Exam of Automatic Controls. February 8th, 2021

Duration: 135 mins

Solve the following problems. Laude is granted if more than 34 points are gained, including the max 3 points from the (optional, upon student's request) oral exam.

1) Sketch the qualitative output response to a unitary step of the system

$$G(s) = \frac{10}{s^2 + 2s + 10},$$

writing the expression of the maximum output, the period of the oscillations, if any, and the time it takes to reach the final output value. **Pts: 2**

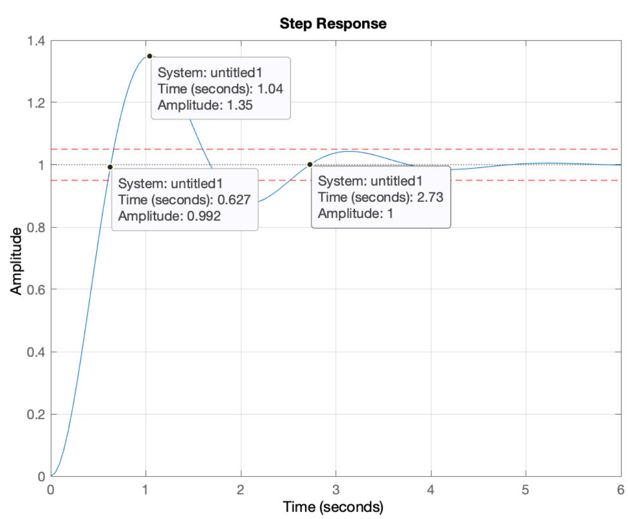
Solution.
In this case $k = 1$, $\delta = 0.32$, $\omega_n = 3.16$. The resulting overshoot, period of oscillation (since $\delta = 0.32 < 1$) and settling time to 5% are

$$S\% = 100\, e^{\frac{-\pi\delta}{\sqrt{1-\delta^2}}} = 34.60\%, \Rightarrow y_{\max} = 1 * 1.3460 = 1.346,$$

$$T = \frac{2\pi}{\omega_n\sqrt{1-\delta^2}} = 2.09s,$$

$$T_{a5} = \frac{3}{\delta\omega_n} = 3.06s.$$

The resulting step response is



2) Calculate the response of the system

$$G(s) = \frac{10}{s^2 + 2s + 10}$$

to the unitary step. **Pts: 2**

Solution.
The response in the Laplace domain to the unitary step is

$$Y(s) = \frac{10}{s^2 + 2s + 10} \cdot \frac{1}{s}.$$

Expand $Y(s)$ in partial fractions

$$Y(s) = \frac{10}{s(s^2 + 2s + 10)} = \frac{A}{s} + \frac{Bs + C}{s^2 + 2s + 10},$$

where

$$A = [s\,Y(s)]\big|_{s=0} = 1,$$

while $B,\ C$ can be found by equating the coefficients of the terms with the same degree at both sides

$$B = -1,\ C = -2.$$

$Y(s)$ can then be written as

$$Y(s) = \frac{1}{s} - \frac{s+1}{(s+1)^2 + 3^2} - \frac{1}{3}\frac{3}{(s+1)^2 + 3^2}.$$

