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A Stochastic Control Approach to Reciprocal Diffusion Processes

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Abstract. The problem of forcing a nondegenerate diffusion process to a given final configuration is considered. Using the logarithmic transformation approach developed by Fleming, it is shown that the perturbation of the drift suggested by Jamison solves an optimal stochastic control problem. Such perturbation happens to have minimum energy between all controls that "bring" the diffusion to the desired final distribution. A special property of the change of measure on the path-space that corresponds to the aforesaid perturbation of the drift is also shown.

1. Introduction

The problem of assigning an initial and a final condition to a Markov process was first formulated by Schrödinger in [27]. Schrödinger's consideration can be synthetized as follows. Let us assume that a Markov process $x(t)$ in $\mathbf{R}^n$ has a transition density $q(s, y, t, x)$, $0 \leq s \leq t \leq T$, and that the density of this process can be observed. If we start with an initial density $p_0(x)$, we would like to conclude that at time T we observe $p_T(x) = \int q(s, y, T, x)p_0(y)\, dy$. Suppose, however, that we observe a density π_T that is different from p_T: this means that our assumption about the Markov kernel of $x(t)$ is wrong. Thus the following question arises: what is the transition density $\tilde{q}(s, y, t, x)$ for which $\int \tilde{q}(s, y, T, x)p_0(y)\, dy = \pi_T(x)$ and which is the closest to $q(s, y, t, x)$ in some suitable sense? As a consequence of this question, Bernstein in [3] introduced a class of stochastic processes that is wider than the Markov one.

Let $x(t)$ be a stochastic process in the interval $[0, T]$ defined on a probability space $(\Omega, \mathscr{F}, P)$. For $0 \leq s < t \leq T$, we define the following σ-algebras:

$$\mathscr{B}_{s,t} = \sigma\{x(\tau)\colon s \leq \tau \leq t\},$$

$$\mathscr{F}_{s,t} = \sigma\{x(\tau)\colon \tau \leq s \text{ or } \tau \geq t\}.$$

We say that $x(t)$ is a *reciprocal process* if, for any $0 \leq s < t \leq T$ and $A \in \mathscr{B}_{s,t}$, $B \in \mathscr{F}_{s,t}$,

$$P\{A \cap B | x(s), x(t)\} = P\{A | x(s), x(t)\} P\{B | x(s), x(t)\}. \tag{1.1}$$

From the time reversibility of the Markov property [26], it follows that any Markov process is reciprocal, but the reverse is false. For a more detailed discussion about reciprocal processes see [4]–[9]. In particular, interesting results were obtained by Krener [21]–[23], who constructed a second-order stochastic calculus as a general machinery for the study of a large class of reciprocal processes.

Under suitable technical assumptions, a reciprocal process admits an *intermediate density* $p(s, x, t, y, u, z)$, $0 \leq s < t < u \leq T$, which expresses the conditional density of $x(t)$ given $x(s) = x$ and $x(u) = z$. In particular, if $x(t)$ is a Markov process with a transition density $q(s, y, t, x)$, the following relation holds:

$$p(s, x, t, y, u, z) = \frac{q(s, x, t, y) q(t, y, u, z)}{q(s, x, u, z)}. \tag{1.2}$$

It is shown in [18] that, given two measures μ_0 and μ_T and the intermediate density (1.2), we can construct a reciprocal process $x(\cdot)$ having (1.2) as the intermediate density, and such that $x(0)$ and $x(T)$ are distributed according to μ_0 and μ_T, respectively. Furthermore, this process $x(t)$ has the Markov property. If, in particular, $q(s, x, t, y)$ is the Markov kernel of a diffusion process, the foregoing construction corresponds to a change of the drift or, equivalently, to a change of measure on the path-space. This suggests the following version of Schrödinger's problem: given a controlled diffusion

$$dx^u = (b + u)\, dt + \sigma\, dw \tag{1.3}$$

and two measures μ_0 and μ_T, what is the *minimum energy control* u^* for which x^{u^*} evolves from μ_0 to μ_T?

This problem is rigorously stated and solved in this paper. The method of the logarithmic transformation developed by Fleming [10]–[13] is used to obtain a stochastic control problem from a parabolic partial differential equation (p.d.e.). Recently, the relevance of Fleming's variational principles in the framework of canonical structures for dissipative systems has been emphasized by F. Guerra (unpublished notes). The connection with Schrödinger's problem is established through results of Beurling [4] and Jamison [17]–[19], which we recall in Section 2. In Section 3 we solve our version of Schrödinger's problem. An explicit computation for a linear case is presented in Section 4. In Section 5 a special property of the optimal control u^* is exhibited. Namely, the measure transformation that changes the drift of a diffusion $x(t)$ from b to $b + u^*$, leaves invariant the family of the "most probable paths" for $x(t)$ [30].

2. Preliminaries

Given a probability space $(\Omega, \mathscr{F}, P)$, we consider the following n-dimensional stochastic differential equation:

$$dx(t) = b(x(t), t)\, dt + \sigma(x(t), t)\, dw(t), \tag{2.1}$$

where we assume that, for any ξ, $x \in \mathbf{R}^n$, $t \in [0, T]$, $\xi^t a(x, t)\xi \geq \eta \xi^t \xi$ for some constant $\eta > 0$, where $a := \sigma\sigma^t$.

Some further assumptions about (2.1) are needed for our purposes.

A1. For any square integrable random variable x_0 independent of $\{w(t); 0 \leq t \leq T\}$, (2.1) admits a weak solution in $[0, T]$ with initial condition $x(0) = x_0$.

