Exam of Automatic Controls. July 16th, 2021
Duration: 150 mins

Solve the following problems. Laude is granted if more than 35 points are gained, including the max 3 points from the (optional, upon student's request) oral exam and the 3 points from the assignments.

1) Calculate the Laplace transforms of the following signals
   (a) $x_1(t) = 5\delta(t-2) + 2cos(3t)$,
   (b) $x_2(t) = (e^{-t} + \sin t)^2$

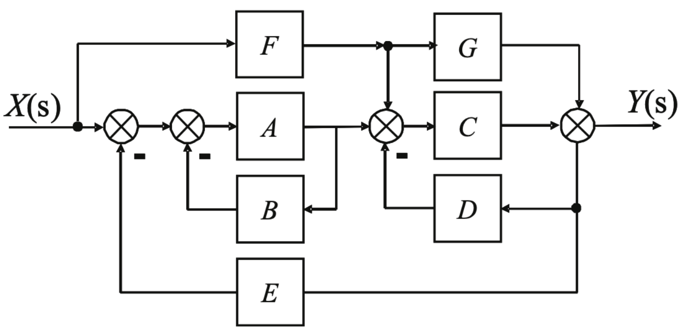
Pts: 2

Solution.

(a) $\mathcal{L}[x_1(t)] = \mathcal{L}[5\delta(t-2)] + \mathcal{L}[2cos(3t)] = 5e^{-2s} + \frac{2s}{s^2+9}$,
(b)

$$\begin{aligned}
\mathcal{L}[x_2(t)] &= \mathcal{L}\left[e^{-2t} + 2e^{-t}\sin t + \sin^2 t\right] \\
&= \mathcal{L}\left[e^{-2t}\right] + \mathcal{L}\left[2e^{-t}\sin t\right] + \mathcal{L}\left[1-\cos 2t\right] \\
&= \frac{1}{s+2} + \frac{2}{(s+1)^2+1} + \frac{1}{2}\left(\frac{1}{s} - \frac{s}{s^2+4}\right) \\
&= \frac{1}{s+2} + \frac{2}{s^2+2s+2} + \frac{2}{s(s^2+4)}
\end{aligned}$$

2) Derive the transfer function $G(s) = \frac{Y(s)}{X(s)}$ of the block diagram below, by using the block diagrams reduction rules. Show the intermediate steps. **Pts: 2**



Solution.

$$G(s) = \frac{AC + F(C+G)(1+AB)}{1 + AB + CD + ACE + ABCD}.$$

3) Assume zero initial conditions for $y(t)$, $u(t)$ and their higher order derivatives and calculate the transfer function corresponding to the following difference equation

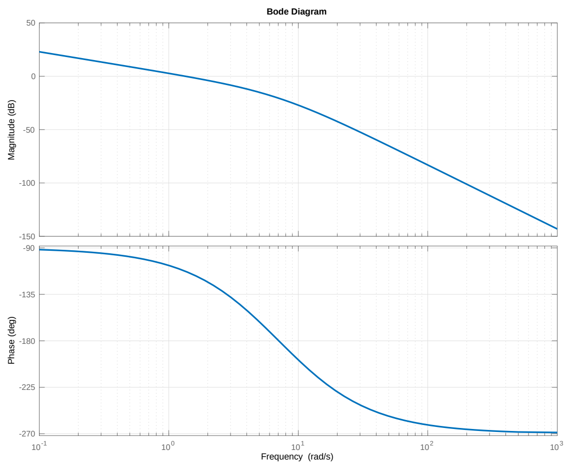
$$2\dddot{y} + 4\dot{y} + 3y = \ddot{u} + 5\dot{u} + 2u.$$

Pts: 2

Solution.

