Exam of Automatic Controls. September 7$^{\text{th}}$, 2020
Duration: 135 mins

Solve the following problems. Laude is granted if more than 34 points are gained, including the max 3 points from the (optional, upon student's request) oral exam.

1) Sketch the qualitative output response to a unitary step of the system

$$G(s) = \frac{50}{s^2 + 2s + 25},$$

writing the expression of the maximum output, the period of the oscillation, if any, and the time it takes to reach the final output value. **Pts: 2**

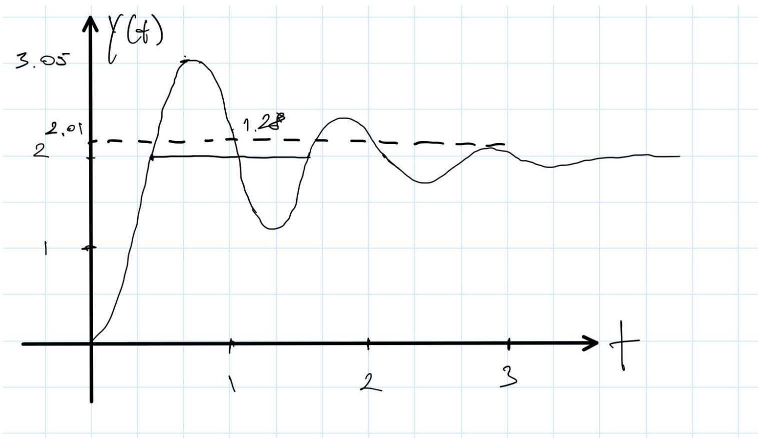
Solution.
In this case $k = 2,\ \delta = 0.2,\ \omega_n = 5$. The resulting overshoot, period of oscillation (since $\delta = 0.2 < 1$) and settling time to 5% are

$$S\% = 100\, e^{\frac{-\pi\delta}{\sqrt{1-\delta^2}}} = 52.66\%, \ \Rightarrow\ y_{\max} = 2 * 1.5266 = 3.05,$$

$$T = \tfrac{2\pi}{\omega_n\sqrt{1-\delta^2}} = 1.28s,$$

$$T_{a5} = \tfrac{3}{\delta\omega_n} = 3s.$$

The resulting step response is



2) Calculate the inverse Laplace transform of the signal

$$Y(s) = \frac{2}{(s+2)(s+3)^2}.$$

Pts: 2

Solution.
Since the pole $s = -3$ is double, we need to find the coefficients $K_1,\ K_2,\ K_3$ in

$$Y(s) = \frac{2}{(s+2)(s+3)^2} = \frac{K_1}{s+2} + \frac{K_2}{s+3} + \frac{K_3}{(s+3)^2}.$$

The coefficients of the fractions are:

$$K_1 = [(s+2)\,Y(s)]\big|_{s=-2} = 2$$

$$K_2 = \frac{d}{ds}\left[(s+3)^2\,Y(s)\right]\Big|_{s=-3} = \frac{d}{ds}\left[\frac{2}{s+2}\right]\Big|_{s=-3} = -2$$

$$K_3 = [(s+3)^2\,Y(s)]\big|_{s=-3} = -2.$$

Note that $n = 3,\ m = 1$. In this case the sum of the coefficients of the terms with degree one is zero. Hence, K_2 could have been found as $K_2 = -K_1$. The resulting output response is

$$y(t) = 2[e^{-2t} + e^{-3t} - te^{-3t}].$$

3) Plot the Bode diagrams of the following system

$$G(s) = \frac{s(s+10)}{s^2 + 0.4s + 1}$$

Pts: 2

Solution.
The initial part of the asymptotic diagram of the modulus intersects the axis $\omega = 1$ at $K_{\mathrm{dB}} = 20$. The frequencies where the phase changes slope due to the pole and the zero are $\omega_{aP} = 0.73$, $\omega_{bP} = 1.36$ and $\omega_{aZ} = 2$, $\omega_{bZ} = 48.1$, respectively. The resulting Bode diagrams are

4) Plot the Nyquist diagrams of the system in Problem 3). Clearly explain the plotting rules you used and how. *(Hint: you can make use of the Bode plots you plotted in Problem 3.)* **Pts: 2**

Solution.

