Chapter I

Structure and Spaces

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1 Set-Theoretic Digression: Sets

[1´1] Sets and Elements

[1´2] Equality of Sets

[1´3] The Empty Set

[1´4] Basic Set of Numbers

[1´5] Describing a Set By Listing Its Elements

1.1. *What is $\{\varnothing\}$? How many elements does it contain?*

Answer. This is a set made up of the empty set, it contains one element. □

1.2. *Which of the following formulas are correct:*

1. $\varnothing \in \{\varnothing, \{\varnothing\}\}$ *is correct*
2. $\{\varnothing\} \in \{\{\varnothing\}\}$ *is correct*
3. $\varnothing \in \{\{\varnothing\}\}$ *is incorrect*

Primary author: Willie Kaufman

1.3. *Yes, $\{\{\varnothing\}\}$ contains just one element.*
Primary author: Willie Kaufman

1.4. *How many elements do the following sets contains?*

1. $\{1, 2, 1\}$; 2 elements

2. $\{1, 2, \{1, 2\}\}$; 3 elements
3. $\{\{2\}\}$; 1 element
4. $\{\{1\}, 1\}$; 2 elements
5. $\{1, \varnothing\}$; 2 elements
6. $\{\{\varnothing\}, \varnothing\}$; 2 elements
7. $\{\{\varnothing\}, \{\}\}$; 1 element
8. $\{x, 3x - 1\}$for some $x \in \mathbb{R}$; 1 element if $x = \frac{1}{2}$, and 2 elements for all for all values of x

Primary author: Reilly Noonan Grant

1.5. *(a) :* $\{0, 1, 2, 3, 4\}$
(b) : $\varnothing$
(c) : $\{-1, -2, -3, \cdots\}$

[1´6] Subsets

1.A. *Let a set A have a elements, and let a set B have b elements. Prove that if $A \subset B$, then $a \leq b$.*

Proof. Suppose A is a set made up of a elements, and B is a set made up of b elements. Suppose $a > b$; then there must be some element $x \in A$ that satisfies $x \notin B$. This implies that $A \not\subset B$, as x is not a member of B. By the contrapositive, we obtain the desired result. □

[1´7] Properties of Inclusion

1.B (Reflexivity of Inclusion)**.** *Any set includes itself: $A \subset A$ holds true for any A.*

Proof. We first let A be an arbitrary Set. Suppose that A is the empty set. We would then see that as A doesn't have any elements, that it is vacuously true that every element of A is in A, and thus $A \subset A$. Now suppose that A is non empty. Let $a \in A$ be arbitrary. We see that $a \in A$, and as a was arbitrary, we know that this is true for all elements of A, and thus $A \subset A$. As A was arbitrary, and $A \subset A$ for all cases, we see that $A \subset A$ holds true for any A. □

Primary author: Reilly Noonan Grant

1.C. ***The Empty Set Is Everywhere.*** *The inclusion $\varnothing \subset A$ holds true for any A. In other words, the empty set is present in each set as a subset.*

Proof. Assume that $\varnothing \subset A$ does not hold, by defintion of inclusion, there exist at least one element $a \in \varnothing$ such that $a \notin A$. Since $\varnothing$ does not contain any element, we have a contradiction, and the statement above is true. □

Primary author: Jimin Tan

1.D. *If A , B, and C are sets, $A \subset B$, and $B \subset C$, then $A \subset C$.*

Proof. Let A, B, and C be arbitrary sets such that $A \subset B$ and $B \subset C$. If A is the empty set, it is true that $A \subset C$. If A is nonempty, choose an arbitrary element $a \in A$. Because $A \subset B$, we know that $a \in B$, and similarly because $B \subset C$, $a \in C$. Since a was arbitrary, $A \subset C$.

□

Primary author: Willie Kaufman

[1′8] To Prove Equality of Sets, Prove Two Inclusions

1.E (Criterion of Equality for Sets.)**.** *$A = B$ if and only if $A \subset B$ and $B \subset A$.*

Proof. Let A and B be arbitrary sets. Let us first suppose that $A = B$. We show that $A \subset B$ and $B \subset A$. Let $x \in A$ be arbitrary. Since $A = B$, they have the same elements; since x is an element of A, x is an element of B. Hence, $A \subset B$. A similar argument establishes that $B \subset A$.

We now establish the converse of the previous claim via contraposition. Suppose that it is not the case that $A \subset B$ and $B \subset A$; without loss of generality, we shall assume that $A \not\subset B$. Then there necessarily exists an element $x \in A$ satisfying $x \notin B$. As such, it is not the case that A and B contain the same elements, since B does not contain x. Hence, $A \neq B$. By the contrapositive, this establishes the converse. □

[1′9] Inclusion Versus Belonging

1.F. *$x \in A$ if and only if $\{x\} \subset A$.*

Proof. We will first show that if $\{x\} \subset A$, then $x \in A$. We can see that as $\{x\}$ is a set described by listing all of its elements, that $x \in \{x\}$. We also see that as $\{x\} \subset A$, that all of the elements of $\{x\}$ are also elements of A, and thus $x \in A$. We thus know that if $\{x\} \subset A$, then $x \in A$.

We will now show that if $x \in A$, then $\{x\} \subset A$. We can see that as $\{x\}$ is a set described by listing all of its elements, that x is the only element in $\{x\}$. We also see that as $x \in A$, that all elements of $\{x\}$ belong to A, and thus $\{x\} \subset A$. We now can see that $x \in A$ if and only if $\{x\} \subset A$. □

Primary author: Reilly Noonan Grant

1.G. ***Non-Reflexivity of Belonging.*** *Construct a set A such that $A \notin A$. The example, $\{1\} \notin \{1\}$ shows the statement above. The set that contains $\{1\}$ is $\{\{1\}\}$. Primary author: Jimin Tan*

1.H. ***Non-Transitivity of Belonging.*** *Construct three sets A, B, and C such that $A \in B$ and $B \in C$, but $A \notin C$.*
$A = \{1\}$

$B = \{\{1\}, 2\}$
$C = \{\{\{1\}, 2\}, 3\}$

Primary author: Willie Kaufman

[1′10] Defining a Set by a Condition (Set-Builder Notation)

[1′11] Intersection and Union

1.I (Commutativity of $\cap$ and $\cup$)**.** *For any two sets A and B, we have*

$$A \cap B = B \cap A \quad \textit{and} \quad A \cup B = B \cup A.$$

Proof. For our proof we rely on the commutativity of logical operators, which can be verified via truth tables. Namely, we have

$$\alpha \text{ and } \beta = \beta \text{ and } \alpha \quad \text{and} \quad \alpha \text{ or } \beta = \beta \text{ or } \alpha,$$

where α and β are arbitrary statements. We will show that the statements about intersections and unions reduce to statements with "and" and "or" operators, respectively.

Let A and B be arbitrary sets. Then $x \in A \cap B$ if and only if $x \in A$ and $x \in B$, which holds if and only if $x \in B$ and $x \in A$, which holds if and only if $x \in B \cap A$. This establishes via double-containment that $A \cap B = B \cap A$.

Similarly, $x \in A \cup B$ if and only if $x \in A$ or $x \in B$, which holds if and only if $x \in B$ or $x \in A$, which holds if and only if $x \in B \cup x \in A$. This establishes via double-containment that $A \cup B = B \cup A$. □

Primary author: David Kraemer.

1.6. *Prove that for any set A we have $A \cap A = A$, $A \cup A = A$, $A \cup \varnothing = A$, and $A \cap \varnothing = \varnothing$.*

Proof. Let A be an arbitrary set. If A is the empty set, each of these is true. So we consider when A is not the emptyset. Choose an arbitrary element $a \in A$. This element is in $A \cap A$ by the definition of intersection, as it belongs to both A and A. This element is also in $A \cup A$ by the definition of union, as it belongs to at least one of A and A. This element is in $A \cup \varnothing$, as it belongs to at least one of A and $\varnothing$. This element is not in $A \cap \varnothing$, as it does not belong to both A and $\varnothing$. Since a was arbitrary, each element of A will be in $A \cup A$, $A \cap A$, and $A \cup \varnothing$, and no elements of A will be in $A \cap \varnothing$. No elements that do not belong to A could be in any of these sets, so combining these two facts requires that $A \cup A$, $A \cap A$, and $A \cup \varnothing$ each equal A and $A \cup \varnothing = \varnothing$. □

Primary author: Willie Kaufman

1.7. *Prove that for any sets A and B we have*

$$A \subset B, \quad \text{iff} \quad A \cap B = A, \quad \text{iff} \quad A \cup B = B$$

Proof. We will break this chain if and only if statement into two parts and then prove them separately.

To begin with, we want to show that $A \subset B$, if and only if $A \cap B = A$. For if and only if statement, we need to prove it in both directions. For the forward direction, assume that $A \subset B$ and let $x \in A$, since $A \subset B$, we know that $x \in B$ by definition of inclusion. We have $x \in A$ and $x \in B$, so $x \in A \cap B$. Since $A \cap B \subset A$ by definition of intersection, we have

$$A \cap B = A$$

Then, we consider the backward direction, assume that $A = A \cap B$ and let $x \in A$, since $A \subset A \cap B$, we have $x \in A$ and $x \in B$. Hence, we have

$$A \subset B$$

Now we want to prove the second if and only if statement which is $A \cap B = A$ iff $A \cup B = B$.

We start with the forward. Assume that $A \cap B = A$ and let $x \in A \cup B$, by definition, we know $x \in A$ or $x \in B$. If $x \in A$, since $A \subset A \cap B$, $x \in B$. Since $B \subset A \cup B$, we have:

$$A \cup B = B$$

Backward direction: Let $x \in A$, since $A \cup B \subset B$, $x \in B$. We have whenever $x \in A$, $x \in B$, so $x \in A \cap B$ and $A \subset A \cap B$. Since $A \cap B \subset A$, we have

$$A \cap B = A$$

□

Primary author: Jimin Tan

1.J. *Associativity of $\cap$ and $\cup$. For any sets A, B, and C, we have*

$$(A \cap B) \cap C = A \cap (B \cap C)$$

and

$$(A \cup B) \cup C = A \cup (B \cup C)$$

Proof. Let A, B, and C be arbitrary sets. First, consider the associativity of $\cap$. In the case of intersection, if any of A, B, or C are $\varnothing$, the top claim is true, as both evaluate to the emptyset. So consider the case where none of A, B, or C are the emptyset. Choose an arbitrary element $a \in A \cup B \cup C$. Only elements in $A \cup B \cup C$ will be in the intersection of these sets, so we can ignore other elements; if the two sets we are considering contain exactly the same elements from these sets, they will be the same. $a \in (A \cap B) \cap C$ iff $a \in A \cap B$ and $a \in C$ by the definition of intersection. $a \in A \cap B$ iff $a \in A$ and $a \in B$ by the definition

of intersection. Combining these logically, $a \in (A \cap B) \cap C$ iff it is an element of A, B, and C. Now considering the right side of the equation, $a \in A \cap (B \cap C)$ iff it is in both A and $B \cap C$ by the definition of intersection. a is in $B \cap C$ iff it is in B and C by the definition of intersection. Combining these logically, we have that $a \in A \cap (B \cap C)$ iff $a \in A$, $a \in B$ and $a \in C$. We know that $a \in (A \cap B) \cap C$ iff $a \in A$ and $a \in B$ and $a \in C$ iff $a \in A \cap (B \cap C)$, or $a \in (A \cap B) \cap C$ iff $a \in A \cap (B \cap C)$. Since a was arbitrary, these sets must be the same.
Second, consider the associativity of $\cup$. If all of A, B, and C are $\varnothing$, both the left and right hand sides of the equation evaluate to $\varnothing$, and so the bottom claim is true. So consider the case where $A \cup B \cup C$ is nonempty. Choose an arbitrary element $a \in A \cup B \cup C$. Only elements in $A \cup B \cup C$ will be in the union of these sets, so we can ignore other elements; if the two sets we are considering contain exactly the same elements from these sets, they will be the same. $a \in (A \cup B) \cup C$ iff $a \in (A \cup B)$ or $a \in C$ by the definition of union. $a \in (A \cup B)$ iff $a \in A$ or $a \in B$ by the definition of union. Combining these logically, $a \in (A \cup B) \cup C$ iff $a \in A$, $a \in B$ or $a \in C$. Now consider the right hand side of the equation. $a \in A \cup (B \cup C)$ iff $a \in A$ or $a \in B \cup C$ by the definition of union. $a \in B \cup C$ iff $a \in B$ or $a \in C$ by the definition of union. Combining these logically, $a \in A \cup (B \cup C)$ iff $a \in A$, $a \in B$, or $a \in C$. We know that $a \in (A \cup B) \cup C$ iff $a \in A$, $a \in B$, or $a \in C$ iff $a \in (A \cup B) \cup C$. Since a was arbitrary, these sets must be the same. ☐

Primary author: Willie Kaufman

1.K. *The notions of intersection and union of an arbitrary collection of sets generalize the notions of intersection and union of two sets: for $\Gamma = \{A, B\}$, we have*

$$\bigcap_{C \in \Gamma} C = A \cap B \text{ and } \bigcup_{C \in \Gamma} C = A \cup B$$

Proof. We will discuss the intersection of multiple sets first.

Let $x \in \bigcap_{C \in \Gamma}$ and assume we can list all element like $C_0, C_1 \cdots, C_n \cdots$, we have $x \in C_0 \cap C_1 \cdots \cap C_n$ by definition. Since there are only two sets in Γ, which are A, B, $x \in A \cap B$. We have $\bigcap_{C \in \Gamma} \subset A \cap B$. Let $y \in A \cap B$. Since A and B are the only two elements in Γ, we have $y \in \bigcap_{C \in \Gamma}$, and $A \cap B \subset \bigcap_{C \in \Gamma}$. Then we have

$$A \cap B = \bigcap_{C \in \Gamma}$$

Union:

Let $x \in \bigcup_{C \in \Gamma}$ and assume we can list all element like $C_0, C_1 \cdots, C_n \cdots$, we have $x \in C_0 \cup C_1 \cdots \cup C_n$. Since A and B are the only two sets in Γ, we have $x \in A \cup B$ and $\bigcup_{C \in \Gamma} \subset A \cup B$. Let $y \in A \cup B$, since A and B are the only two sets in Γ, we have $y \in \bigcup_{C \in \Gamma}$, and $A \cup B \subset \bigcup_{C \in \Gamma}$. Then we have

$$A \cup B = \bigcup_{C \in \Gamma}$$

☐

Primary author: Jimin Tan

1.8 (Riddle)**.** *How are the notions of system of equations and intersection of sets related to each other?*

Answer. If $E_1, E_2, \dots, E_n$ are a system of equations and $S_1, S_2, \dots, S_n$ are the solution sets corresponding to each equation, then the set

$$S = \bigcap_{i=1}^{n} S_i$$

is the solution to the system of equations, as any solution $s \in S$ solves each equation E_i simultaneously. □

Primary author: David Kraemer

1.L (Two Distributitivites)**.** *For any sets A, B, and C, we have*

$$\begin{aligned}(A \cap B) \cup C &= (A \cup C) \cap (B \cup C)\\ (A \cup B) \cap C &= (A \cap C) \cup (B \cap C)\end{aligned}$$

Proof. These properties follow from unpacking the definitions of set union and intersection, as well as from recalling the distributive properties of logical operators:

$$\begin{aligned}(\alpha \text{ and } \beta) \text{ or } \gamma &\Longleftrightarrow (\alpha \text{ or } \gamma) \text{ and } (\beta \text{ or } \gamma)\\ (\alpha \text{ or } \beta) \text{ and } \gamma &\Longleftrightarrow (\alpha \text{ and } \gamma) \text{ or } (\beta \text{ and } \gamma).\end{aligned}$$

We first show $(A \cap B) \cup C = (A \cup C) \cap (B \cup C)$. We have that

$$x \in (A \cap B) \cup C$$

if and only if either

$$x \in A \cap B \text{ or } x \in C,$$

which holds if and only if

$$(x \in A \text{ and } x \in B) \text{ or } x \in C,$$

which holds if and only if

$$(x \in A \text{ and } x \in C) \text{ or } (x \in B \text{ and } x \in C),$$

which holds if and only if

$$(x \in A \cap C) \text{ or } (x \in B \cap C),$$

which holds if and only if

$$x \in (A \cap C) \cup (B \cap C).$$

We now show $(A \cup B) \cap C = (A \cap C) \cup (B \cap C)$. We have that

$$x \in (A \cup B) \cap C$$

if and only if either

$$x \in A \cup B \text{ and } x \in C,$$

which holds if and only if

$$(x \in A \text{ or } x \in B) \text{ and } x \in C,$$

which holds if and only if

$$(x \in A \text{ or } x \in C) \text{ and } (x \in B \text{ or } x \in C),$$

which holds if and only if

$$(x \in A \cup C) \text{ and } (x \in B \cup C),$$

which holds if and only if

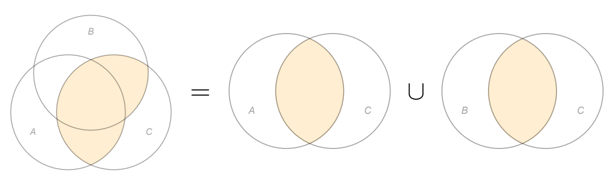
$$x \in (A \cup C) \cap (B \cup C).$$

These equivalencies establish the claim. □

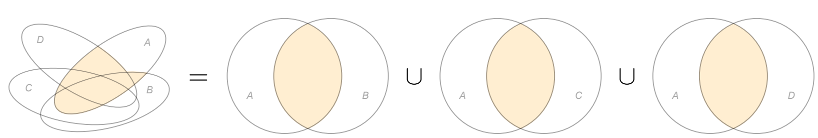
Primary author: David Kraemer

1.M. *Draw a Venn diagram illustrating (2). Prove (1) and (2) by tracing all details of the proofs in the Venn diagrams. Draw Venn diagrams illustrating all formulas below in this section.*

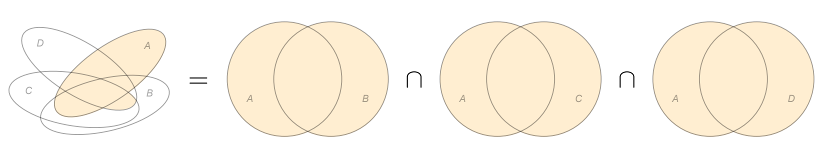
Answer. Demonstration of $(A \cup B) \cap C = (A \cap C) \cup (B \cap C)$:



Demonstration of $A \cap \bigcup_{B \in \Gamma} B = \bigcup_{B \in \Gamma} (A \cap B)$, with $|\Gamma| = 3$.



Demonstration of $A \cup \bigcap_{B \in \Gamma} B = \bigcap_{B \in \Gamma} (A \cup B)$, with $|\Gamma| = 3$.



□

Primary author: David Kraemer.

1.9 (Riddle)**.** *Generalize Theorem 1.L to the case of arbitrary collections of sets.*

Proof. (See 1.N) Let A be a set and let Γ be a set consisting of sets, then we have
$$A \cap \bigcup_{B \in \Gamma} B = \bigcup_{B \in \Gamma} (A \cap B) \text{ and } A \cup \bigcap_{B \in \Gamma} B = \bigcap_{B \in \Gamma} (A \cup B)$$
□

Primary author: Reilly Noonan Grant

1.N (Yet Another Pair of Distributivities)**.** *Let A be a set and let Γ be a set consisting of sets, then we have*

$$A \cap \bigcup_{B \in \Gamma} B = \bigcup_{B \in \Gamma} (A \cap B) \textit{ and } A \cup \bigcap_{B \in \Gamma} B = \bigcap_{B \in \Gamma} (A \cup B)$$

Proof. We will first show that $A \cap \bigcup_{B \in \Gamma} B = \bigcup_{B \in \Gamma} (A \cap B)$ using double containment.