$$G(s) = \frac{s^2 + 5s + 2}{2s^3 + 4s + 3}.$$

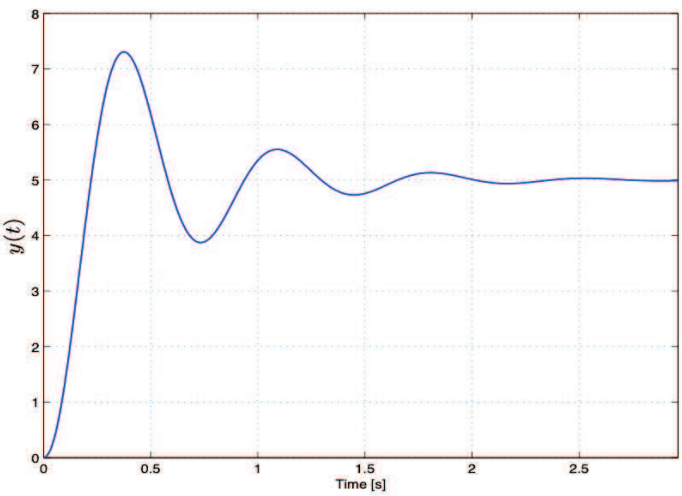
4) Find the stability margins of the system with the Bode diagrams in the figure below and use them to calculate the damping of the closed-loop poles. **Pts: 2**



Solution.

$$M_\varphi \approx 67^\circ, \; M_\alpha \approx 20\text{db}, \; \delta \approx 0.67.$$

5) A dynamical system has the following output response to the step $u(t) = 2$. Calculate the dominant poles, the static gain and the natural frequency. **Pts: 2**



Solution.

- Since the settling time is approximatively 1.5 s, then the real part of the dominant poles is $\sigma = -2$. The overshoot is 48%. Hence, the damping is $\delta = 0.23$ and the natural frequency is $\omega_n \approx 8.7$ rad/s. The resulting imaginary part is $\omega = \omega_n\sqrt{1-\delta^2} = 8.46$.
- The static gain of the system is 2.5.

6) Calculate the $\mathcal{Z}$-transform of the signals

$$x_1(t) = 3^{-t},\ x_2(t) = 2t,\ t = kT.$$

Pts: 2

Solution.

$$X_1(z) = \frac{z}{z-3^{-T}},\ X_2(z) = \frac{2Tz}{(z-1)^2}.$$

7) Calculate the steady-state value $x(\infty)$ of the sequence $x(k)$ with $\mathcal{Z}$-transform $X(z) = \frac{z+1}{(z-1)(z-0.5)}$. **Pts: 2**

Solution. $x(\infty) = 4$.

8) Assume $y(0) = 0$ and calculate the response of the system $y(k+1) = 2y(k) + 3u(k)$ to the signal $u(k) = 0.5^k$. **Pts: 2**

Solution. By $\mathcal{Z}$-transforming the system and the input signal, we obtain

$$Y(z) = \frac{3}{z-2}U(z) = \frac{3z}{(z-2)(z-0.5)} = 2\left[\frac{z}{z-2} - \frac{z}{z-0.5}\right].$$

By inverse transforming

$$y(k) = 2(2^k - 0.5^k).$$

9) Calculate the response $y_\infty(t)$ of the system $G(s) = \frac{2}{s+3}$ to the signal $u(t) = 2+5sin(4t)$. **Pts: 2**

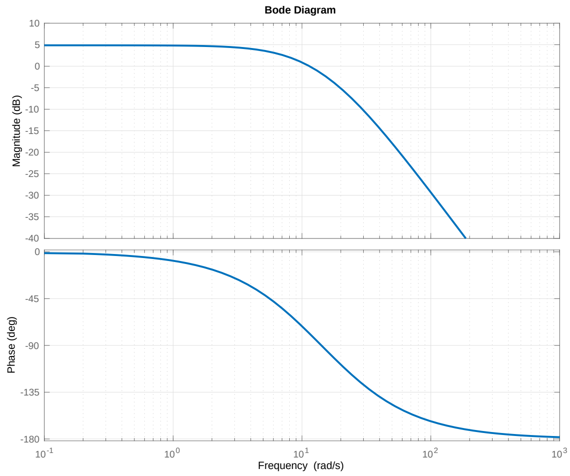
Solution.
The response $y_\infty(t)$ can be calculated with the frequency response function $G(j\omega)$

$$y_\infty(t) = 2G(0) + 5|G(4j)|\sin(4t + \angle G(4j)),$$

with

$$G(0) = \frac{2}{3},\ |G(4j)| = \frac{2}{5},\ \angle G(4j) = \arctan\frac{4}{3}.$$

10) Consider the dynamical system with the Bode diagrams in the figure below. Calculate the bandwidth of the system and an approximation of the closed-loop bandwidth. Specify the type of approximation (e.g., lower/upper bound, other) you considered. **Pts: 2**



Solution.
The system bandwidth is $\omega_f \approx 8$rad/s. A *lower-bound* of the closed-loop bandwidth is $\omega_{f0} \approx 11$rad/s.