Now, let

$$L := b \cdot \nabla + \tfrac{1}{2} \sum_{ij=0}^{n} a_{ij} \frac{\partial^2}{\partial x_i\, \partial x_j}$$

be the generator of any diffusion process solving (2.1). We need the following:

A2. The p.d.e. $(\partial/\partial t)f + Lf = 0$ has a fundamental solution $q(s, y, t, x)$ which is twice continuously differentiable with respect to y, and continuously differentiable with respect to s. Moreover, for any positive measurable function g such that $\int q(0, x, T, z)g(z)\, dz < +\infty$ for some $x \in \mathbf{R}^n$,

$$h(x, t) := \int q(t, x, T, z)g(z)\, dz$$

belongs to $C^{2,1}(\mathbf{R}^n \times [0, T))$ and satisfies $(\partial/\partial t + L)h = 0$.

Under rather classical assumptions on the coefficients of (2.1), it can be shown that A1 and A2 hold.

Proposition 2.1 [18], [29]. *The following conditions imply* A1 *and* A2:

(i) *The coefficients $b_i(x, t)$ are bounded and continuous, and satisfy Holder conditions with respect to x.*
(ii) *The coefficients $a_{ij}(x, t)$ are in $C^1(\mathbf{R}^n \times [0, T])$; a_{ij} and $\partial a_{ij}/\partial x_k$ are bounded and satisfy Holder conditions both with respect to x and t.*

Proof. A1 is proved in Section 6.1 of [29].

Using the estimates in [8] for $q(s, y, t, x)$ A2 can be established if g is bounded [14]. Otherwise, we can choose a sequence $\{f_n\}$ of bounded functions such that $f_n \uparrow f$ a.s. Defining $h_n(x, t) = \int q(t, x, T, z)f_n(z)\, dz$ and using a special Harnack inequality [2], it can be shown that $h_n \to h$ uniformily on $K \times [0, T]$ for any compact K. Therefore $h(x, t)$ is in $C^{2,1}(\mathbf{R}^n \times [0, T))$. For further details see [18] and references therein. □

Remark. If b and σ are constant, then A1 and A2 hold. Indeed, the previous proof can be easily adapted to this case, since the estimations of the transition kernel are easily satisfied (we know it explicitly).

In the following, for a positive definite $n \times n$ matrix α, we use the notation

$$\|\xi\|_\alpha^2 := \sum_{ij=1}^{n} \alpha_{ij}\xi_i\xi_j.$$

It is well known [15] that an absolutely continuous change of measure on the space of trajectories is related to a change of the drift of a diffusion. A special case was considered by Jamison in [18] leading to the following theorem.

Theorem 2.1. *Let $x(t)$ be a weak solution of (2.1) and let $h(x, t)$ be a strictly positive classical solution ($h \in C^{2,1}(\mathbf{R}^n \times [0, T])$) of the equation*

$$\frac{\partial h}{\partial t} + Lh = 0 \qquad \text{in } [0, T) \tag{2.2}$$

such that $E\{h(x(t), t)\} < +\infty$ and $E_{x,s}\{h(x(t), t)\} = h(x, s)$ for any $0 \le s < t < T$. Then the stochastic differential equation

$$dx = (b + a\nabla \log h)\, dt + \sigma\, dw(t) \tag{2.3}$$

admits a weak solution in $[0, T)$ (i.e. in $[0, T - \varepsilon]$ for any $\varepsilon > 0$). Moreover, if there exists a measurable function $g(x)$ such that $E_{x,s}(g(x(T))) = h(x, s)$, then (2.3) has solution in $[0, T]$. Finally, the transition density of a process solving (2.3) is

$$q^h(s, y, t, x) = \frac{q(s, y, t, x)h(x, t)}{h(y, s)}. \tag{2.4}$$

Proof. The process $Z(t) = h(x(t), t)/h(x(0), 0)$ is a martingale with respect to the natural filtration $F_t = \sigma\{x(s): 0 \le s \le t\}$. Since $E[Z(t)] = 1$, we can define the following change of probability measures on Ω through

$$\frac{dQ}{dP} = Z(T - \varepsilon)$$

for a fixed $\varepsilon > 0$.

Let $0 \le s < t \le T - \varepsilon$ and $f \in C_0^\infty(\mathbf{R}^n)$. Using the fact that $Z(t)$ is a martingale we get

$$\begin{aligned}
&E^Q_{x,s}\{f(x(t))\} - f(x)\} \\
&= \frac{1}{h(x, s)} [E_{x,s}\{h(x(t), t)f(x(t)) - h(x, s)f(x)\}] \\
&= \frac{1}{h(x, s)} \left[E_{x,s}\left\{ \int_s^t [hLf + a\nabla h \cdot \nabla f]\, d\tau \right\} \right] \\
&= \frac{1}{h(x, s)} E_{x,s}\left\{ \int_s^t h(x(\tau), \tau) \left[(b + a\nabla \log h) \cdot \nabla f + \tfrac{1}{2} \sum_{ij=1}^{n} a_{ij} \frac{\partial^2 f}{\partial x_i \partial x_j} \right](x(\tau), \tau)\, d\tau \right\} \\
&= \frac{1}{h(x, s)} E_{x,s}\left\{ h(x(t), t) \int_s^t \left[(b + a\nabla \log h) \cdot \nabla f + \tfrac{1}{2} \sum_{ij=1}^{n} a_{ij} \frac{\partial^2 f}{\partial x_i \partial x_j} \right](x(\tau), \tau)\, d\tau \right\} \\
&= E^Q_{x,s}\left\{ \int_s^t \left[(b + a\nabla \log h) \cdot \nabla f + \tfrac{1}{2} \sum_{ij=1}^{n} a_{ij} \frac{\partial^2 f}{\partial x_i \partial x_j} \right](x(\tau), \tau)\, d\tau \right\},
\end{aligned}$$

where we have applied Ito's rule to $h(x, t)f(x)$. This means that the law of $x(\cdot)$, as a process defined in (Ω, F, Q), solves the martingale problem for

$$L^h = (b + a\nabla \log h)\cdot\nabla + \tfrac{1}{2}\sum_{ij} a_{ij}\frac{\partial^2}{\partial x_i \partial x_j} \tag{2.5}$$

in $[0, T-\varepsilon]$. This is equivalent to saying that (2.3) has a weak solution in $[0, T-\varepsilon]$.

