Notes on Mathematical Logic

1 Sentential Logic

In order to *describe a formal language* we have to give three pieces of information:

(a) We should specify the set of symbols (the *alphabet*).

(b) We should specify the rules for forming the *grammatically correct* finite sequences of symbols, called **well-formed formulas** or **wffs**.

(c) We may need to indicate the allowable translations between English and the formal language. *This information is dispensable for "formal or symbolic logic"*[1].

1.1 The Language of Sentential Logic

- Alphabet of sentential logic
  - **Logical symbols**
    - *Sentential connectives*: $\neg, \vee, \wedge, \rightarrow, \leftrightarrow$
    - *Parentheses*: $($, $)$
  - **Sentence symbols** or non-logical symbols: $A_1, A_2, \cdots$
    - Only countably many sentence symbols exist.
- We assume that no symbol is a finite sequence of other symbols. This assumption aims to assure that finite sequences of symbols be *uniquely decomposable*. That is, if

$$\langle a_1, a_2, \cdots, a_m \rangle = \langle b_1, b_2, \cdots, b_n \rangle$$

  and each a_i and b_i is a symbol, then $m = n$ and $a_i = b_i$.
- An **expression** is a finite sequence of symbols.
- A well-formed formula for sentential logic is defined as follows:
  (a) Every sentence symbol is a wff.
  (b) If α and β are wffs, then so are $(\neg\alpha)$, $(\alpha \vee \beta)$, $(\alpha \wedge \beta)$, $(\alpha \rightarrow \beta)$, $(\alpha \leftrightarrow \beta)$.
  (c) No expression is a wff unless it is compelled to be one by (a) and (b).
- There are two equivalent ways to make the property (c) precise.
  1. **Bottom-up**: using the notion of *inductive sets*
     - An expression is a wff iff it is a member of inductive set.
  2. **Top-down**: using the operational notion of *formula-building operations*
     - An expression is a wff iff it can be built up from the sentence symbols by applying some finite number of times the formula-binding operations (on expressions) defined by the equations

$$\begin{aligned} \mathfrak{E}_\neg(\alpha) &= (\neg\alpha), \\ \mathfrak{E}_\vee(\alpha,\beta) &= (\alpha \vee \beta), \\ \mathfrak{E}_\wedge(\alpha,\beta) &= (\alpha \wedge \beta), \\ \mathfrak{E}_\rightarrow(\alpha,\beta) &= (\alpha \rightarrow \beta), \\ \mathfrak{E}_\leftrightarrow(\alpha,\beta) &= (\alpha \leftrightarrow \beta) \end{aligned}$$

[1] Translations between the formal language and the mathematical structures are studied in *Model Theory*.

1.2 Induction

- Let's consider an initial set $B \subseteq U$ and a class $\mathcal{F}$ of functions containing just two members f and g, where

$$f : U \times U \rightarrow U, \quad g : U \rightarrow U.$$

- A set S is **inductive** iff $B \subseteq S$ and S is *closed* under f and g.
  - A set $S \subseteq U$ is **closed** under f and g iff

$$x, y \in S \Rightarrow f(x,y),\ g(x) \in S.$$

- Let C^* be the intersection of all the inductive subsets of U; thus $x \in C^*$ iff x belongs to every inductive subset of U.
  - C^* is itself inductive.
  - C^* is the smallest set that is included in all the other inductive sets.
  - C^* is called the **inductive closure** w.r.t. B, f, g.
- Let C_* be the set of things which can be obtained from B by applying f and g a finite number of times.
  - A **construction sequence** is a finite sequence $\langle x_0, \cdots, x_n \rangle$ of elements of U s.t. for each $i \leq n$ we have at least one of

$$\begin{aligned} x_i &\in B, \\ x_i &= f(x_j, x_k) \quad \text{for } j < i \text{ and } k < i, \\ x_i &= g(x_j) \quad \text{for } j < i. \end{aligned}$$

  - C_* is defined as $\{x :$ some construction sequence that with $x\}$.
    - If C_i is the set of all points x s.t. some construction sequence of length i ends with x, then

$$C_* = \bigcup_i C_i.$$

- $C^* = C_*$; this means the above two definitions are equivalent.
- Since $C^* = C_*$, we call the set simply C and refer to it as the **set generated from B by the functions in $\mathcal{F}$**.
- (**Induction Principle**) Assume that C is the set generated from B by applying the functions in $\mathcal{F}$. If S satisfies $B \subseteq S \subseteq C$ and S is closed under the functions of $\mathcal{F}$, then $S = C$.