- **Starting point.**

$$G_0(s) \simeq G(s)|_{s\simeq 0} = 0 \ \Rightarrow \ \begin{cases} M_0 = 0, \\ \varphi_0 = \frac{\pi}{2}. \end{cases}$$

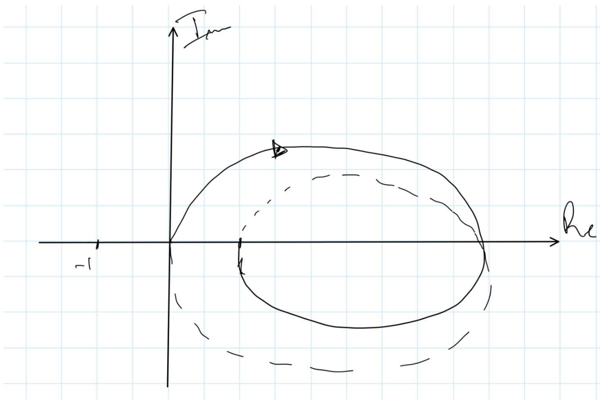
- **Final point.**

$$G_\infty(s) \simeq G(s)|_{s\simeq \infty} = 0 \ \Rightarrow \ \begin{cases} M_\infty = 1, \\ \varphi_\infty = 0. \end{cases}$$

The final phase is approached from below (smaller phase, negative in this case) since $\Delta p = -10 - (-0.4) < 0$.

- **Path direction.** The plot starts clockwise since $\Delta\tau = 0.1 - 0.4 = -0.3 < 0$.

All the above information could have been obtained by inspecting the Bode diagrams in Problem 3. The resulting Nyquist plot is



5) Consider the system in Problem 3 in closed loop with a proportional gain $K \in \mathbb{R}$. Use the Routh criterion to study the closed-loop stability. **Pts: 2**

Solution.
The characteristic equation is

$$(K+1)s^2 + (10K+0.4)s + 1 = 0.$$

Since the characteristic equation has order two, the Descartes rue of signs is enough. It follows that

$$K > -1,\ K > -0.04 \rightarrow \underline{K^* = -0.04}.$$

6) Consider the system in Problem 3 in closed loop with a proportional gain $K \in \mathbb{R}$. Use the Nyquist criterion to study the closed-loop stability. **Pts: 2**

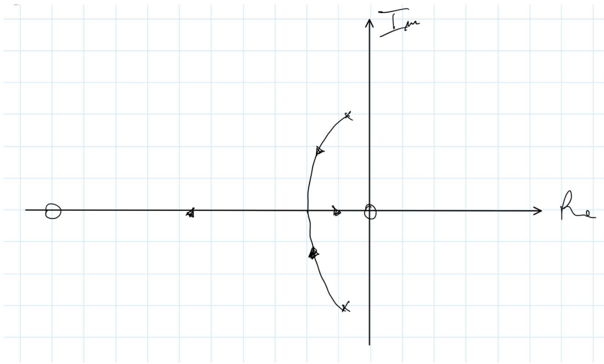
Solution.
According to the Nyquist criterion, the CL system is asymptotically stable for all $K > 0$ as the complete Nyquist diagram cannot encircle the critical point. In fact, as K increases, the Nyquist diagram expands and the intersection points with the real axis move to the right, while the leftmost intersection point with the real axis remains at 0.

For negative gains, the Nyquist plot is obtained by flipping the diagram wrt the imaginary axis. Hence, the CL system will be unstable below a minimum gain, say K^*, because the Nyquist diagram will encircle the critical point twice clockwise.

The value of K^* can be found by setting the real part equal to -1. More conveniently, K^* can be found with the Routh (Descartes) criterion.

