = \frac{1}{M_A} = 0.15, \ \varphi = \varphi_B - \varphi_A = -30^\circ \text{ at } \omega_A = 0.58.$$

By applying the inversion formulas

$$C(s) = \frac{1 + 2.47s}{1 + 20s}.$$

15) Consider a car with mass $M = 1000$ Kg. Its aerodynamic is such that the drag force can be sufficiently well described by $F_{\text{drag}} = bv$, where v is the vehicle speed in m/s and $b = 200$ Ns/m. Design a speed controller such that

(a) a zero steady-state speed error is achieved in presence of a constant speed reference and of a road grade, while a steady-state speed error smaller than 5 m/s is achieved for acceleration references of max 2 m/s^2.

(b) the closed-loop system exhibits a maximum overshoot $S_{\max} = 10\%$ and a settling time at 5% not larger than 2s.

Pts: 5

Solution.
We start from the static specs. Apply the final value theorem

$$e_\infty = \lim_{s\to 0} s \frac{1}{1 + \dfrac{C(s)}{Ms+b}} V_{\text{ref}}(s) = \begin{cases} \dfrac{b}{b + C(s)}_{|s=0} V_{\text{ref}} & \text{constant speed,} \\ \\ \dfrac{b}{s(b + C(s))}_{|s=0} A_{\text{ref}} & \text{constant acceleration.} \end{cases}$$

With $C(s) = \dfrac{C_1(s)}{s}$ we obtain

$$e_\infty = \begin{cases} 0, & \text{constant speed} \\ \dfrac{bA_{\text{ref}}}{C_1(0)} & \text{constant acceleration.} \end{cases}$$

In order for the error to be bounded by $\bar{e} = 5$, C_1 must be designed such that

$$\frac{A_{\text{ref}}^{\max} b}{C_1(0)} \Rightarrow C_1(0) > b\frac{A_{\text{ref}}^{\max}}{\bar{e}}, \; A_{\text{ref}}^{\max} = 2.$$

With such controller structure the steady state error due to the road grade is

$$e_\infty = \lim_{s \to 0} s \cdot \frac{1}{Ms + b - C(s)} \cdot \frac{1}{s} = \lim_{s \to 0} \frac{s}{Ms^2 + bs - C_1(s)} = 0.$$

Hence, $C(s) = \dfrac{C_1(s)}{s}$ also rejects the effects of the road grade on the steady state tracking error.
Consider the loop transfer function

$$L(s) = \frac{C_1(s)}{s(Ms + b)}.$$

$C_1(s)$ must be designed to satisfy also the following requirements

- $S < 10\% \Rightarrow \delta > 0.6$,
- $T_a < \bar{T}_a = 2, \; \Rightarrow |\sigma_a| > \dfrac{3}{\bar{T}_a} = 1.5.$

where δ and σ_a are the damping and the real part of the CL dominant poles.
The requirements on damping and real part of the dominant poles provide a region where the poles are allowed to be, determined by a vertical line passing by $\sigma_a = -1.5$ and the lines passing by the origin and forming with the real axis an angle $\alpha = \arccos\delta = 53°$. The locus has to pass by the intersection between these two lines, that is, the points

$$P_{1,2} = (\sigma_a, \sigma_a \tan\arccos\delta) = (-1.5, \pm 2.04).$$

By denoting with $\alpha_1, \; \alpha_2$ the angles formed by P with the poles in 0 and in $-\frac{b}{M}$, respectively, and with β the angle formed with the zero of $C_1(s)$ to be placed, the following must hold

$$(2\nu + 1)\pi = \beta - \alpha_1 - \alpha_2,$$

for some integer ν. Since $\alpha_1 = 126.24°, \; \alpha_2 = 122.42°$, with $\nu = -1$ $\beta = -\pi + \alpha_1 + \alpha_2 = 68.66°$.
By simple geometrical arguments

$$|z| - \sigma_a = P_y \cot\beta \;\Rightarrow\; z = -(\sigma_a + P_y \cot\beta) = -2.29.$$

The controller is then

$$C(s) = K\frac{(s + 2.29)}{s},$$

with K to be chosen such that the poles are at $P_{1,2}$. The loop TF is

$$L(s) = K_1 \frac{s + 2.29}{s(s + 0.2)} \; K_1 = \frac{K}{M}.$$

K_1 is found as

$$K_1 = \frac{\sum \eta_i}{\sum \lambda_i},$$

where η_i and λ_i are the distances of P from the poles and zeros.

$$\eta_1 = \sqrt{P_x^2 + P_y^2} = 2.54,\ \ \eta_2 = \frac{P_y}{\sin(\pi - \alpha_2)} = 2.42,\ \ \lambda = \frac{P_y}{\sin\beta} = 2.20\ \Rightarrow\ K_1 = 2.80.$$

The resulting K satisfies the boundedness requirement on the steady state tracking error for constant acceleration reference

$$K = 2.8 \cdot 10^3 > b\frac{A_{\rm ref}^{\rm max}}{\bar{e}}.$$