1.3 Recursion

- A set C is **freely generated from B by f and g** iff in addition to the requirements for being generated we have
  (a) $f \upharpoonright C$ and $g \upharpoonright C$ are one-to-one (injective), and
  (b) The range of $f \upharpoonright C$, the range of $g \upharpoonright C$, , and the set B are pairwise disjoint.
- (**Recursion Theorem**) Assume that a subset C of U is freely generated from B by f and g, where

$$f : U \times U \rightarrow U, \quad g : U \rightarrow U.$$

Further assume that V is a set and F, G, and h functions such that

$$\begin{array}{rcl} h & : & B \to V, \\ F & : & V \times V \to V, \\ G & : & V \to V. \end{array}$$

Then there exists a unique function

$$\bar{h} : C \to V$$

such that

(a) For all $x \in B$, $\bar{h}(x) = h(x)$.

(b) For all $x, y \in B$,

$$\begin{array}{rcl} \bar{h}(f(x,y)) & = & F(\bar{h}(x), \bar{h}(y)), \\ \bar{h}(g(x)) & = & G(\bar{h}(x)). \end{array}$$

- Pseudo-commutative diagram & commutative diagram:

$$\begin{tikzcd} x \in B \arrow[r, "\bar{h}:U\to V"] \arrow[d, "g:U\to U"'] & \bar{h}(x) \in V \arrow[d, "G:V\to V"] \\ g(x) \in C \arrow[r, "\bar{h}"] & \bar{h}(g(x)) = G(\bar{h}(x)) \end{tikzcd}$$

$$\begin{tikzcd} B \arrow[r, "\bar{h}\equiv h"] \arrow[d, "g"'] & V \arrow[d, "G"] \\ C \arrow[r, "\bar{h}"] & V \end{tikzcd}$$

- Algebraically, *recursion theorem means that any map h of B into V can be extended to a homomorphism $\bar{h}$ from C (with operations f and g) into V (with operations F and G).*

- The wffs are freely generated from the sentence symbols by the five formula-building operations, $\mathfrak{E}_\neg$, $\mathfrak{E}_\vee$, $\mathfrak{E}_\wedge$, $\mathfrak{E}_\to$, $\mathfrak{E}_\leftrightarrow$.

1.4 Truth Assignments

- A **truth assignment** v for a set $\mathcal{S}$ of sentence symbols is a function

$$v : \mathcal{S} \to \{\mathrm{T}, \mathrm{F}\}.$$

  - A truth assignment is an **interpretation**.

- Let $\overline{\mathcal{S}}$ be the set of wffs generated from the five formula building operations. We want an extension $\bar{v}$ of v,

$$\bar{v} : \overline{\mathcal{S}} \to \{\mathrm{T}, \mathrm{F}\},$$

which assigns the correct truth value to each wff in $\overline{\mathcal{S}}$. It should meet the following conditions:

(a) For any $A \in \mathcal{S}$, $\bar{v}(A) = v(A)$.

(b) For any $\alpha \in \overline{\mathcal{S}}$,

(1) $\bar{v}((\neg\alpha)) = \begin{cases} \mathrm{T} & \text{if } \bar{v}(\alpha) = \mathrm{F}, \\ \mathrm{F} & \text{otherwise.} \end{cases}$

(2) $\bar{v}((\alpha \vee \beta)) =$
$\begin{cases} \mathrm{T} & \text{if } \bar{v}(\alpha) = \mathrm{T} \text{ or } \bar{v}(\alpha) = \mathrm{T}, \\ \mathrm{F} & \text{otherwise.} \end{cases}$

(3) $\bar{v}((\alpha \wedge \beta)) =$
$\begin{cases} \mathrm{T} & \text{if } \bar{v}(\alpha) = \mathrm{T} \text{ and } \bar{v}(\alpha) = \mathrm{T}, \\ \mathrm{F} & \text{otherwise.} \end{cases}$

(4) $\bar{v}((\alpha \to \beta)) =$
$\begin{cases} \mathrm{F} & \text{if } \bar{v}(\alpha) = \mathrm{T} \text{ and } \bar{v}(\alpha) = \mathrm{F}, \\ \mathrm{T} & \text{otherwise.} \end{cases}$

(5) $\bar{v}((\alpha \leftrightarrow \beta)) = \begin{cases} \mathrm{T} & \text{if } \bar{v}(\alpha) = \bar{v}(\alpha), \\ \mathrm{F} & \text{otherwise.} \end{cases}$

- Situation explained:

$$\begin{tikzcd} \mathcal{S} \arrow[r, "\bar{v}\equiv v"] \arrow[d, "\{\mathfrak{E}_{\cdot}\}"'] & \{T, F\} \arrow[d, "F{,}G"] \\ \overline{\mathcal{S}} \arrow[r, "\bar{v}"] & \{T, F\} \end{tikzcd}$$

  - In the diagram, the meanings of F and G are indicated in (b.1)–(b.5).