First let $x \in A \cap \bigcup_{B \in \Gamma} B$ be arbitrary. We see that

$$x \in A \text{ and } x \in B$$

for some $B \in \Gamma$. We thus know that for some B,

$$x \in (A \cap B)$$

We now can see that as
$$\bigcup_{B \in \Gamma} (A \cap B)$$
contains every $(A \cap B)$, we know that

$$x \in \bigcup_{B \in \Gamma} (A \cap B)$$

As x was arbitrary, we know that

$$A \cap \bigcup_{B \in \Gamma} B \subseteq \bigcup_{B \in \Gamma} (A \cap B).$$

We now let $x \in \bigcup (A \cap B)$ be arbitrary. We see that

$$x \in A \text{ and } x \in B$$

for some $B \in \Gamma$. We also see that as $B \in \Gamma$, that

$$B \subset \bigcup_{B \in \Gamma} B$$

and thus $x \in A$ and $x \in \bigcup_{B\in\Gamma}$. We now can see that this only holds if

$$x \in A \cap \bigcup_{B\in\Gamma}$$

and thus as x was arbitrary

$$A \cap \bigcup_{B\in\Gamma} B \supseteq \bigcup_{B\in\Gamma} (A \cap B).$$

We now can see by double containment, that $A \cap \bigcup_{B\in\Gamma} B = \bigcup_{B\in\Gamma}(A \cap B)$.

We will now show that

$$A \cup \bigcap_{B\in\Gamma} B = \bigcap_{B\in\Gamma} (A \cup B)$$

by double containment.

We first let $x \in A \cup \bigcap_{B\in\Gamma} B$ be arbitrary. We see that

$$x \in A \text{ or } x \in \bigcap_{B\in\Gamma} B$$

We now suppose that $x \in A$. We would then see that $x \in A \cup B$ for any B, and thus

$$x \in \bigcap_{B\in\Gamma} (A \cap B)$$

Now suppose that $x \notin A$. We would then see that

$$x \in \bigcap_{B\in\Gamma} B$$

and if x is in some B, we know that it would also be in $A \cup B$, so thus, we would know that

$$x \in \bigcap_{B\in\Gamma} (A \cup B)$$

As x was arbitrary, we know that

$$A \cup \bigcap_{B\in\Gamma} B \subseteq \bigcap_{B\in\Gamma} (A \cup B)$$

We now let $x \in \bigcap_{B\in\Gamma}(A \cup B)$ be arbitrary. We can see that $x \in A$ or $x \in B$ for every $B \in \Gamma$. Suppose $x \in A$. It would then be the case that $x \in A \cup \bigcap_{B\in\Gamma} B$ as $x \in A$. Now, suppose that $x \notin A$ we would then have that $x \in \bigcap_{B\in\Gamma} B$ as $x \in \bigcap_{B\in\Gamma}(A \cap B)$, and $x \notin A$. We thus see that as x was arbitrary, that

$$A \cup \bigcap_{B\in\Gamma} B \supseteq \bigcap_{B\in\Gamma} (A \cup B).$$

and thus

$$A \cup \bigcap_{B \in \Gamma} B = \bigcap_{B \in \Gamma} (A \cup B)$$

We can now see that

$$A \cap \bigcup_{B \in \Gamma} B = \bigcup_{B \in \Gamma} (A \cap B) \text{ and } A \cup \bigcap_{B \in \Gamma} B = \bigcap_{B \in \Gamma} (A \cup B)$$

□

Primary author: Reilly Noonan Grant

[1′12] Different Differences

1.10. *Prove that for any two sets A and B their union $A \cup B$ is the union of the following three sets: $A \setminus B$, $B \setminus A$, and $A \cap B$, which are pairwise disjoint.*

Proof. Let A, B and C be arbitrary sets. For an arbitrary value a, $a \in A \cup B$ iff $a \in A$ or $a \in B$. This is equivalent to $a \in A \cup B$ iff one of the three following conditions are met; $a \in A$ and $a \notin B$, $a \notin A$ and $a \in B$, or $a \in A$ and $a \in B$. $a \in A \setminus B$ iff $a \in A$ and $a \notin B$, $a \in B \setminus A$ iff $a \notin A$ and $a \in B$, and $a \in A \cap B$ iff $a \in A$ and $a \in B$. So a is in the union of these sets iff it meets one of those criteria by the definition of union. These criteria are the same as those enumerated about for $A \cup B$. Since a was arbitrary, these sets must be the same. □

Primary author: Willie Kaufman

1.11. *Prove that $A \setminus (A \setminus B) = A \cap B$ for any sets A and B.*

Proof. Let $x \in A \setminus (A \setminus B)$, by definition of set difference, we have $x \in A$ and $x \notin A \setminus B$. By basic set operation, $x \notin A \setminus B$ is the same as $x \in (A \setminus B)^c$ which is equal to $B \cup A^c$. By distribution rule, $(A \cap (A^c \cup B)) = (A \cap A^c) \cup (A \cap B) = A \cap B$, so $x \in A \cap B$ and we have $A \setminus (A \setminus B) \subset A \cap B$. Since this process is reversible, we have $A \cap B \subset A \setminus (A \setminus B)$, and we have:

$$A \setminus (A \setminus B) = A \cap B$$

□

Primary author: Jimin Tan

1.12. *Prove that $A \subset B$ if and only if $A \setminus B = \varnothing$.*

Proof. We have $A \subset B$ if and only if there does not exist an $x \in A$ with $x \notin B$; which holds if and only if it is not the case that $A \setminus B \neq \varnothing$, which holds if and only if $A \setminus B = \varnothing$. □

1.13. *Prove that $A \cap (B \setminus C) = A \cap B \setminus A \cap C$ for any sets A,B, and C.*

Proof. We will show this by using double containment. Let $x \in (A \cap (B \setminus C)$ be arbitrary. We see that x is in A, and in B, but not in C and thus $x \in (A \cap B)$, and as x is not in C, that $x \notin (A \cap C)$. We thus see that $x \in (A \cap B) \setminus (A \cap C)$. We thus see that as x is arbitrary, we know that $A \cap (B \setminus C) \subseteq A \cap B \setminus A \cap C$. We now let $x \in (A \cap B) \setminus (A \cap C)$. We see because of this, that x is in A and B, but that x is not in A and C.We thus know that $x \in A$, and because x is in A, and x is in B, but x is not in A and C, that x is not in B and C, and thus $x \in (B \setminus C)$. We thus see that $x \in A \cap (B \setminus C)$

□

Primary author: Reilly Noonan Grant

1.14. *Prove that for any sets A and B we have*

$$A \vartriangle B = (A \cup B) \setminus (A \cap B)$$

Proof. Let A and B be arbitrary sets. $A \vartriangle B$ denotes the set of values a for which it is true either that $a \in A$ and $a \notin B$ or $a \notin A$ and $a \in B$. $(A \cup B) \setminus (A \cap B)$ denotes the set of values b for which $b \in A \cup B$ and $b \notin A \cap B)$. We then know $(A \cup B) \setminus (A \cap B)$ denotes the set of values b for which $b \in A$ or $b \in B$ and $b \notin A \cap B$. This is the same as the set of values b for which b belongs to exactly one of A or B, i.e. the set of values for which it is true either that $b \in A$ and $b \notin B$ or $b \notin A$ and $b \in B$. We then have the exact same characterizations of $A \vartriangle b$ and $(A \cup B) \setminus (A \cap B)$, so the sets are the same. □

1.15 (Associativity of Symmetric Difference.)**.** *Prove that for any sets A, B and C we have*

$$(A \vartriangle B) \vartriangle C = A \vartriangle (B \vartriangle C)$$

Proof. To prove this equation we need to reinterpret the formula.

LHS =

$$\begin{aligned}
&(A \vartriangle B) \vartriangle C \\
=&((A \cup B) \setminus (A \cap B)) \cup C \setminus ((A \cup B) \setminus ((A \cap B)) \cap C) \\
=&A \cup B \cup C \setminus (A \cap B) \setminus (((A \cup B) \cap C) \setminus A \cap B \cap C) \\
=&A \cup B \cup C \setminus (A \cap B) \setminus (((A \cap C) \cup (B \cap C) \setminus A \cap B \cap C) \\
=&A \cup B \cup C \setminus ((A \cap B) \cup (A \cap C) \cup (B \cap C) \setminus A \cap B \cap C))
\end{aligned}$$

RHS:

$$\begin{aligned}
=&(A \vartriangle (B \vartriangle C) \\
=&A \vartriangle (B \cup C \setminus B \cap C) \\
=&A \cup (B \cup C \setminus B \cap C) \setminus A \cap (B \cup C \setminus B \cap C) \\
=&(A \cup B \cup C \setminus B \cap C) \setminus (A \cap (B \cup C)) \setminus (A \cap B \cap C) \\
=&A \cup B \cup C \setminus (((B \cap C) \cup (A \cap B) \cup (A \cap C)) \setminus (A \cap B \cap C)) = LHS
\end{aligned}$$

We have:

$$(A \vartriangle B) \vartriangle C = A \vartriangle (B \vartriangle C)$$

□

Primary author: Jimin Tan

1.16. *Riddle. Find a symmetric definition of the symmetric difference* $(A \vartriangle B) \vartriangle C$ *of three sets and generalize it to arbitrary finite collections of sets.*

Proof. We will start with the definition of symetric difference for three sets. By the definition of symmetric difference of two sets, we can see that the symmetric difference between two sets is the result of removing their intersection from their union. We can then see that by iteratively applying this definition to a set C, that we have that all the elements of A, B, and C that don't belong another set are are included, that all the elements which belong to 2 sets are excluded, and all the elements which belong to both A, B, and C are included. From this, we can see that for 3 sets, the symmetric difference is composed of all elements which belong to an odd number of sets. By continuing to apply the symmetric diffence opperator, we would see that this remains to be the pattern, and thus the general pattern for the definition of symmetric difference will be the collection of elements of all sets which belong to an odd number of sets. □

Primary author: Reilly Noonan Grant

1.17 (Distributivity)**.** *Prove that* $(A \vartriangle B) \cap C = (A \cap C) \vartriangle (B \cap C)$ *for any sets A,B ,and C*

Proof. Let A, B, and C be arbitrary sets. We will show $(A \vartriangle B) \cap C = (A \cap C) \vartriangle (B \cap C)$ by double containment.

We first let $x \in (A \vartriangle B) \cap C$ be arbitrary. We can see that $x \in C$ and $x \in (A \vartriangle B)$ by the properties of intersection. Because $x \in (A \vartriangle B)$, we know that $x \in A$ but $x \notin B$ or $x \in A$ but $x \notin B$. We thus know that $x \in C$ and $x \in A$, but $x \notin B$ or $x \in C$ and $x \in B$, but $x \notin A$. We can see that this is true if and only if $x \in (C \cap A) \setminus B \cup (C \cap B) \setminus A$, and thus $x \in (A \cap C) \vartriangle (B \cap C)$. As x was arbitrary, we can see that $(A \vartriangle B) \cap C \subseteq (A \cap C) \vartriangle (B \cap C)$.

We now let $x \in (A \cap C) \vartriangle (B \cap C)$ be arbitrary. We can see that $x \in (A \cap C)$, but $x \notin (B \cap C)$ or $x \in (B \cap C)$ but $x \notin (A \cap C)$. We thus can see that equivalently, $x \in A$ and $x \in C$ but $x \notin (B \cap C)$ or $x \in B$ and $x \in C$ but $x \notin (A \cap C)$. We can see that in every case that $x \in C$, and thus $x \in C$ and $x \in A$ but $x \notin B$, or $x \in B$ but $x \notin A$. We can see that this is equivilent to $x \in C \cap ((A \setminus B) \cup (B \setminus A))$, and by the definition of symmetric difference, we can see that $x \in (A \vartriangle B) \cap C$. As x was arbitrary, we can now see that $(A \vartriangle B) \cap C \supseteq (A \cap C) \vartriangle (B \cap C)$, and thus $(A \vartriangle B) \cap C = (A \cap C) \vartriangle (B \cap C)$ □

Primary author: Reilly Noonan Grant

1.18. *Does the following inequality hold true for any sets A, B, and C?*

$$(A \vartriangle B) \cup C = (A \cup C) \vartriangle (B \cup C)$$

Proof. For any sets A, B, C where $C \subset A \cap B$, $(A \triangle B) \cup C = (A \cup C) \triangle (B \cup C)$. $\square$

Primary author: Willie Kaufman

2 Topology on a Set

[2′1] Definition of Topological Space

[2′2] Simplest Examples

2.A. *Check that the discrete topological space is a topological space, i.e., all axioms of topological structure hold true.*

Proof. Let the discrete topological space be given by (X, Ω) We will first show that Axiom (1) holds true. Let A be the union of an arbitrary collection of sets in Ω. If $A = \varnothing$, we know by 1.C, and the definition of the discrete topological space that A belongs to Ω. Now suppose that A is non empty. Let $a \in A$ be arbitrary. We see that because $a \in A$, that by properties of a union, that a is in at least one set in Ω, and every set in Ω is a subset of X, we know that $a \in X$. As a was arbitrary, we see that $A \subset X$, and thus A belongs to Ω. As A was arbitrary, we know that Axiom (1) holds.

We will now show that Axiom (2) holds true. Let A be an arbitrary intersection of a finite collection of sets that are elements of Ω. If $A = \varnothing$, we know by 1.C that $A \subset X$ and the definition of the discrete topological space that A belongs to Ω. Now suppose that A is non empty. Let $a \in A$ be arbitrary. We see by the definition of intersection, and the definition of the discrete topological space that a is an element of a subset of X, and thus $a \in X$. As $a \in X$, and a was arbitrary, we see that $A \subset X$, and thus A belongs to Ω. As A was arbitrary, we know that Axiom (2) holds.

By 1.B and 1.C we see that $\varnothing$ and X are subsets of X, and thus by the definition of the discrete topological space, we have that $\varnothing$ and X belong to Ω, and thus Axiom (3) holds true.

As all Axioms of topological structure hold true, we know that a discrete topological space is a topological space. $\square$

Primary author: Reilly Noonan Grant

2.B. *The indiscrete topological space is a topological structure, is it not?*

Proof. We show that the indiscrete topology $\Omega_I = \{X, \varnothing\}$ is indeed a topological structure. We see immediately that $X \in \Omega_I$ and $\varnothing \in \Omega_I$, so Axiom 3 is satisfied.

To see that Ω_I is closed under arbitrary unions, let $\Omega_I' \subseteq \Omega_I$ be arbitrary. Then since $A \cup A = A$ and since

$$\bigcup_{A \in \Omega_I'} A$$

$\square$

2.1. *Let X be the ray $[0, +\infty)$, and let Ω consist of $\varnothing$, X, and all rays $(a, +\infty)$ with $a \geq 0$. Prove that Ω is a topological space.*

Proof. Let $S_1, S_2, \cdots, S_n, \cdots$ be a union of collection of arbitrary number of subsets in X. Since every ray with notation $(a, +\infty) \in X$, and we know that the union of open interval can only be a open interval, we know that the union of these collections of set still belongs to Ω. Since the finite intersection of open interval can only be open interval, so the finite intersection of an arbitrary collection belongs to Ω. We know that $\varnothing$ and X belongs to Ω, so Ω is a topological structure on X. □

Primary Author: Jimin Tan

2.2. *Let X be a plane. Let Σ consist of $\varnothing$, X, and all open disks centered at the origin. Is Σ a topological structure?*

Answer. Yes. We see immediately that Axiom 3 is satisfied, since $\varnothing, X \in \Sigma$. In general, we will represent D_r to indicate the open disk of radius $r > 0$ centered at the origin.

To see that Axiom 1 holds, let $\Sigma' \subseteq \Sigma$ be arbitrary. To proceed, we observe that $D_r \subseteq D_{r'}$ whenever $r \leq r'$ (and equality holding exactly when $r = r'$). Let $f : \Sigma' \to \mathbb{R}$ be defined by $f(D_r) = r$, and consider the set $R = r|f(D_r) = r for some D_r \in \Sigma'$. Either this set is bounded above, or it is unbounded. If it is bounded above, we have

$$s = \sup R.$$

In this case,

$$\bigcup_{D_r \in \Sigma'} D_r = D_s \in \Sigma.$$

Otherwise, if we have that R is unbounded, every element in X is contained in $\bigcup_{D_r \in \Sigma'} D_r$. We see this by choosing an arbitrary element $x \in X$; $\rho(x, origin) = r$. If $x \notin \Sigma'$, it must be the case that $r > R$ by the definition of disk, but this cannot be the case as R is unbounded. No elements in D_r can be outside of X, of course, so we have that

$$\bigcup_{D_r \in \Sigma'} D_r = X \in \Sigma.$$

In both cases, we see that the unions are elements of Σ.

To see that Axiom 2 holds, let $\Sigma' \subseteq \Sigma$ be an arbitrary finite subset of Σ, and consider

$$\bigcap_{D_r \in \Sigma'} D_r.$$

Now, since $D_r \subseteq D_{r'}$ whenever $r \leq r'$, we have that $D_r \cap D_{r'} = D_r$ whenever $r \leq r'$. In this case, $f(\Sigma')$ is finite, so by taking

$$m = \min(f(\Sigma')),$$

we have that

$$\bigcap_{D_r \in \Sigma'} D_r = D_m.$$

□

Primary authors: Reilly Noonan Grant and Willie Kaufman

2.3. *Let X consist of four elements: $X = \{a, b, c, d\}$. Which of the following collections of its subsets are topological structures in X, i.e., satisfy the axioms of topological structure:*

1. $\varnothing, X, \{a\}, \{b\}, \{a, c\}\{a, b, c\}, \{a, b\}$;
2. $\varnothing, X, \{a\}, \{b\}, \{a, b\}\{b, d\}$;
3. $\varnothing, X, \{a, c, d\}, \{b, c, d\}$?

We see that (1) is a topological structure, however (2) and (3) both fail to satisfy the axioms of topological structure. For (1), we see that any union of the elements of the set is also included, and that any intersection of the elements is also included. Because $\varnothing$ and X are also in (1), we know that (1) satisfies the axioms of topological struture, and thus is a topological structure.

We see that $\{a, b\} \cup \{a, d\} = \{a, b, d\}$. As $\{a, b\}$ and $\{a, d\}$ belong to (2), but $\{a, b, d\}$ does not we see that (2) does not satisfy Axiom 1

We see that $\{a, c, d\} \cap \{b, c, d\} = \{c, d\}$. As $\{a, c, d\}$ and $\{b, c, d\}$ belong to (3), but $\{c, d\}$ does not, we know that (3) does not satisfy Axiom 2

Primary Author: Reilly Noonan Grant

[2′3] The Most Important Example: Real Line

2.C. *Check whether Ω satisfies the axioms of topological structure*

Proof. $\mathbb{R}$ and $\varnothing$ are both open sets, so they belong to the collection of sets Ω. Then, we will prove the first two properties of topology. For the first property, we assume that there is a union of a collection C with arbitrary n sets, and we name them $S_1, S_2, \cdots, S_n$. We can show that the union of two open set is still a open set. So we know that $S_1 \cup S_2$ is still open. Then if S_i and S_{i+1} are open sets, then $S_i + S_{i+1}$ is an open set. By induction we know that

$$\bigcup_{S_i \in C} S_i \in \Omega$$

Then, we prove the other property with induction. Let C be a collection of arbitrary n sets, and we label them $S_1, S_2, \cdots, S_n$. We can show that $S_1 \cap S_2 \in \Omega$, and we can also show that $S_i \cap S_{i+1} \in \Omega$. By induction we know that

$$\bigcap_{S_i \in C} S_i \in \Omega$$

So Ω satisfy the axioms of topological structure. □

Primary Author: Jimin Tan

[2′4] Additional Examples

2.4. *Let X be $\mathbb{R}$, and let Ω consist of the empty set and all infinte subsets of $\mathbb{R}$. Is Ω a topological structure*

No, Ω is not a topology. Consider the sets $\{x|x \in [0,1] or x = -3\}$ and $\{x|x \in [2,3] or x = -3\}$. The intersection of these two sets is simply 3, which does not belong to Ω, violating the definition of a topology.