The resulting response in the time domain is

$$y(t) = 1 - e^{-t}\left[\cos(3t) + \frac{1}{3}\sin(3t)\right].$$

3) Calculate the Laplace transform of the signal

$$y(t) = (e^{-t} + \sin t)^2.$$

Pts: 2

Solution.
The Laplace transform $Y(s)$ of $y(t)$ is

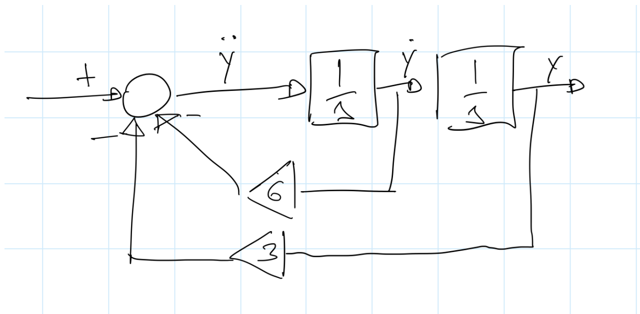
$$\begin{aligned}
Y(s) &= \mathcal{L}\left[e^{-2t} + 2e^{-t}\sin t + \sin^2 t\right] \\
&= \mathcal{L}\left[e^{-2t}\right] + \mathcal{L}\left[2e^{-t}\sin t\right] + \mathcal{L}\left[1 - \cos 2t\right] \\
&= \frac{1}{s+2} + \frac{2}{(s+1)^2 + 1} + \frac{1}{2}\left(\frac{1}{s} - \frac{s}{s^2+4}\right) \\
&= \frac{1}{s+2} + \frac{2}{s^2 + 2s + 2} + \frac{2}{s(s^2+4)}
\end{aligned}$$

4) Sketch the block diagram of the system

$$\ddot{y} + 6\dot{y} + 3y = u.$$

Pts: 2

Solution.
The block diagram is



5) Consider the system

$$G(s) = \frac{100}{s^3 + 11s^2 + 31s + 21}.$$

Design a lag network such that the phase margin is 60° **Pts: 2**

Solution.
The qualitative plot of the Nyquist diagram clearly shows that the intersection of the diagram with the imaginary axis is within the feasibility region of the lag network and can be conveniently chosen as the point A. We have to look for the $\omega_A : \mathcal{R}e[G(j\omega_A)] = 0$.

$$\begin{aligned} G(j\omega) &= \frac{100}{(1+j\omega)(3+j\omega)(7+j\omega)} = \frac{100}{(21-11\omega^2)+j(31\omega-\omega^3)} \\ &= 100\frac{(21-11\omega^2)-j(31\omega-\omega^3)}{(21-11\omega^2)^2+(31\omega-\omega^3)^2}, \\ \mathcal{R}e[G(j\omega)] &= \frac{100(21-11\omega^2)}{(21-11\omega^2)^2+(31\omega-\omega^3)^2}. \\ \mathcal{R}e[G(j\omega)] &= 0, \ \rightarrow \ \omega_A = \sqrt{\frac{21}{11}} = 1.38. \end{aligned}$$

Modulus and phase at ω_A are, respectively,

$$\varphi_A = 270°, \ M_A = \frac{100}{\sqrt{(21-11\omega_A^2)^2+(31\omega_A-\omega_A^3)^2}} = 2.49.$$

Point A must be moved to a point B with $\varphi_B = 240°, \ M_B = 1$. Hence,

$$M = \frac{M_B}{M_A} = \frac{1}{M_A} = 0.40, \ \varphi = \varphi_B - \varphi_A = -30° \text{ at } \omega_A = 1.38.$$

By applying the inversion formulas

$$C(s) = \frac{1+0.67s}{1+2.37s}.$$

6) Consider the second order system

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -25 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 25 \end{bmatrix} u, \ y = [1 \ 0] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

Calculate damping coefficient and natural frequency of the system. **Pts: 2**

Solution.
A second order system with natural frequency ω_n and damping δ

$$\ddot{y} + 2\delta\omega_n\dot{y} + \omega_n^2 y(t) = \omega_n^2\, u(t)$$

can be written in the state space as

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\omega_n^2 & -2\delta\omega_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ \omega_n^2 \end{bmatrix} u, \; y\; C = [1\; 0] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

$\omega_n = 5,\; \delta = 0.1$ follows. The same result could have been obtained by extracting ω_n and δ from the transfer function.

