Chapter I

Structure and Spaces

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1 Set-Theoretic Digression: Sets

[1´1] Sets and Elements

[1´2] Equality of Sets

[1´3] The Empty Set

[1´4] Basic Set of Numbers

[1´5] Describing a Set By Listing Its Elements

1.1. *What is $\{\varnothing\}$? How many elements does it contain?*

Answer. This is a set made up of the empty set, it contains one element. □

1.2. *Which of the following formulas are correct:*

1. *$\varnothing \in \{\varnothing, \{\varnothing\}\}$ is correct*
2. *$\{\varnothing\} \in \{\{\varnothing\}\}$ is correct*
3. *$\varnothing \in \{\{\varnothing\}\}$ is incorrect*

Primary author: Willie Kaufman

1.3. *Yes, $\{\{\varnothing\}\}$ contains just one element.*
Primary author: Willie Kaufman

1.4. *How many elements do the following sets contains?*

1. $\{1, 2, 1\}$; 2 elements

2. $\{1, 2, \{1, 2\}\}$; 3 elements
3. $\{\{2\}\}$; 1 element
4. $\{\{1\}, 1\}$; 2 elements
5. $\{1, \varnothing\}$; 2 elements
6. $\{\{\varnothing\}, \varnothing\}$; 2 elements
7. $\{\{\varnothing\}, \{\}\}$; 1 element
8. $\{x, 3x - 1\}$for some $x \in \mathbb{R}$; 1 element if $x = \frac{1}{2}$, and 2 elements for all for all values of x

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1.5. *(a) :* $\{0, 1, 2, 3, 4\}$
(b) : $\varnothing$
(c) : $\{-1, -2, -3, \cdots\}$

[1´6] Subsets

1.A. *Let a set A have a elements, and let a set B have b elements. Prove that if $A \subset B$, then $a \leq b$.*

Proof. Suppose A is a set made up of a elements, and B is a set made up of b elements. Suppose $a > b$; then there must be some element $x \in A$ that satisfies $x \notin B$. This implies that $A \not\subset B$, as x is not a member of B. By the contrapositive, we obtain the desired result. □

[1´7] Properties of Inclusion

1.B (Reflexivity of Inclusion)**.** *Any set includes itself: $A \subset A$ holds true for any A.*

Proof. We first let A be an arbitrary Set. Suppose that A is the empty set. We would then see that as A doesn't have any elements, that it is vacuously true that every element of A is in A, and thus $A \subset A$. Now suppose that A is non empty. Let $a \in A$ be arbitrary. We see that $a \in A$, and as a was arbitrary, we know that this is true for all elements of A, and thus $A \subset A$. As A was arbitrary, and $A \subset A$ for all cases, we see that $A \subset A$ holds true for any A. □

Primary author: Reilly Noonan Grant

1.C. ***The Empty Set Is Everywhere.*** *The inclusion $\varnothing \subset A$ holds true for any A. In other words, the empty set is present in each set as a subset.*

Proof. Assume that $\varnothing \subset A$ does not hold, by defintion of inclusion, there exist at least one element $a \in \varnothing$ such that $a \notin A$. Since $\varnothing$ does not contain any element, we have a contradiction, and the statement above is true. □

Primary author: Jimin Tan

1.D. *If A , B, and C are sets, $A \subset B$, and $B \subset C$, then $A \subset C$.*

Proof. Let A, B, and C be arbitrary sets such that $A \subset B$ and $B \subset C$. If A is the empty set, it is true that $A \subset C$. If A is nonempty, choose an arbitrary element $a \in A$. Because $A \subset B$, we know that $a \in B$, and similarly because $B \subset C$, $a \in C$. Since a was arbitrary, $A \subset C$.

□

Primary author: Willie Kaufman

[1′8] To Prove Equality of Sets, Prove Two Inclusions

1.E (Criterion of Equality for Sets.)**.** *$A = B$ if and only if $A \subset B$ and $B \subset A$.*

Proof. Let A and B be arbitrary sets. Let us first suppose that $A = B$. We show that $A \subset B$ and $B \subset A$. Let $x \in A$ be arbitrary. Since $A = B$, they have the same elements; since x is an element of A, x is an element of B. Hence, $A \subset B$. A similar argument establishes that $B \subset A$.

We now establish the converse of the previous claim via contraposition. Suppose that it is not the case that $A \subset B$ and $B \subset A$; without loss of generality, we shall assume that $A \not\subset B$. Then there necessarily exists an element $x \in A$ satisfying $x \notin B$. As such, it is not the case that A and B contain the same elements, since B does not contain x. Hence, $A \neq B$. By the contrapositive, this establishes the converse. □

[1′9] Inclusion Versus Belonging

1.F. *$x \in A$ if and only if $\{x\} \subset A$.*

Proof. We will first show that if $\{x\} \subset A$, then $x \in A$. We can see that as $\{x\}$ is a set described by listing all of its elements, that $x \in \{x\}$. We also see that as $\{x\} \subset A$, that all of the elements of $\{x\}$ are also elements of A, and thus $x \in A$. We thus know that if $\{x\} \subset A$, then $x \in A$.

We will now show that if $x \in A$, then $\{x\} \subset A$. We can see that as $\{x\}$ is a set described by listing all of its elements, that x is the only element in $\{x\}$. We also see that as $x \in A$, that all elements of $\{x\}$ belong to A, and thus $\{x\} \subset A$. We now can see that $x \in A$ if and only if $\{x\} \subset A$. □

Primary author: Reilly Noonan Grant

1.G. ***Non-Reflexivity of Belonging.*** *Construct a set A such that $A \notin A$. The example, $\{1\} \notin \{1\}$ shows the statement above. The set that contains $\{1\}$ is $\{\{1\}\}$. Primary author: Jimin Tan*

1.H. ***Non-Transitivity of Belonging.*** *Construct three sets A, B, and C such that $A \in B$ and $B \in C$, but $A \notin C$.*
$A = \{1\}$

$B = \{\{1\}, 2\}$
$C = \{\{\{1\}, 2\}, 3\}$

Primary author: Willie Kaufman

[1′10] Defining a Set by a Condition (Set-Builder Notation)

[1′11] Intersection and Union

1.I (Commutativity of $\cap$ and $\cup$)**.** *For any two sets A and B, we have*

$$A \cap B = B \cap A \quad \textit{and} \quad A \cup B = B \cup A.$$

Proof. For our proof we rely on the commutativity of logical operators, which can be verified via truth tables. Namely, we have

$$\alpha \text{ and } \beta = \beta \text{ and } \alpha \quad \text{and} \quad \alpha \text{ or } \beta = \beta \text{ or } \alpha,$$

where α and β are arbitrary statements. We will show that the statements about intersections and unions reduce to statements with "and" and "or" operators, respectively.

Let A and B be arbitrary sets. Then $x \in A \cap B$ if and only if $x \in A$ and $x \in B$, which holds if and only if $x \in B$ and $x \in A$, which holds if and only if $x \in B \cap A$. This establishes via double-containment that $A \cap B = B \cap A$.

Similarly, $x \in A \cup B$ if and only if $x \in A$ or $x \in B$, which holds if and only if $x \in B$ or $x \in A$, which holds if and only if $x \in B \cup x \in A$. This establishes via double-containment that $A \cup B = B \cup A$. □

Primary author: David Kraemer.

1.6. *Prove that for any set A we have $A \cap A = A$, $A \cup A = A$, $A \cup \varnothing = A$, and $A \cap \varnothing = \varnothing$.*

Proof. Let A be an arbitrary set. If A is the empty set, each of these is true. So we consider when A is not the emptyset. Choose an arbitrary element $a \in A$. This element is in $A \cap A$ by the definition of intersection, as it belongs to both A and A. This element is also in $A \cup A$ by the definition of union, as it belongs to at least one of A and A. This element is in $A \cup \varnothing$, as it belongs to at least one of A and $\varnothing$. This element is not in $A \cap \varnothing$, as it does not belong to both A and $\varnothing$. Since a was arbitrary, each element of A will be in $A \cup A$, $A \cap A$, and $A \cup \varnothing$, and no elements of A will be in $A \cap \varnothing$. No elements that do not belong to A could be in any of these sets, so combining these two facts requires that $A \cup A$, $A \cap A$, and $A \cup \varnothing$ each equal A and $A \cup \varnothing = \varnothing$. □

Primary author: Willie Kaufman

1.7. *Prove that for any sets A and B we have*

$$A \subset B, \quad \text{iff} \quad A \cap B = A, \quad \text{iff} \quad A \cup B = B$$

Proof. We will break this chain if and only if statement into two parts and then prove them separately.

To begin with, we want to show that $A \subset B$, if and only if $A \cap B = A$. For if and only if statement, we need to prove it in both directions. For the forward direction, assume that $A \subset B$ and let $x \in A$, since $A \subset B$, we know that $x \in B$ by definition of inclusion. We have $x \in A$ and $x \in B$, so $x \in A \cap B$. Since $A \cap B \subset A$ by definition of intersection, we have

$$A \cap B = A$$

Then, we consider the backward direction, assume that $A = A \cap B$ and let $x \in A$, since $A \subset A \cap B$, we have $x \in A$ and $x \in B$. Hence, we have

$$A \subset B$$

Now we want to prove the second if and only if statement which is $A \cap B = A$ iff $A \cup B = B$.