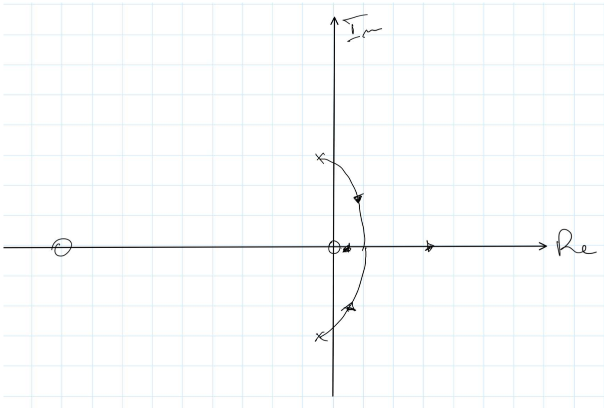
7) Consider the system in Problem 3 in closed loop with a proportional gain $K \in \mathbb{R}$. Use the root locus to study the closed-loop stability. **Pts: 2**

Solution.
The system has two zeros $z_1 = 0$, $z_2 = -10$ and a pair of conjugate complex poles $p_{1,2} = -0.2 \pm 0.98j$. The segment between the two zeros is part of the direct locus. The two branches leaving the conjugate complex poles intersect the real axis and then they end in the two zeros as shown in the plot below



Hence, the CL system is stable for every positive gain.

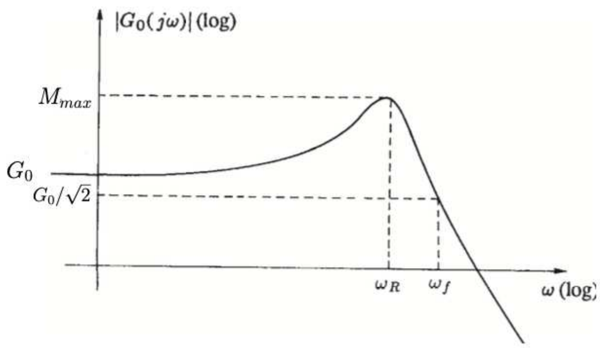
The inverse locus in the plot below



shows that, beyond a limit value K^*, the CL system becomes unstable. K^* can be conveniently found using the Routh criterion.

8) Sketch the Bode diagrams of a second order system and show the *Resonance peak*, the *resonance frequency* and the *bandwidth*. How the bandwidth is connected to the dominant time constants? What does the resonance peak depend on? **Pts: 2**

Solution.
The *Resonance peak*, the *resonance frequency* and the *bandwidth* are shown in the figure below



The bandwidth in inversely proportional to the dominant time constant, while the resonance peak depends on the damping coefficient according to

$$M_R = \frac{1}{2\,\delta\,\sqrt{1-\delta^2}}.$$

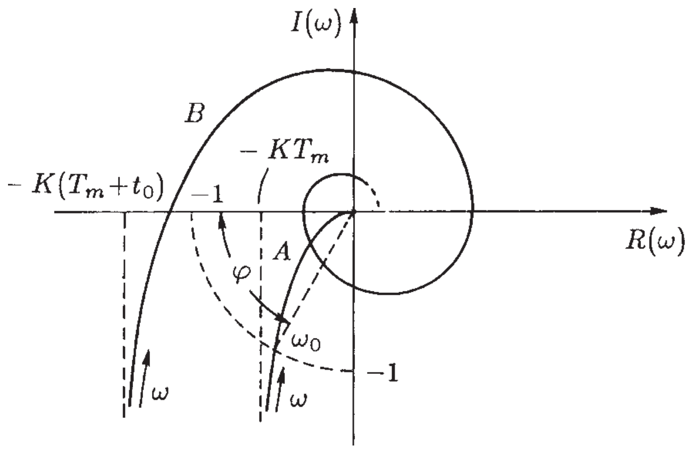
9) Show that, in feedback systems with high loop gain, i.e. $H(s)G(s) \gg 1$, the bandwidth ω_{f0} of feedback system $G_0(s)$ can be easily obtained (in an approximate way) from the module Bode diagram of function $H(s)G(s)$. **Pts: 2**

Solution.
Let us suppose, for example, that $H(s) = 1$. In this case, it follows that:

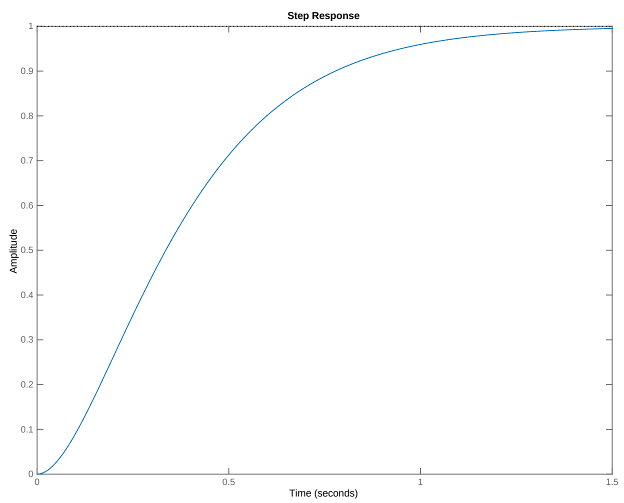
$$G_0(j\omega) = \frac{G(j\omega)}{1 + G(j\omega)} \quad \simeq \begin{cases} 1 & \text{if} \quad |G(j\omega)| \gg 1 \\ G(j\omega) & \text{if} \quad |G(j\omega)| \ll 1 \end{cases}$$

10) With the help of the Nyquist diagram, explain how a delay can introduce closed-loop instability. **Pts: 2**

Solution.
As shown in the figure below, the effect of a delay is "twisting" the diagram, thus, possibly, encircling the critical point and inducing instability.



11) Consider the following unitary step response.



Is this the response of first or a second order system? Rigorously motivate your answer. **Pts: 3**

Solution.
This is the response of a second order system with unitary damping ($\delta = 1$). This follows from the slope of the response at $t = 0$. In fact, the initial slope for a second order system is 0 as

shown next.

$$Y(s) = \frac{k}{1 + 2\frac{\delta}{\omega_n}s + \frac{s^2}{\omega_n^2}} \cdot \frac{1}{s},$$

$$\mathcal{L}\left[\dot{y}(t)\right] = \frac{k}{1 + 2\frac{\delta}{\omega_n}s + \frac{s^2}{\omega_n^2}},$$

$$\dot{y}(0) = \lim_{s\to\infty} s \cdot \frac{k}{1 + 2\frac{\delta}{\omega_n}s + \frac{s^2}{\omega_n^2}} = 0.$$

12) The system

$$G(s) = \frac{1}{s(s+1)},$$

is controlled in closed loop with a unitary gain. Calculate the phase margin with the approximated Bode diagrams. Calculate the phase margin also in the case the controller is implemented in discrete time with a sampling time $T = 0.2s$. **Pts: 3**

Solution.
From the Bode diagrams it results that the system has a phase margin $M_F \simeq 180 - 90 - 45 \cdot 0.9° = 50$ at $\omega \simeq 0.8$ rad/s. At this frequency, the delay introduced by the zero order hold is $\frac{0.2}{2} \cdot \omega \cdot \frac{180}{\pi} = 4.6°$. The resulting phase margin is $45.4°$.

13) For the system

$$G(s) = \frac{1}{s+10},$$

design a PI controller such that the tracking error at steady state is not larger than 0.1 when the reference signal is a ramp and the settling time is not larger than 1 s. **Pts: 3**

Solution.
The specification on the steady-state error imposes that

$$e_\infty = \lim_{s\to 0} s\frac{1}{1+CG}R(s) = \lim_{s\to 0} \frac{s^2(s+10)}{s^2 + (K_P+10)s + K_I} \cdot \frac{1}{s^2},$$

$$= \frac{10}{K_I} \leq 0.1 \ \rightarrow K_I \geq 100.$$

Set $K_I = 100$. The specification on the settling time imposes that the real part of the dominant poles should be smaller than -3. By placing the zero at -5, the loop gain must be larger than

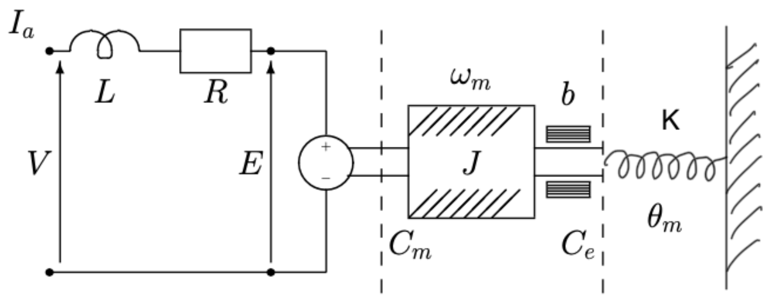
$$K^* = \frac{7}{2} = 3.5.$$

This is achieved with $K_P = 20$. The resulting loop gain is

$$K = \frac{K_I}{10} = 10 > K^*,$$

thus satisfying the requirement on the settling time as well.