7) Consider the system

$$G(s) = \frac{k}{s(s+3)(s+5)}.$$

Use the root locus to study the stability of the closed-loop system as k varies. **Pts: 2**

Solution.
We start by studying the closed-loop poles for $k > 0$ (direct locus)

- The points $x \in [-\infty, -5] \bigcup [-3, 0]$ belong to the direct locus.
- The three poles go to infinity along three asymptotes that form with the real negative semi-axis angles

$$\vartheta_{a,\nu} = \frac{(2\,\nu + 1)\,\pi}{3} \qquad k > 0, \qquad\qquad \nu = 0,\,1,\,2.$$

$$\vartheta_{a,0} = \frac{\pi}{3},\; \vartheta_{a,1} = \pi,\; \vartheta_{a,2} = 5\frac{\pi}{3}.$$

  Two branches cross the imaginary axis. Hence, there is a value $\bar{k}$ such that the CL system is unstable for $k > \bar{k}$.
- The three asymptotes intersect the real axis at the point

$$\sigma_a = \frac{1}{n-m}\left(\sum_{i=1}^{m} z_i - \sum_{i=1}^{n} p_i\right) = \frac{1}{3}(-3-5) = -\frac{8}{3}.$$

- Denote by $\bar{k}$ the gain such that the poles of the CL system enters the rhp. Since the relative degree is 3, the sum of the poles do not depend on k. Hence, for $k = \bar{k}$ two poles will be on the imaginary axis, while the third will move to x

$$\underbrace{-3-5}_{k=0} = \underbrace{x}_{k=\bar{k}}.$$

  The value of $\bar{k}$ such that the third pole is in -8 can be found by using the $\bar{k} = \dfrac{\prod_{i=1}^{n} \eta_i}{\prod_{i=1}^{m} \lambda_i}$, where $\eta_i,\; \lambda_i$ are the distances of $s = -8$ from poles and zeros, respectively. Hence,

$$\bar{k} = 3 \cdot 5 \cdot 8 = 120.$$

The points $x \in [-5, -3] \bigcup [0, \infty]$ belong to the inverse locus. Hence the closed loop system is stable for

$$0 \le k \le 120.$$

8) Use the Routh criterion to solve the previous problem. **Pts: 2**

Solution.
The characteristic equation is

$$s^3 + 8s^2 + 15s + k = 0.$$

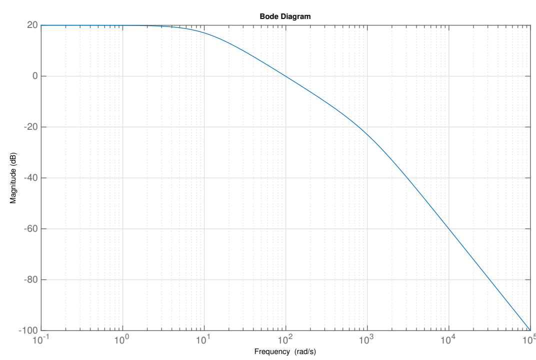
From the corresponding Routh table

$$\begin{array}{c|cc} 3 & 1 & 15 \\ 2 & 8 & k \\ 1 & \dfrac{120-k}{8} & 0 \\ 0 & \dfrac{8k}{120-k} & \end{array}$$

it follows that

$$0 < k < 120.$$

9) Consider the following Bode diagrams.



Find a 1st order approximation. **Pts: 2**

Solution.
The system has clearly two poles in 10 and 1000. By considering the dominant pole only, the resulting transfer function is

$$G(s) = \frac{10}{1 + 0.1s}.$$

10) Plot the Nyquist diagrams of the system

$$G(s) = \frac{s(s+10)}{s^2 + 0.4s + 1}.$$

Clearly explain the plotting rules you used and how. **Pts: 2**

Solution.