Primary author: Willie Kaufman

2.5. *Let X be $\mathbb{R}$, and let Ω consist of the empty set and complements of all finite subsets of $\mathbb{R}$. Is Ω a topological structure?*

Proof. We know that finite subsets of $\mathbb{R}$ is a closed set, and the complement must be an open set. Since the union of any collection of open sets is still an open set that belongs to $\mathbb{R}$, and the intersection of any finite collection of open sets is an open set that belongs to $\mathbb{R}$. Ω also contains the empty set and $\mathbb{R}$, so it is a topological structure on X. □

Primary Author: Jimin Tan

2.6. *Let (X, Ω) be a topological space, Y the set obtained from X by adding a single element a. Is*

$$\Omega' = \{\{a\} \cup U : U \in \Omega\} \cup \{\varnothing\}$$

a topological structure in Y?

Answer. Yes. To see that Axiom 3 is satisfied, notice that since $X \in \Omega$, $\{a\} \cup X \in \Omega'$. We also have that $\varnothing \in \Omega'$. Thus, Axiom 3 holds.

A brief aside on the relationship of Ω and Ω'. Let $f : \Omega' \to \Omega$ be defined by $f(A) = A \setminus \{a\}$. We show that f is a bijection. To see that f is onto, let $U \in \Omega$ be arbitrary. Since $U \cup \{a\} \in \Omega'$ by definition, we have $f(U \cup \{a\}) = U$. Since U was arbitrary, we have that f is onto. To see that f is one-to-one, consider the function $g : \Omega \to \Omega'$ defined by $g(U) = U \cup \{a\}$, which is well-defined. We have

$$\begin{aligned}(f \circ g)(U) &= f(U \cup \{a\}) \\ &= U,\end{aligned}$$

while

$$\begin{aligned}(g \circ a)(A) &= g(A \setminus \{a\}) \\ &= A,\end{aligned}$$

showing that $g = f^{-1}$, which establishes that f is one-to-one. Thus, f is a bijection.

To see that Axiom 1 is satisfied, let $S \subseteq \Omega'$ be arbitrary, and consider $\bigcup_{A \in S} A$.

$$\begin{aligned}\bigcup_{A\in S} A &= \bigcup_{U \in f(S)} U \cup \{a\} \\ &= \{a\} \cup \left(\bigcup_{U \in f(S)} U \right).\end{aligned}$$

Since Ω is a topology, $\bigcup_{U \in f(S)} U = V$ for some $V \in \Omega$, so

$$\bigcup_{A \in S} A = \{a\} \cup V,$$

which is by definition an element of Ω'. This verifies Axiom 1.

To see that Axiom 2 is satisfied, let $S \subseteq \Omega'$ be an arbitrary finite set. Then

$$\begin{aligned}\bigcap_{A\in S} A &= \bigcap_{U \in f(S)} U \cup \{a\} \\ &= \{a\} \cup \left(\bigcup_{U \in f(S)} U \right).\end{aligned}$$

Since Ω is a topology, there is a $V \in \Omega$ such that

$$V = \bigcap_{U \in f(S)} U.$$

$\square$

2.7. *Is the set* $\{\varnothing, \{0\}, \{0,1\}\}$ *a topological structure in* $\{0,1\}$*?*

The set $\{\varnothing, \{0\}, \{0,1\}\}$ a topological structure in $\{0,1\}$. We the union of any collection of elements in the topology will be either $\varnothing$, $\{0\}$ or $\{0,1\}$ and thus Axiom 1 is satisfied. We similarly see that an intersection of any collection of elements in the set will result in an element that already exists in the topology, and thus Axiom 2 is satisfied. Finally, we see that as $\varnothing$ and $\{0,1\}$ are in the set, that Axiom 3 is satisfied.

2.8. *List all topological structures in a two-element set, say, in* $\{0,1\}$

$\{\varnothing, \{0,1\}\}\{\varnothing, \{0\}, \{1\}, \{0,1\}\}\{\varnothing, \{1\}, \{0,1\}\}\{\varnothing, \{0\}, \{0,1\}\}$
are all topologies.

[2′5] Using New Words: Points, Open Sets, Closed Sets

2.D. *Reformulate the axioms of topological structure using the words open set wherever possible.*

Answer. Let X be a set. Let Ω be a collection of sets (called *open sets*) such that:

1. the union of any collection of open sets is open;
2. the intersection of any finite collection of open sets is open;
3. the empty set $\varnothing$ and the whole X are open.

Then (X, Ω) is a topological space. $\square$

[2′6] Set-Theoretic Digression: De Morgan Formulas

2.E. *Let Γ ba an arbitrary collection of subsets of a set X. Then*

$$X \setminus \bigcup_{A\in\Gamma} A = \bigcap_{A\in\Gamma} (X \setminus A) \tag{I.1}$$

$$X \setminus \bigcap_{A\in\Gamma} A = \bigcup_{A\in\Gamma} (X \setminus A) \tag{I.2}$$

Proof. We will first show that $X \setminus \bigcup_{A\in\Gamma} A = \bigcap_{A\in\Gamma}(X \setminus A)$ Let $x \in X \setminus \bigcup_{A\in\Gamma} A$ be arbitrary. We see by definition of union, that $x \in X$ and $x \notin A$ for any $A \in \Gamma$. We now can see that for any $A \in \Gamma$, $x \in (X \setminus A)$ as $x \in X$, and $x \notin A$. Because for any $A \in \Gamma$ we know $x \in (X \setminus A)$, we thus also know that $x \in \bigcap_{A\in\Gamma}(X \setminus A)$, and as x was arbitrary, we know that $X \setminus \bigcup_{A\in\Gamma} A \subset \bigcap_{A\in\Gamma}(X \setminus A)$.

Now let $y \in \bigcap_{A\in\Gamma}(X \setminus A)$ be arbitrary. We see that $y \in X$, and $y \notin A$ for each A. We can now see that if it were true that $y \in \bigcup_{A\in\Gamma}$, then for at least one $A \in \Gamma$, it would not be true that $y \in X$, and $y \notin A$ for each A, and thus $y \notin \bigcup_{A\in\Gamma}$. As it is also true that $y \in X$ we know that $y \in (X \setminus \bigcup_{A\in\Gamma})$, and as y was arbitrary, we know that $X \setminus \bigcup_{A\in\Gamma} A \supset \bigcap_{A\in\Gamma}(X \setminus A)$. We can now see that by 1.E that $X \setminus \bigcup_{A\in\Gamma} A = \bigcap_{A\in\Gamma}(X \setminus A)$.

We will now show that $X \setminus \bigcap_{A\in\Gamma} A = \bigcup_{A\in\Gamma}(X \setminus A)$. Let $x \in X \setminus \bigcap_{A\in\Gamma} A$ be arbitrary. We see by definition of set difference and intersection that $x \in X$, and for some A $x \notin A$. We thus can see that for at least one A, $x \in (X \setminus A)$, and thus that $x \in \bigcup_{A\in\Gamma}(X \setminus A)$. As x was arbitrary, we see that $X \setminus \bigcap_{A\in\Gamma} A \subset \bigcup_{A\in\Gamma}(X \setminus A)$.

Now let $y \in \bigcup_{A\in\Gamma}(X \setminus A)$ be arbitrary. We see that for some $A \in \Gamma$ $y \in X$ and $y \notin A$. Because $y \notin A$ we know that $y \notin \bigcap_{A\in\Gamma} A$, as by definition of intersection, if $y \in \bigcap_{A\in\Gamma} A$ we would have that $y \in A$. We now see that as $y \in X$, and $y \notin \bigcap_{A\in\Gamma} A$ we know that $y \in X \setminus \bigcap_{A\in\Gamma} A$. As y was arbitrary, we know that $X \setminus \bigcap_{A\in\Gamma} A \supset \bigcup_{A\in\Gamma}(X \setminus A)$. We now can see by 1.E that $X \setminus \bigcap_{A\in\Gamma} A = \bigcup_{A\in\Gamma}(X \setminus A)$. We thus know that

$$X \setminus \bigcup_{A\in\Gamma} A = \bigcap_{A\in\Gamma} (X \setminus A)$$

and

$$X \setminus \bigcap_{A\in\Gamma} A = \bigcup_{A\in\Gamma} (X \setminus A)$$

□

2.9 (Riddle)**.** *Find such a formulation.*

Answer. We will first show that $(\bigcup_{A\in\Gamma} A) \cup \bigcap_{A\in\Gamma}(X \setminus A) = X$. Let $x \in (\bigcup_{A\in\Gamma} A) \cup \bigcap_{A\in\Gamma}(X \setminus A)$. We see as each A is a subset of X, and $X \setminus A \subset X$, that $x \in X$, and thus $(\bigcup_{A\in\Gamma} A) \cup \bigcap_{A\in\Gamma}(X \setminus A) \subset X$. Now let $a \in X$ be arbitrary. We see that if $a \notin (\bigcup_{A\in\Gamma} A)$, that for any A, $a \notin A$, and thus $a \in X \setminus A$. We thus have that $a \in \bigcap_{A\in\Gamma}(X \setminus A)$ by definition of intersection, and thus $a \in (\bigcup_{A\in\Gamma} A) \cup \bigcap_{A\in\Gamma}(X \setminus A)$ and that if $a \in (\bigcup_{A\in\Gamma} A)$, we have that $a \in (\bigcup_{A\in\Gamma} A) \cup \bigcap_{A\in\Gamma}(X \setminus A)$, and as $(\bigcup_{A\in\Gamma} A) \cup \bigcap_{A\in\Gamma}(X \setminus A) \subset X$, and a was arbitrary, we have that $(\bigcup_{A\in\Gamma} A) \cup \bigcap_{A\in\Gamma}(X \setminus A) = X$

We now will show that $(\bigcap_{A\in\Gamma} A) \cup \bigcup_{A\in\Gamma}(X \setminus A) = X$. Let $x \in (\bigcap_{A\in\Gamma} A) \cup \bigcup_{A\in\Gamma}(X \setminus A)$. We see as each A is a subset of X, and $X \setminus A \subset X$, that $x \in X$, and thus $(\bigcap_{A\in\Gamma} A) \cup \bigcup_{A\in\Gamma}(X \setminus A) \subset X$. We now let $a \in X$ be arbitrary. We see that if $a \notin (\bigcap_{A\in\Gamma} A)$, that for at least one A, $a \notin A$, and thus for some A, $a \in (X \setminus A)$, and thus $a \in \bigcup_{A\in\Gamma}(X \setminus A)$, and thus $a \in (\bigcap_{A\in\Gamma} A) \cup \bigcup_{A\in\Gamma}(X \setminus A)$. We also see that if $a \in (\bigcap_{A\in\Gamma} A)$, that $a \in (\bigcap_{A\in\Gamma} A) \cup \bigcup_{A\in\Gamma}(X \setminus A)$. We thus have that as $(\bigcap_{A\in\Gamma} A) \cup \bigcup_{A\in\Gamma}(X \setminus A) \subset X$, and a was arbitrary, we have that $(\bigcap_{A\in\Gamma} A) \cup \bigcup_{A\in\Gamma}(X \setminus A) = X$

We now see that $(\bigcup_{A\in\Gamma} A) \cup \bigcap_{A\in\Gamma}(X \setminus A) = X$, and $(\bigcap_{A\in\Gamma} A) \cup \bigcup_{A\in\Gamma}(X \setminus A) = X$, and thus $(\bigcup_{A\in\Gamma} A) \cup \bigcap_{A\in\Gamma}(X \setminus A) = (\bigcap_{A\in\Gamma} A) \cup \bigcup_{A\in\Gamma}(X \setminus A)$ and we have found such a formulation.

□

[2′7] Properties of Closed Sets

2.F. *Prove that:*

1. *the intersection of any collection of closed sets is closed;*
2. *the union of any finite number of closed sets is closed;*
3. *the empty set and the whole space (i.e., the underlying set of the topological structure) are closed.*

Proof. Define $f : \mathcal{P}(X) \to \mathcal{P}(X)$ by $f(S) = X \setminus S$. This function will be used throughout the proof.

1. Let $K \subset \mathcal{P}(X)$ be an arbitrary collection of closed sets of X. Notice that for all $C \in K$, $X \setminus C$ is open, so $f(K) \subseteq \Omega$. We have, then, that

$$\bigcap_{C\in K} C = \bigcap_{E\in f(K)} X \setminus E.$$

By De Morgan's law (2.E.3), we have

$$\bigcap_{E\in f(K)} X \setminus E = X \setminus \bigcup_{E\in f(K)} E.$$

As the arbitrary union of open sets is open, there is some $O \in \Omega$ such that

$$\bigcup_{E \in f(K)} E = O.$$

Thus,

$$\bigcap_{C \in K} C = X \setminus O,$$

which is therefore closed.

2. Let $K \subset \mathcal{P}(X)$ be an arbitrary finite collection of closed sets of X. Notice that for all $C \in K$, $X \setminus C$ is open, so $f(K) \subseteq \Omega$. We have, then, that

$$\bigcup_{C \in K} C = \bigcap_{E \in f(K)} X \setminus E.$$

By De Morgan's law (2.E.4), we have

$$\bigcup_{E \in f(K)} X \setminus E = X \setminus \bigcap_{E \in f(K)} E.$$

As the finite intersection of open sets is open, there is some $O \in \Omega$ such that

$$\bigcap_{E \in f(K)} E = O.$$

Thus,

$$\bigcup_{C \in K} C = X \setminus O,$$

which is therefore closed.

3. Since $X \in \Omega$ by Axiom 3, X is open; thus, $X \setminus X = \varnothing$ is closed. Similarly, since $\varnothing \in \Omega$, $X \setminus \varnothing = X$ is closed.

□

[2′8] Being Open or Closed

2.G. *Find Examples of sets that are*

1. both open and closed simultaneously

2. neither open, nor closed

(1) Example: $\varnothing$ and $\mathbb{R}$ are both open and closed at the same time in $\mathbb{R}$.

Proof. They are all open since they belong to the topology on $\mathbb{R}$, and we know that they are complements of each other, so they are both open and closed. □

(2) Example: $(0, 1]$ is neither closed or open on $\mathbb{R}$

Proof. $(0,1]$ is not open because we have to have infinite unions or intersections of opens to approach 1. Then $(0,1]^c = (-\infty, 0] \cup (1, +\infty)$ is still not open since we have to take infinite intersections or unions to approach 0. Thus, this set is neither open nor closed. $\square$

Primary Author: Jimin Tan

2.H. *Is a closed segment $[a,b]$ closed in $\mathbb{R}$? Yes, it is. The complement of a closed segment $[a,b]$ is the union of the open sets $\{x|x < a\}$ and $\{x|x > b\}$, and so the closed segment is closed by the definition of a closed set and the standard topology on RR.*

2.10. *Give an explicit description of closed sets in*

1. *a discrete space;*
2. *an indiscrete space;*
3. *the arrow;*
4. *the weird one;*
5. $\mathbb{R}_{T_1}$.

Answer. 1. Since $A \in \Omega$ whenever $A \subseteq X$, it follows that both $X \setminus A \in \Omega$ as well; hence, for all $A \subseteq X$, we have that A is closed. The closed sets are therefore comprised of $\mathcal{P}(X) = \Omega$.

2. Since $X = X \setminus \varnothing$ and $\varnothing = X \setminus X$, the closed sets are therefore $\{X, \varnothing\} = \Omega$.

3. We have that
$$\mathcal{F} = \{[0,a] : a \geq 0\} \cup \{\mathbb{R}_+, \varnothing\}.$$

4. We have that
$$\mathcal{F} = \{\{a,b,c,d\}, \varnothing, \{b,c,d\}, \{a,c,d\}, \{b,d\}, \{d\}, \{c,d\}\}.$$

5. We have that
$$\mathcal{F} = \{A : |A| = n \text{ for some } n \geq 0\} \cup \{\mathbb{R}\}.$$ $\square$

2.11. *Prove that the half-open interval $[0,1)$ is neither open nor closed, but is both a union of closed sets and an intersection of open sets.*

Proof. We will first show that $[0,1)$ is not open. By the definition of an open set, we know that an open set is an arbitrary union of open sets. We now consider 0. We see that as 0 is in $[0,1)$, for $[0,1)$ to be open, there would have to be an open set which contains it, however, if any set (a,b) contains 0, it would also contain the points less than 0 and greater than a, and $[0,1)$ does not contains these points. We thus see that $[0,1)$ is not open.

We now will show that $[0,1)$ is not closed. Consider the complement of $[0,1)$, $(-\infty,0)\cup[1,\infty)$. We see that for this set to be open, some open set would have to contain 1, however, if any open set (a,b) contained 1, then we would have that the points less than 1 and greater than 0 would be in the set, and we know that these points are not in the set, and thus $(-\infty,0)\cup[1,\infty)$ is not open, and its complement is $[0,1)$ not closed.

We will now construct $[0,1)$ out of a union of closed sets. Consider a set of the form $[0,x]$ where $x\in(0,1)$. We see that its complement is $(-\infty,0)\cup(x,\infty)$ which is open, and thus $[0,x]$ is closed. We also see that if we take the union of every set of this form, that it will have all the same points as $[0,1)$. We will now prove this. Let A be the union of every set of this form. Because for every x $x<1$, we know that for any x $[0,x]\subset[0,1)$, and thus for any $a\in A$, as $a\in[0,x]$ for some x, $a\in[0,1)$, and thus $A\subset[0,1)$. Now, let $b\in[0,1)$ be arbitrary. We see that if $b=0$, that $b\in A$, and that if $b\neq 0$, that $b\in(0,1)$, and thus $[0,b]$ is a set in A, and thus $b\in A$. As b was arbitrary $A\supset[0,1)$. We now see that by 1.E that $A=[0,1)$.

We will now construct $[0,1)$ out of an intersection of open sets. Consider a set of the form $(-\frac{1}{n},1)$ where $n\in\mathbb{N}^{>0}$. Because both $-\frac{1}{n}$ and 1 are real numbers, we know that $(-\frac{1}{n},1)$ is an open set. We will call the intersection of all sets of this form B. Let $a\in[0,1)$ be arbitrary. We see that for any n, $a\geq 0>-\frac{1}{n}$, and $a<1$, and thus $a\in B$, and $[0,1)\subset B$. Now let $b\in B$ be arbitrary. Suppose $b<0$. We would then see that for some n, $b>-\frac{1}{n}>0$, and thus b would not be in B. We thus can see that $b\geq 0$. We also see that as $b\in(-1,1)$, that $b<1$. We thus see that as $b\geq 0$ and $b<1$, that $b\in[0,1)$, and thus as b was arbitrary, $B\subset[0,1)$. We now can see that by 1.E $B=[0,1)$ □

Primary author: Reilly Noonan Grant

2.12. *Prove that the set* $A=\{0\}\cup\{1/n|n\in N\}$ *is closed in* $\mathbb{R}$.

Proof. To show this, we will show that its complement is the arbitrary union of open sets. Note $A=\{0,1/2,1/3,1/4,...\}$. Consider the collection of open intervals $\{A_i,A_i+1)|i\in\mathbb{N}\}$; call this collection B. This set contains no values in A, as they will only be endpoints of the open intervals. There is no way for a value in A to be in an interval of B, because we're creating our collection such that every time we reach a value in A, we create a new interval in our collection. The union of all the sets in B, the open set $\{x|x<0\}$ and the open set $\{x|x>1\}$ is the complement of A in $\mathbb{R}$. To see this, choose an arbitrary element $r\in R$. If this element is in A, it must not be in B, as previously discussed, and the values in A are bounded between 0 and 1, so we know the element being in A precludes it from being in this union. The closure of this union is $\mathbb{R}$, so if it is not in this union, it must be a limit point of $\mathbb{R}$ that is not contained in the union. The only values that meet these criteria are exactly the values of A, so we know the element must be in A. We've shown that an element of $\mathbb{R}$ is in A if and only if it is not in the union, so the union must be the complement. Then by the definition of closed, A is closed. □

Primary author: Willie Kaufman

[2′9] Characterization of Topology in Terms of Closed Sets

2.13. *Suppose a collection $\mathcal{F}$ of subsets of X satisfies the followings conditions:*

1. *the intersection of any family of sets from $\mathcal{F}$ belongs to $\mathcal{F}$;*
2. *the union of any finite number sets from $\mathcal{F}$ belongs to $\mathcal{F}$;*
3. *$\varnothing$ and X belongs to $\mathcal{F}$.*

Prove that then $\mathcal{F}$ is the set of all closed sets of a topological structure (which one?).

