Generic Finitude for Dziobek Central Configurations

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Understanding the Title

What is a Configuration?

A *configuration* is a vector $x = (x_1, \ldots, x_n) \in (\mathbb{R}^d)^n$, such that $x_1, \ldots, x_n \in \mathbb{R}^d$ and $x_i \neq x_j, \forall i \neq j$.

When is it central?

Given a such that $2a \in \mathbb{Z}$, we say that x is a *central configuration associated with a semi-integer* a if there are $\lambda \neq 0$ and $c \in \mathbb{R}^d$ such that

$$\sum_{j \neq i} m_j r_{ij}^{2a} (x_i - x_j) = \lambda (x_i - c), \quad i = 1, \ldots, n. \tag{1}$$

What makes her Dziobek's?

The configuration x is said *Dziobek* if the dimension of the smallest affine subspace containing $x_1, \ldots, x_n$ measures $n - 2$.

What is a Configuration?

Central Configurations are Isometries and Homotheties Invariant.

By rotating, translating and dilating a central configuration, we still get a central configuration. In this way, let us consider the configuration classes module isometries and homotheties.

What do we mean by finitude?

The problem of finitude consists in knowing whether the number of classes of central configurations associated with n positive and arbitrary masses $m_1, \ldots m_n$ is finite.

And why Generic?

What we were able to prove is that, with the exertion of a Zariski closure in the real-positive n-dimensional Euclidean space of the masses, the finiteness is verified.

Knowing the History of the Problem

The Importance of Central Configurations.

They play an important role in the study of the Newtonian n-body problem. They are initial conditions for the only explicit solutions of the equations of motion.

The Newtonian case.

When $a = \frac{-3}{2}$ the equations that determine the central configurations are deduced from the relations of forces of attraction between bodies.

Knowing the History of the Problem

Some results about Finitude.

Regardless of the number of fields, Moulton proved that there is a unique collinear central configuration, for $a = -3/2$. For three bodies $(n = 3)$ Lagrange showed that there is only one planar central configuration, namely the equilateral triangle. More generally, Saari generalized by noting that given n positive masses, there exists a single $(n-1)$-dimensional central configuration which is the regular simplex.

Moeckel and Generic Finitude.

The first knowledge about it is due to Moeckel when he demonstrated the generic finitude in the Newtonian case. Later, Dias extended it to $a \in \mathbb{Z}/2$, where this is the result that we will present.

Rewriting the Problem Using Mutual Distances

To avoid rid of rotations and translations, using the distances between bodies to describe their positions in space proved to be a useful technique.

A configuration $x = (x_1, \ldots, x_n)$ can be associated with a matrix $A(x) = (a_{ij})$ called *Cayley-Menger* whose main diagonal is zero , its first row and column is completed with 1 and the remaining entries are given by $a_{ij} = r^2_{i-1,j-1}$. If x is from Dziobek, then

$$F(r) := \det A(x) = 0. \tag{2}$$

Let $r_0^{2a} = \frac{\lambda}{M}$, where M is the total mass. It can be proved that there exist constants $z_1, \ldots, z_n$ such that

$$g_{ij}(r, z) := r_{ij}^{2a} - r_0^{2a} - z_i z_j = 0, \quad \forall\, i \neq j. \tag{3}$$

Rewriting the Problem Using Mutual Distances

Such z_i together with the mutual distances also satisfy the equations

$$\Gamma_0(z,m) := m_1 z_1 + \cdots + m_n z_n = 0, \tag{4}$$

$$\Gamma_i(r,\Delta_0,z,m) := \sum_{j=1}^{n} m_j z_j ||x_j||^2 - \sum_{j=1}^{n} m_j z_j ||r_{ij}||^2 = 0, \qquad i = 1,\ldots n. \tag{5}$$

Taking $q = n(n-1)/2$, the following affine manifold in the space $\mathbb{R}^{q+2n+1}$ is such that its elements describe Dziobek's central configurations:

$$\widetilde{V}^a := \left\{ (r,\Delta_0,z,m) \in \mathbb{C}^{q+2n+1}; \begin{array}{c} F(r) = 0 \\ g_{ij}^a(r,z) = 0 \\ \Gamma_0(z,m) = 0 \\ \Gamma_i(r,\Delta_0,z,m) = 0 \end{array} \right\}. \tag{6}$$

Attacking with Algebraic Geometry

The previous affine variety $\widetilde{V}^a$ can be reduced to a quasi-affine variety V^a so that in this process no point that is associated with a Dziobek configuration is discarded.

In order to obtain the finiteness it is necessary to show that $\dim V^a \leq n$. For this we use the following proposition where we get such an estimate by calculating only the rank of the Jacobian matrix.

Jacobian inequality

Let $V = \{P \in \mathbb{C}^N ; f_1(P) = \cdots = f_m(P) = 0\}$ an affine manifold and $P \in V$. Then

$$\dim_P V \leq N - \text{posto } (J(f_1, \ldots, f_m)(P)).$$

By proving that rank $J(g_{ij}^a, F, \Gamma_0, \Gamma_i)(P) \geq n + q + 1$ we obtain

$$\dim V^a \leq \max_{P \in V^a} \dim_P V^a \leq (q + 2n + 1) - (n + q + 1) = n.$$

Attacking with Algebraic Geometry

The regular projection map in mass space

$$\begin{array}{rccc} \pi: & V^a & \longrightarrow & \mathbb{C}^n \\ & (r, \Delta_0, z, m) & \longmapsto & m \end{array}$$

immediately gives us generic finiteness if we find an affine variety $\widetilde{B} \subset \mathbb{C}^n$ such that if $m \in \mathbb{C}^n \backslash \widetilde{B}$, then the fiber $\pi^{-1}(m)$ is finite. This is done through the previous result and the following Theorem:

Fiber Dimension Theorem

If $f: V_1 \longrightarrow V_2$ is a regular surjective function with V_1 and V_2 irreducible then

i) $\dim f^{-1}(y) \geq |\dim V_1 - \dim V_2|, \forall\, y \in V_2$;

ii) There is a non-empty open set U of V_2 such that

$$\dim f^{-1}(y) = |\dim V_1 - \dim V_2|, \forall y \in U.$$

A little Topology and here's the Result

From $\widetilde{B}$ we can obtain a proper real affine submanifold B, where if $m \in \mathbb{R}^n \backslash B$, then there is a finite amount of central Dziobek configurations associated with m.
Furthermore, it is also true that there is an upper bound for this finiteness that is independent of the chosen mass. This follows from:

Thom-Milnor Theorem

Consider $f_1, \ldots, f_m \in \mathbb{R}[T_1, \ldots, T_n]$. Let the affine variety $V = V(f_1, \ldots, f_m)$ with the induced Zariski topology. Then the number of connected components of V is less than or equal to $\beta(2\beta - 1)^{n-1}$, where $\beta = \max\{\text{grau}\, f_1, \ldots, \text{grau}\, f_m\}$.

A little Topology and here's the Result

Hence we get

Theorem

If $n \geq 4$, for all $m = (m_1, \ldots, m_n) \in \mathbb{R}^n \backslash B$, the number of Dziobek configurations with masses $m_1, \ldots, m_n$ with $a \in \mathbb{Z}/2$ is less than or equal to $\beta(2\beta - 1)^{q-1}$ where
$\beta = \max\{-2a + 2, 2n\}$
if $a < 0$ and $\max\{2a, 2n\}$ if $a > 0$.

References

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