- For any truth assignment v for a set $\mathcal{S}$ there is a unique function $\bar{v} : \overline{\mathcal{S}} \to \{\mathrm{T}, \mathrm{F}\}$ meeting the above conditions (a) and (b.1)–(b.5) by the *Recursion Theorem*.

- A truth assignment v **satisfies** a sentence φ iff $\bar{v}(\varphi) = \mathrm{T}$.
  - $\bar{v}$ is said to be a **model** of φ.

- A set of wffs Σ **tautologically implies** τ, written as $\Sigma \vDash \tau$, iff every truth assignment for the sentence symbols in Σ and τ which satisfies every member of Σ also satisfies τ.
  - Σ can be thought of as hypotheses and τ can be thought of as a *possible* conclusion.
  - $\varnothing \vDash \tau$ iff every truth assignment satisfies τ; in this case we call τ a **tautology** and write this as $\vDash \tau$.

- (**Compactness Theorem**) Let Σ be an infinite set of wffs such that for any finite subset Σ_0 of Σ, there is a truth assignment which satisfies every member of Σ_0. Then there is a truth assignment which satisfies every member of Σ.

- $\Sigma; \alpha \vDash \beta$ iff $\Sigma \vDash (\alpha \to \beta)$.

1.5 Unique Readability

- Every wff has the same number of left as right parentheses.
- Any proper initial segment of a wff contains an excess of left parentheses.
- (**Unique Readability Theorem**) The five formula-building operations, when restricted to the set of wffs,

  (a) have ranges which are disjoint from each other and from the set of sentence symbols, and

  (b) are one-to-one.

- *Unique readability theorem ensures that the set of wffs are* **freely** *generated from the sentence symbols by the formula-building operations.*

1.6 Sentential Connectives

- A k-**place Boolean function** is a function from $\{\mathrm{T},\mathrm{F}\}^k$ to $\{\mathrm{T},\mathrm{F}\}$.
- Suppose that α is a wff whose sentence symbols are at most $A_1, \cdots, A_n$. We define an n-place Boolean function B^n_α (or just B_α), the **Boolean function realized by** α, by

$$B_\alpha(X_1, \cdots, X_n) = \bar{v}(\alpha) \text{ when } v(A_i) = X_i.$$

- Let $\mathrm{F} \le \mathrm{T}$. And let α and β be wffs whose sentence symbols are among $A_1, \cdots, A_n$. Then
  (a) $\alpha \vDash \beta$ iff all $\vec{X} \in \{\mathrm{T},\mathrm{F}\}^n$, $B_\alpha(\vec{X}) \le B_\beta(\vec{X})$.
  (b) $\alpha \vDash\Dashv \beta$ iff $B_\alpha = B_\beta$.
  (c) $\vDash \alpha$ iff $\mathrm{ran}(B_\alpha) = \{\mathrm{T}\}$.
- Let G be an n-place Boolean function, $n \ge 1$. We can find a wff α such that $G = B^n_\alpha$, i.e., such that α realizes the function G.
- For any wff φ, we can find a tautologically equivalent wff α in disjunctive normal form.
- If every function $G : \{\mathrm{T},\mathrm{F}\}^n \to \{\mathrm{T},\mathrm{F}\}$ can be realized by a wff using only the connective symbols $\{s_i\}$, we say that the set $\{s_i\}$ is **complete**.
  - Both $\{\neg, \vee\}$ and $\{\neg, \wedge\}$ are complete.
  - $\{\wedge, \to\}$ is not complete.