Proof. We show that with $\Omega = \mathcal{P}(X) \backslash \mathcal{F}$, $\mathcal{F}$ is the set of all closed sets of (X, Ω). To do so, we will show that (X, Ω) is, in fact, a topological structure. As a consequence, we shall see that our desired result falls out of this investigation.

Let $\Omega' \subseteq \Omega$ be arbitrary, and consider

$$U = \bigcup_{S \in \Omega'} S.$$

Now, since $S \in \Omega' \subseteq \Omega$, there exists some $K \in \mathcal{F}$ such that $S = X \setminus K$. If $\mathcal{F}' \subseteq \mathcal{F}$ is the collection which corresponds to Ω' through the complement in this way, we may thus write using De Morgan's laws

$$\begin{aligned} \bigcup_{S \in \Omega'} S &= \bigcup_{K \in \mathcal{F}'} X \setminus K \\ &= X \setminus \bigcap_{K \in \mathcal{F}'} K \\ &= X \setminus C, \end{aligned}$$

where $C \in \mathcal{F}$, using property (1). Hence, we have $U = X \setminus C$, so $U \in \Omega$.

Let $\Omega' \subseteq \Omega$ be an arbitrary finite collection of sets, and consider

$$U = \bigcap_{S \in \Omega'} S.$$

Now, since $S \in \Omega' \subseteq \Omega$, there exists some $K \in \mathcal{F}$ such that $S = X \backslash K$. If $\mathcal{F}' \subseteq \mathcal{F}$ is the (finite) collection which corresponds to Ω' through the complement in this way, we may thus write using De Morgan's laws

$$\begin{aligned} \bigcap_{S \in \Omega'} S &= \bigcap_{K \in \mathcal{F}'} X \setminus K \\ &= X \setminus \bigcup_{K \in \mathcal{F}'} K \\ &= X \setminus C, \end{aligned}$$

where $C \in \mathcal{F}$, using property (2). Hence, we have $U = X \setminus C$, so $U \in \Omega$.

Finally, note that $X = X \setminus \varnothing$, so $X \in \Omega$; similarly, $\varnothing \in \Omega$. Hence, (X, Ω) satisfies the topological axioms. As a consequence, $\mathcal{F}$ characterizes the closed sets of this topological space. $\square$

2.14. *List all collections of subsets of a three-element set such that there are topologies where these collections are complete sets of closed sets.*

Answer. - $\varnothing, a, b, c meets the conditions in 2.13 (this is the set of closed sets in the discrete topology)$
- $\{\varnothing, \{a, b, c\}, \{a\}, \{b\}, \{c\}, \{a, b\}, \{b, c\}, \{a, c\}\}$ meets the conditions in 2.13 (this is the set of closed sets in the indiscrete topology) - $\{\varnothing, \{a, b, c\}, \{a\}$ meets the conditions in 2.13, and so do other sets that are the same up to relabelling - $\varnothing, a, b, c, a, b, a, b$ meets the conditions in 2.13, and so do other sets that are the same up to relabelling □

Primary author: Willie Kaufman

[2′10] Neighborhoods

2.15 Give an explicit description of all neighborhoods of a point in

1. a discrete space;
2. an indiscrete space
3. The arrow;
4. X = $\{a, b, c, d\}$
5. a connected pair of points;
6. particular point topology

1. A neighborhood of a point in a discrete space topology (X, Ω) would be all subsets of X which contain that point.
2. A neighborhood of a point in a indiscret space topology (X, Ω) would just be X, as $X$and $\varnothing$ are the only elements of Ω, and all of the points are in X.
3. Let $(a, +\infty)$ be an element of the arrow. We see that a neighborhood of this point would be $[0, +\infty)$ or any ray of the form $(b, +\infty)$ where $b \leq a$, as otherwise, a would not be included, in the set, and thus it would not be a neighborhood
4. We see that if $X = \{a, b, c, d\}$, that the neighborhoods of a are $\{a\}$,$\{a, b\}$,$\{a, c\}$,$\{a, d\}$$\{a, b, c\}$,$\{a, b, d\}$,$\{a, c, d\}$ and$\{a, b, c, d\}$, and we can see that for any other point, we can get their neighborhoods by swaping all occurances of a with that point, and that point with a
5. The neighborhood of a point in a connected pair of points topology would just be the set of the point by itself, and the set of the point with its pair, as no other set includes that point.

6. The neighborhood of a point in a particular point topology would depend on the point in question. If that point is not the particular point, then it would be similar to the discrete topology, except that every set would also include the particular point, and if the point is the particular point, then every set in the topology would be a neighborhood.

[2′11] Open Sets on Line

2.I. *Prove that every open subset of the real line is a union of disjoint open intervals.*

Proof. We will show this by using proof by contradiction. We first notice that by the definiton of the canonical topological space of the real line, that open sets are arbitary unions of open intervals. We assume for sake of contradiction that there exists an open subset X of the real line where it is neccesary for some collection of open intervals to intersect. We will let A be an arbitrary one of these intersections. Let $c \in A$ be arbitrary, and let B be the open interval (a, b) where a is chosen to be the real number closest to c which satisfies $a < c$, and $a \notin X$, or $-\infty$ if there is no such element, and where b is chosed to be the real number closest to c which satisfies $c < b$ and $b \notin X$, or $+\infty$ if there is no such element. As a and b are the closest elements to c which are not in X, we know that $(a, b) \subset X$. We now can see that if $A \subset (a, b)$ that every element in A could have been covered by one open set, and thus A would not have been neccessary, and every element of X in A could have been covered by one open interval, and we would have a contradiction. We now see that as A is the intersection of arbitrary unions of open sets, that A is an open interval, and thus has the form (α, β) from some $\alpha, \beta \in \mathbb{R}$. We can see that if $\alpha < a$, that $a \in A$, and as $a \notin X$ we would see that A is not a subset of X, and thus $\alpha \geq a$. We similarly see that $\beta \leq b$. We now pick an arbitrary element d from A. As $d \in A$, we know $\alpha < d < \beta$, and thus $a < d < b$. We thus see that $d \in (a, b)$, and as d was arbitrary, we have that $A \subset (a, b)$ and thus have a contradiction, and know that there is no open subset of the real line whiere it is neccesary for some collection of open intervals to intersect, and thus every open subset of the real line can be described as a union of disjoint open intervals.

□

[2′12] Cantor Set

2.J. *Find a geometric description of K.*
Prove that

1. *K is contained in $[0, 1]$,*
2. *K does not meet $(\frac{1}{3}, \frac{2}{3})$,*
3. *K does not meet $\left(\frac{3s+1}{3^k}, \frac{3s+2}{3^k}\right)$ for any integers k and s.*

Present K as $[0,1]$ with an infinite family of open intervals removed. Try to sketch K.

Proof. To see that $K \subseteq [0,1]$, we argue that $\min(K) = 0$ and $\max(K) = 1$. Let $\{a_k\}_{k=1}^{\infty}$ be a sequence defined such that $a_k \in \{0,2\}$ for all k. We have

$$0 \le a_k \le 2,$$

so it follows that

$$\sum_{k=1}^{\infty} \frac{0}{3^k} \le \sum_{k=1}^{\infty} \frac{a_k}{3^k} \le \sum_{k=1}^{\infty} \frac{2}{3^k}.$$

The left side simplifies to 0, and for the right side, noting that

$$\sum_{k=0}^{\infty} \frac{1}{3^k} = \frac{1}{1-\frac{1}{3}} = \frac{3}{2},$$

we have

$$\sum_{k=1}^{\infty} \frac{2}{3^k} = 2 \cdot \left(\frac{3}{2} - 1\right) = 1.$$

Hence, for all $\{a_k\}_{k=1}^{\infty}$ properly defined, we have

$$0 \le \sum_{k=1}^{\infty} \frac{a_k}{3^k} \le 2.$$

To see that $(\frac{1}{3}, \frac{2}{3}) \not\subseteq K$, suppose $\{a_k\}_{k=1}^{\infty}$ is such that $a_1 = 0$ and $a_k = 2$ for all $k > 1$. Then

$$\begin{aligned}\sum_{k=1}^{\infty} \frac{a_k}{3^k} &= 0 + 2\sum_{k=2}^{\infty} \frac{1}{3^k}\\ &= 2\sum_{k=0}^{\infty} \frac{1}{3^k} - 2 - \frac{2}{3}\\ &= 3 - 2 - \frac{2}{3}\\ &= \frac{1}{3}.\end{aligned}$$

Now suppose that $\{a_k\}_{k=1}^{\infty}$ is such that $a_1 = 2$ and $a_k = 0$ for all $k > 1$. Then

$$\sum_{k=1}^{\infty} \frac{a_k}{3^k} = \frac{2}{3} + \sum_{k=2}^{\infty} \frac{0}{3^k} = \frac{2}{3}.$$

This shows that, even given the value of a_1, no sequence $\{a_k\}_{k=1}^{\infty}$ generates a series value in $(\frac{1}{3}, \frac{2}{3})$.

Assume that after k terms the sequence $\{a_n\}$ is either all 0s or all 2s. For a given $(\frac{3s+1}{3^k}, \frac{3s+2}{3^k})$, the upper bound is achieved with trailing 0s, and the lower bound is achieved with trailing 1s. Permutations of 0s and 2s in the first k elements bounds the sequence around a given third.

We may write K as

$$K = [0,1] \setminus \bigcap_{k,s\in\mathbb{N}} \left((\frac{3s+1}{3^k}, \frac{3s+2}{3^k}) \right).$$

□

2.K. *Prove that K is a closed set in the real line.*

Proof. We will take the complement of the Cantor set $(\{\Sigma_{k=1}^{\infty} a_k/3^k | a_k \in 0,2\})^c$ and label it C^c. In the construction of the Cantor set, we know that $(\{\Sigma_{k=1}^{n} a_k/3^k | a_k \in 0,2\})^c$ $(n \in \mathbb{R})$ is a open set, and the addition of $(\{\Sigma_{k=1}^{n+1} a_k/3^k | a_k \in 0,2\})^c$ which is also open, is the intersection of two open sets, and will still result in a open set. Thus., $(\{\Sigma_{k=1}^{\infty} a_k/3^k | a_k \in 0,2\})^c$ is a intersection of open sets, and it is still a open set. By definition $\{\Sigma_{k=1}^{\infty} a_k/3^k | a_k \in 0,2\}$ which is the Cantor Set, is closed. □

[2′13] Topology and Arithmetic Progressions

2.L. *2.Lx*. Consider the following property of a subset F of the set N of positive integers: there is n N such that F contains no arithmetic progressions of length n. Prove that subsets with this property together with the whole N form a collection of closed subsets in some topology on N.*

Proof. To show this, we show that following 3 properties hold: (1) F is closed under arbitrary intersections (2) F is closed under finite unions (3) $\varnothing$ and $\mathbb{N}$ are elements of F

First, $\varnothing$ and $\mathbb{N}$ are elements. $\mathbb{N}$ is added artificially and $\varnothing$ is trivially in F, so the third property holds.

Second, consider an arbitrary union of elements in F, $\bigcap f_1, f_2, .$ Let $n_1, n_2,$ be the values satisfying the property in the statement of the problem. If $f_1, f_2,$ contains an arithmetic progression of length n, it must be the case that each f_i contains an arithmetic progression of length n as well, since $\bigcap f_1, f_2, \subseteq f_i$ for all f_i. Then we can just choose any n_i and $\bigcap f_1, f_2,$ cannot contain any arithmetic progressions of length n, i.e. the first property holds.

Now consider a finite union of elements in F, $\bigcup f_1, f_2, , f_n$, where $n_1, n_2, , n_m$ are the values satisfying the property in the statement of the problem. Because of the properties of unions, if we prove $f_1 \cup f_2$ is in F, we know that $\bigcup f_1, f_2, , f_n$ are in F, since all f_i are arbitrary. Let N equal the value given by Van der Waerdens Theorem with $n = max(n_1, n_2)$. Assume that $f_1 \cup f_2$ contains an arithmetic progression of length N. This arithmetic progression must be in the form $mx + b | x \in 0, 1, , N-1$. Let $S = x \in 0, 1, , N-1 | mx + b \in f_1$ and let $S = 0, 1, , N-1 \setminus S$. By Wan der Waerdens theorem, it must be the case that either S or S contains an arithmetic progression of length $max(n_1, n_2)$. Because of the way we constructed S and S, if S contains an arithmetic progression of length N, it must be that f_1 contains an arithmetic progression of length N, and the same is true for S and f_2. So we can rephrase this as either f_1 or f_2 must contain an arithmetic progression of length $max(n_1, n_2)$. Since $max(n_1, n_2) \geq n_1$ and $max(n_1, n_2) \geq n_2$, this is not possible and our assumption that $f_1 \cup f_2$ contains an arithmetic progression of length N must be incorrect. Then the second property holds.

Since the three properties hold, we know this collection is a collection of closed sets.

Primary author: Willie Kaufman □

2.M. *For every $n \in \mathbb{N}$, there is an $N \in \mathbb{N}$ such that for any subset $A \subset \{1, 2, ..., N\}$, either A or $\{1, 2, ..., N\} \setminus A$ contains an arithmetic progression of length n*

Proof. We first examine the case where $n = 1$. We see that if we choose N to be 3, that either A or it's complement must have a set with 2 elements, and those two elements form a arithmetic progression of length 1.

We now examine the case where $n = 2$. We choose N to be 9. We assume for sake of contradiction, that neither A nor $\{1, 2, ..., N\} \setminus A$ contains an arithmetic progression of length 2.

We define the concept of a run, as a collection of adjacent integers. We see that in this context, there cannot be a run of 3. 2 runs cannot be adjacent, as in that case they would only be one run.

We first see that $|A| \geq 3$, as otherwise, there would be some run of 3 numbers in the complement which are all adjacent, and thus an arithmetic progression of length 2. A similar statement holds for A's complement. We now consider the possible runs in A. We see that there are at least 2 runs in a A, and its complement, as a run is less than 3, and we know that $|A| \geq 3$. We consider the case were there are only 2 runs in A. We see that these runs couldn't cover 1 or 9, as otherwise, there would be a run of 3 in the complement, as one run cannot split 7 or 8 up into entirely runs less than 3. We also see that a run cannot cover 5, as otherwise, the side that the other run isn't on would have a run of 3. We see that $1, 5, 9$ form an arithmetic progression, and thus that A and its complement must each have 3 or more runs. We also see that A and its complement must not have 3 runs which are 2 long, as they would either cover either $(1, 2)$, $(4, 5)$ and $(7, 8)$, or $(2, 3)$, $(5, 6)$ and $(8, 9)$, and 2,5,7 and 3,6,9 are progressions. We also see that A and its complement must have less than 5 runs, as the only way to have 5 runs is $1, 3, 5, 7, 9$, and these are each 2 apart. We thus have the there are either 3 or 4 runs. We now see that as $1, 5, 9$ is an arithmetic progression, that without loss of generality, A must contain a run with one of these. From here, we can find that there must be a contradiction, but i'm not sure how. □

Primary author: Reilly Noonan Grant

3 Bases

[3′1] Definition of Base

3.1. *Can two distinct topological structures have the same base?*

We claim that two distinct topological structures must have different bases.

Proof. Assume that two different topological structure Ω_1 and Ω_2 have the same base S_b. Every subset in Ω_1 can be expressed as unions of elements in S_b which will be in Ω_2 by definition of topological structure. So we reach a conclusion that Ω_1 is the same as Ω_2 since they contain the same elements, a contradiction. $\square$

3.2. *Find some bases for the topology of*

1. *a discrete space;*
2. *the weird space;*
3. *an indiscrete space;*
4. *the arrow.*

Try to choose the smallest base possible.

Answer.

1. Let $\Sigma = \{\{x\} : x \in X\} \cup \{\varnothing\}$ be the set of all singleton subsets of X. Then any nonempty subset of X can be represented by the union of all its singleton element sets.
2. Let $\Sigma = \{\{a\}, \{b\}, \{c\}, \{d\}\}$. Every element of Ω is a union of some of these singletons.
3. Let $\Sigma = \{X, \varnothing\}$. This is the smallest basis, since both are needed to generate $\{X, \varnothing\} = \Omega$.
4. Whenever $a < b$, we have $(a, \infty) \cup (b, \infty) = (a, \infty)$, so each open set of the arrow is necessary to generate itself. Hence, $\Sigma = \Omega$.

$\square$

Primary Author: Jimin Tan

3.3 (Riddle)**.** *Prove that any base of the canonical topology on* $\mathbb{R}$ *can be decreased*

Proof. Let Σ be a base of the canonical topology on $\mathbb{R}$. Let A be an element of Σ. We see that A is an open set, and thus an arbitrary union of open intervals. We let (a, b) be an arbitrary open interval such that $(a, b) \cup B = A$ where B is a possibly empty union of open intervals. We see that by the definition of a base, that as (a, b) is in the cannonical topology, and B is in the cannonical topology, that there must already be some collection of elements in Σ such that (a, b) and B are unions of those elements. We thus see that if A is of the form $(a, b) \cup B$, that it can be removed from Σ, and thus Σ can be decreased.

We now suppose that A is not a union of open intervals, and is instead just an open interval (a, b). We now let $c \in (a, b)$, and $d \in (a, c)$, and $e \in (c, b)$. We see that (a, e) and (d, b) are in the canonical topology on $\mathbb{R}$, and thus that they are the union of some set of elements in Σ. This set of elements does not include A, as $d, e \in A$, and thus a union of A with other elements would contain d and e. We now see that as $(a, e) \cup (d, b) = (a, b)$, that a union of elements in Σ will include A, and thus that removing A from Σ would not remove any element from the cannonical topology, and thus that Σ can be decreased. As A was arbitrary, and Σ was arbitrary, and in all cases we were able to remove A from Σ, we see that any base of the cannonical topology can be decreased. $\square$

Primary author: Reilly Noonan Grant

3.4 (Riddle)**.** *What topological structures have exactly one base?*

The first thing to note is that given a base Σ of a topology Ω, if there exists an open set o in Ω that is not in Σ, there exists another base $\Sigma \cup o$. So the only topologies for which there are a single base are those for which the only base is all of Ω. We know the set X must be the union of sets in Ω, so there must be no collection of sets in Ω, the union of which is X, except the entire set. This is the case when, for all sets $a \in \Omega$, there does not exist some collection of sets C in Ω a for which $a \subset C$.

Primary author: Willie Kaufman

[3′2] When a Collection of Sets is a Base

3.A. *A collection Σ of open sets is a base for the topology iff for every open set U and every point $x \in U$ there is a set $V \in \Sigma$ such that $x \in V \subset U$*

Proof. We will first show that if Σ is a base for the topology if for every open set U and every point $x \in U$ there is a set $V \in \Sigma$ such that $x \in V \subset U$. Let $U \in \Sigma$ be arbitrary and let $x \in U$ be arbitrary. By the definition of a basis, we know that because U is an open set that U is a union of sets in Σ. We know because $x \in U$ that for at least one of the sets in this union, x will be in the set. We will call one of these sets V. We see that if V had an element which was not in U, that the union of sets would not be equal to U, and as $x \in V$, we know that $V \subset U$. We thus have that because U and x were arbitrary, that the statement holds true in all cases.