We start with the forward. Assume that $A \cap B = A$ and let $x \in A \cup B$, by definition, we know $x \in A$ or $x \in B$. If $x \in A$, since $A \subset A \cap B$, $x \in B$. Since $B \subset A \cup B$, we have:

$$A \cup B = B$$

Backward direction: Let $x \in A$, since $A \cup B \subset B$, $x \in B$. We have whenever $x \in A$, $x \in B$, so $x \in A \cap B$ and $A \subset A \cap B$. Since $A \cap B \subset A$, we have

$$A \cap B = A$$

□

Primary author: Jimin Tan

1.J. *Associativity of $\cap$ and $\cup$. For any sets A, B, and C, we have*

$$(A \cap B) \cap C = A \cap (B \cap C)$$

and

$$(A \cup B) \cup C = A \cup (B \cup C)$$

Proof. Let A, B, and C be arbitrary sets. First, consider the associativity of $\cap$. In the case of intersection, if any of A, B, or C are $\varnothing$, the top claim is true, as both evaluate to the emptyset. So consider the case where none of A, B, or C are the emptyset. Choose an arbitrary element $a \in A \cup B \cup C$. Only elements in $A \cup B \cup C$ will be in the intersection of these sets, so we can ignore other elements; if the two sets we are considering contain exactly the same elements from these sets, they will be the same. $a \in (A \cap B) \cap C$ iff $a \in A \cap B$ and $a \in C$ by the definition of intersection. $a \in A \cap B$ iff $a \in A$ and $a \in B$ by the definition

of intersection. Combining these logically, $a \in (A \cap B) \cap C$ iff it is an element of A, B, and C. Now considering the right side of the equation, $a \in A \cap (B \cap C)$ iff it is in both A and $B \cap C$ by the definition of intersection. a is in $B \cap C$ iff it is in B and C by the definition of intersection. Combining these logically, we have that $a \in A \cap (B \cap C)$ iff $a \in A$, $a \in B$ and $a \in C$. We know that $a \in (A \cap B) \cap C$ iff $a \in A$ and $a \in B$ and $a \in C$ iff $a \in A \cap (B \cap C)$, or $a \in (A \cap B) \cap C$ iff $a \in A \cap (B \cap C)$. Since a was arbitrary, these sets must be the same.
Second, consider the associativity of $\cup$. If all of A, B, and C are $\varnothing$, both the left and right hand sides of the equation evaluate to $\varnothing$, and so the bottom claim is true. So consider the case where $A \cup B \cup C$ is nonempty. Choose an arbitrary element $a \in A \cup B \cup C$. Only elements in $A \cup B \cup C$ will be in the union of these sets, so we can ignore other elements; if the two sets we are considering contain exactly the same elements from these sets, they will be the same. $a \in (A \cup B) \cup C$ iff $a \in (A \cup B)$ or $a \in C$ by the definition of union. $a \in (A \cup B)$ iff $a \in A$ or $a \in B$ by the definition of union. Combining these logically, $a \in (A \cup B) \cup C$ iff $a \in A$, $a \in B$ or $a \in C$. Now consider the right hand side of the equation. $a \in A \cup (B \cup C)$ iff $a \in A$ or $a \in B \cup C$ by the definition of union. $a \in B \cup C$ iff $a \in B$ or $a \in C$ by the definition of union. Combining these logically, $a \in A \cup (B \cup C)$ iff $a \in A$, $a \in B$, or $a \in C$. We know that $a \in (A \cup B) \cup C$ iff $a \in A$, $a \in B$, or $a \in C$ iff $a \in (A \cup B) \cup C$. Since a was arbitrary, these sets must be the same. □

Primary author: Willie Kaufman

1.K. *The notions of intersection and union of an arbitrary collection of sets generalize the notions of intersection and union of two sets: for* $\Gamma = \{A, B\}$, *we have*

$$\bigcap_{C \in \Gamma} C = A \cap B \text{ and } \bigcup_{C \in \Gamma} C = A \cup B$$

Proof. We will discuss the intersection of multiple sets first.

Let $x \in \bigcap_{C \in \Gamma}$ and assume we can list all element like $C_0, C_1 \cdots, C_n \cdots$, we have $x \in C_0 \cap C_1 \cdots \cap C_n$ by definition. Since there are only two sets in Γ, which are A, B, $x \in A \cap B$. We have $\bigcap_{C \in \Gamma} \subset A \cap B$. Let $y \in A \cap B$. Since A and B are the only two elements in Γ, we have $y \in \bigcap_{C \in \Gamma}$, and $A \cap B \subset \bigcap_{C \in \Gamma}$. Then we have

$$A \cap B = \bigcap_{C \in \Gamma}$$

Union:

Let $x \in \bigcup_{C \in \Gamma}$ and assume we can list all element like $C_0, C_1 \cdots, C_n \cdots$, we have $x \in C_0 \cup C_1 \cdots \cup C_n$. Since A and B are the only two sets in Γ, we have $x \in A \cup B$ and $\bigcup_{C \in \Gamma} \subset A \cup B$. Let $y \in A \cup B$, since A and B are the only two sets in Γ, we have $y \in \bigcup_{C \in \Gamma}$, and $A \cup B \subset \bigcup_{C \in \Gamma}$. Then we have

$$A \cup B = \bigcup_{C \in \Gamma}$$

□

Primary author: Jimin Tan

1.8 (Riddle)**.** *How are the notions of system of equations and intersection of sets related to each other?*

Answer. If $E_1, E_2, \ldots, E_n$ are a system of equations and $S_1, S_2, \ldots, S_n$ are the solution sets corresponding to each equation, then the set

$$S = \bigcap_{i=1}^{n} S_i$$

is the solution to the system of equations, as any solution $s \in S$ solves each equation E_i simultaneously. □

Primary author: David Kraemer

1.L (Two Distributitivites)**.** *For any sets A, B, and C, we have*

$$(A \cap B) \cup C = (A \cup C) \cap (B \cup C)$$
$$(A \cup B) \cap C = (A \cap C) \cup (B \cap C)$$

Proof. These properties follow from unpacking the definitions of set union and intersection, as well as from recalling the distributive properties of logical operators:

$$(\alpha \text{ and } \beta) \text{ or } \gamma \Longleftrightarrow (\alpha \text{ or } \gamma) \text{ and } (\beta \text{ or } \gamma)$$
$$(\alpha \text{ or } \beta) \text{ and } \gamma \Longleftrightarrow (\alpha \text{ and } \gamma) \text{ or } (\beta \text{ and } \gamma).$$

We first show $(A \cap B) \cup C = (A \cup C) \cap (B \cup C)$. We have that

$$x \in (A \cap B) \cup C$$

if and only if either

$$x \in A \cap B \text{ or } x \in C,$$

which holds if and only if

$$(x \in A \text{ and } x \in B) \text{ or } x \in C,$$

which holds if and only if

$$(x \in A \text{ and } x \in C) \text{ or } (x \in B \text{ and } x \in C),$$

which holds if and only if

$$(x \in A \cap C) \text{ or } (x \in B \cap C),$$

which holds if and only if

$$x \in (A \cap C) \cup (B \cap C).$$

We now show $(A \cup B) \cap C = (A \cap C) \cup (B \cap C)$. We have that

$$x \in (A \cup B) \cap C$$

if and only if either

$$x \in A \cup B \text{ and } x \in C,$$

which holds if and only if

$$(x \in A \text{ or } x \in B) \text{ and } x \in C,$$

which holds if and only if

$$(x \in A \text{ or } x \in C) \text{ and } (x \in B \text{ or } x \in C),$$

which holds if and only if

$$(x \in A \cup C) \text{ and } (x \in B \cup C),$$

which holds if and only if

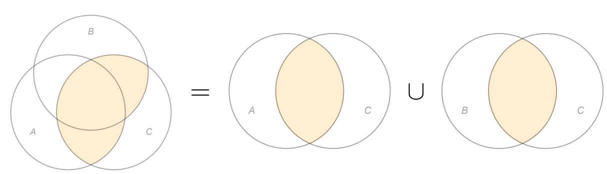
$$x \in (A \cup C) \cap (B \cup C).$$

These equivalencies establish the claim. $\square$

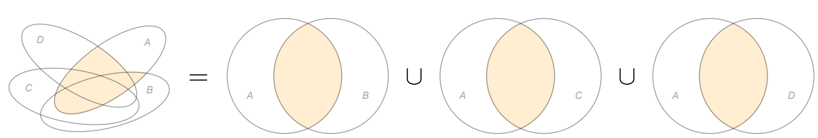
Primary author: David Kraemer

1.M. *Draw a Venn diagram illustrating (2). Prove (1) and (2) by tracing all details of the proofs in the Venn diagrams. Draw Venn diagrams illustrating all formulas below in this section.*

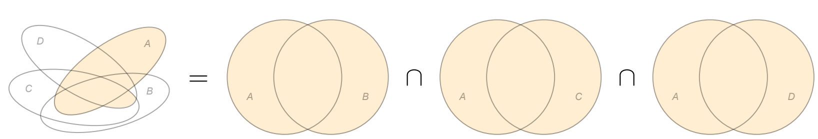
Answer. Demonstration of $(A \cup B) \cap C = (A \cap C) \cup (B \cap C)$:



Demonstration of $A \cap \bigcup_{B \in \Gamma} B = \bigcup_{B \in \Gamma} (A \cap B)$, with $|\Gamma| = 3$.



Demonstration of $A \cup \bigcap_{B \in \Gamma} B = \bigcap_{B \in \Gamma} (A \cup B)$, with $|\Gamma| = 3$.



$\square$

Primary author: David Kraemer.