11) Consider a double integrator

$$G(s) = \frac{1}{s^2}.$$

Can the closed-loop system be made asymptotically stable with a proportional controller? If not, propose a suitable control structure (you don't need to calculate the exact position of the controller's poles and zeros). **Pts: 3**

Solution.
By using the root locus one can see that with a proportional controller the closed-loop poles move on the imaginary axis. Hence, a proportional controller cannot make the closed-loop system asymptotically stable.

A pole+zero controller can make the closed-loop AS if a zero is placed at a lower frequency than the pole.

12) Consider the system

$$G(s) = \frac{10}{s^3 + 3s^2 + 3s + 1}.$$

Design a lead/lag network to be implemented in discrete-time with a sampling time $T = 1s$ such that, the closed-loop system has a damping $\delta = 0.6$ and a bandwidth not smaller than 0.6 rad/s. **Pts: 3**

Solution.
The spec on the CL damping imposes a phase margin $M_\alpha = 60°$. At $\omega_{f0} = 0.6$, a ZH reconstructor introduces a lag

$$\angle e^{-\frac{1}{2}0.6j} = -17.2.$$

Hence, the controller has to be designed in order to introduce a phase margin $M_\alpha = 77°$. Set $\omega_A = \omega_{f0} = 0.6$. By observing that $\mathcal{R}e[G(j\omega_A)] = 0$, we conclude that ω_A is within the feasibility region of a lag network.

Modulus and phase at ω_A are, respectively,

$$\varphi_A = 270°,\ M_A = \frac{10(3\omega_A - \omega_A^3)}{(1-3\omega_A^2)^2 + (3\omega_A - \omega_A^3)^2} = 6.49.$$

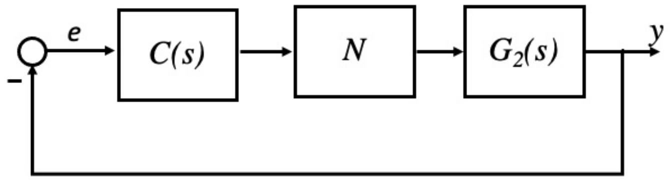
Point A must be moved to a point B with $\varphi_B = 257°,\ M_B = 1$. Hence,

$$M = \frac{M_B}{M_A} = \frac{1}{M_A} = 0.15,\ \varphi = \varphi_B - \varphi_A = -13° \text{ at } \omega_A = 0.58.$$

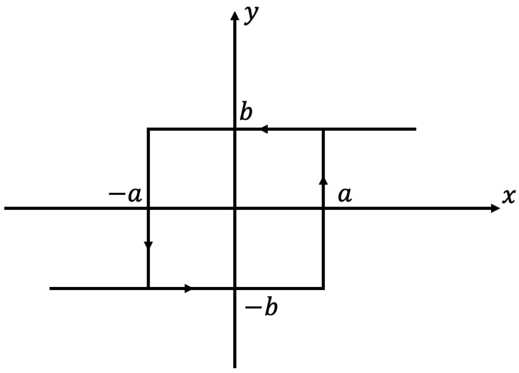
By applying the inversion formulas

$$C(s) = \frac{1+6.32s}{1+43.63s}.$$

13) Consider the system



where the nonlinearity N is the relay with hysteresis



with $a = b = 1$, and the plant $G(s)$ has transfer function

$$G(s) = \frac{5}{(1+s)^2}.$$

Design the regulator $C(s)$ such that permanent oscillations are established with

$$e(t) = 2\cos 0.6t.$$

Pts: 3

Hint. The describing function of the relay with hysteresis is

$$F(X) = \frac{4b}{\pi X}\left[\sqrt{1-\left(\frac{a}{X}\right)^2} - j\frac{a}{X}\right],\quad X > a.$$

