Notes on Lattices and Order

1 Ordered Sets

1.1 Ordered sets

- Let P be a set. An **order** (or **partial order**) on P is a binary relation $\leq$ on P such that, for all $x, y, z \in P$,
  - (a) $x \leq x$ (reflexivity)
  - (b) $x \leq y$ and $y \leq x$ imply $x = y$ (antisymmetry)
  - (c) $x \leq y$ and $y \leq z$ imply $x \leq z$ (transitivity)
- A set P equipped with an order relation $\leq$ is said to be an **ordered set** (or **partially ordered set** or **poset**).
- On any set, $=$ is an order, the **discrete order**.
- A relation $\leq$ on a set P which is reflexive and transitive but not necessarily antisymmetric is called a **quasi-order** (or **pre-order**).
- A subset Q of an ordered set P inherits an order relation from P, which we say that Q has the **induced order**.

1.2 Chains and antichains

- An ordered set P is a **chain** (or **linearly ordered set** or **totally ordered set**) if, for all $x, y \in P$, either $x \leq y$ or $y \leq x$.
- The ordered set P is an **antichain** if $x \leq y$ in P only if $x = y$.

1.3 Order isomorphisms

- Two ordered set P and Q are **order-isomorphic** (or **isomorphic**), written $P \cong Q$, if there exists a map φ from P onto Q such that

$$x \leq y \text{ in } P \iff \varphi(x) \leq \varphi(y) \text{ in } Q.$$

- $\varphi : P \to Q$ is necessarily bijective and there exists a well-defined inverse $\varphi^{-1} : Q \to P$.

1.4 The covering relation

- Let P be an ordered set and let $x, y \in P$. We say x is **covered by** y (or y **covers** x), and write $x \prec y$, if $x < y$ and $x \leq z < y$ implies $z = x$.
- Using the covering relation, we can draw the **Hasse diagram** for an ordered set, which is a *canonical* graphical representation of the set.
- **Lemma**: Let P and Q be finite ordered sets and let $\varphi : P \to Q$ be a bijective map. Then the following are equivalent:
  - (a) φ is an order-isomorphism.
  - (b) $x < y$ in P if and only if $\varphi(x) < \varphi(y)$ in Q.
  - (c) $x \prec y$ in P if and only if $\varphi(x) \prec \varphi(y)$ in Q.

1.5 The dual of an ordered set

- Given an ordered set P we can form a new ordered set P^∂ (the **dual** of P) by defining $x \leq y$ to hold in P^∂ if and only if $y \leq x$ in P.
- An ordered set P has a bottom element if there exists $\bot \in P$ (called **bottom**) with the property that $\bot \leq x$ for all $x \in P$.
- An ordered set P has a top element if there exists $\top \in P$ (called **top**) with the property that $x \leq \top$ for all $x \in P$.
- A finite chain always have top and bootom elements.
- Top and bottom elements are unique when they exists.
- **The duality principle**: Given a statement Φ about ordered sets which is true in all ordered sets, the dual statement Φ^∂ is also true in all ordered sets.

1.6 Lifting

- When an ordered set P (with or without a bottom element), we can form $P_\bot$ with a bottom $\bot$ by **lifting** P, where lifting is done as follows: take an element $\mathbf{0} \in P$ and define $\leq$ on $P_\bot \triangleq P \cup \{\mathbf{0}\}$ by

$$x \leq y \text{ in } P_\bot \text{ if and only if } x = \mathbf{0} \text{ or } x \leq y \text{ in } P.$$

- A **flat** ordered set can be constructed from any *set* S as follows: get an antichain $\overline{S}$, and then form $\overline{S}_\bot$.

1.7 Maximal and minimal elements

- Let P be an ordered set and let $Q \subseteq P$. Then $a \in Q$ is a **maximal element** of Q if

$$(a \leq x \text{ and } x \in Q) \Rightarrow a = x.$$

- We denote the set of maximal elements of Q by $\max Q$.
- If Q has a top elment, $\top_Q$, then $\max Q = \{\top_Q\}$; in this case $\top_Q$ is called the **greatest element** (or **maximum**) of Q and we write $\top_Q = \max Q$.
- A **minimal element** of $Q \subseteq P$ and $\min Q$, and the **least element** (or **minimum**) of Q (when these exists) are defined dually.

1.8 Sum of ordered sets

- Let P and Q be disjoint ordered sets. The **disjoint union** $P \cup Q$ of P and Q is the ordered set formed by defining $x \leq y$ in $P \cup Q$ if and only if either $x, y \in P$ and $x \leq y$ in P or $x, y \in Q$ and $x \in y$ in Q.
- Let P and Q be disjoint ordered sets. The **linear sum** $P \oplus Q$ is defined by taking the following order relation on $P \cup Q$: $x \leq y$ if and only if
  - (a) $x, y \in P$ and $x \in y$ in P, or
  - (b) $x, y \in Q$ and $x \in y$ in Q, or
  - (c) $x \in P$ and $y \in Q$.
- The lifting construction is a special case of linear sum: $P_\bot$ is just $1 \oplus P$.