14) Write the state space model of the following DC motor, where the inertia is connected to a spring. **Pts: 3**



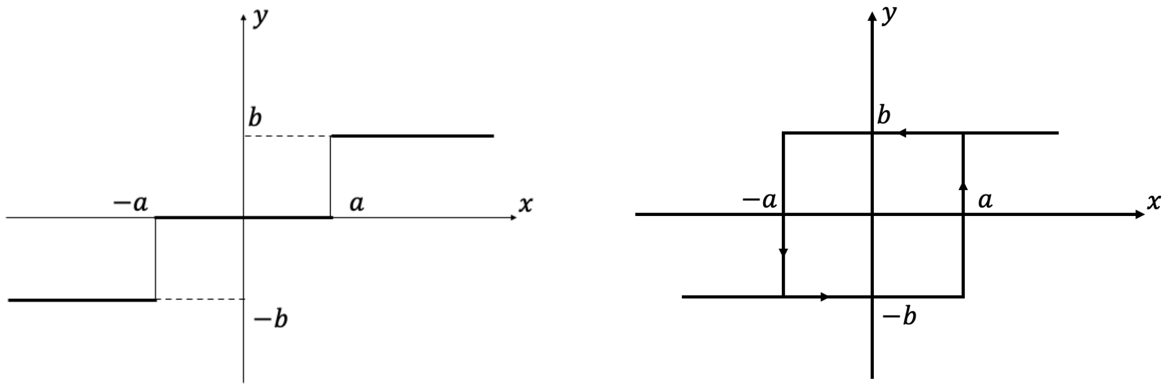
Solution.
The system is described by the following three differential equations:

$$\begin{cases} L\dot{I}_a &= -RI_a - K_e\,\omega_m + V, \\ \dot{\theta}_m &= \omega_m, \\ J\dot{\omega}_m &= K_e I_a - b\,\omega_m - C_e - K\theta_m. \end{cases}$$

15) Consider the system

$$G(s) = \frac{1}{s(s+3)}, \tag{1}$$

and the following two nonlinearities.



(a) Relay with dead zone (without hysteresis)

(b) Relay with hysteresis

Figura 1: Nonlinearities

(a) Which nonlinearity, in closed loop with $G(s)$, can enable a limit cycle? Rigorously motivate your answer with plots. Pts: 2

(b) Design the parameters of the nonlinearity you have chosen, in order to obtain a limit cycle with frequency $\omega = 5.19$ rad/s and unitary amplitude. Pts: 3

Hint The describing functions of the relay with dead zone and the relay with hysteresis in Figure 1 are

$$F_1(X) = \frac{4b}{\pi X}\left[\sqrt{1-\left(\frac{a}{X}\right)^2}\right], \quad X > a,$$

$$F_2(X) = \frac{4b}{\pi X}\left[\sqrt{1-\left(\frac{a}{X}\right)^2} - j\frac{a}{X}\right], \quad X > a,$$

respectively. **Pts: 5**

Solution.

(a) Since $\underline{/G(j\omega)} = -\pi$, $\omega \to \infty$, while $-\frac{1}{F_1(X)}$, with $F_1(X)$ the DF of the relay with dead zone, completely lies on the negative semi-axis, a limit cycle cannot be established with such nonlinearity.

(b) The relay with hysteresis enables a limit cycle if $\bar{X} = 1$, $\bar{\omega} = 5.19$ solve the pseudo-characteristic equation

$$G(j\bar{\omega}) = -\frac{1}{F_2(\bar{X})},$$

where $F_2(X)$ is the DF of the relay with hysteresis. By equating the real and imaginary parts of

$$G(j\bar{\omega}) = -\frac{1+3j}{\bar{\omega}^3 + 9\bar{\omega}}, \quad -\frac{1}{F_2(\bar{X})} = -\frac{\pi\bar{X}}{4b}\left(\sqrt{1-\left(\frac{a}{\bar{X}}\right)^2} + j\frac{a}{\bar{X}}\right),$$

it follows that

$$a = \frac{3}{\sqrt{10}} = 0.95, \ \ b = \frac{\pi a}{4}\frac{\bar{\omega}^3 + 9\bar{\omega}}{3} = 46.32.$$