We will now show that if for every open set U and every point $x \in U$ there is a set $V \in \Sigma$ such that $x \in V \subset U$ that Σ is a base for the topology. Let U be an arbitrary open set. We define A to be the union of all V that correspond to some point in U. We see that as for each V, $V \subset U$, we know that $A \subset U$, as if any element of A were not in U, then for some set V, V would not be a subset of U. We also see that $U \subset A$ as for each $x \in U$, we have that $x \in V$ for some V, and thus $x \in A$ by the definition of a union. We thus have that by 1.E that $A = U$, and as U was arbitrary, we know that any U can is a union of sets in Σ, and thus Σ is a base for the topology. $\square$

3.B. *A collection Σ of subsets of a set X is a base for a certain topology on X iff X is the union of all sets in Σ and the intersection of any two sets in Σ is the union of some sets in Σ.*

First we will prove that if Σ is the base for a topology on X, $X = \bigcup_(s \in \Sigma)s$ and the intersection of any two sets in Σ is the union of some sets in Σ. First, let Ω be the topology for which Σ is a base. We have that $X \in \Omega$ and, by the definition of base, X must the union of some sets in Σ. Since every element of each set in Σ must be in X, X being the union of some sets in Σ implies it is the union of all sets in Σ. Now let s_1 and s_2 be two sets in Σ. Because the sets are in Σ, they are open, and so their intersection is also open. Now we know that their intersection is equal to the union of some open sets in Ω, and since all open sets in Ω are equal to the union of some open sets in Σ by the definition of base, we can replace those open sets with the elements of Σ whose union equals them and the intersection of s_1 and s_2 must be the union of some sets in Σ. Now we prove that if $X = \bigcup_(s \in \Sigma)s$ and the intersection of any two sets in Σ is the union of some sets in Σ, Σ is a base for some topology on X. Let Ω be the set of arbitrary unions of elements in Σ and $\varnothing. Wewillshowthat \Omega$ is a topology. First, it is the case that X and $\varnothing \in \Omega$, by the way we defined Ω and the assumptions we started with. So the third axiom of topology is satisfied. Second, consider the finite intersection of sets in Ω. Let $u_1, u_2, ..., u_n \in \Omega$ to be arbitrary. First, notice that the intersection of u_1 and u_2 is the union of some sets in Σ, i.e. is in Ω. Let $u_1 = \bigcup_(i \in I)\sigma_i$, $u_2 = \bigcup_(j \in J)\sigma_j$, etc. Then their intersection is equal to $\bigcup_(i \in I, j \in J)(\sigma_i \cap \sigma_j)$. By our assumptions, each $(\sigma_i \cap \sigma_j)$ is the union of some sets in Σ, i.e. is in Ω. We can apply intersections in any order, so the fact that the intersection of two arbitrary sets in Ω is in Ω assures us that, for any finite number of sets in Ω, their intersections are also in Ω, and the second axiom of topology is satisfied. Third, consider the arbitrary union of elements in Ω. Let $u_1, u_2, ... \in \Omega$ be arbitrary. The properties of unions assure us that the union of all u_i is the union of each element in Σ such that there exists u_i where u_i was formed by a union including that element of Σ. Then we have that the union of $u_1, u_2, ...$ is the union of some number of sets in Σ, i.e. is in Ω, and the first axiom of topology is satisfied.

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3.C. *Show that the second condition in 3.B (on the intersection) is equivalent to the following one: the intersection of any two sets in Σ contains, together with any its points, a certain set in Σ containing this point (cf. Theorem 3A).*

Let Σ be a collection of subsets of a set X such that the intersection of any two sets in Σ is the union of some sets in Σ. Let $A, B \in \Sigma$ be arbitrary. If $c \in A \cap B$, it must be the case that there exists some set in Σ contained in $A \cap B$ that contains c, because otherwise we will be unable to find a collection of sets in Σ whose union equals $A \cap B$; we would have no way to include c in this union without some set that "overflows". Now we consider the other direction of the equivalence; let Σ be a collection of subsets of a set X such that, for any $A, B \in \Sigma$ and $c \in A \cap B$, there exists $S \in \Sigma$ such that $c \in S$ and $S \subseteq A \cap B$. Consider the set $= S|forsomec \in A \cap B, Sisasetthatsatisfiesthepreviousassumption.$ Consider $\bigcup_(\alpha \in)\alpha$. For each points $c \in A \cap B$, c is in this union, and because each $\alpha \subseteq A \cap B$, $_(\alpha \in)\alpha = A \cap B$ by double containment. Each $\alpha \in$ is in Σ by the way we chose them, and so for an arbitrary $A, B \in \Sigma$, their intersection is the union of some sets - those in - in Σ.

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[3'3] Bases for Plane

3.5. *Prove that every element of* Σ^2 *is a union of elements in* Σ^∞*.*

3.6. *Prove that the intersection of any two elements of* Σ^1 *is a union of elements in* Σ^1*.*

Proof. Let $\sigma_1, \sigma_2 \in \Sigma^1$. Then $\sigma_1 \cap \sigma_2$ is a rectangle with width w and length ℓ. Without loss of generality, assume $w \leq \ell$. Then if we take $B_w(c_1)$ and $B_w(c_2)$, where c_i corresponds to the square of length w which coincides with one of the edges of $\sigma_1 \cap \sigma_2$ of length w, we claim that $B_w(c_1) \cup B_w(c_2) = \sigma_1 \cap \sigma_2$.

If $x \in \sigma_1 \cap \sigma_2$, then □

¡++¿

3.7. *Prove that each of the collections* Σ^2,Σ^∞*, and* Σ^1 *is a base for some topological structure in* $\mathbb{R}^2$*, and that the structures determined by these collections coincide*

Proof. We will first show that Σ^2 is a base for the topology of all shapes in $\mathbb{R}^2$ which don't include their boundries. Let U be an arbitrary element of this topology which is not the emptyset, and let $x \in U$ be arbitrary. We now let c be a point not in U such that the distance between x and c is as short as the shortest distance between x and any other point not in U. If no such point exists, then U contains every $x \in \mathbb{R}^2$, and thus $U = \mathbb{R}^2$. We see in this case, that that as Σ^2 contains all open discs, that the disk V centered at x with a radius of 1 is in Σ^2, and as $V \in \mathbb{R}^2$, we know $V \in U$, and thus for some $V \in \Sigma^2$ $x \in V \subset U$. We now suppose that c exists. We see that as the distance is the shortest, that the disk centered at x, V, with a radius equal to the distance between c and x will be contained within U. Because Σ^2 contains all open discs, we see that $V \in \Sigma^2$ and as it is centered at x, we know that $x \in V$. We thus have that for some V, $x \in V \subset U$. We now can see that as x and U were arbitrary, that by 3.A, we know that Σ^2 is a base for the topology of all open shapes in $\mathbb{R}^2$. We can also see that if with each V in the above aproach we inscribe either an element of Σ^1 or Σ^∞ which would also contain x, and all be contained within the open set. Thus by a similar argument, and 3.A, we know that Σ^1 and Σ^∞ are also bases for the topology of all open shapes in $\mathbb{R}^2$.

□

[3′4] Subbases

3.8. *Let X be a set, $\triangle$ a collection of subsetsof X. Prove that $\triangle$ is a subbase for a topology on X iff $X = \cup_{(W\in\triangle)W}$.*

Proof. First, we prove that if $\triangle$ is a subbase for a topology on X, $X = \cup_{W\in\triangle)W}$. By the definition of topology, $X \in \Omega$, and so X must be a subset of the union of any base of Ω by the definition of base. This implies each element of X must be in at least 2 sets in $\triangle$. Since $\triangle$ is a collection of subsets of X, i.e. will not contain elements that X does not, it follows that $X = \cup_{W\in\triangle}W$. Now we consider the other direction. Assume $X = \cup_{W\in\triangle)W}$ for some set $\triangle$. The collection of finite intersections of $\triangle$ is a set for the which union of all the elements equals X; this is obvious if we just consider the set of intersections of a single set in $\triangle$. Then we have that $\triangle$ is a subbase for the indiscrete topology on X; the only nonempty set is X, which we have assured ourselves is equal to the union of some number of finite intersections of sets in $\triangle$. $\square$

[3′5] Infiniteness of the Set of Prime Numbers

3.9. *Prove that all (infinite) arithmetic progressions consisting of positive integers form a base for some topology on $\mathbb{N}$.*

Proof. Denote $AP(c, k)$ the arithmetic progression starting at c with increment k. We have $AP(1, 1) = \mathbb{N}$. Moreover, $AP(1, 2) \cap AP(1, 3) = \varnothing$, so both $\mathbb{N}$ and $\varnothing$ are open sets, verifying axiom 3.

Let

$$U = \bigcap_{B\in\Omega'} B$$

be an arbitrary union of open sets. Let $x \in U$ be arbitrary; hence there is some B such that $x \in B$. But then any $c \in \mathbb{N}$ such that $AP(x, c) \subseteq B$ establishes that $AP(x, c) \subseteq U$. Thus U is open.

Now, let

$$U = B_1 \cap B_2$$

be an intersection of two open sets with $u \in U$ fixed. Then if $x_1 \in B_1$ and $x_2 \in B_2$, we can let $x = (x_1, x_2)$, and it follows that $AP(u, x) \subseteq B_1$ and $AP(u, x) \subseteq B_2$). Hence, $AP(u, x) \subseteq U$, so U is open. $\square$

3.10. *Using this topology, prove that the set of all prime numbers is infinite.*

Proof. Let P denote the set of prime numbers, and suppose P is finite. Now, because each nonempty $U \in \Omega$ contains an arithmetic progression, every nonempty open set is infinite. Each $AP(c, k)$ is open, since the arithmetic progressions all form a base for Ω. In addition, since

$$AP(c, k) = \mathbb{N} \setminus (AP(c + 1, k) \cup AP(c + 2, k) \cup \cdots \cup AP(c + k - 1, k)),$$

each arithmetic progression is also closed. For each $n > 1$, n can be expressed as some multiple of a prime number; hence,

$$\bigcup_{p \in P} AP(p,p) = \mathbb{N} \setminus \{1\}.$$

Now, since $AP(p,p)$ is closed, and P is finite, we have that $\bigcup_{p \in P} AP(p,p)$ is closed; hence, $\mathbb{N} \setminus \{1\}$ is closed. But this implies that $\{1\}$ is open, which contradicts our observation that nonempty open sets are infinite. Hence, P must be infinite. □

[3′6] Hierarchy of Topologies

3.11. *Show that the $T_1 - topology$ on the real line is coarser than the canonical topology.*

Proof. We first let A be an arbitrary set in the $T_1 - topology$. We will show that A is in the canonical topology. Because A is in the $T_1 - topology$, we know that it is the complement of some finite set $\{a_1, \cdots, a_n\}$. We thus see that A is of the form $(-\infty, a_1) \cup (a_1, a_2) \cup (a_2, a_3) \cdots \cup (a_n, \infty)$. By defintion of the cannonical topology, we know that any set (a_n, a_{n+1}) is in the cannonical topology, and as $(-\infty, a_1)$ and (a_n, ∞) are open intervals, we see that they are also in the cannonical topology. We thus see that as A is a union of elements in the canonical topology, that it is in the canonical topology, and as A was arbitrary, we see that the canonical topology is coarser than the complement topology. □

3.D. *3.D. Riddle. Formulate a necessary and sufficient condition for two bases to be equivalent without explicitly mentioning the topological structures determined by the bases.*

In order for two bases to describe the same topology, it must be that if an open set O is the union of sets in B_1, it is also equal to the union of sets in B_2. WLOG, assume a set A is equal to the union of some sets in B_1. For an arbitrary element $x \in A$, we must have a set $b \in B_2$ such that $b \subset A$ and $x \in b$ for an arbitrary x. In order to guarantee that this is the case without referring to A, for each set $b \in B_1$ and element $x \in b$, there exists a set $c \in B_2$ such that $c \subset b$ and $x \in c$, as well as this being the case in the other direction.

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4 Metric Spaces

[4′1] Definition and First Examples

4.A. *Prove that the function*

$$\rho : X \times X \to \mathbb{R}_+ : (x, y) \mapsto$$

$$\begin{cases} 0 & \textit{if } x = y \\ 1 & \textit{if } x \neq y \end{cases}$$

is a metric for any set X

Proof. We first see that as if $x = y$, then $\rho(x, y)$, and if $x \neq y$, that $\rho(x, y) = 1 \neq 0$, that we have that the first property of metrics holds.

We now see that as $x = y$ implies $y = x$, that if $x = y$, $\rho(x, y) = \rho(y, x) = 0$, and if $x \neq y$, $\rho(x, y) = \rho(y, x) = 1$, and thus the second property of metrics holds.

Finally, we have that as equality is a equivalence relation, that if $x = y$, and $y = z$, that $x = z$, and thus $\rho(x, y) = \rho(x, z)+\rho(z, y) = 0$, and that otherwise, as $\rho(x, y) \geq 0$ for any x, y, we see that if $x \neq y$, that $\rho(x, y) \leq \rho(x, z) + \rho(z, y)$ □

4.B. *Prove that* $\mathbb{R} \times \mathbb{R} \to \mathbb{R}_+ : (x, y) \mapsto |x - y|$ *is a metric*

Proof. We first can see that for any $x, y \in \mathbb{R}, |x - y| = 0$ is only true when $x - y = 0$, and thus when $x = y$. We thus have that $\rho(x, y) = 0$ is true if and only if $x = y$

We also can see that as for any $x, y \in \mathbb{R}$, we have that if $x - y = a$, for some $a \in \mathbb{R}$ that $y - x = -a$. By the definition of absolute value, we know that $|a| = |-a|$, and thus for any x, y, we have that $\rho(x, y) = \rho(y, x)$. As x, y were arbitrary, we have that for all x, y $\rho(x, y) = \rho(y, x)$.

We finally will prove that $\rho(x, y) \leq \rho(x, z) + \rho(z, y)$. Let $x, y, z \in \mathbb{R}$ be arbitrary. We see that $\rho(x, y) = |x - y| = |x - z + z - y|$, and by the triangle inequality, we know that $|x - z + z - y| \leq |x - z| + |z - y| = \rho(x, z) + \rho(z, y)$. We thus have that $\rho(x, y) \leq \rho(x, z) + \rho(z, y)$, and as x, y, z were arbitrary, that this is true for any x, y, z. □

4.C. *Prove that* $\mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}_+ : (x, y) \mapsto \sqrt{\sum_{i=1}^n (x_i - y_i)^2}$ *is a metric.*

[4′2] FurtherExamples

4.1. *Prove that* $\mathbb{R}^n \times \mathbb{R}^n :\to \mathbb{R}_+ : (x, y) \mapsto max_{i=1,\dots,n}|x_i - y_i|$ *is a metric.*

Proof. We first see that if $\rho(x, y) = 0$, that for some i, we know by $4.B$ that $x_i = y_i$. As this is the maximum value, and the absolute value function is never negative, we thus know that this must be true for every i, and thus for every i $x_i = y_i$. We thus have that $\rho(x, y) = 0$ if and only if $x = y$.

We will now show that for any $x, y \in \mathbb{R}^n$ that $\rho(x, y) = \rho(y, x)$. Let $x, y \in \mathbb{R}^n$ be arbitrary. We see that for some i, $\rho(x, y) = |x_i - y_i|$. By $4.B$, we have that $|x_i - y_i| = |y_i - x_i|$. By the definition of max, we know that i is such that $|x_i - y_i| \geq |x_j - y_j|$ for any j, and thus, $|y_i - x_i| \geq |y_j - x_j|$, and we have that $\rho(y, x) = |y_i - x_i|$ We thus have that $\rho(x, y) = |x_i - y_i| = |y_i - x_i| = \rho(y, x)$. As x, ywere arbitrary, we see that for all x, y, $\rho(x, y) = \rho(y, x)$

We will finally show that for any $x, y, z \in \mathbb{R}^n$, that $\rho(x,y) \leq \rho(x,z)+\rho(z,y)$. Let $x, y, z \in \mathbb{R}^n$ be aribtrary.We see that for some i, $\rho(x,y) = |x_i - y_i|$, and by $4.B$, we know that $|x_i - y_i| \leq |x_i - z_i| + |z_i - y_i|$, and by definition of max, we have that $\rho(x,z) \geq |x_i - z_i|$, and $\rho(z,y) \geq |z_i - y_i|$. We thus see that $\rho(x,y) \leq \rho(x,z)+\rho(z,y)$, and as x, y, z were arbitrary, we have that this is true for any $x, y, z \in \mathbb{R}^n$

As we have show that all properties of a metric hold, we know that $\mathbb{R}^n \times \mathbb{R}^n :\to \mathbb{R}_+ : (x,y) \mapsto max_{i=1,\ldots,n}|x_i - y_i|$ is a metric. □

4.2. *Prove that* $\mathbb{R}^n \times \mathbb{R}^n :\to \mathbb{R}_+ : (x,y) \mapsto \Sigma_{i=1,\ldots,n}|x_i - y_i|$ *is a metric.*

Proof. We will first show that if $\rho(x,y) = 0$, that $x = y$. Let $x, y \in \mathbb{R}^n$ be arbitrary. We see that for any i, $|x_i - y_i| \geq 0$ by the definition of absolute value, and thus as$\rho(x,y) = 0$ we have that for every i, $|x_i - y_i| = 0$, and thus by $4.B$, $x_i = y_i$, and thus $x = y$. As x, y were arbitrary, we see that this is true for any $x, y \in \mathbb{R}^n$. We can also see that if $x = y$, that for each i $x_i = y_i$, and thus $|x_i - y_i| = 0$, and thus $\rho(x,y) = 0$. We see as both directions hold that $\rho(x,y) = 0$ if and only if $x = y$.

We will now show that for any x, y, $\rho(x,y) = \rho(y,x)$. Let $x, y \in \mathbb{R}^n$ be arbitrary. We see that $\rho(x,y) = \Sigma_{i=1}^n |x_i - y_i| = \Sigma_{i=1}^n |y_i - x_i| = \rho(y,x)$, and as $x, y \in \mathbb{R}^n$ were arbitrary, we have that $\rho(x,y) = \rho(y,x)$ for all $x, y \in \mathbb{R}^n$

Finally, we will show that for any $x, y, z \in \mathbb{R}^n$ that $\rho(x,y) \leq \rho(x,z)+\rho(z,y)$. We see that

$$\rho(x,y) = \Sigma_{i=1}^n |x_i - y_i| = \Sigma_{i=1}^n |x_i - z_i + z_i - y_i|$$

and by the triangle inequality,

$$\Sigma_{i=1}^n |x_i - z_i + z_i - y_i| \leq \Sigma_{i=1}^n |x_i - z_i| + |z_i - y_i|$$

We now see by spliting up the sums, that we have that

$$\Sigma_{i=1}^n |x_i - z_i| + |z_i - y_i| = \Sigma_{i=1}^n |x_i - z_i| + \Sigma_{i=1}^n |z_i - y_i| = \rho(x,z) + \rho(z,y)$$

and thus $\rho(x,y) \leq \rho(x,z) + \rho(z,y)$. As x, y, z were arbitrary, we have that this is true for any $x, y, z \in \mathbb{R}^n$.

As we have shown all properties of a metrics hold, we know that $\mathbb{R}^n \times \mathbb{R}^n :\to \mathbb{R}_+ : (x,y) \mapsto \Sigma_{i=1,\ldots,n}|x_i - y_i|$ is a metric. □

4.3. *Prove that* ρ^p *is a metric for any* $p \geq 1$, *where*

$$\rho^{(p)} : (x,y) \mapsto (\sum_{i=1}^n |x_i - y_i|^p)^{1/p},\ p \geq 1$$

and then show the Holder Inequality.

Let $x_1, ..., x_n, y_1, ..., y_n \geq 0$, *let* $p, q > 0$, *and let* $1/p + 1/q = 1$. *Prove that*

$$\sum_{i=1}^n x_i y_i \leq (\sum_{i=1}^n x_i^p)^{1/p} (\sum_{i=1}^n y_i^q)^{1/q}$$

Proof. We will first show that if $\rho^{(p)}(x,y)=0$ then $x=y$. Let $x,y\in\mathbb{R}^n$ be arbitrary, such that $\rho^{(p)}(x,y)=0$. We then have that

$$(\sum_{i=1}^{n}|x_i-y_i|^p)^{1/p}=0$$

and thus

$$\sum_{i=1}^{n}|x_i-y_i|^p=0$$

As $|x_i-y_i|^p$ is never negative for any x_i,y_i, we thus see that for every i

$$|x_i-y_i|^p=0$$

and the exponent function is injective, we know that $|x_i-y_i|=0$, and thus $x_i=y_i$ for all i, and thus by definition of equality for $\mathbb{R}^n$, we have $x=y$. As x,y were arbitrary, we know have that if $\rho^{(p)}(x,y)=0$ then $x=y$.