Solution.
The following condition must be satisfied

$$C(0.6j)G(0.6j) = -\frac{1}{F(2)}.$$

It follows that

$$C(0.6j) \approx 0.43e^{-j1.54}.$$

Such condition can be satisfied with

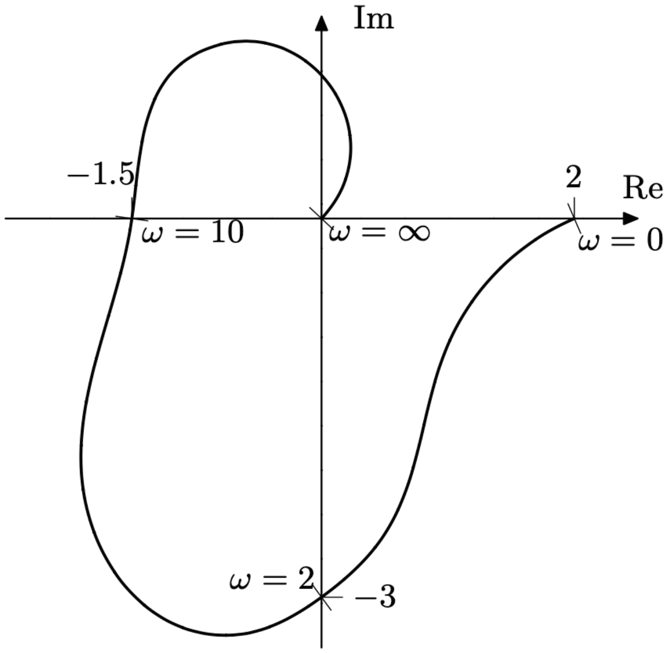
$$C(s) = \frac{k}{1+s\tau},$$

provided that

$$|C(0.6j)| = \frac{k}{\sqrt{1+(0.6\tau)^2}} = 0.43, \ \arg C(0.6j) = -\arctan 0.6\tau = -1.54.$$

From the second condition, $\tau = 54.1$ follows and $k = 14$. By plotting the Nyquist diagram of $C(s)G(s)$ and $-\frac{1}{F(X)}$ it can be verified that the oscillations are also stable.

14) An asymptotically stable system $G(s)$ has the Nyquist diagram in the figure below.



(a) For which values of $K > 0$ the closed-loop system is asymptotically stable?
(b) Determine the steady-state error as a function of K when the reference signal is a unitary step.
(c) Assume $G(s)$ is controlled with an integral controller $\frac{K}{s}$. For which values of K is the CL system asymptotically stable?

Pts: 3

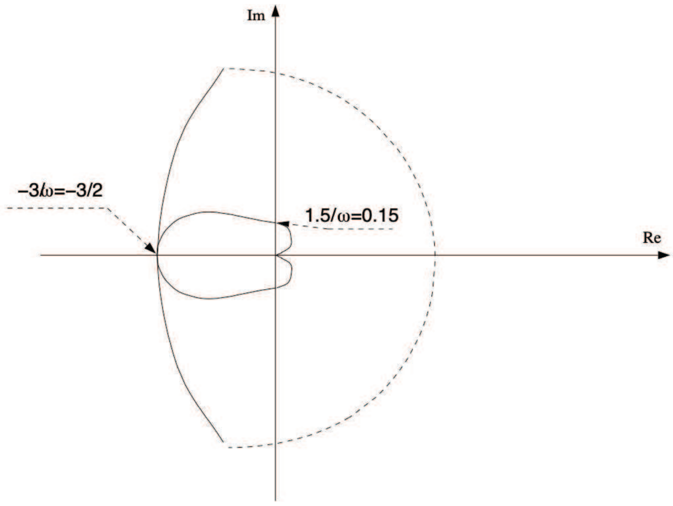
Solution.

(a) According to the Nyquist criterion, the point (-1,0) must not be encircled. That is, $1.5K < 1 \Rightarrow K < \frac{2}{3}$.

(b)

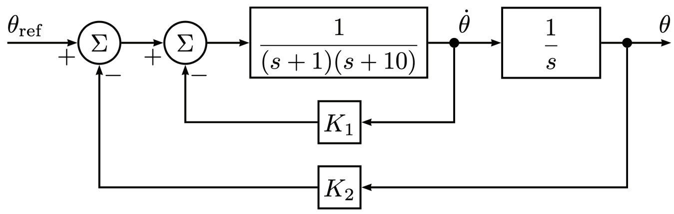
$$\lim_{t\to\infty} e(t) = \lim_{s\to 0} sE(s) = \lim_{s\to 0} s\frac{1}{1+KG(s)}\frac{1}{s} = \frac{1}{1+2K}, \ K < \frac{2}{3}.$$

(c) The term $\frac{1}{s}$ rotates 90° clockwise the Nyquist diagrams and scales it by a factor $\frac{1}{\omega}$. The resulting Nyquist diagram is



By applying the Nyquist criterion, it must hold that $\frac{3}{2}K < 1 \Rightarrow K < \frac{2}{3}$.