If $h(x, t) = E_{x,t}(g(x(T))$, we can define $dQ/dP = g(x(T))$ and we obtain again

$$\begin{aligned} &E^Q_{x,s}\{f(x(t))\} - f(x)\} \\ &\quad = E^Q_{x,s}\left\{\int_s^t \left[(b + a\nabla \log h)\cdot\nabla f + \tfrac{1}{2}\sum_{ij=1}^n a_{ij}\frac{\partial^2 f}{\partial x_i \partial x_j}\right](x(\tau), \tau)\, d\tau\right\}. \end{aligned}$$

Since f has compact support, and $x(t) \to x(T)$ a.e., we can let $t \to T$ and conclude by the Dominate Convergence Theorem.

To show (2.4) we just have to check that $E^Q_{x,s}\{f(x(t))\} = \int q^h(s, x, t, y)f(y)\, dy$ for any $f \in C_0^\infty(\mathbf{R}^n)$. Indeed,

$$\begin{aligned} E^Q_{x,s}\{f(x(t))\} &= \frac{1}{h(x, s)} E_{x,s}\{h(x(t), t)f(x(t))\} \\ &= \frac{1}{h(x, s)}\int q(s, x, t, y)h(y, t)f(y)\, dy = \int q^h(s, x, t, y)f(y)\, dy. \quad \square \end{aligned}$$

With a particular choice of h, we obtain the generalization of the well-known brownian bridge (see [16]).

Corollary 2.1. *Assume* A1 *and* A2. *For fixed* $x_1 \in \mathbf{R}^n$, *consider the strictly positive function*

$$h(x, t) = q(t, x, T, x_1).$$

Then there exists a weak solution $y(t)$ *of the stochastic differential equation*

$$dy = [b + a\nabla \log h]\, dt + \sigma\, dw,$$

$$y(0) = x_0 \in \mathbf{R}^n$$

in $[0, T)$. *If* $x(t)$ *is a weak solution of* (2.1) *with initial condition* x_0, *then* $y(t)$ *has the "conditioned law" of* $x(t)$, *in the following sense*:

$$E\{g(y(t))\} = E\{g(x(t)) \mid x(T) = x_1\}$$

for any $t \in [0, T)$ *and any bounded measurable function* g.

Beurling in [4] and Jamison in [17] and [18] showed that the possibility of "forcing" a Markov process to an assigned final distribution is related to an automorphism on the space of the product measures on $\mathbf{R}^n \times \mathbf{R}^n$. Now we state a version of a theorem of Beurling [4] that is particularly useful for our purposes. An application of this theorem is discussed in Section 3.

Theorem 2.2. *Given two probability measures μ_0 and μ_T on $\mathbf{R}^n$ and the Markov kernel $q(s, y, t, x)$, there exists a unique pair of σ-finite measures (ν_0, ν_T) on $\mathbf{R}^n$ such that the measure μ on $\mathbf{R}^n \times \mathbf{R}^n$ defined by*

$$\mu(E) = \int_E q(0, y, T, x)\nu_0(dy)\nu_T(dx) \tag{2.6}$$

has marginals μ_0 and μ_T. Furthermore, $\nu_T \approx \mu_T$ ($\nu_0 \approx \mu_0$) where "$\approx$" denotes mutual absolute continuity of measures.

3. A Stochastic Control Problem with a Final Constraint

Consider the controlled diffusion process that solves the following stochastic differential equation (in a weak sense):

$$dx^u(t) = [b(x^u(t), t) + u(t)]\,dt + \sigma(x^u(t), t)\,dw(t). \tag{3.1}$$

A control $u(t)$ will be said to be *admissible* if:

(i) $u(t)$ is $\sigma\{x^u(t)\}$-measurable;
(ii) (3.1) admits a weak solution in $[0, T]$;
(iii) $E \int_0^T \|u(t)\|_{a^{-1}}^2\,dt < \infty$.

Given two probability measures μ_0 and μ_T, the following problem can be considered as a rigorous stochastic control version of the one formulated by Schrödinger in [27].

Problem (P). Find an admissible control $u^*(t)$ such that:

(1) $x^{u^*}(0)$ is distributed according to μ_0, and $x^{u^*}(T)$ according to μ_T;
(2) between all admissible control satisfying (1), u^* minimizes the cost function

$$J(u) = E \int_0^T \tfrac{1}{2}\|u(t)\|_{a^{-1}}^2\,dt. \tag{3.2}$$

Our technique for solving Problem (P) includes the use of the logarithmic transformation introduced by Fleming in [10], [11], and [13]. Throughout this section we assume A1 and A2.