- **Starting point.**

$$G_0(s) \simeq G(s)|_{s \simeq 0} = 0 \;\Rightarrow\; \begin{cases} M_0 = 0, \\ \varphi_0 = \frac{\pi}{2}. \end{cases}$$

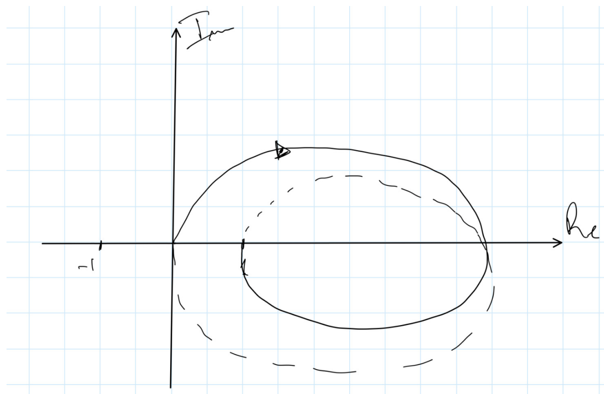
- **Final point.**

$$G_\infty(s) \simeq G(s)|_{s\simeq\infty} = 0 \;\Rightarrow\; \begin{cases} M_\infty = 1, \\ \varphi_\infty = 0. \end{cases}$$

  The final phase is approached from below (smaller phase, negative in this case) since $\Delta p = -10 - (-0.4) < 0$.

- **Path direction.** The plot starts clockwise since $\Delta\tau = 0.1 - 0.4 = -0.3 < 0$.

The resulting Nyquist plot is



11) For the system

$$G(s) = \frac{1}{s+10},$$

design a controller such that the tracking error at steady state is not larger than 0.1 when the reference signal is a ramp and the settling time is not larger than 1 s. **Pts: 3**

Solution.
Choose the controller as a PI. The specification on the steady-state error imposes that

$$e_\infty = \lim_{s\to 0} s\frac{1}{1+CG}R(s) = \lim_{s\to 0} \frac{s^2(s+10)}{s^2+(K_P+10)s+K_I}\cdot\frac{1}{s^2},$$

$$= \frac{10}{K_I} \leq 0.1 \;\to K_I \geq 100.$$

Set $K_I = 100$. The specification on the settling time imposes that the real part of the dominant poles should be smaller than -3. By placing the zero at -5, the loop gain must be larger than

$$K^* = \frac{7}{2} = 3.5.$$

This is achieved with $K_P = 20$. The resulting loop gain is

$$K = \frac{K_I}{10} = 10 > K^*,$$

thus satisfying the requirement on the settling time as well.

12) Consider the system

$$G(s) = \frac{(s+5)}{s(s-1)(s^2+s+25)}.$$

(a) Plot the asymptotic Bode diagrams of modulus and phase (stepped diagram). **Pts: 1**
Solution.
The initial slope of the modulus diagram is -20 dB/dec. The initial branch crosses the line $\omega = 1$ at -14 dB.
The initial and final phases are $-\frac{\pi}{2}$.
The zero $z_2 = -5$ introduces a phase shift of $+90°$.
The pole $p_1 = 1$ introduces a phase shift of $+90°$.
The conjugate complex poles with $\omega_n = 5,\ \delta = 0.1$ introduce a phase shift of $-180°$.

(b) Plot the linear approximation of the phase diagram. **Pts: 1**
Solution.
The zero $z_2 = -5$ introduces a slope change between $\omega_a^{z_2} = \frac{1}{4.81 \cdot 0.2} = 1.03$ and $\omega_b^{z_2} = \frac{4.81}{0.2} = 24.05$.
The pole $p_1 = 1$ introduces a slope change between $\omega_a^{p_1} = \frac{1}{4.81} = 0.21$ and $\omega_b^{p_1} = 4.81$.
The conjugate complex poles with $\omega_n = 5,\ \delta = 0.1$ introduce a slope change between $\omega_a^{\omega_2} = \frac{5}{4.81^{0.1}} = 4.27$ and $\omega_b^{\omega_2} = 5 \cdot 4.81^{0.1} = 5.85$.