1.7 Compactness and Effectiveness

- A set Σ of wffs is **satisfiable** iff there is a truth assignment which satisfies every member of Σ.
- (**Compactness Theorem**) A set of wffs is satisfiable iff every finite subset is satisfiable.
- If $\Sigma \vDash \tau$, then there is a finite Σ_0 such that $\Sigma_0 \vDash \tau$.
- A set Σ of expressions is **decidable** iff there exists an effective procedure which, given an expression α, will decide whether or 'not' $\alpha \in \Sigma$.
  - Some infinite sets are undecidable since there are $2^{\aleph_0}$, i.e. uncountably infinite, sets of expressions but only countably many effective procedures.
- There is an effective procedure which, given a finite set $\Sigma; \tau$ of wffs, will decide whether $\Sigma \vDash \tau$.
  - The truth-table method enables us the effective decision.
  - Note that this theorem is for "sentential logic."
- For a finite set Σ, the set of tautological consequences of Σ is decidable. In particular, the set of tautologies is decidable.
- A set A of expressions is **effectively enumerable** iff there is an effective procedure which lists, in some order, the members of A.
  - When A is infinite, the procedure may not halt but *for any specific member of A, it should eventually* (*in a finite length of time*) *appear on the list.*
- A set A of expressions is effectively enumerable iff there is an effective procedure which, given any expression ⊠, produces the answer "yes" iff ⊠ $\in A$.
  - Note that this procedure may not produce an answer when ⊠ $\notin A$.
- *A set of expressions is decidable iff both it and its complement (w.r.t the set of all expressions) are effectively enumerable.*
- If Σ is a decidable set of wffs, then the set of tautological consequences of Σ is effectively enumerable.

2 First-Order Logic

2.1 First-Order Languages

- Alphabet of first-order logic
  - **Logical symbols**
    * *Parentheses*: (,), [,]
    * *Sentential connective symbols*: $\to, \neg$
    * *Variables* (one for each positive integer n): $v_1, v_2, \cdots$
    * *Equality symbol* (optional): $\approx$
  - **Parameters**
    * *Quantifier symbol*: $\forall$
    * *Predicate symbols*: for each n, some set of symbols, called n-**place predicate symbols**
    * *Function symbols*: for each n, some set of symbols, called n-**place function symbols**
    * *Constant symbols*: some set of symbols; this can be treated as 0-place function symbols
- Example: Language of elementary number theory
  - Equality: =
  - Predicate parameter: two-place predicate symbol <
  - Constant symbol: symbol 0
  - One-place function symbol: S for successor
  - Two-place function symbol: $+, \cdot$ and E for exponentiation
- A **term** is an expression that can be built up from the *constant symbols* and the *variables* by prefixing the *function symbols*.
  - Formally, if $\mathfrak{F}_f$ is a n-place term-building operation for function symbol f such that

$$\mathfrak{F}_f(x_1, \cdots, x_n) = f(x_1, \cdots, x_n)$$

  then the set of terms is the set of expressions generated from the constant symbols and variable by the $\mathfrak{F}_f$ operations.
- An **atomic formula** is an expression of the form

$$P(t_1, \cdots, t_n)$$

  where P is an *n-place predicate symbol* and $t_1, \cdots, t_n$ are *terms*.

- The set of **well-formed formulas** is the set of expressions generated from the *atomic formulas* by the operations $\mathfrak{E}_\neg$, $\mathfrak{E}_\to$, and $\mathfrak{A}_i$ $(i = 1, 2, \cdots)$, where

$$\begin{aligned} \mathfrak{E}_\neg(\gamma) &= (\neg\gamma) \\ \mathfrak{E}_\to(\gamma, \delta) &= (\gamma \to \delta) \\ \mathfrak{A}_i(\gamma) &= (\forall v_i)[\gamma] \end{aligned}$$

- We define, for each wff α, what it means a variable x **occur free** in α recursively:
  (a) For atomic α, x occurs free in α iff x is a symbol of α.
  (b) x occurs free in $\neg\alpha$ iff x occurs free in α.
  (c) x occurs free in $\alpha \to \beta$ iff x occurs free in α or in β.
  (d) x occurs free in $(\forall v_i)[\alpha]$ iff x occurs free in α and $v_i \neq x$.
- If no variable occurs free in the wff α, then α is a **sentence**.

2.2 Truths and Models

- In sentential logic, **truth assignments** enables us to answer the true/false questions and, in first-order logic, **structures** enables us to do.
- A **structure** U for our first-order logic is a *function* whose domain is the set of parameters and such that
  (a) U assigns to the quantifier symbol $\forall$ a nonempty set $|U|$, called the **universe** of U,
  (b) U assigns to each n-place predicate symbol P an n-ary relation $P^U \subseteq |U|^n$, i.e., P^U is a set of n-tuples of members of the universe,
  (c) U assigns to each constant symbol c a member c^U of the universe $|U|$, and
  (d) U assigns to each n-place function symbol f an n-ary (total) operation f^U on $|U|$, i.e., $f^U : |U|^n \to |U|$.
- Setup
  - Let φ be a wff of our language.
  - Let U a structure for the language.
  - Let $s : V \to |U|$ be a **valuation** function from the set V of all variables into the universe $|U|$ of U.
- Definition for what is meant by "U **satisfies** φ **with** s", written as $\models_U \varphi[s]$
  1. **Terms**
     - Extension of s,

$$\bar{s} : T \to |U|,$$

       a function from the set T of all terms into the universe of U is defined as

$$\bar{s}(x) = \begin{cases} s(x) & x \text{ variable}, \\ c^U & x = \text{constant } c, \\ f^U(\bar{s}(t_1), \cdots, \bar{s}(t_n)) & x = f(t_1, \cdots, t_n). \end{cases}$$