Let $q(s, y, t, x)$ be the fundamental solution of the equation corresponding to the operator

$$\frac{\partial}{\partial s} + L = \frac{\partial}{\partial s} + b\cdot\nabla + \tfrac{1}{2}\sum_{ij} a_{ij}\frac{\partial^2}{\partial x_i \partial x_j}. \tag{3.3}$$

Then, let S_t be the operator, acting on the set of the σ-finite measures on $\mathbf{R}^n$, defined by

$$\frac{dS_t\mu}{d\lambda}(x) = \int q(0, y, t, x)\mu(dy), \tag{3.4}$$

where the notation above denotes the Radon–Nikodym derivative with respect to the Lebesgue measure λ.

It is useful to recall the following definition.

Definition 3.1. If μ and ν are σ-finite measures defined in the same measurable space and $\mu \ll \nu$, we define the *Kullback distance* between μ and ν [24] to be

$$K(\mu, \nu) = \int \left(\log \frac{d\mu}{d\nu}\right) d\mu. \tag{3.5}$$

Although $K(\cdot, \cdot)$ is not really a metric, it is of central importance in various fields of probability and statistics, for instance, in the study of detection problems [7] and time series [1].

In the rest of this section we assume that $x(t)$ is a weak solution in $[0, T]$ of

$$\begin{aligned} &dx = b\,dt + \sigma\,dw, \\ &x(0) = x_0 \quad \text{distributed according to } \mu_0, \qquad E|x_0|^2 < +\infty. \end{aligned} \tag{3.6}$$

First we solve Problem (P) in the case of a deterministic initial condition. Then we discuss the proof for the general case.

Theorem 3.1. *Let μ_0 be the Dirac measure concentrated in a point $x_0 \in \mathbf{R}^n$, and assume that*

$$K(\mu_T, S_T\mu_0) < +\infty.$$

Let h be defined by

$$h(x, t) := \int q(t, x, T, z) \frac{d\mu_T}{dS_T\mu_0}(z)\,dz. \tag{3.7}$$

Then $u^*(x, t) = a\nabla \log h(x, t)$ solves Problem (P) *and $J(u^*) = K(\mu_T, S_T\mu_0)$.*

Proof. First, let us observe that

$$h(x_0, 0) = \int q(0, x_0, T, z) \frac{d\mu_T}{dS_T\mu_0}(z)\,dz = 1.$$

So, A2 implies that $h(x, t)$ is in $C^{2,1}(\mathbf{R}^n \times [0, T))$ and satisfies

$$\frac{\partial h}{\partial t} + Lh = 0, \qquad h(x, T) = \frac{d\mu_T}{dS_T\mu_0}(x).$$

By definition $h(x(t), t)$ is a martingale and $Eh(x(t), t) = 1$. By Theorem 2.1 the stochastic differential equation

$$dx = (b + a\nabla \log h)\,dt + \sigma\,dw \tag{3.8}$$

has a weak solution in $[0, T]$ and, by (2.4), $x(T) \sim \mu_T$.

Let $x^*(t)$ denote a weak solution of (3.8), and let P_x and P_{x^*} be the measures induced by x and x^* respectively on $C[0, T]$. In the proof of Theorem 2.1 we obtained

$$\frac{dP_{x^*}}{dP_x}(\xi(\cdot)) = h(\xi(T), T).$$

Using that fact, we can show that

$$E(\log h(x^*(t), t)) \leq K(\mu_T, S_T\mu_0). \tag{3.9}$$

In fact,

$$\begin{aligned} E(\log h(x^*(t), t) &= E(h(x(t), t) \log h(x(t), t)) \\ &\leq E(h(x(T), T) \log h(x(T), T)) = K(\mu_T, S_T\mu_0), \end{aligned}$$

where we have used the fact that, since $\phi(x) = x \log x$ is convex and bounded below, $h(x(t), t) \log h(x(t), t)$ is a submartingale.

Let us define the following sequence of stopping times:

$$\tau_n = \inf\{s \colon |x(s)| > n\}$$

and

$$\tau_n(\omega) = T \qquad \text{if} \quad |x(t, \omega)| \leq n \quad \text{for every} \quad 0 \leq t \leq T.$$

If $t \leq T$, Ito's formula gives

$$\begin{aligned} &E\{h(x(t \wedge \tau_n), t \wedge \tau_n) \log h(x(t \wedge \tau_n), t \wedge \tau_n)\} \\ &\quad = E \int_0^{t \wedge \tau_n} \tfrac{1}{2}\|u^*(x(s), s)\|^2_{a^{-1}} h(x(s), s)\, ds. \end{aligned} \tag{3.10}$$

By the Optional Sampling Theorem we have

$$\begin{aligned} &E\{h(x(t \wedge \tau_n), t \wedge \tau_n) \log h(x(t \wedge \tau_n), t \wedge \tau_n)\} \\ &\quad \leq E\{h(x(t), t) \log h(x(t), t)\} \end{aligned} \tag{3.11}$$

and, by (3.9),

$$E\{h(x(t), t) \log h(x(t), t)\} < \infty.$$

On the other hand, $t \wedge \tau_n \uparrow t$ as $n \uparrow +\infty$. So, (3.10) and (3.11) give

$$E \int_s^t \tfrac{1}{2}\|u^*(x(s), s)\|^2_{a^{-1}} h(x(s), s)\, ds \leq E\{h(x(t), t) \log h(x(t), t)\}$$

and Fatou's lemma applied to the left-hand side of (3.10) gives the opposite inequality. Therefore we get

$$E \int_0^t \tfrac{1}{2}\|u^*(x(s), s)\|^2_{a^{-1}} h(x(s), s)\, ds = E\{h(x(t), t) \log h(x(t), t)\}.$$