(c) Show the deviation of the exact modulus diagram from the asymptotic at the breaking points and sketch the exact modulus diagrams. **Pts: 1**
Solution.
The deviation of the exact modulus diagram from the asymptotic is

- 3+0.2+0.5=-2.3 dB at $\omega = 1$,
- 3+14+0.1≈17 dB at $\omega = 5$.

13) Write the equation in the discrete-time domain of a digital filter that attenuates the components of the input signal with frequencies $\omega \in [10, 50]$rad/s, of at least a factor 10, without introducing a delay larger than $\tau_{\max} = 0.2s$ in $[10, 50]$rad/s. Discretize the filter with the pole-zero matching method. **Pts: 3**

Solution.
The modulus of the TF of the filter must be at least -20 dB at $\omega = 10$rad/s. Set

$$G(s) = \frac{1}{s+1}.$$

It follows that

$$\underline{/G}j(\bar{\omega}) = -90° = \underline{/G}(j\underline{\omega}).$$

Such a phase shift introduces a delay in the considered frequency range between

$$\underline{\tau} = \frac{\pi}{2} \cdot \frac{1}{\underline{\omega}}, \text{ and } \bar{\tau} = \frac{\pi}{2} \cdot \frac{1}{\bar{\omega}}.$$

The ZOH introduces a phase shift of $-\frac{T}{2}\omega$, where T is the sampling time. Then it must be $\max\left\{\underline{\tau} + \frac{T}{2}, \bar{\tau} + \frac{T}{2}\right\} \leq \tau_{\max}$. It follows $T = 0.1$.
By discretizing the filter with the pole-zero matching method, we obtain

$$G_d(z) = \frac{k}{z - e^{aT}} = \frac{k}{z - 0.9048},$$

where the gain k needs to be adjusted such that

$$G_d(1) = 10.50k = G(0) = 1.$$

It follows $k = 0.0952$. The equation of the filter in the discrete time domain is

$$y(t) = 0.9048y(t-1) + 0.095u(t-1).$$

14) Consider the system

$$G(s) = \frac{1}{s(s+3)}. \tag{1}$$

Design the parameters of a relay with hysteresis (i.e., a and b), in order to obtain a limit cycle with amplitude $X = 1$ and frequency $\omega = 5.19$ rad/s.

Hint. The relay with hysteresis has the following describing function

$$F(X) = \frac{4b}{\pi X}\left[\sqrt{1-\left(\frac{a}{X}\right)^2} - j\frac{a}{X}\right], \quad X > a.$$

Pts: 3

Solution.
The relay with hysteresis enable a limit cycle if $\bar{X} = 1,\ \bar{\omega} = 5.19$ solve the pseudo-characteristic equation

$$G(j\bar{\omega}) = -\frac{1}{F_2(\bar{X})},$$

where $F_2(X)$ is the DF of the relay with hysteresis. By equating the real and imaginary parts of

$$G(j\bar{\omega}) = -\frac{1+3j}{\bar{\omega}^3 + 9\bar{\omega}}, \quad -\frac{1}{F_2(\bar{X})} = -\frac{\pi\bar{X}}{4b}\left(\sqrt{1-\left(\frac{a}{\bar{X}}\right)^2} + j\frac{a}{\bar{X}}\right),$$

it follows that

$$a = \frac{3}{\sqrt{10}} = 0.95, \ \ b = \frac{\pi a}{4}\frac{\bar{\omega}^3 + 9\bar{\omega}}{3} = 46.32.$$

15) Consider the system

$$1000\dot{y} + 200y = u + d.$$

Design a controller such that

(a) a zero steady-state error is achieved in presence of a constant output reference y_{ref} and a constant disturbance d, while a steady-state output error smaller than 5 is achieved for ramp references $2t$.

(b) the closed-loop system exhibits a maximum overshoot $S_{\max} = 10\%$ and a settling time at 5% not larger than 2s.