     - Commutative diagram:

$$\begin{tikzcd} T \arrow[r, "\bar{s}"] \arrow[d, "\mathfrak{F}_f"'] & {|U|} \arrow[d, "f^U"] \\ T \arrow[r, "\bar{s}"] & {|U|} \end{tikzcd}$$

  2. **Atomic formulas**
     2.1 $\models_U (t_1 \approx t_2)[s]$ iff $\bar{s}(t_1) = \bar{s}(t_2)$.
     2.2 For an n-place predicate parameter P,

$$\models_U P(t_1, \cdots, t_n)[s] \text{ iff } \langle \bar{s}(t_1), \cdots, \bar{s}(t_n)\rangle \in P^U.$$

  3. **Other wffs**
     3.1 $\models_U \neg\varphi[s]$ iff $\not\models_U \varphi[s]$.
     3.2 $\models_U \varphi \to \psi[s]$ iff either $\not\models_U \varphi[s]$ or $\models_U \psi[s]$ or both.
     3.3 $\models_U (\forall x)[\varphi][s]$ iff for every $d \in |U|$, we have $\models_U \varphi[s[d/x]]$
- Let Γ be a set of wffs, φ a wff. Then Γ **logically implies** φ, $\Gamma \models \varphi$, iff for every structure U for the language and every function $s : V \to |U|$ such that U satisfies every member of Γ with s, U also satisfies φ with s.
  - We write "$\gamma \models \varphi$" instead of "$\{\gamma\} \models \varphi$".
  - φ and ψ are **logically equivalent**, $\varphi \models\!\dashv \psi$, iff $\varphi \models \psi$ and $\psi \models \varphi$.
- A wff φ is **valid** iff $\varnothing \models \varphi$ (written just "$\models \varphi$").
  - Thus φ is valid iff for every U and every $s : V \to |U|$, U satisfies φ with s.
  - *First-order analog of the tautologies are the valid formulas.*
- Assume that s_1, s_2 are functions from V into $|U|$ which agree at all variables (if any) which occur free in the wff φ. Then

$$\models_U \varphi[s_1] \text{ iff } \models_U \varphi[s_2].$$

- For a *sentence*[2] σ, either
  (a) U satisfies σ with *every* valuation s from V into $|U|$ (in this case we say that σ is **true** in U, $\models_U \sigma$, or that U is a **model** of σ), or
  (b) U does not satisfy σ with any such function.
- For a set $\Sigma; \sigma$ of sentences, $\Sigma \models \tau$ iff every model of Σ is a model of σ, i.e., at a "meta"-level

$$(\forall U)(\forall s : V \to |U|)[\models_U \Sigma[s] \Rightarrow \models_U \sigma[s]].$$

- *In contrast to the sentential logic, the set of valid formulas is **undecidable**. But the notion of validity turns out to be equivalent to the notion of **deducibility** and using this equivalence we will be able to show that the set of valid wffs is **effectively enumerable**.*
- **Definability of a class of structures**

[2] Note that a sentence is a wff with no free variables.

- **Question**: *given a mathematical object, can we define it in first-order logic?*
- For a set Σ of sentences, let Mod Σ be the class of all models of Σ, i.e., the class of all structures for the language in which every member of Σ is true.
- A class $\mathcal{K}$ of structures for our language is an **elementary class** (EC) iff $\mathcal{K} = \text{Mod}\ \tau$ for some sentence τ ("elementary" is synonymous with "first-order").
- $\mathcal{K}$ is an **elementary class in the wider sense** (EC_Δ) iff $\mathcal{K} = \text{Mod}\ \Sigma$ for some set Σ of sentences.
- Example:
  * Let's consider a language with equality and the parameters $\forall$ and P, where P is a 2-place predicate symbol. A structure (A, R) for the language consists of a nonempty set A and a binary relation R on A.
  * (A, R) is an *ordered set* iff R is transitive and satisfies the trichotomy condition.
  * Since transitivity and trichotomy conditions can be translated into a sentence of the formal language, the *class of nonempty ordered sets is an elementary class.*
  * Essentially the class of nonempty ordered sets is Mod τ, where τ is the *conjunction* of the three sentences

$$(\forall x, y, z)[xPy \to yPz \to xPz]$$
$$(\forall x, y)[xPy \lor x \approx y \lor yPx]$$
$$(\forall x, y)[xPy \to \neg yPx]$$