Taking the limit $t \to T$, again using Fatou's lemma and the submartingale property of $h(x(t), t) \log h(x(t), t)$, we finally get

$$K(\mu_T, S_T\mu_0) = E(h(x(T), T) \log h(x(T), T))$$
$$= E\int_0^T \tfrac{1}{2}\|u^*(x(s), s)\|^2_{a^{-1}} h(x(s), s)\, ds = E\int_0^T \tfrac{1}{2}\|u^*(x^*(s), s)\|^2_{a^{-1}}\, ds$$

which proves a part of our statement. We still need to show that, for any admissible control u, the following holds:

$$E\int_0^t \tfrac{1}{2}\|u(x^u(s), s)\|^2_{a^{-1}}\, ds \geq K(\mu_T, S_T\mu_0). \tag{3.12}$$

This can be obtained through a Girsanov transformation. If u is an admissible control we get

$$1 = E\{h(x(T), T)\}$$
$$= E\left\{h(x^u(T), T)\exp\left[-\int_0^T \sigma^{-1}(x^u(t), t)u(t)\, dw(t) - \tfrac{1}{2}\int_0^T \|u(t)\|^2_{a^{-1}}\, dt\right]\right\}$$
$$\geq \exp\left\{E\left[\log h(x^u(T), T) - \int_0^T \sigma^{-1}(x^u(t), t)u(t)\, dw(t) - \tfrac{1}{2}\int_0^T \|u(t)\|^2_{a^{-1}}\, dt\right]\right\}$$
$$= \exp\left\{K(\mu_T, S_T\mu_0) - \tfrac{1}{2}E\int_0^T \|u(t)\|^2_{a^{-1}}\, dt\right\}$$

since $x^u(T) \sim \mu_T$. The above inequality clearly implies

$$K(\mu_T, S_T\mu_0) \leq \tfrac{1}{2}E\int_0^T \|u(t)\|^2_{a^{-1}}\, dt. \qquad \square$$

Remark. The "energy" $J(u) = \frac{1}{2}E\int_0^T \|u(t)\|^2_{a^{-1}}\, dt$ has a nice interpretation, again in terms of Kullback distance. If P_x *and* P_u are the measures generated by x and x^u on $C[0, T]$, by the Girsanov formula we have

$$K(P_u, P_x) = \int \log\frac{dP_u}{dP_x}\, dP_u = -\int \log\frac{dP_x}{dP_u}\, dP_u$$
$$= -E\left\{\int_0^T \sigma^{-1}(x^u(t), t)u(t)\, dw(t) - \tfrac{1}{2}\int_0^T \|u(t)\|^2_{a^{-1}}\, dt\right\} = J(u).$$

Therefore, an optimal control satisfies the equation

$$K(P_u, P_x) = K(\mu_T, S_T\mu_0),$$

i.e., *the "global" Kullback distance equals the Kullback distance between the final densities.*

Theorem 2.2 allows us to extend the above result to the case of nondeterministic initial condition. We assume that x_0 is a random variable distributed according to a probability measure μ_0, and A1 and A2 hold.

From Theorem 2.2, given μ_0 and $\mu_T \ll S_T\mu_0$, there exist two σ-finite measures ν_0 and ν_T for which (2.6) holds. In particular, $\nu_T \ll \lambda$. Letting $\rho_T(x) = d\nu_T/d\lambda$, from (2.6) we have

$$\begin{aligned} \frac{d\mu_T}{d\lambda} &:= \pi_T(x) = \rho_T(x) \int q(0, y, T, x)\nu_0(dy), \\ \frac{d\mu_0}{d\nu_0} &= \int q(0, x, T, z)\rho_T(z)\, dz. \end{aligned} \tag{3.13}$$

We can establish the following theorem.

Theorem 3.2. *Let* $\int x^2\, d\mu_0 < +\infty$, $K(\mu_T, S_T\nu_0) < +\infty$, $\int (d\mu_0/d\nu_0)\, d\mu_0 < +\infty$, *and* h *be defined by*

$$h(x, t) = \int q(t, x, T, z)\rho_T(z)\, dz. \tag{3.14}$$

Then $u^*(x, t) = a(x, t)\nabla \log h(x, t)$ *solves Problem* (P) *and*

$$E\tfrac{1}{2} \int_0^T \|u^*(t)\|^2_{a^{-1}}\, dt = K(\mu_T, S_T\nu_0) - K(\mu_0, \nu_0). \tag{3.15}$$

Proof. First we need to prove that $h(x, t) = E_{x,t}\{\rho_T(x(T))\}$. By (3.14), this is true if we show that $E\rho_T(x(T)) < +\infty$. Indeed,

$$\begin{aligned} E\rho_T(x(T)) &= \int \rho_T(x)\, dS_T\mu_0 \\ &= \int \rho_T(x)\left(\int q(0, y, T, x)\, d\mu_0(y)\right) dx \\ &= \int\left(\int q(0, y, T, x)\rho_T(x)\, dx\right) d\mu_0(y) = \int \frac{d\mu_0}{d\nu_0}\, d\mu_0 < +\infty. \end{aligned}$$