Pts: 5

Solution.
We start from the static specs. Apply the final value theorem

$$e_\infty = \lim_{s\to 0} s\frac{1}{1+\dfrac{C(s)}{100s+200}}Y_{\text{ref}}(s) = \begin{cases} \dfrac{200}{200+C(s)}_{|s=0}Y_{\text{ref}} & \text{constant output reference,} \\[2ex] \dfrac{200}{s(200+C(s))}_{|s=0}2 & \text{ramp output reference.} \end{cases}.$$

With $C(s) = \dfrac{C_1(s)}{s}$ we obtain

$$e_\infty = \begin{cases} 0, & \text{constant speed} \\ \dfrac{400}{C_1(0)} & \text{constant acceleration.} \end{cases}$$

In order for the error to be bounded by $\bar{e} = 5$, C_1 must be designed such that

$$\frac{400}{C_1(0)} \Rightarrow C_1(0) > \frac{400}{\bar{e}}.$$

With such controller structure the steady state error due to the disturbance is

$$e_\infty = \lim_{s\to 0} s \cdot \frac{1}{1000s + 200 - C(s)} \cdot \frac{1}{s} = \lim_{s\to 0} \frac{s}{1000s^2 + 200s - C_1(s)} = 0.$$

Hence, $C(s) = \dfrac{C_1(s)}{s}$ also rejects the effects of the disturbance on the steady state tracking error.
Consider the loop transfer function

$$L(s) = \frac{C_1(s)}{s(1000s + 200)}.$$

$C_1(s)$ must be designed to satisfy also the following requirements

- $S < 10\% \Rightarrow \delta > 0.6$,
- $T_a < \bar{T}_a = 2, \ \Rightarrow |\sigma_a| > \dfrac{3}{\bar{T}_a} = 1.5.$

where δ and σ_a are the damping and the real part of the CL dominant poles.
The requirements on damping and real part of the dominant poles provide a region where the poles are allowed to be, determined by a vertical line passing by $\sigma_a = -1.5$ and the lines passing by the origin and forming with the real axis an angle $\alpha = \arccos\delta = 53°$. The locus has to pass by the intersection between these two lines, that is, the points

$$P_{1,2} = (\sigma_a, \sigma_a \tan\arccos\delta) = (-1.5, \pm 2.04).$$

By denoting with $\alpha_1, \ \alpha_2$ the angles formed by P with the poles in 0 and in $-\frac{b}{M}$, respectively, and with β the angle formed with the zero of $C_1(s)$ to be placed, the following must hold

$$(2\nu + 1)\pi = \beta - \alpha_1 - \alpha_2,$$

for some integer ν. Since $\alpha_1 = 126.24°, \ \alpha_2 = 122.42°$, with $\nu = -1$ $\beta = -\pi + \alpha_1 + \alpha_2 = 68.66°$.
By simple geometrical arguments

$$|z| - \sigma_a = P_y \cot\beta \ \Rightarrow \ z = -(\sigma_a + P_y \cot\beta) = -2.29.$$

The controller is then

$$C(s) = K\frac{(s + 2.29)}{s},$$

with K to be chosen such that the poles are at $P_{1,2}$. The loop TF is

$$L(s) = K_1 \frac{s + 2.29}{s(s + 0.2)} \;\; K_1 = \frac{K}{M}.$$

K_1 is found as

$$K_1 = \frac{\sum \eta_i}{\sum \lambda_i},$$

where η_i and λ_i are the distances of P from the poles and zeros.

$$\eta_1 = \sqrt{P_x^2 + P_y^2} = 2.54,\;\; \eta_2 = \frac{P_y}{\sin(\pi - \alpha_2)} = 2.42,\;\; \lambda = \frac{P_y}{\sin \beta} = 2.20 \;\Rightarrow\; K_1 = 2.80.$$

The resulting K satisfies the boundedness requirement on the steady state tracking error for ramp reference

$$K = 2.8 \cdot 10^3 > b \frac{A_{\text{ref}}^{\max}}{\bar{e}}.$$