1.9 (Riddle)**.** *Generalize Theorem 1.L to the case of arbitrary collections of sets.*

Proof. (See 1.N) Let A be a set and let Γ be a set consisting of sets, then we have

$$A \cap \bigcup_{B \in \Gamma} B = \bigcup_{B \in \Gamma} (A \cap B) \text{ and } A \cup \bigcap_{B \in \Gamma} B = \bigcap_{B \in \Gamma} (A \cup B)$$

□

Primary author: Reilly Noonan Grant

1.N (Yet Another Pair of Distributivities)**.** *Let A be a set and let Γ be a set consisting of sets, then we have*

$$A \cap \bigcup_{B \in \Gamma} B = \bigcup_{B \in \Gamma} (A \cap B) \text{ and } A \cup \bigcap_{B \in \Gamma} B = \bigcap_{B \in \Gamma} (A \cup B)$$

Proof. We will first show that $A \cap \bigcup_{B \in \Gamma} B = \bigcup_{B \in \Gamma} (A \cap B)$ using double containment.

First let $x \in A \cap \bigcup_{B \in \Gamma} B$ be arbitrary. We see that

$$x \in A \text{ and } x \in B$$

for some $B \in \Gamma$. We thus know that for some B,

$$x \in (A \cap B)$$

We now can see that as

$$\bigcup_{B \in \Gamma} (A \cap B)$$

contains every $(A \cap B)$, we know that

$$x \in \bigcup_{B \in \Gamma} (A \cap B)$$

As x was arbitrary, we know that

$$A \cap \bigcup_{B \in \Gamma} B \subseteq \bigcup_{B \in \Gamma} (A \cap B).$$

We now let $x \in \bigcup (A \cap B)$ be arbitrary. We see that

$$x \in A \text{ and } x \in B$$

for some $B \in \Gamma$. We also see that as $B \in \Gamma$, that

$$B \subset \bigcup_{B \in \Gamma} B$$

and thus $x \in A$ and $x \in \bigcup_{B\in\Gamma}$. We now can see that this only holds if

$$x \in A \cap \bigcup_{B\in\Gamma}$$

and thus as x was arbitrary

$$A \cap \bigcup_{B\in\Gamma} B \supseteq \bigcup_{B\in\Gamma} (A \cap B).$$

We now can see by double containment, that $A \cap \bigcup_{B\in\Gamma} B = \bigcup_{B\in\Gamma}(A \cap B)$.

We will now show that

$$A \cup \bigcap_{B\in\Gamma} B = \bigcap_{B\in\Gamma} (A \cup B)$$

by double containment.

We first let $x \in A \cup \bigcap_{B\in\Gamma} B$ be arbitrary. We see that

$$x \in A \text{ or } x \in \bigcap_{B\in\Gamma} B$$

We now suppose that $x \in A$. We would then see that $x \in A \cup B$ for any B, and thus

$$x \in \bigcap_{B\in\Gamma} (A \cap B)$$

Now suppose that $x \notin A$. We would then see that

$$x \in \bigcap_{B\in\Gamma} B$$

and if x is in some B, we know that it would also be in $A \cup B$, so thus, we would know that

$$x \in \bigcap_{B\in\Gamma} (A \cup B)$$

As x was arbitrary, we know that

$$A \cup \bigcap_{B\in\Gamma} B \subseteq \bigcap_{B\in\Gamma} (A \cup B)$$

We now let $x \in \bigcap_{B\in\Gamma}(A \cup B)$ be arbitrary. We can see that $x \in A$ or $x \in B$ for every $B \in \Gamma$. Suppose $x \in A$. It would then be the case that $x \in A \cup \bigcap_{B\in\Gamma} B$ as $x \in A$. Now, suppose that $x \notin A$ we would then have that $x \in \bigcap_{B\in\Gamma} B$ as $x \in \bigcap_{B\in\Gamma}(A \cap B)$, and $x \notin A$. We thus see that as x was arbitrary, that

$$A \cup \bigcap_{B\in\Gamma} B \supseteq \bigcap_{B\in\Gamma} (A \cup B).$$

and thus

$$A \cup \bigcap_{B \in \Gamma} B = \bigcap_{B \in \Gamma} (A \cup B)$$

We can now see that

$$A \cap \bigcup_{B \in \Gamma} B = \bigcup_{B \in \Gamma} (A \cap B) \text{ and } A \cup \bigcap_{B \in \Gamma} B = \bigcap_{B \in \Gamma} (A \cup B)$$

□

Primary author: Reilly Noonan Grant

[1′12] Different Differences

1.10. *Prove that for any two sets A and B their union $A \cup B$ is the union of the following three sets: $A \setminus B$, $B \setminus A$, and $A \cap B$, which are pairwise disjoint.*

Proof. Let A, B and C be arbitrary sets. For an arbitrary value a, $a \in A \cup B$ iff $a \in A$ or $a \in B$. This is equivalent to $a \in A \cup B$ iff one of the three following conditions are met; $a \in A$ and $a \notin B$, $a \notin A$ and $a \in B$, or $a \in A$ and $a \in B$. $a \in A \setminus B$ iff $a \in A$ and $a \notin B$, $a \in B \setminus A$ iff $a \notin A$ and $a \in B$, and $a \in A \cap B$ iff $a \in A$ and $a \in B$. So a is in the union of these sets iff it meets one of those criteria by the definition of union. These criteria are the same as those enumerated about for $A \cup B$. Since a was arbitrary, these sets must be the same. □

Primary author: Willie Kaufman

1.11. *Prove that $A \setminus (A \setminus B) = A \cap B$ for any sets A and B.*

Proof. Let $x \in A \setminus (A \setminus B)$, by definition of set difference, we have $x \in A$ and $x \notin A \setminus B$. By basic set operation, $x \notin A \setminus B$ is the same as $x \in (A \setminus B)^c$ which is equal to $B \cup A^c$. By distribution rule, $(A \cap (A^c \cup B)) = (A \cap A^c) \cup (A \cap B) = A \cap B$, so $x \in A \cap B$ and we have $A \setminus (A \setminus B) \subset A \cap B$. Since this process is reversible, we have $A \cap B \subset A \setminus (A \setminus B)$, and we have:

$$A \setminus (A \setminus B) = A \cap B$$

□

Primary author: Jimin Tan

1.12. *Prove that $A \subset B$ if and only if $A \setminus B = \varnothing$.*

Proof. We have $A \subset B$ if and only if there does not exist an $x \in A$ with $x \notin B$; which holds if and only if it is not the case that $A \setminus B \neq \varnothing$, which holds if and only if $A \setminus B = \varnothing$. □

1.13. *Prove that $A \cap (B \setminus C) = A \cap B \setminus A \cap C$ for any sets A,B, and C.*

Proof. We will show this by using double containment. Let $x \in (A \cap (B \setminus C)$ be arbitrary. We see that x is in A, and in B, but not in C and thus $x \in (A \cap B)$, and as x is not in C, that $x \notin (A \cap C)$. We thus see that $x \in (A \cap B) \setminus (A \cap C)$. We thus see that as x is arbitrary, we know that $A \cap (B \setminus C) \subseteq A \cap B \setminus A \cap C$. We now let $x \in (A \cap B) \setminus (A \cap C)$. We see because of this, that x is in A and B, but that x is not in A and C.We thus know that $x \in A$, and because x is in A, and x is in B, but x is not in A and C, that x is not in B and C, and thus $x \in (B \setminus C)$. We thus see that $x \in A \cap (B \setminus C)$

□

Primary author: Reilly Noonan Grant

1.14. *Prove that for any sets A and B we have*

$$A \triangle B = (A \cup B) \setminus (A \cap B)$$

Proof. Let A and B be arbitrary sets. $A \triangle B$ denotes the set of values a for which it is true either that $a \in A$ and $a \notin B$ or $a \notin A$ and $a \in B$. $(A \cup B) \setminus (A \cap B)$ denotes the set of values b for which $b \in A \cup B$ and $b \notin A \cap B$). We then know $(A \cup B) \setminus (A \cap B)$ denotes the set of values b for which $b \in A$ or $b \in B$ and $b \notin A \cap B$. This is the same as the set of values b for which b belongs to exactly one of A or B, i.e. the set of values for which it is true either that $b \in A$ and $b \notin B$ or $b \notin A$ and $b \in B$. We then have the exact same characterizations of $A \triangle b$ and $(A \cup B) \setminus (A \cap B)$, so the sets are the same. □

1.15 (Associativity of Symmetric Difference.)**.** *Prove that for any sets A, B and C we have*

$$(A \triangle B) \triangle C = A \triangle (B \triangle C)$$

Proof. To prove this equation we need to reinterpret the formula.