By Theorem 2.1, we get that (3.8) has a weak solution in $[0, T]$. We can repeat the proof of Theorem 3.1, taking into account that

$$\frac{dP_{x^*}}{dP_x}(\xi(\cdot)) = \frac{\rho_T(\xi(T))}{h(\xi(0), 0)}.$$

For instance, in place of (3.9) we get

$$\begin{aligned} E\{\log h(x^*(t), t)\} &= E\left\{\frac{1}{h(x(0), 0)}\, E\{h(x(t), t)\log h(x(t), t) | x(0)\}\right\} \\ &\le E\left\{\frac{\rho_T(x(T))}{h(x(0), 0)} \log h(x(T), T)\right\} \\ &= E\{\log h(x^*(T), T)\} = K(\mu_T, S_T\nu_0) \end{aligned}$$

and in place of (3.10) we get

$$E\left\{\frac{1}{h(x(0),0)}E\{h(x(t\wedge\tau_n),t\wedge\tau_n)\log h(x(t\wedge\tau_n),t\wedge\tau_n)|x(0)\}\right\}-E\log h(x(0),0)$$
$$=E\left\{\frac{1}{h(x(0),0)}\int_0^{t\wedge\tau_n}\tfrac{1}{2}\|u^*(s)\|^2_{a^{-1}}h(x(s),s)\,ds\right\},$$

where

$$E\log h(x(0),0)=\int\log\frac{d\mu_0}{d\nu_0}\,d\mu_0=K(\mu_0,\nu_0)<+\infty$$

since

$$0\le\int\log\frac{d\mu_0}{d\nu_0}\,d\mu_0\le\log\int\frac{d\mu_0}{d\nu_0}\,d\mu_0<+\infty$$

by the Jensen inequality.

Using limit arguments, as in Theorem 3.1, we get

$$E\tfrac{1}{2}\int_0^T\|u^*(t)\|^2_{a^{-1}}\,dt=K(\mu_T,S_T\nu_0)-K(\mu_0,\nu_0).$$

As we did for (3.12), the inequality

$$E\tfrac{1}{2}\int_0^T\|u(t)\|^2_{a^{-1}}\,dt\ge K(\mu_T,S_T\nu_0)-K(\mu_0,\nu_0)$$

for any admissible control u, can be proved by using Girsanov's formula. □

The conditions under which Theorem 2.2 holds, namely $K(\mu_T, S_T\nu_0) < +\infty$ and $\int (d\mu_0/d\nu_0)\, d\mu_0 < +\infty$, are rather difficult to check practically, since it is usually hard to determine ν_0 and ν_T. However, when μ_0 has compact support, we can replace them with nicer conditions.

Proposition 3.1. *If μ_0 has compact support, and $K(\mu_T, S_T\mu_0) < +\infty$, then*

$$K(\mu_T,S_T\nu_0)<+\infty\quad\textit{and}\quad\int\frac{d\mu_0}{d\nu_0}\,d\mu_0<+\infty.$$

Proof. If μ_0 has compact support, since h is smooth we get

$$\int\frac{d\mu_0}{d\nu_0}\,d\mu_0=\int h(0,x)\,d\mu_0(x)<+\infty.$$

Furthermore,

$$\mathrm{K}(\mu_T,S_T\nu_0)=\int\log\frac{d\mu_T}{dS_T\mu_0}\,d\mu_T+\int\log\frac{dS_T\mu_0}{dS_T\nu_0}\,d\mu_T$$
$$=K(\mu_T,S_T\nu_0)+\int\log\frac{dS_T\mu_0}{dS_T\nu_0}\,d\mu_T.$$

Hence, we only have to show that $\int\log(dS_T\mu_0/dS_T\nu_0)\,d\mu_T$ exists.

First, if $\log^- x = \sup(-\log x, 0)$, by Jensen inequality we get

$$\log^- \frac{dS_T\mu_0}{dS_T\nu_0}(y) = \log^- \int \frac{d\mu_0}{d\nu_0}(x) \frac{q(0, x, T, y)}{(dS_T\nu_0/d\lambda)(y)} d\nu_0(x)$$
$$\leq \int \left(\log^- \frac{d\mu_0}{d\nu_0}(x)\right) \frac{q(0, x, T, y)}{(dS_T\nu_0/d\lambda)(y)} d\nu_0(x).$$

Thus,

$$\int \log^- \frac{dS_T\mu_0}{dS_T\nu_0}(y)\, d\mu_T \leq \int \left(\int \left(\log^- \frac{d\mu_0}{d\nu_0}(x)\right) q(0, x, T, y)\, d\nu_0(x)\right) d\nu_T(y)$$
$$= \int \log^- \frac{d\mu_0}{d\nu_0}\, d\mu_0 = \int \log^- h(x, 0)\, d\mu_0(x) < +\infty.$$

Moreover,

$$\int \log \frac{dS_T\mu_0}{dS_T\nu_0}(y)\, d\mu_T \leq \log \int \frac{dS_T\mu_0}{dS_T\nu_0}(y)\, d\mu_T$$
$$= \log \int h_T(y)\, dS_T\mu_0(y) = \log \int h(x, 0)\, d\mu_0(x) < +\infty. \quad \square$$

Remark. A natural generalization of Problem (P) consists in adding a state dependent cost in (3.2):

$$J(u) = E \int_0^T [\tfrac{1}{2}\|u(t)\|^2_{a^{-1}} + V(x^u(t))]\, dt, \qquad V \geq 0.$$

It can be shown that we can treat this problem like the previous one if we take $h(x, t)$ in the kernel of the operator $\partial/\partial t + L - V$ rather than $\partial/\partial t + L$. In this case, (2.4) holds if $q(s, y, t, x)$ is the fundamental solution of the equation $(\partial/\partial t)f + Lf - Vf = 0$. This means that $q(s, y, t, x)$ is the Markov kernel of the *killed* diffusion process which satisfies (3.6) and with *killing rate* $V(x)$ [5], [20]. Further details concerning this point of view are presented in [6].

