Гамов Павел М8О-407Б-18 ②.

$\frac{du}{dt} = a\frac{d^2u}{dx^2}$; $u(x,0) = \sin(\frac{n x}{l})$

$u(0,t) = u(l,t) = 0$

метод разделение переменных.

$u(x,t) = X(x)\,T(t)$; ~~$u(x,t) = 1$~~

$X(x)\,T'(t) = a X''(x) T(t)$; $\frac{l \cdot T'(x)}{aT(t)} = \frac{X''(x)}{X(x)} = -\lambda^2$

найдем X: $X''(x) + \lambda^2 X(x) = 0$

$X(x) = C_1 \cos\lambda x + C_2 \sin\lambda x$;

Краевые условия:

$u(x,t) = X(0)\,T(t) = 0 \Rightarrow X(0) = 0$

$X(0) = C_1 = 0$; $u(l,t) = X(l)\,T(t) = 0$; $X(l) = 0$

$X(l) = C_2 \sin\lambda l = \lambda = \frac{\pi n}{l}$;

получаем собств. решение

$X_n(x) = C \sin(\frac{\pi n}{l} x)$;

Найдем T:

Найдем T:

$T'(t) = a\lambda^2 T(t)$; $T_n(t) = C\exp\frac{a\pi^2 n^2 t}{l^2}$;

$u_n(x,t) = C\exp\left(\frac{a\pi^2 n^2 t}{l^2}\right)$; $u_n(x,t) = C\sin\left(\frac{\pi n x}{l}\right)\exp\left(\frac{a\pi^2 n^2 t}{l^2}\right)$

$$u(x,t) = \sum_{n=1}^{\infty} C_n \sin\left(\frac{\pi n x}{l}\right)\exp\left(\frac{a\pi^2 n^2 t}{l^2}\right)$$

Начальное условие:

$$u(x,0) = \sum_{n=1}^{\infty} C_n \sin\frac{\pi n x}{l} = \sin\left(\frac{\pi x}{l}\right)$$

Следовательно все C_n кроме C_1 равны нулю; $C_1 = 1$

$$u(x,t) = \sin\left(\frac{\pi x}{l}\right)\exp\left(\frac{a\pi^2 t}{l^2}\right)$$

Проверка:

$u_x(x,t) = \frac{\pi}{l}\cos\left(\frac{\pi x}{l}\right)\exp\left(\frac{a\pi^2 t}{l^2}\right)$;

$u_{xx}(x,t) = \frac{a\pi^2}{l^2}\sin\left(\frac{\pi x}{l}\right)\exp\left(\frac{a\pi^2 t}{l^2}\right)$

$u_t(x,t) + a u_{xx}(x,t) = \frac{a\pi^2}{l^2}\sin\left(\frac{\pi x}{l}\right)\exp\left(\frac{a\pi^2 t}{l}\right) -$

$- a\frac{\pi^2}{l^2}\sin\left(\frac{\pi x}{l}\right)\exp\left(\frac{a\pi^2 t}{l^2}\right) = 0$