- **Definability within a structure**
  - **Question**: *Given a mathematical object and a relation on (or other mathematical object based on) that object, can we define the relation (it) in first-order logic?*
  - Given a structure U and a formula φ such that all variables occurring *free* in φ are included among $v_1, \cdots, v_k$. Then for elements $a_1, \cdots, a_k$ of $|U|$,

$$\vDash_U \varphi[[a_1, \cdots, a_k]]$$

    means that U satisfies φ with some function $s : V \to |U|$ for which $s(v_i) = a_i$ for $1 \le i \le k$.
  - For each such φ and U, the k-ary relation

$$\{\langle a_1, \cdots, a_k\rangle : \vDash_U \varphi[[a_1, \cdots, a_k]]\}$$

    is said to be **defined by φ in** U.
  - A k-ary relation on $|U|$ is **definable in** U iff there a formula which defines it in U.
  - Example:
    * Given a structure $\mathcal{N} = (\mathbb{N}, 0, S, +, \cdot)$, some *relations on* $\mathbb{N}$ are definable in $\mathcal{N}$ and some are not; Note that there are uncountably many relations on $\mathbb{N}$ but only $\aleph_0$ possible defining formulas.
    * Ordering relation $\{\langle m, n\rangle : m < n\}$ is defined in $\mathcal{N}$ by the formula

$$(\exists v_3) v_1 + Sv_3 \approx v_2$$

- Let U, B be structures for the language. A **homomorphism** h **of** U **into** B is a function $h : |U| \to |B|$ such that
  (a) For each n-place predicate symbol P and each n-tuple $\langle a_1, \cdots, a_n\rangle$ of elements of $|U|$,

$$\langle a_1, \cdots, a_n\rangle \in P^U \text{ iff } \langle h(a_1), \cdots, h(a_n)\rangle \in P^B,$$

  (b) For each n-place function symbols f and each n-tuple

$$h(f^U(a_1, \cdots, a_n)) = f^B(h(a_1), \cdots, h(a_n)),$$

  (c) For each constant symbol c,

$$h(c^U) = c^B.$$

- If h a homomorphism of U into B and ***one-to-one*** then h is called an **isomorphism** of U into B.
  - In this case, U is said to be **isomorphic to** B.
- Given two structures U and B ($U \subseteq B$), when a homomorphism where
  (a) P^U is the restriction of P^B to $|U|$, for each predicate symbol P and
  (c) f^U is the restriction of f^U to $|U|$, for each function symbol f, and $c^U = c^B$ for each constant symbol c

  exists, U is said to be a **substructure** of B and B an **extension** of U.
- (**Homomorphism Theorem**) Let h be a homomorphism of U into B, and let s map the set of variables into $|U|$.
  (a) For any term t,

$$h(\bar{s}(t)) = \overline{h \circ s}(t)$$

  where $\bar{s}(t)$ is computed in U and $\overline{h \circ s}(t)$ is computed in B.
  (b) for any quantifier-free formula α not containing the equality symbol,

$$\vDash_U \alpha[s] \text{ iff } \vDash_B \alpha[h \circ s].$$

  (c) If h is an isomorphism, then we may delete the restriction "not containing the equality symbol" in (b).
  (d) If h is a homomorphism of U ***onto*** B, then in b we may delete the restriction "quantifier-free".
- Two structures U and B for the language are **elementarily equivalent**, written as $U \equiv B$, iff for any sentence σ,

$$\vDash_U \sigma \Leftrightarrow \vDash_B \sigma.$$

- An **automorphism** of the structure U is an isomorphism of U onto U.
  - *An automorphism preserves the definable relations.* Formally, let h be an automorphism of the structure U and R be an n-ary relation on $|U|$. Then for any $a_1, \cdots, a_n$ in $|U|$,

$$\langle a_1, \cdots, a_n\rangle \in R \Leftrightarrow \langle h(a_1), \cdots, h(a_n)\rangle \in R.$$

  - Above corollary is useful in showing that some relations are *not definable*.

2.3 Unique Readability

- In order to be able to apply Recursion Theorem, we need unique readability results as in sentential logic.
- (**Unique Readability Theorem for Terms**) The set of terms is freely generated from the set of variables and constant symbols by the $\mathfrak{F}_f$ operations.
- (**Unique Readability Theorem for Formulas**) The set of wffs is freely generated from the set of atomic formulas by the operations $\mathfrak{E}_\neg$, $\mathfrak{E}_\to$, $\mathfrak{A}_i$ $(i = 1, 2, \cdots)$.