4. An Example: the One-Dimensional Linear Case

To give a simple and explicit example of the machinery developed in the previous section, we present here the one-dimensional linear case. For further simplicity, we assume a deterministic initial condition, although the case of Gaussian initial distribution could easily be treated too. The notations introduced for Theorem 3.1 are used throughout this section.

Let $x^u(t)$ be a controlled diffusion process solving, in a weak sense, the following stochastic differential equation:

$$\begin{aligned} dx^u(t) &= [Ax^u(t) + u(x^u(t), t)]\, dt + dw(t), \\ x^u(0) &= 0, \end{aligned} \tag{4.1}$$

where $A \neq 0$.

Our goal consists in reaching, with the minimum cost, a Gaussian measure whose density is

$$\pi(x) = \frac{1}{(2\pi\sigma^2)^{1/2}} \exp\left\{-\frac{(x-m)^2}{2\sigma^2}\right\}. \tag{4.2}$$

The uncontrolled transition density for (4.1) is given by

$$q(t, x, T, z) = \frac{1}{(2\pi P(T-t))^{1/2}} \exp\left\{-\frac{(z - e^{A(T-t)}x)^2}{2P(T-t)}\right\}, \tag{4.3}$$

where $P(t) = (1/2A)[e^{2At} - 1]$. In particular,

$$q(0, 0, T, z) = \frac{1}{(2\pi P(T))^{1/2}} \exp\left\{-\frac{z^2}{2P(T)}\right\}. \tag{4.4}$$

We recall that $h(x, t) = \int q(t, x, T, z)(\pi(z)/q(0, 0, T, z))\, dz$. A tedious computation gives

$$h(x, t) \propto \exp\left\{\frac{x^2}{2P(t)}\right\} \exp\left\{-\frac{1}{2}\frac{(m - (P(T)/e^{A(T-t)}P(t))x)^2}{\sigma^2 + (P(T)P(T-t)/P(T) - P(T-t))}\right\}, \tag{4.5}$$

where $\propto$ denotes proportionality. From (4.5) we have

$$\begin{aligned} u^*(x, t) &= \nabla \log h(x, t) \\ &= \frac{e^{2A(T-t)}}{e^{2A(T-t)}P(t)\sigma^2 + P(T-t)P(T)} [(\sigma^2 - P(T))x + e^{-A(T-t)}P(T)m]. \end{aligned} \tag{4.6}$$

It is easy to verify that if $\sigma = 0$, then $u^*(x, t) = \nabla \log q(t, x, T, m)$. This corresponds to the case of a "tied down" diffusion described in Corollary 2.1. However, as observed in [13], this u^* does not have finite energy.

Now, we want to chose the most convenient time T to reach the configuration $\pi(x)$. We already know that, for a given T, the minimal cost of forcing the diffusion to that configuration at time T is

$$\mathscr{E}_T = K(\pi, q(0, 0, T, x)), \tag{4.7}$$

where, with abuse of notation, we indicate a measure by its density. Our goal is to find T^* that minimizes $\mathscr{E}_T$.

First let us assume $A > 0$. From (4.7) we obtain

$$\mathscr{E}_T = \frac{1}{2}\left[\frac{2A\sigma^2}{e^{2AT} - 1} - 1 - \log 2A\sigma^2 + \log(e^{2AT} - 1)\right] + \frac{Am^2}{e^{2AT} - 1}.$$

Elementary calculus shows that

$$\lim_{T \to 0} \mathscr{E}_T = \lim_{T \to +\infty} \mathscr{E}_T = +\infty \tag{4.8}$$

and $\mathscr{E}_T$ has a unique absolute minimum point at

$$T^* = \frac{1}{2A} \log[2A(\sigma^2 + m^2) + 1]. \tag{4.9}$$

Note that if $m = 0$, $P(T^*) = \sigma^2$ and so $\mathscr{E}_{T^*} = 0$ (the configuration $\pi(x)$ is spontaneously reached by the system).

Consider, now, the case $A = -k < 0$. We have

$$\mathscr{E}_T = \frac{1}{2}\left[\frac{2k\sigma^2}{1 - e^{-2kT}} - 1 - \log 2k\sigma^2 + \log(1 - e^{-2kT})\right] + \frac{km^2}{1 - e^{-2kT}}.$$

In this case $\lim_{T\to 0} \mathscr{E}_T = +\infty$ but $\lim_{T\to+\infty} \mathscr{E}_T = k(m^2 + \sigma^2) - \frac{1}{2} - \frac{1}{2}\log 2k\sigma^2 := \mathscr{E}_\infty$. If the condition $2k(m^2 + \sigma^2) < 1$ is satisfied, then $\mathscr{E}_T$ has a unique absolute minimum point at

$$T^* = -\frac{1}{2k}\log[1 - 2k(m^2 + \sigma^2)]. \tag{4.10}$$

Otherwise $\mathscr{E}_T$ is decreasing and converges to $\mathscr{E}_\infty$.

Observe that $2k\sigma^2 < 1$ means that the configuration π can be reached by the system without control. Furthermore, $\mathscr{E}_\infty = 0$ if and only if π is the invariant measure for the free system.