We also see by the definition of equality on $\mathbb{R}^n$, that if for some $x,y\in\mathbb{R}^n$ $x=y$, that for every i, $x_i=y_i$, and thus $|x_i-y_i|=0$ for every i, and thus $(\sum_{i=1}^{n}|x_i-y_i|^p)^{1/p}=0$ and thus $\rho^{(p)}(x,y)=0$. We thus have that as x,y were arbitrary, that if $x=y$, then $\rho^{(p)}(x,y)=0$, and thus $\rho^{(p)}(x,y)=0$ iff $x=y$.

We will now show that for any $x,y\in\mathbb{R}^n$ that $\rho(x,y)=\rho(y,x)$. Let $x,y\in\mathbb{R}^n$ be arbitrary. We see that

$$\rho(x,y)=(\sum_{i=1}^{n}|x_i-y_i|^p)^{1/p}=(\sum_{i=1}^{n}|y_i-x_i|^p)^{1/p}=\rho(x,y)$$

and as x,y were arbitray, we have that for any $x,y\in\mathbb{R}^n$ that $\rho(x,y)=\rho(y,x)$.

We now will show that $\rho(x,y)\leq\rho(x,z)+\rho(z,y)$ for any x,y,z. We see by refering to "When Cauchy and Holder Met Minkowski: a Tour through well-known inequalities" that this is true. □

4.4 (Riddle)**.** *How is this related to Σ^2, Σ^∞, and Σ^1 from Section 3?*

Answer. As Σ^2,Σ^∞, and Σ^1 were respectively the set of all possible open disks, the set of all possible open squares with sides parallel to the coordinate axes, and the set of all possible open squares with sides parallel to the bisectors of the coordinate angles, and the metrics ρ^2,ρ^∞, and ρ^1 define open balls that are open disks, open squares parallel to the coordinate axes, and open squares parallel to the bisectors of the coordinate angles, that each element of these bases can be defined as a ball around a certain point with a certain radius, that each metric defines their corresponding basis, in $\mathbb{R}^2$. □

4.5. *Let $p\geq 1$. Prove that for any two sequences $x,y\in l^{(p)}$ the series $\sum_{i=1}^{\infty}|x_i-y_i|^p$ converges and that*

$$(x,y)\mapsto\left(\sum_{i=1}^{\infty}|x_i-y_i|^p\right)^{1/p}$$

is a metric on $l^{(p)}$.

[4′3] Balls and Spheres

[4′4] Subspaces of a Metric Space

4.D. *Check that D^1 is the segment $[-1,1]$, D^2 is a plane disk, S^0 the pair of points $-1, 1$, S^1 is a circle, S^2 is a sphere, and D^3 is a ball*

Answer. D^1 is, by definition, the set of values $x \in mathdsR$ for which $\rho(x, 0) \leq 1$. In $\mathbb{R}$, this is the set of values $[-1, 1]$ for the Euclidean metric. D^2 is, by definition, the set of values $x \in \mathbb{R}^2$ for which $\rho(x, (0, 0)) \leq 1$. In $\mathbb{R}^2$, this is a disk - we are assured it is closed by the non-strict inequality - centered around (0, 0). S^0 is, by definition, the set of values $x \in \mathbb{R}$ for which $\rho(x, 0) = 1$. In $\mathbb{R}$, this is exactly the set of values with an absolute value of 1, i.e. $1, -1$. S^1 is, by definition, the set of values $x \in \mathbb{R}^2$ for which $\rho(x, (0, 0)) = 1$. In $\mathbb{R}^2$, this is the set of values on the exterior of D^2, i.e. a sphere centered around (0,0) with radius 1. D^3 is, by definition, the set of values $x \in \mathbb{R}^2$ for which $\rho(x, (0, 0, 0)) \leq 1$. In $\mathbb{R}^3$ this takes the shape of a closed ball, because of the way that $rho(x, (0, 0, 0))$ scales sublinearly with each component of x. □

4.E. *Prove that for any points x and a of any metric space and any $r > \rho(a, x)$ we have*

$$B_{r-\rho(a,x)}(x) \subset B_r(a) \text{ and } D_{r-\rho(a,x)}(x) \subset D_r(a)$$

Proof. Let X be an arbitrary metric space, and let $a, x \in X$ be arbitrary such that $r > \rho(a, x)$ for some $r \in \mathbb{R}$. We will first show that $B_{r-\rho(a,x)}(x) \subset B_r(a)$. Let $b \in B_{r-\rho(a,x)}(x)$ be arbitrary. Because $b \in B_{r-\rho(a,x)}(x)$, we know that $\rho(b, x) < r - \rho(a, x)$, and thus $\rho(b, x) + \rho(a, x) < r$. We now can see by properties of a metric, that $\rho(b, a) < r$, and thus by definition of $B_r(a)$, we know that $b \in B_r(a)$. As b was arbitrary, we know that $B_{r-\rho(a,x)}(x) \subset B_r(a)$.

We will now show that $D_{r-\rho(a,x)}(x) \subset D_r(a)$. Let $d \in D_{r-\rho(a,x)}(x)$ be arbitrary. Because $d \in D_{r-\rho(a,x)}(x)$, we know that $\rho(d, x) \leq r - \rho(a, x)$, and thus $\rho(d, x) + \rho(a, x) < r$. We now can see by properties of a metric, that $\rho(d, a) < r$, and thus by definition of $D_r(a)$, we know that $d \in D_r(a)$. As d was arbitrary, we know that $D_{r-\rho(a,x)}(x) \subset D_r(a)$. As we have proven both claims, we know that for any points x and a of any metric space and any $r > \rho(a, x)$ we have

$$B_{r-\rho(a,x)}(x) \subset B_r(a) \text{ and } D_{r-\rho(a,x)}(x) \subset D_r(a)$$

□

4.6. *What if $r < \rho(x, a)$? What is an analog for the statement of Problem 4.E in this case?*

Answer. We will consider the analog for Balls around x and a, if $r < \rho(x, a)$. We will show that $B_r(a) \cap B_{\rho(x,a)-r}(x) = \varnothing$. Let $y \in B_r(a)$ be arbitrary. We see by definition of an open ball, that $\rho(a, y) < r$. We now consider $\rho(a, x)$. We have that $\rho(a, x) \leq \rho(a, y) + \rho(y, x)$ by the triangle inequality, and thus

$\rho(x,a) - \rho(a,y) \leq \rho(y,x)$. We also see that as $\rho(a,y) < r$, that $\rho(x,a) - r \leq \rho(y,x)$, and by definition of an open ball, we know that $y \notin B_{\rho(x,a)-r}$. As y was arbitrary, we see that if a point is an element of $B_r(a)$, that it cannot be an element of $B_{\rho(x,a)-r}(x)$. We thus see that $B_r(a) \cap B_{\rho(x,a)-r}(x) = \varnothing$ □

[4′5] Surprising Balls

4.7. *What are balls and spheres in* $\mathbb{R}^2$ *equipped with the metrics of* 4.1 *and* 4.2 *?*

Answer. Let us first consider a ball $B_r(a)$ with the metric of 4.1. We see that this ball will be defined by all the points in which $|x_1 - a_1| < r$, and $|x_2 - a_2| < r$ where $a = (a_1, a_2)$, and an arbitrary point is given by $x = (x_1, x_2)$. We can see from this that a ball with the metric of 4.1 will be a square centered around a, with sides parallel to the x-axis and the y-axis with a height, and width of $2r$, with the boundry not included. We can similarly see that a sphere with the metric 4.1 would be all points in which $|x_1 - a_1| = r$, and $|x_2 - a_2| = r$, and thus would be the boundry of a square with the same dimensions as the ball with radius r.

We will now consider a ball $B_r(a)$ with the metric of 4.2. We see that this ball will be defined by all the points in which $|x_1 - a_1| + |x_2 - a_2| < r$, where $a = (a_1, a_2)$, and an arbitrary point is given by $x = (x_1, x_2)$. We can see from this that a ball with the metric of 4.2 will be the inside of a square centered around a, rotated such that the diagonals are parallel to the x-axis and the y-axis, and with diagonals of length $2r$. We would also see that a sphere would have the same dimensions as a ball, but would include the boundry instead of the inside. □

4.8. *Find* $D_1(a)$*,* $D_(1/2)(a)$*, and* $S_(1/2)(a)$ *in the space of 4.A.*

Answer. For a set X with the metric described in 4.A, $D_1(a)$ is the entire set X, $D_(1/2)(a)$ is a singleton with just the element a, and $S_(1/2)(a)$ is the emptyset. □

4.9. *Find a metric space and two balls in it such that the ball with the smaller radius contains the ball with the bigger one and does not coincide with it.*

Answer. Consider the metric space on the subset of $\mathbb{R}$, $\{-1, 0, 1\}$. We see that the ball $B_{1.2}(0) = \{-1, 0, 1\}$, and that the ball $B_{1.5}(-1) = \{-1, 0\}$. We thus see that $B_{1.5}(-1) \subset B_{1.2}(0)$, and $B_{1.5}(-1) \not\subset B_{1.2}(0)$, and thus this space has a ball with a smaller radius which contains a ball with a larger radius, which does not coincidde with it □

4.10. *What is the minimal number of points in the space which is required to be constructed in 4.9?*

Answer. The minimal number of points in the space which is required is 3. We first consider a space with one point a we see that for any r, r', where

$r > 0, r' > 0$. We see for any metric ρ that $\rho(a,a) = 0$, and thus by the definition of a ball, $B_r(a) = B_{r'}(a) = \{a\}$, and thus all balls must coincide . We will now consider a space with two points a, b. Let $r > r' \in \mathbb{R}_+$ be arbitrary. Suppose that without loss of generality $B_{r'}(a)$ contains $B_r(b)$, and thus $B_{r'}(a) = \{a,b\}$. We see that $b \in B_{r'}(a)$, and thus $\rho(a,b) < r'$, and thus $\rho(a,b) < r$, and finally, we can see that $\rho(b,a) < r$ by properties of a metric, and thus by the definition of a ball,$B_r(b) = \{a,b\}$, and thus $B_r(b)$, and $B_{r'}(a)$ coincide. We also can see that as $\rho(a,b) < r'$, implies $\rho(a,b) < r$, that we know that $B_{r'}(a) \subset B_r(a)$. $r > r' \in \mathbb{R}_+$ were arbitrary, we know that a ball with a smaller radius cannot contain a ball with a larger radius if there are only two points. □

4.11. *Prove that the largest radius in 4.9 is at most twice the smaller radius.*

Answer. Let a, b, c be arbitrary points such that $B_r(a) \subset B_{r'}(b)$, and $c \in B_{r'}(b)$, and $c \notin B_r(a)$. Because $c \in B_{r'}(b)$ we know that $\rho(c,b) < r'$, and because $c \notin B_r(a)$, we know that $\rho(a,c) \geq r$. We also have that as $B_r(a) \subset B_{r'}(b)$, that $\rho(b,a) < r'$. We now see by the triangle inequality, that $\rho(a,c) \leq \rho(c,b) + \rho(b,a)$, and thus $r \leq r' + r'$, and we have that $r \leq 2r'$. We thus know that the largest radius in 4.9 is at most twice the smaller radius. □

[4′6] Segments (What Is Between)

4.12. *Prove that the segment with endpoints $a, b \in \mathbb{R}^n$ can be described as*

$$\{x \in \mathbb{R}^n : \rho(a,x) + \rho(x,b) = \rho(a,b)\},$$

where ρ is the Euclidean metric.

Proof. Let $a, b \in \mathbb{R}^n$ be arbitrary, and let them define a segment. Let c be an arbitrary point on that segment. We see that the distance between c and a is given by $\rho(a,c)$, and the distance between c and b is given by $\rho(c,b)$. We also see that as c is on the segment, that the shortest distance between a and b, given by $\rho(a,b)$, can also be given by moving from a to c, and then from c to b. We see that the distance given from moving from a to c, and then c to b is also given by $\rho(a,c) + \rho(c,b)$, and thus $\rho(a,c) + \rho(c,b) = \rho(a,b)$. We thus have that if c is on the segment, that $\rho(a,c) + \rho(c,b) = \rho(a,b)$. We can also see that if we know that $\rho(a,c) + \rho(c,b) = \rho(a,b)$, that we know that going from a to c, and then c to b is the shortest distance between a to b, and thus c is on the segment between a and b. As c was arbitrary, and we showed both that $\rho(a,c) + \rho(c,b) = \rho(a,b)$ implies c is on the segment, and that c being on the segment implies that $\rho(a,c) + \rho(c,b) = \rho(a,b)$, we know the segment with endpoints $a, b \in \mathbb{R}^n$ can be described as

$$\{x \in \mathbb{R}^n : \rho(a,x) + \rho(x,b) = \rho(a,b)\},$$

where ρ is the Euclidean metric. □

4.13. *How does the set defined as in Problem 4.12 look if ρ is the metric defined in Problems 4.1 or 4.2? (Consider the case where $n = 2$ if it seems to be easier.)*

Answer. We first consider the set defined in Problem 4.12 using the metric defined in problem 4.2. We see that this metric can be thought to measure the distance from one point to the next by traveling only on the axes. We thus would see that the set would be a rectangle defined with the corner points being the points a, b in the segment. We also see that this generalizes from $\mathbb{R}^2$ to $\mathbb{R}^3$, defining a rectangular box, and similarly generalizes into higher dimensions.

We now consider the set using the metric defined in problem 4.1. We see that this metric is the set of all points c, where the biggest difference in the components of a and b is equal to the biggest difference in the components a and c plus the biggest difference in the components of c and b. We see that this seems to form a dimond like shape in $\mathbb{R}^2$ and $\mathbb{R}^n$ in cases where the points do not define a box. When the points define a box, a segment is identical to a segment with the standard topology of $\mathbb{R}^n$. $\square$

[4′7] Bounded Sets and Balls

4.F. *Prove that a set A is bounded iff A is contained in a ball.*

Proof. We will first show that if a set A is bounded then it is contained in a ball. Because A is bounded, we know that there is a number $d > 0$ such that for any $x, y \in A$, that $\rho(x, y) < d$. We thus have that for an arbitrary point a that $B_d(a)$ will contain every point in A, by the definition of a ball.

We will now show that if a set A is contained in a ball, that it is bounded. Let $B_r(a)$ be a ball which contains A. Let $x, y \in A$ be arbitrary. We see that by the triangle inequality, that $\rho(x, y) \leq \rho(x, a) + \rho(y, a)$, and by the definition of a ball, we have that $\rho(x, a) < r$, and $\rho(y, a) < r$ as A is contained within the ball. We thus would see that $\rho(x, y) < 2r$, and thus as $x, y \in A$ were arbitrary, that for any x, y, we have that $\rho(x, y) < 2r$, and as $2r > 0$, we know that A is bounded by the definition of bounded. $\square$

4.14. *What is the relation between the minimal radius of such a ball and* $\operatorname{diam}(A)$*?*

Answer. $2r \geq \operatorname{diam}(A) \geq r$, where r is the minimal radius of such a ball. We see that $2r \geq \operatorname{diam}(A)$ above, as for an arbitrary ball with radius r, for any $x, y \in A$, $\rho(x, y) < 2r$. We will now show that $\operatorname{diam}(A) \geq r$. We will show this using proof by contradiction. We assume for sake of contradiction, that $\operatorname{diam}(A) < r$. Let a ball which contains A be $B_r(a)$. We would then have that for any $x \in B_r(a)$ have that $\rho(a, x) < \operatorname{diam}(A) < r$, and thus r is not the minimal radius, and we have a contradiction. We thus have that $2r \geq \operatorname{diam}(A) \geq r$. $\square$

[4′8] Norms and Normed Spaces

4.15. *Prove that if $x \to \|x\|$ is a norm, then*

$$\rho : X \times X \to \mathbb{R}_+ : (x, y) \mapsto \|x - y\|$$

Proof. We will first show that $\rho(x,y) = 0$ if and only if $x = y$.Let $x, y \in X$ be arbitrary. We see that as $\rho(x,y) = \|x - y\|$, that $\rho(x,y) = 0$ if and only if $x - y = 0$, and thus if $x = y$. We thus have that $\rho(x,y) = 0$ if and only if $x = y$, and as x, y were arbitrary, this is true for all $x, y \in X$.

We will now show that $\rho(x,y) = \rho(y,x)$. Let $x, y \in X$ be arbitrary. We see that $\rho(x,y) = \|x - y\| = \|(-1)(y - x)\| = \|y - x\| = \rho(y,x)$. As x, y were arbitrary, we see that $\rho(x,y) = \rho(y,x)$ for all x, y.

We will now show that $\rho(x,y) \leq \rho(x,z)+\rho(z,y)$ for all $x, y, z \in X$. Let x, y, z be arbitrary. We see that $\rho(x,y) = \|x-y\| = \|x-z+z-y\| \leq \|x-z\|+\|z-y\| = \rho(x,z) + \rho(z,y)$. As $x, y, z \in X$ were arbitrary, we see that for any $x, y, z \in X$, that $\rho(x,y) \leq \rho(x,z) + \rho(z,y)$ □

4.16. *Look through the problems of this section and figure out which of the metric spaces involved are, in fact, normed vector spaces.*

Answer. We will look through the metric spaces defined by 4.A, 4.B, 4.C, 4.1, 4.2 and 4.3 to determine which of the metric spaces defined by these metrics are normed vector spaces. We will do this, by determining for each metric ρ, if $\rho(x,y) = \|x - y\|$ for some norm. Before examining any specific metric, we see that for any metric space defined as in 4.15, that $\rho(x,0) = \|x\|$, and thus to examine the properties 1 and 2 of a norm, we can examine $\rho(x,0)$ for every $x \in X$. We see that as $\rho(x,0) = 0$ if and only if $x = 0$, that property 1 will hold for every metric. We also see that if property 2 of a norm holds, that $\|x + y\| = \rho(x,-y) \leq \|x\| + \| - y\| = \|x\| + \|y\|$ and thus property 3 of a norm holds. We thus see that we must only examine property 2 of a norm to determine if a metric is defined by a norm as in 4.15.

We first examine ρ defined by 4.A. We see that $2 \cdot \rho(1,0) = 1$, and $\rho(2,0) = 1$, and thus property 2 of norms does not hold.

We now examine ρ defined by 4.B. We see that $|\lambda x| = |\lambda||x|$ by definition of absolute value, and thus that the metric space defined by the metric in 4.B is a normed vector space, as absolute value is a norm.

We now examine ρ defined by 4.C. We let $x \in \mathbb{R}^n$ be arbitrary for an arbitrary n. We see that for an arbitrary $\lambda \in \mathbb{R}$ $\rho(\lambda x, 0) = \sqrt{\Sigma_{i=1}^n (\lambda x_i)^2} = \sqrt{\Sigma_{i=1}^n \lambda^2 x_i^2} = \sqrt{\lambda^2 \Sigma_{i=1}^n x_i^2} = \lambda \sqrt{\Sigma_{i=1}^n x_i^2} = |\lambda| \rho(x,0)$. As x, n and λ were arbitrary, we have that property 2 of a norm holds, and as ρ is a metric, we know that the metric space defined by ρ is a normed vector space.

We now examine ρ defined by 4.1. We let $x \in \mathbb{R}^n$ be arbitrary for an arbitrary n. We see that for an arbitrary $\lambda \in \mathbb{R}$, $\rho(\lambda x, 0) = max_{i=1,\ldots,n}|\lambda x_i| = max_{i=1,\ldots,n}|\lambda||x_i| = |\lambda| max_{i=1,\ldots,n}|x_i| = |\lambda|\rho(x,0)$. As x n and λ were arbitrary, we have that property 2 of a norm holds, and as ρ is a metric, we know that the metric space defined by ρ is a normed vector space.