LHS =

$$\begin{aligned}
&(A \triangle B) \triangle C \\
=&((A \cup B) \setminus (A \cap B)) \cup C \setminus ((A \cup B) \setminus ((A \cap B)) \cap C) \\
=&A \cup B \cup C \setminus (A \cap B) \setminus (((A \cup B) \cap C) \setminus A \cap B \cap C) \\
=&A \cup B \cup C \setminus (A \cap B) \setminus (((A \cap C) \cup (B \cap C) \setminus A \cap B \cap C) \\
=&A \cup B \cup C \setminus ((A \cap B) \cup (A \cap C) \cup (B \cap C) \setminus A \cap B \cap C))
\end{aligned}$$

RHS:

$$\begin{aligned}
=&(A \triangle (B \triangle C) \\
=&A \triangle (B \cup C \setminus B \cap C) \\
=&A \cup (B \cup C \setminus B \cap C) \setminus A \cap (B \cup C \setminus B \cap C) \\
=&(A \cup B \cup C \setminus B \cap C) \setminus (A \cap (B \cup C)) \setminus (A \cap B \cap C) \\
=&A \cup B \cup C \setminus (((B \cap C) \cup (A \cap B) \cup (A \cap C)) \setminus (A \cap B \cap C)) = LHS
\end{aligned}$$

We have:

$$(A \vartriangle B) \vartriangle C = A \vartriangle (B \vartriangle C)$$

□

Primary author: Jimin Tan

1.16. *Riddle. Find a symmetric definition of the symmetric difference* $(A \vartriangle B) \vartriangle C$ *of three sets and generalize it to arbitrary finite collections of sets.*

Proof. We will start with the definition of symetric difference for three sets. By the definition of symmetric difference of two sets, we can see that the symmetric difference between two sets is the result of removing their intersection from their union. We can then see that by iteratively applying this definition to a set C, that we have that all the elements of A, B, and C that don't belong another set are are included, that all the elements which belong to 2 sets are excluded, and all the elements which belong to both A, B, and C are included. From this, we can see that for 3 sets, the symmetric difference is composed of all elements which belong to an odd number of sets. By continuing to apply the symmetric diffence opperator, we would see that this remains to be the pattern, and thus the general pattern for the definition of symmetric difference will be the collection of elements of all sets which belong to an odd number of sets. □

Primary author: Reilly Noonan Grant

1.17 (Distributivity)**.** *Prove that* $(A \vartriangle B) \cap C = (A \cap C) \vartriangle (B \cap C)$ *for any sets A,B ,and C*

Proof. Let A, B, and C be arbitrary sets. We will show $(A \vartriangle B) \cap C = (A \cap C) \vartriangle (B \cap C)$ by double containment.

We first let $x \in (A \vartriangle B) \cap C$ be arbitrary. We can see that $x \in C$ and $x \in (A \vartriangle B)$ by the properties of intersection. Because $x \in (A \vartriangle B)$, we know that $x \in A$ but $x \notin B$ or $x \in A$ but $x \notin B$. We thus know that $x \in C$ and $x \in A$, but $x \notin B$ or $x \in C$ and $x \in B$, but $x \notin A$. We can see that this is true if and only if $x \in (C \cap A) \setminus B \cup (C \cap B) \setminus A$, and thus $x \in (A \cap C) \vartriangle (B \cap C)$. As x was arbitrary, we can see that $(A \vartriangle B) \cap C \subseteq (A \cap C) \vartriangle (B \cap C)$.

We now let $x \in (A \cap C) \vartriangle (B \cap C)$ be arbitrary. We can see that $x \in (A \cap C)$, but $x \notin (B \cap C)$ or $x \in (B \cap C)$ but $x \notin (A \cap C)$. We thus can see that equivalently, $x \in A$ and $x \in C$ but $x \notin (B \cap C)$ or $x \in B$ and $x \in C$ but $x \notin (A \cap C)$. We can see that in every case that $x \in C$, and thus $x \in C$ and $x \in A$ but $x \notin B$, or $x \in B$ but $x \notin A$. We can see that this is equivilent to $x \in C \cap ((A \setminus B) \cup (B \setminus A))$, and by the definition of symmetric difference, we can see that $x \in (A \vartriangle B) \cap C$. As x was arbitrary, we can now see that $(A \vartriangle B) \cap C \supseteq (A \cap C) \vartriangle (B \cap C)$, and thus $(A \vartriangle B) \cap C = (A \cap C) \vartriangle (B \cap C)$ □

Primary author: Reilly Noonan Grant

1.18. *Does the following inequality hold true for any sets A, B, and C?*

$$(A \vartriangle B) \cup C = (A \cup C) \vartriangle (B \cup C)$$

Proof. For any sets A, B, C where $C \subset A \cap B$, $(A \triangle B) \cup C = (A \cup C) \triangle (B \cup C)$. □

Primary author: Willie Kaufman

2 Topology on a Set

[2′1] Definition of Topological Space

[2′2] Simplest Examples

2.A. *Check that the discrete topological space is a topological space, i.e., all axioms of topological structure hold true.*

Proof. Let the discrete topological space be given by (X, Ω) We will first show that Axiom (1) holds true. Let A be the union of an arbitrary collection of sets in Ω. If $A = \varnothing$, we know by 1.C, and the definition of the discrete topological space that A belongs to Ω. Now suppose that A is non empty. Let $a \in A$ be arbitrary. We see that because $a \in A$, that by properties of a union, that a is in at least one set in Ω, and every set in Ω is a subset of X, we know that $a \in X$. As a was arbitrary, we see that $A \subset X$, and thus A belongs to Ω. As A was arbitrary, we know that Axiom (1) holds.

We will now show that Axiom (2) holds true. Let A be an arbitrary intersection of a finite collection of sets that are elements of Ω. If $A = \varnothing$, we know by 1.C that $A \subset X$ and the definition of the discrete topological space that A belongs to Ω. Now suppose that A is non empty. Let $a \in A$ be arbitrary. We see by the definition of intersection, and the definition of the discrete topological space that a is an element of a subset of X, and thus $a \in X$. As $a \in X$, and a was arbitrary, we see that $A \subset X$, and thus A belongs to Ω. As A was arbitrary, we know that Axiom (2) holds.

By 1.B and 1.C we see that $\varnothing$ and X are subsets of X, and thus by the definition of the discrete topological space, we have that $\varnothing$ and X belong to Ω, and thus Axiom (3) holds true.

As all Axioms of topological structure hold true, we know that a discrete topological space is a topological space. □

Primary author: Reilly Noonan Grant

2.B. *The indiscrete topological space is a topological structure, is it not?*

Proof. We show that the indiscrete topology $\Omega_I = \{X, \varnothing\}$ is indeed a topological structure. We see immediately that $X \in \Omega_I$ and $\varnothing \in \Omega_I$, so Axiom 3 is satisfied.

To see that Ω_I is closed under arbitrary unions, let $\Omega_I' \subseteq \Omega_I$ be arbitrary. Then since $A \cup A = A$ and since

$$\bigcup_{A \in \Omega_I'} A$$

□

2.1. *Let X be the ray $[0, +\infty)$, and let Ω consist of $\varnothing$, X, and all rays $(a, +\infty)$ with $a \geq 0$. Prove that Ω is a topological space.*

Proof. Let $S_1, S_2, \cdots, S_n, \cdots$ be a union of collection of arbitrary number of subsets in X. Since every ray with notation $(a, +\infty) \in X$, and we know that the union of open interval can only be a open interval, we know that the union of these collections of set still belongs to Ω. Since the finite intersection of open interval can only be open interval, so the finite intersection of an arbitrary collection belongs to Ω. We know that $\varnothing$ and X belongs to Ω, so Ω is a topological structure on X. □

Primary Author: Jimin Tan

2.2. *Let X be a plane. Let Σ consist of $\varnothing$, X, and all open disks centered at the origin. Is Σ a topological structure?*

Answer—NEEDS FLESHING OUT. Yes. We see immediately that Axiom 3 is satisfied, since $\varnothing, X \in \Sigma$. In general, we will represent D_r to indicate the open disk of radius $r > 0$ centered at the origin.

To see that Axiom 1 holds, let $\Sigma' \subseteq \Sigma$ be arbitrary. To proceed, we observe that $D_r \subseteq D_{r'}$ whenever $r \leq r'$ (and equality holding exactly when $r = r'$). Let $f : \Sigma' \to \mathbb{R}$ be defined by $f(D_r) = r$, and consider $f(\Sigma')$. Either $f(\Sigma')$ is bounded above, or it is unbounded. If it is bounded above, we have

$$s = \sup f(\Sigma').$$

In this case,

$$\bigcup_{D_r \in \Sigma'} D_r = D_s \in \Sigma.$$

Otherwise,

$$\bigcup_{D_r \in \Sigma'} D_r = X \in \Sigma.$$

In both cases, we see that the unions are elements of Σ.

To see that Axiom 2 holds, let $\Sigma' \subseteq \Sigma$ be an arbitrary finite subset of Σ, and consider

$$\bigcap_{D_r \in \Sigma'} D_r.$$

Now, since $D_r \subseteq D_{r'}$ whenever $r \leq r'$, we have that $D_r \cap D_{r'} = D_r$ whenever $r \leq r'$. In this case, $f(\Sigma')$ is finite, so by taking

$$m = \min(f(\Sigma')),$$

we have that

$$\bigcap_{D_r \in \Sigma'} D_r = D_m.$$

□

2.3. *Let X consist of four elements: $X = \{a, b, c, d\}$. Which of the following collections of its subsets are topological structures in X, i.e., satisfy the axioms of topological structure:*

1. $\varnothing, X, \{a\}, \{b\}, \{a, c\}\{a, b, c\}, \{a, b\}$;

2. $\varnothing, X, \{a\}, \{b\}, \{a, b\}\{b, d\}$;

3. $\varnothing, X, \{a, c, d\}, \{b, c, d\}$?

We see that (1) is a topological structure, however (2) and (3) both fail to satisfy the axioms of topological structure. For (1), we see that any union of the elements of the set is also included, and that any intersection of the elements is also included. Because $\varnothing$ and X are also in (1), we know that (1) satisfies the axioms of topological struture, and thus is a topological structure.