1.9 Products

- Let $P_1, \cdots, P_n$ be ordered sets. The Cartesian product $P_1 \times \cdots \times P_n$ can be made into an ordered set by imposing the coordinatewise order defined by

$$(x_1, \cdots, x_n) \le (y_1, \cdots, y_n) \iff (\forall i) x_i \le y_i \text{ in } P_i.$$

- P^n is a shorthand for an n-fold product $P \times \cdots \times P$.
- Let P and Q be ordered sets. A **lexicographic order** is defined by $(x_1, x_2) \le (y_1, y_2)$ if $x_1 < x_2$ or $(x_1 = y_1$ and $x_2 \le y_2)$.
- **Proposition**: Let $X = \{1, 2, \cdots, n\}$ and define $\varphi : \mathcal{P}(X) \to 2^n$ by $\varphi(A) = (\boxtimes_1, \cdots, \boxtimes_n)$ where

$$\boxtimes_i = \begin{cases} 1 & \text{if } i \in A, \\ 0 & \text{if } i \notin A, \end{cases}$$

  Then φ is an order-isomorphism.

1.10 Down-sets and up-sets

- Let P be an ordered set and $Q \subseteq P$.
  1. Q is a **down-set** (or **decreasing set** or **order ideal**) if, whenever $x \in Q$, $y \in P$ and $y \le x$, we have $y \in Q$.
  2. Dually, Q is an **up-set** (or **increasing set** or **order filter**) if, whenever $x \in Q$, $y \in P$ and $y \ge x$, we have $y \in Q$.
- Loosely speaking, a down-set is one which is closed under going down and up-set is one which is closed under going up.
- Given an arbitrary subset Q of P and $x \in P$, we define

$$\downarrow Q \triangleq \{y \in P : (\exists x \in Q) y \le x\}$$

  and

$$\downarrow x \triangleq \{y \in P : y \le x\}.$$

  $\uparrow Q$ and $\uparrow x$ is defined similarly.
- Q is a down-set if and only if $Q = \downarrow Q$.
- Down-sets (up-sets) of the form $\downarrow x$ ($\uparrow x$) are called **principal**.
- The family of all down-sets of P is denoted by $\mathcal{O}(P)$.
- $\mathcal{O}(P)$ itself is an ordered set, under the inclusion order.
- When P is finite, every non-empty down-set Q of P is expressible in the form $\bigcup_{i=1}^{k} \downarrow x_i$.
- **Lemma**: Let P be an ordered set and $x, y \in P$. Then the following are equivalent.
  (a) $x \le y$.
  (b) $\downarrow x \subseteq \downarrow y$.
  (c) $(\forall Q \in \mathcal{O}(P)) y \in Q \Rightarrow x \in Q$.
- Besides being related by duality, down-sets and up-sets are related by complementation: Q is a down-set of P if and only if $P - Q$ is an up-set of P (equivalently, a down-set of P^{∂}).
- **Proposition**: Let P be an ordered set. Then
  (a) $\mathcal{O}(P \oplus 1) \cong \mathcal{O}(P) \oplus 1$ and $\mathcal{O}(1 \oplus P) \cong 1 \oplus \mathcal{O}(P)$.
  (b) $\mathcal{O}(P_1 \cup P_2) \cong \mathcal{O}(P_1) \times \mathcal{O}(P_2)$.

1.11 Maps between ordered sets

- Let P and Q be ordered sets. A map $\varphi : P \to Q$ is said to be
  (a) **order-preserving** (or **monotone**) if $x \le y$ in P implies $\varphi(x) \le \varphi(y)$ in Q.
  (b) an **order-embedding** (written as $\varphi : P \hookrightarrow Q$) if $x \le y$ in P if and only if $\varphi(x) \le \varphi(y)$ in Q.
  (c) an **order-isomorphism** if it is an order-embedding which maps P onto Q.

2 Lattices and Complete Lattices

2.1 Lattices as ordered sets

- Let P be an ordered set and let $S \subseteq P$. An element $x \in P$ is an **upper bound** of S if $s \le x$ for all $s \in S$. A **lower bound** is defined dually.
- The set of all upper bounds of S is denoted by S^u (read as 'S upper') and the set of all lower bounds by S^l (read as 'S lower').

References

[1] G. Birkhoff. *Lattice Theory*. American Mathematical Society, third edition, 1948.

[2] B. A. Davey and H. A. Priestley. *Introduction to Lattices and Order*. Cambridge University Press, 2002.