We now examine ρ defined by 4.2. We let $x \in \mathbb{R}^n$ be arbitrary for an arbitrary n. We see that for an arbitrary $\lambda \in \mathbb{R}$, $\rho(\lambda x, 0) = \Sigma_{i=1}^n |\lambda x_i| = \Sigma_{i=1}^n |\lambda||x_i| = |\lambda| \Sigma_{i=1}^n |x_i| = |\lambda| \rho(x,0)$. As x n and λ were arbitrary, we have that property 2 of a norm holds, and as ρ is a metric, we know that the metric space defined by ρ is a normed vector space.

We now examine ρ defined by 4.3. We let $p \in \mathbb{N}$ and $x \in \mathbb{R}^n$ be arbitrary for an arbitrary n. We see that for an arbitrary $\lambda \in \mathbb{R}$, $\rho(\lambda x, 0) = (\Sigma_{i=1}^n |\lambda x_i|^p)^{1/p} = (\Sigma_{i=1}^n |\lambda|^p |x_i|^p)^{1/p} = (|\lambda|^p |\Sigma_{i=1}^n |x_i|^p)^{1/p} = (|\lambda|^p)^{1/p} (\Sigma_{i=1}^n |\lambda x_i|^p)^{1/p} = |\lambda| \rho(x, 0)$. As x,p, n and λ were arbitrary, we have that property 2 of a norm holds, and as ρ is a metric, we know that the metric space defined by ρ is a normed vector space. $\square$

4.17. *Prove that every ball in a normed space is a convex set symmetric with respect to the center of the ball.*

Proof. We first let $B_r(a)$ be an arbitary ball in a normed space. We will first show that it is a convex set, and thus if $x, y \in B_r(a)$, that for any z which is in the segment bettween x and y that $z \in B_r(a)$. Let $x, y \in B_r(a)$ be arbitrary, and let z such that $\rho(x, z) + \rho(z, y) = \rho(x, y)$ be arbitrary. We see that $z = \lambda x + (1-)y$ for some $\lambda \in (0, 1)$, by definition of a segment in a vector space. We see that $\|z-a\| = \|\lambda x+(1-\lambda)y-a\| = \|\lambda x+(1-\lambda)y+\lambda a+(1-\lambda)a\| \leq \lambda\|x-a\|+(1-\lambda)\|y-a\| < \lambda r+(1-\lambda)r = r$. We thus have that $\rho(a, z) < r$, and thus that $z \in B_r(a)$, and thus that $B_r(a)$ is convex. As $B_r(a)$ was arbitrary, we know that every ball in a normed space is a convex set.

We will now show that every ball in a normed space is symmetric with respect to the center of the ball. Let $B_r(a)$ be an arbitary ball in a normed space. We will show that for any $x \in B_r(a)$, that the vector reflected around a, $2a - x$ is in $B_r(a)$. We see that $\rho(2a - x, a) = \|2a - x - a\| = \|a - x\| = \rho(x, a) < r$, and thus $2a - x \in B_r(a)$. As $B_r(a)$ was arbitrary, we know that every ball in a normed space is symmetric with respect to the center of the ball.

As we have proven all properties, we see that every ball in a normed space is a convex set symmetric with respect to the center of the ball. $\square$

4.18. *Prove that every convex closed bounded set in $\mathbb{R}^n$ that has a center of symmetry and is not contained in any affine space except $\mathbb{R}^n$ itself is a unit ball with respect to a certain norm, which is uniquely determined by this ball.*

Proof. Let S be a convex closed bounded set in $\mathbb{R}^n$ with center c. We define the set operations

$$\alpha \cdot S = \alpha S = \{\alpha x : x \in S\},$$

and

$$S + y = \{x + y : x \in S\}.$$

We see that $x \in S$ if and only if $\alpha x \in \alpha S$, and $x \in S$ if and only if $x + y \in S + y$. Let $\|\cdot\| : \mathbb{R}^n \to \mathbb{R}_+$ be defined by

$$\|y\| = \inf\{\lambda : y \in \lambda(S - c)\}.$$

It follows that $\|y\| = 0$ exactly when $y = 0$, since for every $\lambda > 0$, $c \in \lambda S$. To see that

$$\|\alpha y\| = |\alpha|\|y\|,$$

it suffices to show that

$$A = \{\lambda : \alpha x \in \lambda(S - c)\} = |\alpha|\{\lambda : x \in \lambda(S - c)\} = B.$$

It would then follow that the infimum of one set is the infimum of the other, arriving at our claim. But $\lambda \in A$ if and only if $\alpha x \in \lambda(S - c)$, which holds if and only if $\lambda \in \alpha\{\lambda : x \in \lambda(S - c)\} = b$, as needed.

Finally, to see the triangle inequality holds, it suffices to show that

$$\frac{x + y}{\|x\| + \|y\|} \in S - c,$$

since this establishes that

$$\frac{\|x + y\|}{\|x\| + \|y\|} \leq 1,$$

proving the claim. Since $S - c$ is convex, we have

$$\frac{\|x\|}{\|x\| + \|y\|}(S - c) + \frac{\|y\|}{\|x\| + \|y\|}(S - c) = S - c.$$

Since $\|x\| \leq \|x\| + \|y\|$, we have $\frac{x}{\|x\|+\|y\|} \in S - c$. Similarly, we have $\frac{y}{\|x\|+\|y\|} \in S - c$. By the convexity of $S - c$, we conclude that

$$\frac{x + y}{\|x\| + \|y\|} \in S - c,$$

which allows us to deduce that

$$\frac{\|x + y\|}{\|x\| + \|y\|} \leq 1,$$

giving the desired result. □

[4′9] Metric Topology

4.G. *The collection of all open balls in the metric space is a base for a certain topology.*

Proof. Let (X, ρ) be an arbitrary metric space. We will show that the set of all unions of open balls, and all finite intersections are a topology, and that X and $\varnothing$ are in the topology. We first see that as for any a, that $\rho(a, a) = 0$, that any point $a \in X$ there is a ball $B_r(a)$ where $r > 0$, which contains it, and by taking the union of all such sets , we have that X is in the topology. We also see that a ball with radius 0 doesn't contain any points, and thus that the empty set is included in the topology.

We will now show that the topology is closed under finite intersection. We will call this intersection A. If this intersection is $\varnothing$, we know that it is in the topology. Let a be an arbitrary point in a finite intersection of sets in the topology. We see that if this intersection is X, that as we showed above, a

has an open ball contained within the set containing it. Otherwise, let $r = inf(\rho(a,b)|b \in X \setminus A)$. We see that if a is contained in some open ball $B_r(c)$, then there would be some $c \in \mathbb{R}$ such that $\rho(c,a) < c < r$ by density of $\mathbb{R}$, and if any open ball is intersected with that open ball, that a similar property must hold if a is contained within it. As intersection is the only way for a element of the topology to decrease in size, we know that $r > 0$, and thus we know that $B_r(a)$ contains only elements of A, by definition of a open ball, and $a \in B_r(a)$. As a is in a open ball contained in A, and a was arbitrary, we see that the topology is closed under finite intersection.

We will now show that the topology is closed under arbitrary union. Let A be a union of arbitrary sets of the topology, and let a be an arbitary point in A. We see that by definition of union, a is either in some open ball within, or in the finite intersection of open balls. Because we showed above that the topology is closed under finite intersection, we know that in both cases, that there is some open ball in A which contains a. As a is in a open ball contained in A, and a was arbitrary, we see that the topology is closed under arbitrary intersection.

As we have proven all axioms of a topology, we know that the topology generated by set of all open balls is a topology. $\square$

4.H. *Prove that the standard topological structure in* $\mathbb{R}$ *introduced in Section 2 is generated by the metric* $(x, y) \mapsto |x - y|$.

Proof. Let (a, b) be an arbitrary open interval. We see that $\frac{a+b}{2} \in (a, b)$, and that $B_r(\frac{a+b}{2})$ where $r = \frac{a+b}{2} - a$ does not contain a or b, but it does contain every point x such that $|x - \frac{a+b}{2}| < \frac{a+b}{2} - a$, and thus $-\frac{a+b}{2} + a < x - \frac{a+b}{2} < \frac{a+b}{2} - a$ which is equivalent to $a < x < b$. We thus have that (a, b) is generated by the metric $(x, y) \mapsto |x - y|$. As (a, b) was arbitrary, we have that any open interval is generated by the metric $(x, y) \mapsto |x - y|$, and thus the standard topological structure in $\mathbb{R}$ introduced in Section 2 is generated by the metric $(x, y) \mapsto |x - y|$. $\square$

4.19. *What topological structure is generated by the metric of* 4.*A?*

Answer. Consider the ball $B_1(a)$ for an arbitrary point a. We see that if $x \in B_1(a)$, that $\rho(x, a) < 1$, and by definition of the metric, we would thus have that $x = a$. We thus see as a was arbitrary, that every element of the set has a set withing the topology containing only it. We see by the axioms of a topology, that this means that every union and every intersection of these elements is in the topology, and as any subset is made up of individual element of the set, we know that we can make any possible subset by taking the union of elements in the set. As any possible subset is in the topology, we thus know that 4.*A* generates the discrete topology. $\square$

4.I. *A set* U *is open in a metric space iff, together with each of its points, the set* U *contains a ball centered at this point.*

Proof. We will first prove that if U is open in a metric space that it contains a ball centered at this point. Let $x \in U$ be arbitrary. By the definition the metric space, we know that U is the union of open balls. As $x \in U$, we know that for some open ball centered at a with radius r,$x \in B_r(a)$. By definition of an open ball, we know that as $x \in B_r(a)$, we have $\rho(a, x) < r$, and thus by 4.E, we have that $B_{r-\rho(a,x)}(x) \subset B_r(a)$, and thus the set U contains a ball centered at x. As x was arbitrary, we see that this is true for every point in U, and thus that if U is open in a metric space that it contains a ball centered at this point.

We will now show that if every point in U contains a ball within U centered at that point, that U is an open set in a metric space. Consider the set formed by taking the union of all the balls centered around points in U. We would then see this union would contain every point in U, and that because each ball is within U, that this union would also be contained by U. We thus have that U is equal to the union of all these balls, and thus that U is an open set. □

[4′10] Openness and Closedness of Balls and Spheres

4.20. *Prove that a closed ball is closed (here and below, we mean the metric topology).*

Proof. Let (X, Ω) be a space under a metric topology generated by ρ. Let $r > 0$ be arbitrary. We show that $D_r(x)$ is closed by showing that $X \setminus D_r(x)$ is open.

For any $y \in X \setminus D_r(x)$, if there is some $\varepsilon > 0$ such that $B_\varepsilon(y) \subseteq X \setminus D_r(x)$, we can apply the result from 4.I to conclude that $X \setminus D_r(x)$ is open. To this end, note that as

$$\rho(x, y) > r,$$

since $y \notin D_r(x)$, there exists $\varepsilon > 0$ such that

$$\rho(x, y) = r + \varepsilon.$$

We show that $B_\varepsilon(y) \subseteq X \setminus D_r(x)$. Let $z \in B_\varepsilon(y)$. We have by the triangle inequality that

$$\rho(x, y) \leq \rho(x, z) + \rho(z, y).$$

But since $\rho(x, y) = r + \varepsilon$, and since $\rho(z, y) < \varepsilon$, we have that

$$r + \varepsilon = \rho(x, y) \leq \rho(x, z) + \rho(z, y) < \rho(x, z) + \varepsilon,$$

which implies that

$$r < \rho(x, z).$$

Hence, $z \in X \setminus D_r(x)$, and the desired result follows. □

4.21. *Find a closed ball that is open.*

Answer. For this and the rest of the exercises in this section, we define $X \subseteq \mathbb{R}$ by

$$X = \{-2, 0, 1\}$$

and let $\rho(x, y) = |x - y|$ be the restricted metric inherited from $\mathbb{R}$.

We have that

$$D_{1/2}(-2) = \{-2\},$$

so $\{-2\}$ is closed. We also have that

$$D_1(1) = \{0, 1\},$$

so $\{0, 1\}$ is closed. But $\{-2\} = X \setminus \{0, 1\}$, so $\{-2\}$ is open. Hence, $D_{1/2}(-2)$ is open. $\square$

4.22. *Find an open ball that is closed.*

Answer. Let X be as defined in 4.21. We have that

$$B_1(-2) = \{-2\},$$

so $\{-2\}$ is open. We also have that

$$B_2(0) = \{0, 1\},$$

so $\{0, 1\}$ is open. But $\{-2\} = X \setminus \{0, 1\}$, implying that $\{-2\}$ is closed. Hence, $B_1(-2)$ is closed. $\square$

4.23. *Prove that a sphere is closed.*

Proof. Let (X, Ω) be a topology generated by the metric ρ. Let $x \in X$ and $r \in \mathbb{R}_{++}$ be arbitrary, and consider $S_r(x)$. To show that $S_r(x)$ is closed, it suffices to show that $X \setminus S_r(x)$ is open. But

$$X \setminus S_r(x) = B_r(x) \cup (X \setminus D_r(x)),$$

following the defined constructions of spheres, balls, and disks. From the definition of the metric topology Ω, we have that $B_r(x)$ is open. From 4.20, we have that $D_r(x)$ is closed; hence, $X \setminus D_r(x)$ is closed. As the union of two open sets is open, we find that $X \setminus S_r(x)$ is open. Thus, $S_r(x)$ is closed. $\square$

4.24. *Find a sphere that is open.*

Answer. Let X be defined as in 4.21. We have that

$$S_1(0) = \{1\} = B_1(1),$$

so $S_1(0)$ is open. $\square$

[4′11] Metrizable Topological Space

4.J. *An indiscrete space is not metrizable if it is not a singleton (otherwise, it has too few open sets).*

Proof. Let X be an arbitary nonempty indiscrete space. We will show that if X metrizable by some metric ρ, that it is a singleton. Let $a, b \in X$ be arbitrary. We see that if $\rho(a,b) > 0$, that $b \notin B_{\rho(a,b)}(a)$, and $a \notin B_{\rho(a,b)}(a)$, and as $a \in B_{\rho(a,b)}(a)$, and $b \in B_{\rho(a,b)}(b)$, we would have that $B_{\rho(a,b)}(a) \neq B_{\rho(a,b)}(b) \neq \varnothing$, and thus there would be three sets in the topology. We thus have that $\rho(a,b) = 0$, and thus that $a = b$. As a, b were arbitrary, we see that there is only one element in X, and thus that X is a singleton. □

4.K. *A finite space X is metrizable iff it is discrete.*

Proof. Let X be an arbitrary finite Topological space. Suppose X is discrete. We would then see that the metric $4.A$ would generate it, by 4.19. Now suppose that X is metrizable for some metric ρ. Let $a \in X$ be arbitrary such that a. We see that as X is a finite set, that $B_r(a)$ where $r = inf(\rho(a,b) | b \neq a, b \in X)$ is in the metric topology of X, and will only contain a by the definition of an open ball. As a was arbitrary, we see that every individual element of X has a set which contains only it in the topology, and thus we see that X is discrete, as any supset of X can be made from the union of individual elements. □

4.25. *Which of the topological spaces described in Section 2 are metrizable?*

Answers. To determine this, we will examine the topological spaces described in 2.1, 2.3, 2.C , 2.6.

We first assume that there is a metric ρ such that the topological space in 2.1 is induced by it. We see that in the arrow topology, that if 1 is in a set, that 2 is also in a set, however, we see that as $1 \neq 2$, that $\rho(1,2) > 0$, and thus $_{\rho(1,2)}(1)$ contains 1, but not 2 we thus have a contradiction and see that 2.1 is not metrizable.

We see that as 2.3 is finite, we know by $4.K$ that it is metrizable, with the metric described in $4.A$.

We see by $4.H$ that the topological space described by $2.C$ is metrizable.

We now assume a metric ρ such that the topological space in 2.6 is induced by it. We see for some point a, a is in every set. However, we see that if for some b, $\rho(a,b) > 0$, that there is a set $B_{\rho(a,b)}(b)$ which does not contain b, but which is not the empty set. We also see that as 2.6 is obtained by adding a point to a topology, that there must be at least one point $b \neq a$, and thus we have a contradiction, and the topological space in 2.6 is not metrizable. □

[4′12] Equivalent Metrics

4.26. *Are the metrics of 4.C, 4.1, and 4.2 equivalent?*

Answer. These metrics are equivalent. See 4.30. □

4.27. *Prove that two metrics ρ_1 and ρ_2 in X are equivalent if there are numbers $c, C > 0$ such that*

$$c\rho_1(x,y) \leq \rho_2(x,y) \leq C\rho_1(x,y)$$

for any $x, y \in X$.

Proof. By 4.I, it suffices to show that every ball in Ω_1 is open in Ω_2 and vice versa.

Let $x \in X$ and $r > 0$ be arbitrary. Let $y \in B^1_r(x)$. Now, define $r' = (1 - \frac{c_2}{c_1})r$, and consider $B^2_{r'}(y)$. If $z \in B^2_{r'}(y)$, we have

$$\begin{aligned}
\rho_1(x,z) &\le \tfrac{1}{c_1}\rho_2(x,z) \\
&\le \tfrac{1}{c_1}(\rho_2(x,y) + \rho_2(y,z)) \\
&\le \tfrac{1}{c_1}(c_2\rho_1(x,y) + \rho_2(y,z)) \\
&< \tfrac{1}{c_1}(c_2 r + r') \\
&= r
\end{aligned}$$

as needed.

Next, let $y \in B^2_r(x)$. Now, define $r' =$ and consider $B^1_{r'}(y)$. If $z \in B^2_{r'}(y)$, we have

$$\begin{aligned}
\rho_2(x,z) &\le c_2\rho_1(x,y) \\
&\le c_2(\rho_1(x,y) + \rho_1(y,z)) \\
&\le c_2(\tfrac{1}{c_1}\rho_2(x,y) + \rho_1(y,z)) \\
&< c_2(\tfrac{1}{c_1} r + r') \\
&= r
\end{aligned}$$

as needed.

Thus, ρ_1 and ρ_2 are equivalent metrics. ☐

4.28. *Generally speaking, the converse is not true.*

Proof. As we shall see in 4.33, if ρ is a metric, then so is $\frac{\rho}{1+\rho}$. Moreover, 4.34 shows that these are equivalent metrics. However, suppose that there are c_1, c_2 such that

$$c_1\rho(x,y) \le \frac{\rho(x,y)}{1+\rho(x,y)} \le c_2\rho(x,y)$$

for all $x, y \in X$. This is identical to the statement

$$(c_1 - 1)\rho(x,y) + c_1\rho^2(x,y) \le 0 \le (c_2 - 1)\rho(x,y) + c_2\rho^2(x,y),$$

or

$$\rho(x,y) \le 1 - \frac{1}{c_1}, \qquad \text{and} \qquad \rho(x,y) \ge 1 - \frac{1}{c_2}.$$

Suppose $c_1 > 0$. If $\rho(x,y) = |x - y|$, for example, then

$$\rho(0, 2 - \tfrac{2}{c_1}) > 1 - \frac{1}{c_1},$$

so c_1 necessarily must not be positive. Hence, the condition does not hold for this combination. ☐

4.29 (Riddle)**.** *Hence the condition of equivalence of metrics formulated in Problem 4.27 can be weakened. How?*

Answer. In our proof, we actually bound x prior to binding c_1 and c_2. Hence, we could weaken the statement of the theorem by saying:

For all $x \in X$ there exists $c, C > 0$ such that

$$c\rho_1(x,y) \leq \rho_2(x,y) \leq C\rho_1(x,y),$$

we could use the same proof to get the result. □

4.30. *The metrics $\rho^{(p)}$ in $\mathbb{R}^n$ defined right before Problem 4.3 are equivalent.*

Proof. It suffices to show that $\rho^{(p)}$ is equivalent to $\rho^{(1)}$, since equivalence is transitive. We want to show that there are $c_1, c_2 > 0$ such that

$$c_1 \sum_{i=1}^{n} |x_i - y_i| \leq \left(\sum_{i=1}^{n} |x_i - y_i|^p \right)^{1/p} \leq c_2 \sum_{i=1}^{n} |x_i - y_i|$$

for all x and y. If $x = y$, this is trivial, so assume $x \neq y$. In this case, we may rewrite our goal as

$$c_1 \leq \frac{\left(\sum_{i=1}^n |x_i - y_i|^p\right)^{1/p}}{\sum_{i=1}^n |x_i - y_i|} \leq c_2.$$

In fact, we have $\left(\sum_{i=1}^n |x_i - y_i|^p\right)^{1/p} \leq \sum_{i=1}^n |x_i - y_i|$, so by choosing $c_2 = 1$ we establish an upper bound. For the lower bound, we note that absolute value and positive powers are continuous functions. Moreover, since $x \neq y$, this term is strictly positive. Hence,

$$0 < \frac{\left(\sum_{i=1}^n |x_i - y_i|^p\right)^{1/p}}{\sum_{i=1}^n |x_i - y_i|} \leq 1$$

for a continuous function of x and y, so the infimum ℓ of all possible values of $\frac{\left(\sum_{i=1}^n |x_i - y_i|^p\right)^{1/p}}{\sum_{i=1}^n |x_i - y_i|}$ exists and is attained by some x and y. Thus, setting $c_1 = \ell$, we are done. □

4.31. *Prove that the following two metrics ρ_1 and ρ_c in the set of all continuous functions $[0,1] \to \mathbb{R}$ are not equivalent:*

$$\rho_1(f,g) = \int_0^1 |f(x) - g(x)| \, dx, \qquad \rho_c(f,g) = \max_{x \in [0,1]} |f(x) - g(x)|.$$

Is it true that one of the topological structures generated by them is finer than the other one?