We see that $\{a, b\} \cup \{a, d\} = \{a, b, d\}$. As $\{a, b\}$ and $\{a, d\}$ belong to (2), but $\{a, b, d\}$ does not we see that (2) does not satisfy Axiom 1

We see that $\{a, c, d\} \cap \{b, c, d\} = \{c, d\}$. As $\{a, c, d\}$ and $\{b, c, d\}$ belong to (3), but $\{c, d\}$ does not, we know that (3) does not satisfy Axiom 2

Primary Author: Reilly Noonan Grant

[2′3] The Most Important Example: Real Line

2.C. *Check whether Ω satisfies the axioms of topological structure*

Proof. $\mathbb{R}$ and $\varnothing$ are both open sets, so they belong to the collection of sets Ω. Then, we will prove the first two properties of topology. For the first property, we assume that there is a union of a collection C with arbitrary n sets, and we name them $S_1, S_2, \cdots, S_n$. We can show that the union of two open set is still a open set. So we know that $S_1 \cup S_2$ is still open. Then if S_i and S_{i+1} are open sets, then $S_i + S_{i+1}$ is an open set. By induction we know that

$$\bigcup_{S_i \in C} S_i \in \Omega$$

Then, we prove the other property with induction. Let C be a collection of arbitrary n sets, and we label them $S_1, S_2, \cdots, S_n$. We can show that $S_1 \cap S_2 \in \Omega$, and we can also show that $S_i \cap S_{i+1} \in \Omega$. By induction we know that

$$\bigcap_{S_i \in C} S_i \in \Omega$$

So Ω satisfy the axioms of topological structure. □

Primary Author: Jimin Tan

[2′4] Additional Examples

2.4. *Let X be $\mathbb{R}$, and let Ω consist of the empty set and all infinte subsets of $\mathbb{R}$. Is Ω a topological structure*

No, Ω is not a topology. Consider the sets $\{x|x \in [0,1] or x = -3\}$ and $\{x|x \in [2,3] or x = -3\}$. The intersection of these two sets is simply 3, which does not belong to Ω, violating the definition of a topology.

2.5. *Let X be $\mathbb{R}$, and let Ω consist of the empty set and complements of all finite subsets of $\mathbb{R}$. Is Ω a topological structure?*

Proof. We know that finite subsets of $\mathbb{R}$ is a closed set, and the complement must be an open set. Since the union of any collection of open sets is still an open set that belongs to $\mathbb{R}$, and the intersection of any finite collection of open sets is an open set that belongs to $\mathbb{R}$. Ω also contains the empty set and $\mathbb{R}$, so it is a topological structure on X. □

Primary Author: Jimin Tan

2.6. *Let (X, Ω) be a topological space, Y the set obtained from X by adding a single element a. Is*

$$\Omega' = \{\{a\} \cup U : U \in \Omega\} \cup \{\varnothing\}$$

a topological structure in Y?

Answer. Yes. To see that Axiom 3 is satisfied, notice that since $X \in \Omega$, $\{a\} \cup X \in \Omega'$. We also have that $\varnothing \in \Omega'$. Thus, Axiom 3 holds.

A brief aside on the relationship of Ω and Ω'. Let $f : \Omega' \to \Omega$ be defined by $f(A) = A \setminus \{a\}$. We show that f is a bijection. To see that f is onto, let $U \in \Omega$ be arbitrary. Since $U \cup \{a\} \in \Omega'$ by definition, we have $f(U \cup \{a\}) = U$. Since U was arbitrary, we have that f is onto. To see that f is one-to-one, consider the function $g : \Omega \to \Omega'$ defined by $g(U) = U \cup \{a\}$, which is well-defined. We have

$$\begin{aligned}(f \circ g)(U) &= f(U \cup \{a\}) \\ &= U,\end{aligned}$$

while

$$\begin{aligned}(g \circ a)(A) &= g(A \setminus \{a\}) \\ &= A,\end{aligned}$$

showing that $g = f^{-1}$, which establishes that f is one-to-one. Thus, f is a bijection.

To see that Axiom 1 is satisfied, let $S \subseteq \Omega'$ be arbitrary, and consider $\bigcup_{A \in S} A$.

$$\begin{aligned}\bigcup_{A \in S} A &= \bigcup_{U \in f(S)} U \cup \{a\} \\ &= \{a\} \cup \left(\bigcup_{U \in f(S)} U \right).\end{aligned}$$

Since Ω is a topology, $\bigcup_{U \in f(S)} U = V$ for some $V \in \Omega$, so

$$\bigcup_{A \in S} A = \{a\} \cup V,$$

which is by definition an element of Ω'. This verifies Axiom 1.

To see that Axiom 2 is satisfied, let $S \subseteq \Omega'$ be an arbitrary finite set. Then

$$\begin{aligned}\bigcap_{A \in S} A &= \bigcap_{U \in f(S)} U \cup \{a\} \\ &= \{a\} \cup \left(\bigcup_{U \in f(S)} U \right).\end{aligned}$$

Since Ω is a topology, there is a $V \in \Omega$ such that

$$V = \bigcap_{U \in f(S)} U.$$

$\square$

2.7. *Is the set $\{\varnothing, \{0\}, \{0,1\}\}$ a topological structure in $\{0,1\}$?*

The set $\{\varnothing, \{0\}, \{0,1\}\}$ a topological structure in $\{0,1\}$. We the union of any collection of elements in the topology will be either $\varnothing$, $\{0\}$ or $\{0,1\}$ and thus Axiom 1 is satisfied. We similarly see that an intersection of any collection of elements in the set will result in an element that already exists in the topology, and thus Axiom 2 is satisfied. Finally, we see that as $\varnothing$ and $\{0,1\}$ are in the set, that Axiom 3 is satisfied.

2.8. *List all topological structures in a two-element set, say, in $\{0,1\}$*

$\{\varnothing, \{0,1\}\}\{\varnothing, \{0\}, \{1\}, \{0,1\}\}\{\varnothing, \{1\}, \{0,1\}\}\{\varnothing, \{0\}, \{0,1\}\}$
are all topologies.

[2′5] Using New Words: Points, Open Sets, Closed Sets

2.D. *Reformulate the axioms of topological structure using the words open set wherever possible.*

[2′6] Set-Theoretic Digression: De Morgan Formulas

2.E. *Let Γ ba an arbitrary collection of subsets of a set X. Then*

$$X \setminus \bigcup_{A\in\Gamma} A = \bigcap_{A\in\Gamma} (X \setminus A) \tag{I.1}$$

$$X \setminus \bigcap_{A\in\Gamma} A = \bigcup_{A\in\Gamma} (X \setminus A) \tag{I.2}$$

Proof. We will first show that $X \setminus \bigcup_{A\in\Gamma} A = \bigcap_{A\in\Gamma}(X \setminus A)$ Let $x \in X \setminus \bigcup_{A\in\Gamma} A$ be arbitrary. We see by definition of union, that $x \in X$ and $x \notin A$ for any $A \in \Gamma$. We now can see that for any $A \in \Gamma$, $x \in (X \setminus A)$ as $x \in X$, and $x \notin A$. Because for any $A \in \Gamma$ we know $x \in (X \setminus A)$, we thus also know that $x \in \bigcap_{A\in\Gamma}(X \setminus A)$, and as x was arbitrary, we know that $X \setminus \bigcup_{A\in\Gamma} A \subset \bigcap_{A\in\Gamma}(X \setminus A)$.

Now let $y \in \bigcap_{A\in\Gamma}(X \setminus A)$ be arbitrary. We see that $y \in X$, and $y \notin A$ for each A. We can now see that if it were true that $y \in \bigcup_{A\in\Gamma}$, then for at least one $A \in \Gamma$, it would not be true that $y \in X$, and $y \notin A$ for each A, and thus $y \notin \bigcup_{A\in\Gamma}$. As it is also true that $y \in X$ we know that $y \in (X \setminus \bigcup_{A\in\Gamma})$, and as y was arbitrary, we know that $X \setminus \bigcup_{A\in\Gamma} A \supset \bigcap_{A\in\Gamma}(X \setminus A)$. We can now see that by 1.E that $X \setminus \bigcup_{A\in\Gamma} A = \bigcap_{A\in\Gamma}(X \setminus A)$.

We will now show that $X \setminus \bigcap_{A\in\Gamma} A = \bigcup_{A\in\Gamma}(X \setminus A)$. Let $x \in X \setminus \bigcap_{A\in\Gamma} A$ be arbitrary. We see by definition of set difference and intersection that $x \in X$, and for some A $x \notin A$. We thus can see that for at least one A, $x \in (X \setminus A)$, and thus that $x \in \bigcup_{A\in\Gamma}(X \setminus A)$. As x was arbitrary, we see that $X \setminus \bigcap_{A\in\Gamma} A \subset \bigcup_{A\in\Gamma}(X \setminus A)$.

Now let $y \in \bigcup_{A\in\Gamma}(X \setminus A)$ be arbitrary. We see that for some $A \in \Gamma$ $y \in X$ and $y \notin A$. Because $y \notin A$ we know that $y \notin \bigcap_{A\in\Gamma} A$, as by definition of intersection, if $y \in \bigcap_{A\in\Gamma} A$ we would have that $y \in A$. We now see that as $y \in X$, and $y \notin \bigcap_{A\in\Gamma} A$ we know that $y \in X \setminus \bigcap_{A\in\Gamma} A$. As y was arbitrary, we know that $X \setminus \bigcap_{A\in\Gamma} A \supset \bigcup_{A\in\Gamma}(X \setminus A)$. We now can see by 1.E that $X \setminus \bigcap_{A\in\Gamma} A = \bigcup_{A\in\Gamma}(X \setminus A)$. We thus know that

$$X \setminus \bigcup_{A\in\Gamma} A = \bigcap_{A\in\Gamma} (X \setminus A)$$

and

$$X \setminus \bigcap_{A\in\Gamma} A = \bigcup_{A\in\Gamma} (X \setminus A)$$

□

[2′7] Properties of Closed Sets

2.F. *Prove that:*

1. *the intersection of any collection of closed sets is closed;*
2. *the union of any finite number of closed sets is closed;*

3. *the empty set and the whole space (i.e., the underlying set of the topological structure) are closed.*

Proof. Define $f : \mathcal{P}(X) \to \mathcal{P}(X)$ by $f(S) = X \setminus S$. This function will be used throughout the proof.

