Гамов Павел М80-407Б-18 ③

$$\frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial x^2}; \quad u(x,0) = \sin(\pi x)$$

$u_t(x,0) = 0 \qquad u(0,t) = u(l,t) = 0$

метод разделения переменных.

$u(x,t) = X(x)T(t); \quad X(x)T''(t) = a^2 X''(x)T(t)$

$$\frac{1}{a^2}\frac{T''(t)}{T(t)} = \frac{X''(x)}{X(x)} = -\lambda^2$$

Найдем X: $\quad X''(x) + \lambda^2 X(x) = 0$

$X(x) = C_1 \cos\lambda x + C_2 \sin\lambda x$

Краевые условия: $u(0,t) = X(0)T(t) = 0$

$X(0) = 0; \quad X(0) = C_1 = 0;$

$u(l,t) = X(1)T(t) = 0; \quad X(1) = 0$

$X(1) = C_2 \sin(\lambda) \Rightarrow \lambda = \pi n$

получ. собств. решения

$X_n(x) = C\sin(\pi n x)$

Найдем T:

$T''(t) + a^2\lambda^2 T(t) = 0$

$T(t) = \cancel{C_0}\ C_1 \cos(a\lambda t) + C_2 \sin(a\lambda t)$

$T_n(t) = C_1 \cos(a\Pi_n t) + C_2 \sin(a\Pi_n t)$

$\cancel{T_n(t)} = \cancel{C_1}$ $u_n(x,t) = \sin(\Pi_n x)\left[C_1 \cos(a\Pi_n t) + C_2 \cdot\right.$

$\left.\cdot \sin(a\Pi_n t)\right]$

$u(x,t) = \sum_{n=1}^{\infty} \sin(\Pi_n x)\left[C_1 \cos(a\Pi_n t) + C_2 \sin(a\Pi_n t)\right]$

Нач. условие: $u(x,0) = \sum_{n=1}^{\infty} \sin(\Pi_n x)\left[C_{1n} \cos(0) + \right.$

$\left. + C_{2n} \cancel{\cos} \sin(0)\right] = \sum^{\infty} C_{1n} \sin(\Pi_n x) = \sin \Pi x$

Следовательно, все C_{1n} кроме C_{11} равны 0, $C_{11} = 1$

$u_t(x,0) = \sum^{\infty} \sin(\Pi_n x)\left[-C_{1n} \sin(0) + C_{2n} \cos(0)\right] = 0$

все C_{2n} равны нулю. $u(x,t) = \sin(\Pi x) \cos(a\Pi t)$

проверка:

$u_x(x,t) = \Pi \cos(\Pi x) \cos(a\Pi t)$

$u_{xx}(x,t) = -\Pi^2 \sin(\Pi x) \cos(a\Pi t)$

$u_t(x,t) = -a\Pi \sin(\Pi x) \sin(a\Pi t)$

$u_{tt}(x,t) = -a^2\Pi^2 \sin(\Pi x) \cos(a\Pi t)$

$u_{tt}(x,t) - a^2 u_{xx}(x,t) = -a^2\Pi^2 \sin(\Pi x) \cos(a\Pi t) +$

$+ a^2\Pi^2 \sin(\Pi x) \cos(a\Pi t) = 0$