2.4 Deductive Calculus

- Let's consider what constitutes a '**proof**' of $\Sigma \vDash \tau$. A proof should satisfy at least the followings.
  (a) A proof should be finitely long.
  (b) A person should be able to check the proof to ascertain that it is correct. This checking process should be *effective*.
- We will select an infinite set Λ of formulas to be called **logical axioms** and a **rule of inference** which will enable us to obtain a new formula from certain others.
- Given Λ and inference rules, for a set Γ of formulas, the **theorems** of Γ will be formulas which can be obtained from $\Lambda \cup \Gamma$ by use of the rule of inference (some finite number of times).
- If φ is a theorem of Γ, written as $\Gamma \vdash \varphi$, then a sequence of formulas which records how φ was obtained from $\Gamma \cup \Lambda$ with the rule of inference will be called a **deduction of φ from** Γ.
  - The choice of Λ and the rule(s) of inference is not unique. For example, we can let $\Lambda = \varnothing$ and have many rules of inference. Or we can have infinite set of logical axioms and just one rule of inference.
  - We will take the latter approach.
- Our one rule of inference is traditionally known as **modus ponens**.
- (**Modus Ponens**) *From the formulas α and $\alpha \to \beta$ we may infer β.*
$$\frac{\alpha, \quad \alpha \to \beta}{\beta}$$
- A set Δ of formulas is **closed under modus ponens** iff whenever two formulas α and $\alpha \to \beta$ is in Δ, then also β is in Δ.
- For a fixed set Γ, Δ is **inductive** iff $\Gamma \cup \Lambda \subseteq \Delta$ and Δ is closed under modus ponens.
  - Then the set of theorems of Γ is simply the *smallest inductive set*.
  - This situation is similar to the one in sentential logic but the only difference is we are closing the initial set under ***partially-defined*** function (its domain consists only of pairs of the form $\langle \alpha, \alpha \to \beta \rangle$).
- *We define φ to be a* **theorem of** Γ, *written as $\Gamma \vdash \varphi$ iff φ belongs to the set generated from $\Gamma \cup \Lambda$ by modus ponens.*
  - The set of theorems of Γ is not ***freely*** generated, which implies that a theorem *never has unique deduction.*
- A **deduction of φ from** Γ is a sequence $\langle \alpha_0, \cdots, \alpha_n \rangle$ of formulas s.t. $\alpha_n = \varphi$ and for each $i \le n$ either
  (a) $\alpha_i \in \Gamma \cup \Lambda$, or
  (b) for some j and k less than i, α_i is obtained by modus ponens from α_j and $\alpha_k = \alpha_j \to \alpha_i$.
- *There exists a deduction of α from Γ iff α is a theorem of Γ.*
  - Note that this is similar to '$C^* = C_*$' situation; inductive closure versus formula-building sequences.
  - We define φ to be **deducible from** Γ iff $\Gamma \vdash \varphi$.
- Say that a wff φ is a **generalization** of ψ iff for some $n \ge 0$ and some variables $x_1, \cdots, x_n$,
$$\varphi = (\forall x_1) \cdots (\forall x_n)[\psi].$$
- The **logical axioms** are all generalizations of wffs of the following forms, where x, y are variables and α, β are wffs:
  (a) Tautologies;
  (b) $(\forall x)[\alpha] \to \alpha^x_t$, where t is substitutable for x in α;
    - α^x_t is the expression obtained from the formula α by replacing the variable x, wherever it occurs free in α, by the term t.
  (c) $(\forall x)[\alpha \to \beta] \to ((\forall x)[\alpha] \to (\forall x)[\beta])$;
  (d) $\alpha \to (\forall x)[\alpha]$, where x does not occur free in α;

  And if the language includes equality, then we add
  (e) $x \approx x$;
  (f) $x \approx y \to (\alpha \to \alpha')$, where α is atomic and α' is obtained from α by replacing x in zero or more places by y;
- $\Gamma \vdash \varphi$ iff $\Gamma \cup \Lambda$ tautologically implies φ.
- A set of formulas Γ is **inconsistent** iff for some $\beta \in \Gamma$, both β and $\neg\beta$ are theorems of the set.
- **Deductions and metatheorems**
  - (**Generalization Theorem**) If $\Gamma \vdash \varphi$ and x does not occur free in any formula in Γ, then $\Gamma \vdash (\forall x)[\varphi]$.
    * Generalization Theorem reflects our informal feeling that if we can prove _x_ without any special assumptions about x, then we are entitled to say that "since x are arbitrary, we have $(\forall x)[\_x\_]$".
  - (**Rule T**) If $\Gamma \vdash \alpha_1, \cdots, \Gamma \vdash \alpha_n$ and $\{\alpha_1, \cdots, \alpha_n\}$ tautologically implies β, then $\Gamma \vdash \beta$.
  - (**Deduction Theorem**) If $\Gamma; \gamma \vdash \varphi$, then $\Gamma \vdash (\gamma \to \varphi)$.
  - (**Contraposition**) $\Gamma; \varphi \vdash \neg\psi$ iff $\Gamma; \psi \vdash \neg\varphi$.
  - (**Reductio Ad Absurdum**) If $\Gamma; \varphi$ is inconsistent then $\Gamma \vdash \neg\varphi$.