1. Let $K \subset \mathcal{P}(X)$ be an arbitrary collection of closed sets of X. Notice that for all $C \in K$, $X \setminus C$ is open, so $f(K) \subseteq \Omega$. We have, then, that
$$\bigcap_{C \in K} C = \bigcap_{E \in f(K)} X \setminus E.$$
By De Morgan's law (2.E.3), we have
$$\bigcap_{E \in f(K)} X \setminus E = X \setminus \bigcup_{E \in f(K)} E.$$
As the arbitrary union of open sets is open, there is some $O \in \Omega$ such that
$$\bigcup_{E \in f(K)} E = O.$$
Thus,
$$\bigcap_{C \in K} C = X \setminus O,$$
which is therefore closed.

2. Let $K \subset \mathcal{P}(X)$ be an arbitrary finite collection of closed sets of X. Notice that for all $C \in K$, $X \setminus C$ is open, so $f(K) \subseteq \Omega$. We have, then, that
$$\bigcup_{C \in K} C = \bigcap_{E \in f(K)} X \setminus E.$$
By De Morgan's law (2.E.4), we have
$$\bigcup_{E \in f(K)} X \setminus E = X \setminus \bigcap_{E \in f(K)} E.$$
As the finite intersection of open sets is open, there is some $O \in \Omega$ such that
$$\bigcap_{E \in f(K)} E = O.$$
Thus,
$$\bigcup_{C \in K} C = X \setminus O,$$
which is therefore closed.

3. Since $X \in \Omega$ by Axiom 3, X is open; thus, $X \setminus X = \varnothing$ is closed. Similarly, since $\varnothing \in \Omega$, $X \setminus \varnothing = X$ is closed.

$\square$

[2′8] Being Open or Closed

2.G. *Find Examples of sets that are*

1. *both open and closed simultaneously*

2. *neither open, nor closed*

(1) Example: $\varnothing$ and $\mathbb{R}$ are both open and closed at the same time in $\mathbb{R}$.

Proof. They are all open since they belong to the topology on $\mathbb{R}$, and we know that they are complements of each other, so they are both open and closed. □

(2) Example: $(0, 1]$ is neither closed or open on $\mathbb{R}$

Proof. $(0, 1]$ is not open because we have to have infinite unions or intersections of opens to approach 1. Then $(0, 1]^c = (-\infty, 0] \cup (1, +\infty)$ is still not open since we have to take infinite intersections or unions to approach 0. Thus, this set is neither open nor closed. □

Primary Author: Jimin Tan

2.9.

2.H. *Is a closed segment $[a, b]$ closed in $\mathbb{R}$? Yes, it is. The complement of a closed segment $[a, b]$ is the union of the open sets $\{x|x < a\}$ and $\{x|x > b\}$, and so the closed segment is closed by the definition of a closed set and the standard topology on RR.*

2.10. *Prove that the half-open interval $[0, 1)$ is neither open nor closed, but is both a union of closed sets and an intersection of open sets.*

Proof. We will first show that $[0, 1)$ is not open. By the definition of an open set, we know that an open set is an arbitrary union of open sets. We now consider 0. We see that as 0 is in $[0, 1)$, for $[0, 1)$ to be open, there would have to be an open set which contains it, however, if any set (a, b) contains 0, it would also contain the points less than 0 and greater than a, and $[0, 1)$ does not contains these points. We thus see that $[0, 1)$ is not open.

We now will show that $[0, 1)$ is not closed. Consider the complement of $[0, 1)$, $(-\infty, 0) \cup [1, \infty)$. We see that for this set to be open, some open set would have to contain 1, however, if any open set (a, b) contained 1, then we would have that the points less than 1 and greater than 0 would be in the set, and we know that these points are not in the set, and thus $(-\infty, 0) \cup [1, \infty)$ is not open, and its complement is $[0, 1)$ not closed.

We will now construct $[0, 1)$ out of a union of closed sets. Consider a set of the form $[0, x]$ where $x \in (0, 1)$. We see that its complement is $(-\infty, 0) \cup (x, \infty)$ which is open, and thus $[0, x]$ is closed. We also see that if we take the union of every set of this form, that it will have all the same points as $[0, 1)$. We will now prove this. Let A be the union of every set of this form. Because for every x $x < 1$, we know that for any x $[0, x] \subset [0, 1)$, and thus for any $a \in A$, as

$a \in [0, x]$ for some x, $a \in [0, 1)$, and thus $A \subset [0, 1)$. Now, let $b \in [0, 1)$ be arbitrary. We see that if $b = 0$, that $b \in A$, and that if $b \neq 0$, that $b \in (0, 1)$, and thus $[0, b]$ is a set in A, and thus $b \in A$. As b was arbitrary $A \supset [0, 1)$. We now see that by 1.E that $A = [0, 1)$.

We will now construct $[0, 1)$ out of an intersection of open sets. Consider a set of the form $(-\frac{1}{n}, 1)$ where $n \in \mathbb{N}^{>0}$. Because both $-\frac{1}{n}$ and 1 are real numbers, we know that $(-\frac{1}{n}, 1)$ is an open set. We will call the intersection of all sets of this form B. Let $a \in [0, 1)$ be arbitrary. We see that for any n, $a \geq 0 > -\frac{1}{n}$, and $a < 1$, and thus $a \in B$, and $[0, 1) \subset B$. Now let $b \in B$ be arbitrary. Suppose $b < 0$. We would then see that for some n, $b > -\frac{1}{n} > 0$, and thus b would not be in B. We thus can see that $b \geq 0$. We also see that as $b \in (-1, 1)$, that $b < 1$. We thus see that as $b \geq 0$ and $b < 1$, that $b \in [0, 1)$, and thus as b was arbitrary, $B \subset [0, 1)$. We now can see that by 1.E $B = [0, 1)$ □

Primary author: Reilly Noonan Grant

2.11. *Prove that the set* $A = \{0\} \cup \{1/n | n \in N\}$ *is closed in* $\mathbb{R}$.

Proof. To show this, we will show that its complement is the arbitrary union of open sets. Note $A = \{0, 1/2, 1/3, 1/4, ...\}$. Consider the collection of open intervals $\{A_i, A_i + 1) | i \in \mathbb{N}\}$; call this collection B. This set contains no values in A, as they will only be endpoints of the open intervals. There is no way for a value in A to be in an interval of B, because we're creating our collection such that every time we reach a value in A, we create a new interval in our collection. The union of all the sets in B, the open set $\{x | x < 0\}$ and the open set $\{x | x > 1\}$ is the complement of A in $\mathbb{R}$. To see this, choose an arbitrary element $r \in R$. If this element is in A, it must not be in B, as previously discussed, and the values in A are bounded between 0 and 1, so we know the element being in A precludes it from being in this union. The closure of this union is $\mathbb{R}$, so if it is not in this union, it must be a limit point of $\mathbb{R}$ that is not contained in the union. The only values that meet these criteria are exactly the values of A, so we know the element must be in A. We've shown that an element of $\mathbb{R}$ is in A if and only if it is not in the union, so the union must be the complement. Then by the definition of closed, A is closed. □

[2′9] Characterization of Topology in Terms of Closed Sets

[2′10] Neighborhoods

2.15 Give an explicit description of all neighborhoods of a point in

1. a discrete space;
2. an indiscrete space
3. The arrow;
4. X = $\{a, b, c, d\}$

5. a connected pair of points;
6. particular point topology

1. A neighborhood of a point in a discrete space topology (X, Ω) would be all subsets of X which contain that point.
2. A neighborhood of a point in a indiscret space topology (X, Ω) would just be X, as $X$and $\varnothing$ are the only elements of Ω, and all of the points are in X.
3. Let $(a, +\infty)$ be an element of the arrow. We see that a neighborhood of this point would be $[0, +\infty)$ or any ray of the form $(b, +\infty)$ where $b \leq a$, as otherwise, a would not be included, in the set, and thus it would not be a neighborhood
4. We see that if $X = \{a, b, c, d\}$, that the neighborhoods of a are $\{a\}$,$\{a, b\}$,$\{a, c\}$,$\{a, d\}\{a, b, c\}$,$\{a, b, d\}$,$\{a, c, d\}$ and$\{a, b, c, d\}$, and we can see that for any other point, we can get their neighborhoods by swaping all occurances of a with that point, and that point with a
5. The neighborhood of a point in a connected pair of points topology would just be the set of the point by itself, and the set of the point with its pair, as no other set includes that point.
6. The neighborhood of a point in a particular point topology would depend on the point in question. If that point is not the particular point, then it would be similar to the discrete topology, except that every set would also include the particular point, and if the point is the particular point, then every set in the topology would be a neighborhood.

[2′11] Open Sets on Line

2.I. *Prove that every open subset of the real line is a union of disjoint open intervals.*

Proof. We will show this by using proof by contradiction. We first notice that by the definiton of the canonical topological space of the real line, that open sets are arbitary unions of open intervals. We assume for sake of contradiction that there exists an open subset X of the real line where it is neccesary for some collection of open intervals to intersect. We will let A be an arbitrary one of these intersections. Let $c \in A$ be arbitrary, and let B be the open interval (a, b) where a is chosen to be the real number closest to c which satisfies $a < c$, and $a \notin X$, or $-\infty$ if there is no such element, and where b is chosed to be the real number closest to c which satisfies $c < b$ and $b \notin X$, or $+\infty$ if there is no such element. As a and b are the closest elements to c which are not in X, we know that $(a, b) \subset X$. We now can see that if $A \subset (a, b)$ that every element in A could have been covered by one open set, and thus A would not have been neccessary, and every element of X in A could have been covered by one open interval, and we would have a contradiction. We now see that as A is the intersection of arbitrary unions of open sets, that A is an open interval, and thus has the form (α, β) from some $\alpha, \beta \in \mathbb{R}$. We can see that if $\alpha < a$, that $a \in A$, and as $a \notin X$ we would see that A is not a subset of X, and thus $\alpha \geq a$. We similarly see that $\beta \leq b$. We now pick an arbitrary element d from A. As $d \in A$, we know $\alpha < d < \beta$, and thus $a < d < b$. We thus see that $d \in (a, b)$, and as d was arbitrary, we have that $A \subset (a, b)$ and thus have a contradiction, and know that there is no open subset of the real line whiere it is neccesary for some collection of open intervals to intersect, and thus every open subset of the real line can be described as a union of disjoint open intervals.