5. A Path-Space Measure Invariance Property

Given a random process $x(t)$ with continuous trajectories, there is a natural way of defining the *measure induced* by $x(t)$ on the *path-space* $C[0, T]$ [15], [8]. This follows from the simple idea that a random process is a random variable with values on a space of functions containing its trajectories. As was observed in [25], we cannot hope that this random variable $x(\cdot)$ admits a density with respect to "uniform" measure on $C[0, T]$, since such a measure does not exist. However, if $x(t)$ is a diffusion process, some "local" information about the measure induced by $x(\cdot)$ can be obtained. In particular, an asymptotic estimate of the probability of a small tube around a C^2-function can be derived (see [8] and [30]). For instance, if $x(t)$ is a solution of (2.1) with σ and b of class C_b^2 (bounded together with their derivatives), given $\varphi \in C^2[0, T]$ and for small ε we have the following estimate:

$$P\{\|x(\cdot) - \varphi\| < \varepsilon\} \approx K_\varepsilon \exp\left\{-\int_0^T L(\varphi, \dot\varphi, t)\,dt\right\} \tag{5.1}$$

with

$$L(\varphi, \dot\varphi, t) = \tfrac{1}{2}\|\mathbf{b}(\varphi, t) - \dot\varphi\|^2_{a^{-1}} + \tfrac{1}{2}\nabla_t \cdot b(\varphi, t) + \tfrac{1}{12}R(\varphi, t), \tag{5.2}$$

where

$$\mathbf{b}_i = b_i - \tfrac{1}{2}|\sigma| \sum_j \frac{\partial(a_{ij}/|\sigma|)}{\partial x_j},$$

$$|\sigma(x, t)| = \det(\sigma_{ij}(x, t)),$$

$R(x, t)$ is the scalar curvature of the Riemannian metric $a^{-1}(x, t)$ [16], and the operator ∇_t is defined by

$$\nabla_t f(x, t) = |\sigma(x, t)| \sum_i \frac{\partial(f(x, t)/|\sigma(x, t)|)}{\partial x_i}.$$

The asymptotic relation (5.1) justifies the definition of "most probable path" for the function which minimizes the functional $\int_0^T L(\varphi, \dot\varphi, t)\,dt$. The *extremal trajectories* are the C^2-functions which solve the Euler–Lagrange equation associated with L:

$$\frac{d}{dt}\frac{\partial L}{\partial \dot\varphi} = \frac{\partial L}{\partial \varphi}. \tag{5.3}$$

The following theorem shows that, perturbing a diffusion with controls of the type introduced in Section 3, we do not change the extremal trajectories.

Theorem 5.1. *Assume* $\sigma(x, t) \in C_b^2(\mathbf{R}^n \times [0, T], \mathbf{L}(\mathbf{R}^n, \mathbf{R}^n))$, $\sigma(x, t) > 0$, *and* $b(x, t) \in C_b^2(\mathbf{R}^n \times [0, T], \mathbf{R}^n)$. *Consider two diffusion process* $x_1(t)$ *and* $x_2(t)$ *with the same diffusion coefficient* $\sigma(x, t)$ *and whose drifts are* $b(x, t)$ *and* $b(x, t) + a(x, t)\nabla \log h(x, t)$, *respectively, where* $a = \sigma^t\sigma$ *and* h *is a strictly positive function belonging to the kernel of the operator* $\partial/\partial t + b \cdot \nabla + \frac{1}{2}\sum_{ij} a_{ij}(\partial^2/\partial x_i \partial x_j)$. *Then* x_1 *and* x_2 *have the same extremal trajectories.*

Proof. Since R depends only on the diffusion coefficient that is the same for the two diffusions, it is not restrictive to assume $R(x, t) \equiv 0$. For further simplicity we prove our result in the one-dimensional case. The proof consists of rather tedious computations that we will only sketch.

The Euler–Lagrange equation for L takes the following form:

$$a^{-1}\left(\ddot\phi - \frac{\partial b}{\partial t} - \mathbf{b}D\mathbf{b}\right) + \frac{\partial a^{-1}}{\partial t}(\dot\phi - \mathbf{b}) + \tfrac{1}{2}Da^{-1}(\dot\phi^2 - \mathbf{b}^2) - \tfrac{1}{2}D\sigma D\frac{\mathbf{b}}{\sigma} = 0, \tag{5.4}$$

where D denotes the derivative with respect to the space variable. Let

$$L^h = \tfrac{1}{2}\|\mathbf{b} + aD\log h - \dot\varphi\|^2 + \tfrac{1}{2}\sigma D\frac{(\mathbf{b} + aD\log h)}{\sigma}.$$

A suitable manipulation to the Euler–Lagrange equation for L^h gives the following form:

$$a^{-1}\left(\ddot\phi - \frac{\partial b}{\partial t} - \mathbf{b}D\mathbf{b}\right) + \frac{\partial a^{-1}}{\partial t}(\dot\phi - \mathbf{b}) + \tfrac{1}{2}Da^{-1}(\dot\phi^2 - \mathbf{b}^2) - \tfrac{1}{2}D\sigma D\frac{\mathbf{b}}{\sigma}$$
$$- D\left[\frac{\partial}{\partial t}(\log h) + (\mathbf{b} + \tfrac{1}{2}\sigma D\sigma)D(\log h) + \tfrac{1}{2}aD^2(\log h) + \tfrac{1}{2}a(D\log h)^2\right] = 0.$$

Since $\mathbf{b} + \frac{1}{2}\sigma D\sigma = b$, the logarithmic transformation of $(\partial h/\partial t) + bDh + \frac{1}{2}D^2 h = 0$ gives

$$\frac{\partial}{\partial t}(\log h) + (\mathbf{b} + \tfrac{1}{2}\sigma D\sigma)D(\log h) + \tfrac{1}{2}aD^2(\log h) + \tfrac{1}{2}a(D\log h)^2 = 0$$

and, therefore, the two Lagrangians yield the same Euler–Lagrange equation. □

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