15) In the cascade controlled DC motor in the figure below, $K_1 > 0, \ K_2 > 0$.



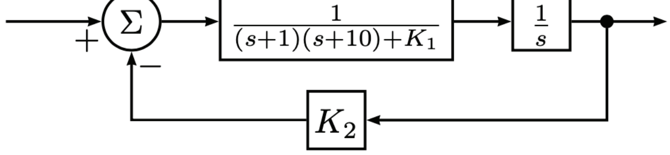
(a) Draw the root locus w.r.t. K_2 and determine the values of K_2 such that the CL system is asymptotically stable.

(b) How is the stability requirement on K_2 affected by K_1?

Pts: 5

Solution.

By manipulating the block diagram, the following is obtained.

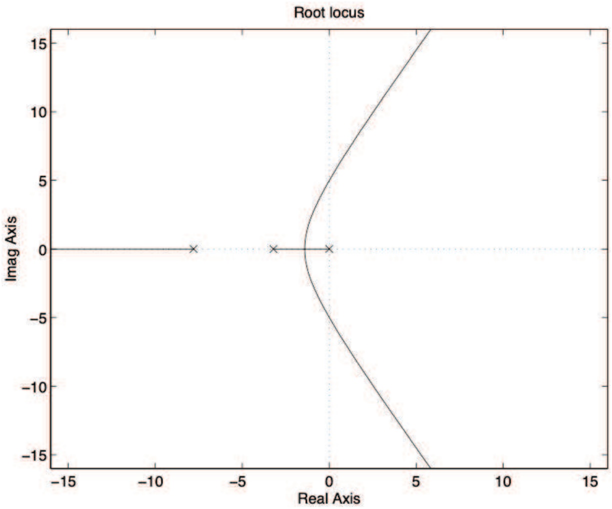


The characteristic equation $s[(s+1)(s+10)+K_1]+K_2=0$ follows with roots in 0 and $s = -5.5 \pm \sqrt{5.5^2 - 10 - K_1}$. It follows that

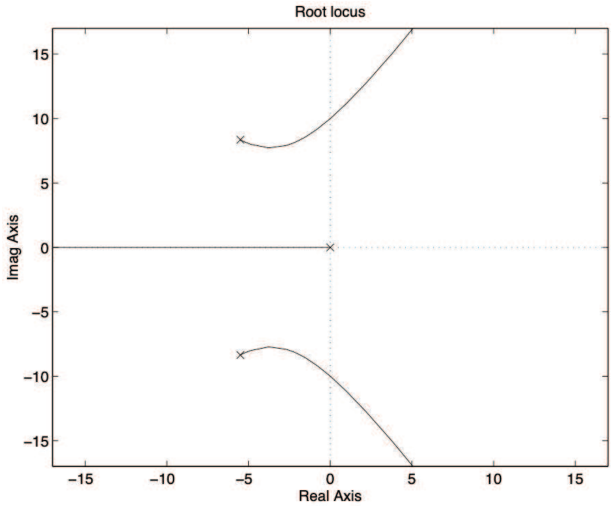
(a) the roots are **real** when $K_1 \leq 20.25$,

(b) the roots are **complex** when $K_1 < 20.25$.

The root loci corresponding to these two cases are reported in the figure below.



(a) Case 1



(b) Case 1

In order to find the limit value of K_2 such that asymptotical stability holds, set $s = j\omega$ in the characteristic equation. It follows that $\omega = 0 \Rightarrow K_2 = 0$, while $\omega^2 = 10 + K_1 \Rightarrow K_2 = 11K_1 + 110$. Hence, the CL system is asymptotically stable for

$$0 < K_2 < 11K_1 + 110.$$

It can be observed that using the inner feedback ($K_1 > 0$) allows a larger value of K_2.