□

[2′12] Topology and Arithmetic Progressions

3 Bases

[3′1] Definition of Base

3.1. *Can two distinct topological structures have the same base?*

We claim that two distinct topological structures must have different bases.

Proof. Assume that two different topological structure Ω_1 and Ω_2 have the same base S_b. Every subset in Ω_1 can be expressed as unions of elements in S_b which will be in Ω_2 by definition of topological structure. So we reach a conclusion that Ω_1 is the same as Ω_2 since they contain the same elements, a contradiction. □

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3.2 (Riddle)**.** *Prove that any base of the canonical topology on* $\mathbb{R}$ *can be decreased*

Proof. Let Σ be a base of the canonical topology on $\mathbb{R}$. Let A be an element of Σ. We see that A is an open set, and thus an aritrary union of open intervals. □

Primary author: Reilly Noonan Grant

3.3 (Riddle)**.** *What topological structures have exactly one base?*

The first thing to note is that given a base Σ of a topology Ω, if there exists an open set o in Ω that is not in Σ, there exists another base $\Sigma \cup o$. SO the only topologies for which there are a single base are those for which the only base is all of Ω. We know the set X must be the union of sets in Ω, so there must be no collection of sets in Ω, the union of which is X, except the entire set. This is the case when, for all sets $a \in \Omega$, there does not exist some collection of sets C in $\Omega \;\; a$ for which $a \subset C$.

[3′2] When a Collection of Sets is a Base

3.A. *A collection Σ of open sets is a base for the topology iff for every open set U and every point $x \in U$ there is a set $V \in \Sigma$ such that $x \in V \subset U$*

Proof. We will first show that if Σ is a base for the topology if for every open set U and every point $x \in U$ there is a set $V \in \Sigma$ such that $x \in V \subset U$. Let $U \in \Sigma$ be arbitrary and let $x \in U$ be arbitrary. By the definition of a basis, we know that because U is an open set that U is a union of sets in Σ. We know because $x \in U$ that for at least one of the sets in this union, x will be in the set. We will call one of these sets V. We see that if V had an element which was not in U, that the union of sets would not be equal to U, and as $x \in V$, we know that $V \subset U$. We thus have that because U and x were arbitrary, that the statement holds true in all cases.

We will now show that if for every open set U and every point $x \in U$ there is a set $V \in \Sigma$ such that $x \in V \subset U$ that Σ is a base for the topology. Let U be an arbitrary open set. We define A to be the union of all V that correspond to some point in U. We see that as for each V, $V \subset U$, we know that $A \subset U$, as if any element of A were not in U, then for some set V, V would not be a subset of U. We also see that $U \subset A$ as for each $x \in U$, we have that $x \in V$ for some V, and thus $x \in A$ by the definition of a union. We thus have that by 1.E that $A = U$, and as U was arbitrary, we know that any U can is a union of sets in Σ, and thus Σ is a base for the topology. □

[3′3] Bases for Plane

3.4. *Prove that each of the collections Σ^2,Σ^∞, and Σ^1 is a base for some topological structure in $\mathbb{R}^2$, and that the structures determined by these collections coincide*

Proof. We will first show that Σ^2 is a base for the topology of all shapes in $\mathbb{R}^2$ which don't include their boundries. Let U be an arbitrary element of this topology which is not the emptyset, and let $x \in U$ be arbitrary. We now let c be a point not in U such that the distance between x and c is as short as the shortest distance between x and any other point not in U. If no such point exists, then U contains every $x \in \mathbb{R}^2$, and thus $U = \mathbb{R}^2$. We see in this case, that that as Σ^2 contains all open discs, that the disk V centered at x with a radius of 1 is in Σ^2, and as $V \in \mathbb{R}^2$, we know $V \in U$, and thus for some $V \in \Sigma^2$ $x \in V \subset U$. We now suppose that c exists. We see that as the distance is the shortest, that the disk centered at x, V, with a radius equal to the distance between c and x will be contained within U. Because Σ^2 contains all open discs, we see that $V \in \Sigma^2$ and as it is centered at x, we know that $x \in V$. We thus have that for some V, $x \in V \subset U$. We now can see that as x and U were arbitrary, that by 3.A, we know that Σ^2 is a base for the topology of all open shapes in $\mathbb{R}^2$. We can also see that if with each V in the above aproach we inscribe either an element of Σ^1 or Σ^∞ which would also contain x, and all be contained within the open set. Thus by a similar argument, and 3.A, we know that Σ^1 and Σ^∞ are also bases for the topology of all open shapes in $\mathbb{R}^2$.

$\square$

[3′4] Subbases

3.5. *Let X be a set, $\triangle$ a collection of subsetsof X. Prove that $\triangle$ is a subbase for a topology on X iff $X = \cup_{(W\in\triangle)W}$.*

Proof. First, we prove that if $\triangle$ is a subbase for a topology on X, $X = \cup_{W\in\triangle)W}$. By the definition of topology, $X \in \Omega$, and so X must be a subset of the union of any base of Ω by the definition of base. This implies each element of X must be in at least 2 sets in $\triangle$. Since $\triangle$ is a collection of subsets of X, i.e. will not contain elements that X does not, it follows that $X = \cup_{W\in\triangle}W$. Now we consider the other direction. Assume $X = \cup_{W\in\triangle)W}$ for some set $\triangle$. The collection of finite intersections of $\triangle$ is a set for the which union of all the elements equals X; this is obvious if we just consider the set of intersections of a single set in $\triangle$. Then we have that $\triangle$ is a subbase for the indiscrete topology on X; the only nonempty set is X, which we have assured ourselves is equal to the union of some number of finite intersections of sets in $\triangle$. $\square$

[3′5] Infiniteness of the Set of Prime Numbers

[3′6] Hierarchy of Topologies

3.6. *Show that the T_1- topology on the real line is coarser than the canonical topology.*

Proof. We first let A be an arbitrary set in the $T_1 -$ *topology*. We will show that A is in the canonical topology. Because A is in the $T_1 -$ *topology*, we know that it is the complement of some finite set $\{a_1, \cdots, a_n\}$. We thus see that A is of the form $(-\infty, a_1) \cup (a_1, a_2) \cup (a_2, a_3) \cdots \cup (a_n, \infty)$. By defintion of the cannonical topology, we know that any set (a_n, a_{n+1}) is in the cannonical topology, and as $(-\infty, a_1)$ and (a_n, ∞) are open intervals, we see that they are also in the cannonical topology. We thus see that as A is a union of elements in the canonical topology, that it is in the canonical topology, and as A was arbitrary, we see that the canonical topology is coarser than the complement topology. $\square$

3.B. *3.D. Riddle. Formulate a necessary and sufficient condition for two bases to be equivalent without explicitly mentioning the topological structures determined by the bases.*

In order for two bases to describe the same topology, it must be that if an open set O is the union of sets in B_1, it is also equal to the union of sets in B_2. WLOG, assume a set A is equal to the union of some sets in B_1. For an arbitrary element $x \in A$, we must have a set $b \in B_2$ such that $b \subset A$ and $x \in b$ for an arbitrary x. In order to guarantee that this is the case without referring to A, for each set $b \in B_1$ and element $x \in b$, there exists a set $c \in B_2$ such that $c \subset b$ and $x \in c$, as well as this being the case in the other direction.

4 Metric Spaces

[4′1] Definition and First Examples

4.A. *Prove that the function*

$$\rho : X \times X \to \mathbb{R}_+ : (x, y) \mapsto$$

$$\begin{cases} 0 & \textit{if } x = y \\ 1 & \textit{if } x \neq y \end{cases}$$

is a metric for any set X

Proof. We first see that as if $x = y$, then $\rho(x, y)$, and if $x \neq y$, that $\rho(x, y) = 1 \neq 0$, that we have that the first property of metrics holds.

We now see that as $x = y$ implies $y = x$, that if $x = y$, $\rho(x, y) = \rho(y, x) = 0$, and if $x \neq y$, $\rho(x, y) = \rho(y, x) = 1$, and thus the second property of metrics holds.

Finally, we have that as equality is a equivalence relation, that if $x = y$, and $y = z$, that $x = z$, and thus $\rho(x, y) = \rho(x, z)+\rho(z, y) = 0$, and that otherwise, as $\rho(x, y) \geq 0$ for any x, y, we see that if $x \neq y$, that $\rho(x, y) \leq \rho(x, z) + \rho(z, y)$ □

[4′2] FurtherExamples

4.1. *Prove that ρ^p is a metric for any $p \geq 1$, where*

$$\rho^{(p)} : (x, y) \mapsto (\sum_{i=1}^{n} |x_i - y_i|^p)^{1/p},\ p \geq 1$$

and then show the Holder Inequality.