- (**Generalization on Constants**) Assume that $\Gamma \vdash \varphi$ and that c is a constant symbol which does not occur in Γ. Then there is a variable y (which does not occur in φ) s.t. $\Gamma \vdash (\forall y)[\varphi^c_y]$. Furthermore, there is a deduction of $(\forall y)[\varphi^c_y]$ from Γ in which c does not occur.
    * Assume that $\Gamma \vdash \varphi^x_c$, where the constant c does not occur in Γ or in φ. Then $\Gamma \vdash (\forall x)[\varphi]$, and there is a deduction of $(\forall x)[\varphi]$ from Γ in which c does not occur.
- (**Rule EI**) Assume that the constant symbol c does not occur in φ, ψ, or Γ, and that

$$\Gamma; \varphi^x_c \vdash \psi.$$

Then

$$\Gamma; (\exists x)[\varphi \vdash \psi]$$

and there is a deduction of ψ from $\Gamma; (\exists x)[\varphi]$ in which c does not occur.

2.5 Soundness and Completeness Theorems

- We need to prove that the information conveyed by our deductive calculus is no more (soundness) and no less (completeness).
- A set of formulas Γ is **satisfiable** iff there is some U and s s.t. U satisfies every member of Γ with s.
- (**Soundness Theorem**) If $\Gamma \vdash \varphi$, then $\Gamma \vDash \varphi$.
    - This theorem can be proved using the following lemmas:
        1. *Every logical axiom is valid.*
        2. $\bar{s}(u^x_t) = \overline{s[\bar{s}(t)/x]}(u)$.
        3. (**Substitution Lemma**) If the term t is substitutable for the variable x in the wff φ, then

$$\vDash_U \varphi^x_t[x] \text{ iff } \vDash_U \varphi[s[\bar{s}(t)/x]].$$

        4. If $\vdash (\varphi \leftrightarrow \psi)$, then φ and ψ are logically equivalent.
        5. If φ' is an alphabetic variant of φ, then φ and φ' are logically equivalent.
        6. ***If Γ is satisfiable, then Γ is consistent.*** (In essence, this is equivalent to the soundness theorem.)
- (**Completeness Theorem**[3])
    (a) If $\Gamma \vDash \varphi$, then $\Gamma \vdash \varphi$.
    (b) Any consistent set of formulas is satisfiable.
- (**Compactness Theorem**)
    (a) If $\Gamma \vDash \varphi$, then for some finite $\Gamma_0 \subseteq \Gamma$ we have $\Gamma_0 \vDash \varphi$.
    (b) If every finite subset Γ_0 of Γ is satisfiable, then Γ is satisfiable.
- (**Enumerability Theorem**) For a reasonable language, the set of valid wffs can be effectively enumerable.
    - By a **reasonable language**, we mean one whose set of parameters can be effectively enumerated and such that the two relations

$$\{\langle P, n\rangle : P \text{ is an } n\text{-place predicate symbol}\}$$

and

$$\{\langle f, n\rangle : f \text{ is an } n\text{-place function symbol}\}$$

are *decidable*.
- Let Γ be a decidable set of formulas in a reasonable language.
    (a) The set of theorems of Γ is effectively enumerable.
    (b) The set $\{\varphi : \Gamma \vDash \varphi\}$ of formulas logically implied by Γ is effectively enumerable.
- Assume that Γ is a decidable set of formulas in a reasonable language, and for any sentence σ either $\Gamma \vDash \sigma$ or $\Gamma \vDash \neg\sigma$. Then the set of sentences implied by Γ is decidable.

2.6 Models of Theories

References

[1] J. Barwise, editor. *Handbook of Mathematical Logic.* North Holland, 1977.

[2] J. Barwise and L. Moss. *Vicious Circles.* CSLI Publications, 1996.

[3] H. B. Enderton. *A Mathematical Introduction to Logic.* Academic Press, 1972.

[3] By Gödel (1930).