Let $x_1, ..., x_n, y_1, ..., y_n \geq 0$, let $p, q > 0$, and let $1/p + 1/q = 1$. Prove that

$$\sum_{i=1}^{n} x_i y_i \leq (\sum_{i=1}^{n} x_i^p)^{1/p} (\sum_{i=1}^{n} y_i^q)^{1/q}$$

Proof. We will first show that if $\rho^{(p)}(x, y) = 0$ then $x = y$. Let $x, y \in \mathbb{R}^n$ be arbitrary, such that $\rho^{(p)}(x, y) = 0$. We then have that

$$(\sum_{i=1}^{n} |x_i - y_i|^p)^{1/p} = 0$$

and thus

$$\sum_{i=1}^{n} |x_i - y_i|^p = 0$$

As $|x_i - y_i|^p$ is never negative for any x_i, y_i, we thus see that for every i

$$|x_i - y_i|^p = 0$$

and the exponent function is injective, we know that $|x_i - y_i| = 0$, and thus $x_i = y_i$ for all i, and thus by definition of equality for $\mathbb{R}^n$, we have $x = y$. As x, y were arbitrary, we know have that if $\rho^{(p)}(x, y) = 0$ then $x = y$.

We also see by the definition of equality on $\mathbb{R}^n$, that if for some $x, y \in \mathbb{R}^n$ $x = y$, that for every i, $x_i = y_i$, and thus $|x_i - y_i| = 0$ for every i, and thus $(\sum_{i=1}^{n} |x_i - y_i|^p)^{1/p} = 0$ and thus $\rho^{(p)}(x, y) = 0$. We thus have that as x, y were arbitrary, that if $x = y$, then $\rho^{(p)}(x, y) = 0$, and thus $\rho^{(p)}(x, y) = 0$ iff $x = y$.

We will now show that for any $x, y \in \mathbb{R}^n$ that $\rho(x, y) = \rho(y, x)$. Let $x, y \in \mathbb{R}^n$ be arbitrary. We see that

$$\rho(x, y) = (\sum_{i=1}^{n} |x_i - y_i|^p)^{1/p} = (\sum_{i=1}^{n} |y_i - x_i|^p)^{1/p} = \rho(x, y)$$

and as x, y were arbitray, we have that for any $x, y \in \mathbb{R}^n$ that $\rho(x, y) = \rho(y, x)$.

We now will show that $\rho(x, y) \leq \rho(x, z) + \rho(z, y)$ for any x, y, z. Let $x, y, z \in \mathbb{R}^n$ be arbitrary. We see that

$$\rho(x, y) = (\sum_{i=1}^{n} |x_i - y_i|^p)^{1/p} =$$

□

[4′3] Balls and Spheres

[4′4] Subspaces of a Metric Space

4.B. *Prove that for any points x and a of any metric space and any $r > \rho(a, x)$ we have*

$$B_{r-\rho(a,x)}(x) \subset B_r(a) \text{ and } D_{r-\rho(a,x)}(x) \subset D_r(a)$$

Proof. Let X be an arbitrary metric space, and let $a, x \in X$ be arbitrary. We will first show that $B_{r-\rho(a,x)}(x) \subset B_r(a)$. Let $b \in B_{r-\rho(a,x)}(x)$ be arbitrary. □

[4′5] Ultrametrics and p-Adic Numbers

4.C. *Check that only one metric in 4.A -4.2 is an ultra metric*

To do this, we must examine the following metrics
4.A: $\rho : X \times X \to \mathbb{R}_+ : (x, y) \mapsto$

$$\begin{cases} 0 & \text{if } x = y \\ 1 & \text{if } x \neq y \end{cases}$$

4.B: $\mathbb{R} \times \mathbb{R} \to \mathbb{R}_+ : (x, y) \mapsto |x - y|$
4.C: $\mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}_+ : (x, y) \mapsto \sqrt{\Sigma_{i=1}^n (x_i - y_i)^2}$
4.1: $\mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}_+ : (x, y) \mapsto max_{i=1,\ldots,n}|x_i - y_i$
4.2 $\mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}_+ : (x, y) \mapsto \Sigma_{i=1}^n |x_i - y_i|$

We will first show that $4.A$ is an ultrametric, and that all other metrics have a counter example.

Proof. We assume for sake of contradiction that $4.A$ is not an ultra metric, and thus that for some x, y, z $\rho(x, y) > max\{\rho(x, z), \rho(z, y)\}$. We thus see that as there are only two possible values for ρ, that $\rho(x, y) = 1$, and $max\{\rho(x, z), \rho(z, y)\} = 0$ and thus $\rho(z, y) = 0$ and $\rho(x, z) = 0$. However, we see that as $\rho(z, y) = 0$, and $\rho(x, z) = 0$ that by definition of the metric, that $z = y$, and $x = z$, and thus $x = y$, and by definitiion of ρ, we must have that $\rho(x, y) = 0$. We thus have a contradiction, and we know that $4.A$ is an an ultra metric □

We now see that if $x = 1, y = 0, z = .5$, that $|x - y| = 1$, $|x - z| = 0.5$ and $|z - y| = 0.5$, and thus for the metric in $4.B$, there is some value of x, y and z such that $\rho(x, y) > max\{\rho(x, z), \rho(z, y)\}$.

We also see that given the same values of x, y, and z that $\sqrt{(x - y)^2} = 1$, $\sqrt{(x - z)^2} = 0.5$ $\sqrt{(y - z)^2} = 0.5$, and thus for the metric in $4.C$, there is some value of x, y and z such that $\rho(x, y) > max\{\rho(x, z), \rho(z, y)\}$.

We also see that in the case of $n = 1$, that 4.1 and 4.2 reduce to the cases above, and thus are not ultra metrics.

4.D. *Prove that all triangles in an ultrametric space are isosceles (i.e., for any three points a, b, and c, at least two of the three distances $\rho(a, b)$, $\rho(b, c)$, and $\rho(a, c)$ are equal).*

Proof. Suppose for the sake of contradiction that ρ is an ultrametric and ab, c do not form an isosceles triangle. Then without loss of generality, we assume that

$$\rho(a, b) < \rho(a, c) < \rho(b, c).$$

We would then have that $\rho(b, c) > \rho(a, c)$, and $\rho(b, c) > \rho(a, b)$, and thus $\rho(b, c) > max\{\rho(a, c), \rho(a, b)\}$. We thus have that this contradicts that ρ is an ultrametrics, and thus we have that there is not a strict inequality on $\rho(a, b)$,$\rho(a, c)$, and $\rho(b, c)$, and thus at least two of the three distances are equal. □

4.E. *Prove that spheres in an ultrametric space are not only closed (see Problem 4.23), but also open.*

Proof. Let $a \in X$, $r \in \mathbb{R}_+$ be arbitrary and consider the sphere $S_r(a)$. It suffices to show that for every $x \in S_r(a)$, there is a ball $B_{r'}(x) \subseteq S_r(a)$.

Let $x \in S_r(a)$ be arbitrary, and choose $r' = \frac{r}{2}$. For any $y \in B_{r'}(x)$, we have

$$\begin{aligned}\rho(a, y) &\leq \max\{\rho(a, x), \rho(x, y)\} \\ &= \max\{r, \frac{r}{2}\} \\ &= r,\end{aligned}$$

and we have

$$\begin{aligned}\rho(a,x) &\leq \max\{\rho(a,y),\rho(x,y)\}\\ r &\leq \max\{\rho(a,y),\frac{r}{2}\}\\ r &\leq \rho(a,y).\end{aligned}$$

Hence,

$$r \leq \rho(a,y) \leq r,$$

so $r = \rho(a,y)$. Thus, $y \in S_r(a)$, and the result follows. □

4.F. *Prove that ρ is an ultrametric.*

Proof. Let $x, y, z \in \mathbb{Q}$ be arbitrary. There exist $r_i, s_i, \alpha_i \in \mathbb{Z}$ ($i = 1,2,3$) such that

$$\begin{aligned}x - y &= \frac{r_1}{s_1}p^{\alpha_1}\\ x - z &= \frac{r_2}{s_2}p^{\alpha_2}\\ y - z &= \frac{r_3}{s_3}p^{\alpha_3}.\end{aligned}$$

Note that each α_i is unique to the particular difference. Without loss of generality, assume $\alpha_2 \geq \alpha_1$. If $\alpha_1 = \alpha_3$, we are done, because $\rho(x,z) \leq \max\{\rho(x,y),\rho(y,z)\}$, the other inequalities following easily. We have

$$\begin{aligned}y - z &= (y - x) + (x - z)\\ \frac{r_3}{s_3}p^{\alpha_3} &= \frac{r_2}{s_2}p^{\alpha_2} - \frac{r_1}{s_1}p^{\alpha_1}\\ \frac{r_3}{s_3}p^{\alpha_3} &= \left(\frac{r_2}{s_2}p^{\alpha_2-\alpha_1} - \frac{r_1}{s_1}\right)p^{\alpha_1}\\ \frac{r_3}{s_3}p^{\alpha_3} &= \left(\frac{r_2p^{\alpha_2-\alpha_1} - r_1}{s_1s_2}\right)p^{\alpha_1}.\end{aligned}$$

It now suffices to show that

$$\gcd\left(\frac{r_2p^{\alpha_2-\alpha_1} - r_1}{s_1s_2}, p\right) = 1.$$

To this end, notice that the denominator is coprime with p, since both s_1 and s_2 are coprime with p. And notice that the numerator is coprime with p, since r_1 is coprime with p. Hence, we can write

$$\frac{r_3}{s_3}p^{\alpha_3} = \frac{r_4}{s_4}p^{\alpha_1}$$

where $r_4 = r_2p^{\alpha_2-\alpha_1} - r_1$ and $s_4 = s_1s_2$ are coprime with p. This implies that $\alpha_1 = \alpha_3$, and the result follows. □

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