Proof. For $r > 0$, let $g_r \in C[0,1]$ be defined by

$$g_r(x) = \begin{cases} 0 & x \in [0, \frac{1}{2} - \frac{1}{2r}] \\ 2r^2(x - (\frac{1}{2} - \frac{1}{2r})) & x \in [\frac{1}{2} - \frac{1}{2r}, \frac{1}{2}] \\ r - 2r^2(x - \frac{1}{2}) & x \in [\frac{1}{2}, \frac{1}{2} + \frac{1}{2r}] \\ 0 & x \in [\frac{1}{2} + \frac{1}{2r}, 1] \end{cases},$$

so that $g_r(x)$ is a "triangular impulse" centered at $\frac{1}{2}$ such that $g(\frac{1}{2}) = r$ and that the total area of the triangle is $\frac{1}{2}$. As

$$\int_0^1 |g_r(x)|\, dx = \frac{1}{2}$$

for all $r > 0$, we have that $g_r \in B_1^1(0)$, where 0 denotes the 0 function. However, for any $f \in C[0,1]$ and $r' > 0$, by choosing r such that

$$r > \max\{\max_{x\in[0,1]} |f(x) - g(x)|, r'\},$$

we guarantee that $g_r \notin B_{r'}^2(f)$. Hence, $B_r^1(0)$ cannot be described as a union of balls with respect to ρ_2, so $B_r^1 \notin \Omega_2$. □

[4′13] Operations with Metrics

4.32. *1. Prove that if ρ_1 and ρ_2 are two metrics in X, then $\rho_1 + \rho_2$ and $\max\{\rho_1, \rho_2\}$ are also metrics.*

2. Are the functions $\min\{\rho_1, \rho_2\}$, $\rho_1\rho_2$, and ρ_1/ρ_2 metrics? (By definition, for $\rho = \rho_1/\rho_2$ we put $\rho(x,x) = 0$.)

Proof. 1. For $\rho = \rho_1 + \rho_2$, the first two metric properties follow simply. To see the triangle inequality holds, note that

$$\begin{aligned} \rho(x,y) &= \rho_1(x,y) + \rho_2(x,y) \\ &\le (\rho_1(x,z) + \rho_1(z,y)) + (\rho_2(x,z) + \rho_2(z,y)) \\ &= (\rho_1(x,z) + \rho_2(x,z)) + (\rho_1(z,y) + \rho_2(z,y)) \\ &= \rho(x,z) + \rho(z,y), \end{aligned}$$

as needed.

For $\rho = \max\{\rho_1, \rho_2\}$, again the first two metric properties follow simply. To see the triangle inequality holds, note that

$$\begin{aligned} \rho(x,y) &= \max\{\rho_1(x,y), \rho_2(x,y)\} \\ &\le \max\{\rho_1(x,z) + \rho_1(z,y), \rho_2(x,z) + \rho_2(z,y)\} \\ &= \max\{\rho_1(x,z), \rho_2(x,z)\} + \max\{\rho_1(z,y), \rho_2(z,y)\} \\ &= \rho(x,z) + \rho(z,y) \end{aligned}$$

as needed.

2. Let $X = \{a, b, c\}$ and suppose

$$\begin{aligned}
\rho_1(a,b) &= 1\\
\rho_1(b,c) &= 3\\
\rho_1(a,c) &= 4\\
\rho_2(a,b) &= 3\\
\rho_2(b,c) &= 1\\
\rho_2(a,c) &= 4.
\end{aligned}$$

The triangle inequalities hold for both respective metrics. However,

$$\min\{\rho_1(a,c), \rho_2(a,c)\} = 4,$$

whereas

$$\min\{\rho_1(a,b), \rho_2(a,b)\} + \min\{\rho_1(b,c), \rho_2(b,c)\} = 1 + 1 = 2,$$

violating the triangle inequality. Hence, the minimum fails to be a metric.

Let $\rho_1 = \rho_2$ be the canonical metric on $\mathbb{R}$. Then

$$\begin{aligned}
\rho_1\rho_2(0,1) &> \rho_1(0,\tfrac{1}{2})\rho_2(0,\tfrac{1}{2}) + \rho_1(\tfrac{1}{2},1)\rho_2(\tfrac{1}{2},1)\\
1 &> \tfrac{1}{4} + \tfrac{1}{4}\\
&= \tfrac{1}{2}.
\end{aligned}$$

Hence, the product fails to be a metric.

Let ρ_1 be the discrete metric on $\mathbb{R}$ and let ρ_2 be the canonical metric on $\mathbb{R}$. Then

$$\begin{aligned}
\frac{\rho_1(0,\frac{1}{2})}{\rho_2(0,\frac{1}{2})} &= \frac{1}{\frac{1}{2}}\\
&= 2,
\end{aligned}$$

while

$$\begin{aligned}
\frac{\rho_1(0,\frac{5}{2})}{\rho_2(0,\frac{5}{2})} + \frac{\rho_1(\frac{5}{2},\frac{1}{2})}{\rho_2(\frac{5}{2},\frac{1}{2})} &= \frac{1}{\frac{5}{2}} + \frac{1}{2}\\
&= \frac{9}{10}.
\end{aligned}$$

Hence, the quotient fails to be a metric.

□

¡++¿

4.33. *Prove that if $\rho : X \times X \to \mathbb{R}_+$ is a metric, then*

1. *the function $(x,y) \mapsto \frac{\rho(x,y)}{1+\rho(x,y)}$ is a metric;*

2. *the function* $(x, y) \mapsto \min\{p(x, y), 1\}$ *is a metric;*
3. *the function* $(x, y) \mapsto f(\rho(x, y))$ *is a metric if* f *satisfies the following conditions:*
   (a) $f(0) = 0$,
   (b) f *is a monotone increasing function, and*
   (c) $f(x + y) \leq f(x) + f(y)$ *for any* $x, y \in \mathbb{R}$.

Proof. 1. Let $d(x, y) = \frac{\rho(x,y)}{1+\rho(x,y)}$. That $d(x, x) = 0$ and $d(x, y) = d(y, x)$ follow from inspection of the construction of d. The triangle inequality emerges less elegantly, as we shall see:

$$\rho(x, y) \leq \rho(x, z) + \rho(z, y)$$

$$\rho(x, y) + \rho(x, y)(\rho(z, y) + \rho(x, z)(1 + \rho(y, z))) \leq \rho(x, z) + \rho(z, y) + \rho(x, y)(\rho(z, y) + \rho(x, z)(1 + 2\rho($$

$$\vdots$$

2. Let $d(x, y) = \min\{p(x, y), 1\}$. That $d(x, x) = 0$ and $d(x, y) = d(y, x)$ follow from inspection. We show that the triangle inequality holds.

   Suppose $\rho(x, y) \leq 1$. For any z, either $\rho(x, z) \leq 1$ and $\rho(z, y) \leq 1$, or not. If so, then

   $$\min\{\rho(x, z), 1\} = \rho(x, z) \qquad \text{and} \qquad \min\{\rho(z, y), 1\} = \rho(z, y),$$

   and applying the the triangle inequality gives the needed result. Otherwise, $\rho(x, y) \leq 1$, so the triangle inequality holds.

   Suppose $\rho(x, y) \geq 1$. If both $\rho(x, z) \leq 1$ and $\rho(z, y) \leq 1$, we have

   $$d(x, y) = 1 \leq \rho(x, y) \leq \rho(x, z) + \rho(z, y) = d(x, z) + d(y, z),$$

   verifying the triangle inequality. Otherwise, since $1 \leq 1$, the inequality holds easily.

3. Let $d(x, y) = f(\rho(x, y))$, where f is defined as above. Since $\rho(x, y) = 0$ if and only if $x = y$, and since $f(0) = 0$, we have $d(x, y) = 0$ if and only if $x = y$. Since $\rho(x, y) = \rho(y, z)$, we have $d(x, y) = f(\rho(x, y)) = f(\rho(y, z)) = d(y, z)$. Finally, since ρ is nonnegative, and since f is monotonically increasing, we have

   $$d(x, y) = f(\rho(x, y)) \leq f(\rho(x, z) + \rho(z, y)).$$

   Since $f(a + b) \leq f(a) + f(b)$, we have

   $$f(\rho(x, z) + \rho(z, y)) \leq f(\rho(x, z) + \rho(z, y)).$$

   Hence,

   $$d(x, y) \leq d(x, z) + d(z, y),$$

   as needed.

$\square$

4.34. *Prove that the metrics ρ and $\frac{\rho}{1+\rho}$ are equivalent.*

Proof. It suffices to show that the function $f : \mathbb{R}_+ \to \mathbb{R}_+$ by $f(x) = \frac{x}{1+x}$ is a bijection. If $d(x,y) = \frac{\rho(x,y)}{1+\rho(x,y)}$, and f is bijective, then for each $r \geq 0$ there exists a unique $r' \geq 0$ such that $B_r(x)$ with respect to ρ coincides with $B_{r'}(x)$ with respect to d. As such, the two topologies coincide.

To see that $\frac{x}{1+x}$ is injective, notice that if $x_1 < x_2$, then

$$x_1 + x_1x_2 < x_2 + x_1x_2,$$

so

$$x_1(1 + x_2) < x_2(1 + x_1),$$

and hence

$$f(x_1) = \frac{x_1}{1 + x_1} < \frac{x_2}{1 + x_2} = f(x_2).$$

Thus, f is injective. To see that f is surjective, let $y \geq 0$ be arbitrary. Take $x = \frac{y}{1-y} \geq 0$, and we have

$$f(x) = f\left(\frac{y}{1-y}\right) = \frac{\frac{y}{1-y}}{1 + \frac{y}{1-y}} = y.$$

Thus, f is surjective, and hence f is bijective. □

[4′14] Ultrametrics and p-Adic Numbers

4.L. *Check that only one metric in 4.A -4.2 is an ultra metric*

To do this, we must examine the following metrics
4.A: $\rho : X \times X \to \mathbb{R}_+ : (x, y) \mapsto$

$$\begin{cases} 0 & \text{if } x = y \\ 1 & \text{if } x \neq y \end{cases}$$

4.B: $\mathbb{R} \times \mathbb{R} \to \mathbb{R}_+ : (x, y) \mapsto |x - y|$
4.C: $\mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}_+ : (x, y) \mapsto \sqrt{\Sigma_{i=1}^n (x_i - y_i)^2}$
4.1: $\mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}_+ : (x, y) \mapsto \max_{i=1,\dots,n} |x_i - y_i$
4.2 $\mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}_+ : (x, y) \mapsto \Sigma_{i=1}^n |x_i - y_i|$
We will first show that $4.A$ is an ultrametric, and that all other metrics have a counter example.

Proof. We assume for sake of contradiction that $4.A$ is not an ultra metric, and thus that for some x, y, z $\rho(x, y) > \max\{\rho(x, z), \rho(z, y)\}$. We thus see that as there are only two possible values for ρ, that $\rho(x, y) = 1$, and $max\{\rho(x, z), \rho(z, y)\} = 0$ and thus $\rho(z, y) = 0$ and $\rho(x, z) = 0$. However, we see that as $\rho(z, y) = 0$, and $\rho(x, z) = 0$ that by definition of the metric, that $z = y$, and $x = z$, and thus $x = y$, and by definitiion of ρ, we must have that $\rho(x, y) = 0$. We thus have a contradiction, and we know that $4.A$ is an an ultra metric □

We now see that if $x = 1, y = 0, z = .5$, that $|x - y| = 1$, $|x - z| = 0.5$ and $|z - y| = 0.5$, and thus for the metric in $4.B$, there is some value of x, y and z such that $\rho(x, y) > max\{\rho(x, z), \rho(z, y)\}$.

We also see that given the same values of x, y, and z that $\sqrt{(x-y)^2} = 1$, $\sqrt{(x-z)^2} = 0.5$ $\sqrt{(y-z)^2} = 0.5$, and thus for the metric in $4.C$, there is some value of x, y and z such that $\rho(x, y) > max\{\rho(x, z), \rho(z, y)\}$.

We also see that in the case of $n = 1$, that 4.1 and 4.2 reduce to the cases above, and thus are not ultra metrics.

4.M. *Prove that all triangles in an ultrametric space are isosceles (i.e., for any three points a, b, and c, at least two of the three distances $\rho(a, b)$, $\rho(b, c)$, and $\rho(a, c)$ are equal).*

Proof. Suppose for the sake of contradiction that ρ is an ultrametric and ab, c do not form an isosceles triangle. Then without loss of generality, we assume that

$$\rho(a, b) < \rho(a, c) < \rho(b, c).$$

We would then have that $\rho(b, c) > \rho(a, c)$, and $\rho(b, c) > \rho(a, b)$, and thus $\rho(b, c) > max\{\rho(a, c), \rho(a, b)\}$. We thus have that this contradicts that ρ is an ultrametrics, and thus we have that there is not a strict inequality on $\rho(a, b)$,$\rho(a, c)$, and $\rho(b, c)$, and thus at least two of the three distances are equal. □

4.N. *Prove that spheres in an ultrametric space are not only closed (see Problem 4.23), but also open.*

Proof. Let $a \in X$, $r \in \mathbb{R}_+$ be arbitrary and consider the sphere $S_r(a)$. It suffices to show that for every $x \in S_r(a)$, there is a ball $B_{r'}(x) \subseteq S_r(a)$.

Let $x \in S_r(a)$ be arbitrary, and choose $r' = \frac{r}{2}$. We also let $y \in B_{r'}(x)$be arbitrary. We see that we have

$$\begin{aligned} \rho(a, y) &\leq \max\{\rho(a, x), \rho(x, y\} \\ &= \max\{r, \frac{r}{2}\} \\ &= r, \end{aligned}$$

and we have

$$\begin{aligned} \rho(a, x) &\leq \max\{\rho(a, y), \rho(x, y)\} \\ r &\leq \max\{\rho(a, y), \frac{r}{2}\} \\ r &\leq \rho(a, y). \end{aligned}$$

Hence,

$$r \leq \rho(a, y) \leq r,$$

so $r = \rho(a, y)$. Thus, $y \in S_r(a)$, and as y was arbitrary, we have that $B_{r'}(x) \subseteq S_r(a)$. As x was arbitrary, we see that this is true for every $x \in S_r(a)$, and thus as the union of open balls is an open set, we have as that every element of $S_r(a)$ is contained in an open set that is a subset of $S_r(a)$, that $S_r(a)$ is an open set. □

4.O. *Prove that ρ is an ultrametric.*

Proof. Let $x, y, z \in \mathbb{Q}$ be arbitrary. There exist $r_i, s_i, \alpha_i \in \mathbb{Z}$ $(i = 1, 2, 3)$ such that

$$x - y = \frac{r_1}{s_1}p^{\alpha_1}$$
$$x - z = \frac{r_2}{s_2}p^{\alpha_2}$$
$$y - z = \frac{r_3}{s_3}p^{\alpha_3}.$$

Note that each α_i is unique to the particular difference. Without loss of generality, assume $\alpha_2 \geq \alpha_1$. If $\alpha_1 = \alpha_3$, we are done, because $\rho(x, z) \leq \max\{\rho(x, y), \rho(y, z)\}$, the other inequalities following easily. We have

$$\begin{aligned} y - z &= (y - x) + (x - z) \\ \frac{r_3}{s_3}p^{\alpha_3} &= \frac{r_2}{s_2}p^{\alpha_2} - \frac{r_1}{s_1}p^{\alpha_1} \\ \frac{r_3}{s_3}p^{\alpha_3} &= \left(\frac{r_2}{s_2}p^{\alpha_2-\alpha_1} - \frac{r_1}{s_1}\right)p^{\alpha_1} \\ \frac{r_3}{s_3}p^{\alpha_3} &= \left(\frac{r_2p^{\alpha_2-\alpha_1} - r_1}{s_1s_2}\right)p^{\alpha_1}. \end{aligned}$$

It now suffices to show that

$$\gcd\left(\frac{r_2p^{\alpha_2-\alpha_1} - r_1}{s_1s_2}, p\right) = 1.$$

To this end, notice that the denominator is coprime with p, since both s_1 and s_2 are coprime with p. And notice that the numerator is coprime with p, since r_1 is coprime with p. Hence, we can write

$$\frac{r_3}{s_3}p^{\alpha_3} = \frac{r_4}{s_4}p^{\alpha_1}$$

where $r_4 = r_2p^{\alpha_2-\alpha_1} - r_1$ and $s_4 = s_1s_2$ are coprime with p. This implies that $\alpha_1 = \alpha_3$, and the result follows. □

5 Subspaces

[5′1] Topology for a Subset of a Space

5.A. *The collection Ω_A is a topological structure in A.*

Proof. We first see that as $X \in \Omega$, that $A \cap X \in \Omega_A$ and thus $A \in \Omega_A$. We also have that as $\varnothing \in \Omega$ that $A \cap \varnothing \in \Omega_A$, and thus $\varnothing \in \Omega_A$. As $A, \varnothing \in \Omega_A$, we have that Axiom 3 of a topological space is satisfied.

We will now show that the union of any collection of sets that are elements of Ω_A belong to Ω_A. Let B be an arbitrary union of elements of Ω_A. We also let $V \in \Omega$ be defined to be the arbitrary union of all sets V' such that $A \cap V' \subset B$. As Ω is a topology, we know that it is closed under arbitrary unions, and thus that $V \in \Omega$. We thus have by the definition of Ω_A that $A \cap V \in \Omega_A$. We will now show that $B = A \cap V$, and thus that $B \in \Omega_A$. Let $b \in B$ be arbitrary. We see that by the definition of B, that $b \in A$, and for some $V' \subset A$, $b \in V'$. We thus have by the definition of V, that $b \in V$, and as $b \in A$, we have that $b \in A \cap V$. As b was arbitrary, we have that $B \subset A \cap V$. We now let $a \in A \cap V$ be arbitrary. We see that $a \in A$, and by the definition of V, for some $A \cap V' \subset B$, $a \in A \cap V'$. We thus have that $a \in B$, and as a was arbitrary, and $B \subset A \cap V$, we have that $B = A \cap V$. As $A \cap V$ is in Ω_A, and B was an arbitrary union of elements of Ω_A, we see that Ω_A is closed under arbitrary union.

We will now show that Ω_A is closed under finite intersection. Let B be an arbitrary finite intersection of sets in Ω_A. We also let V be defined such that V is the intersection of every V' such that $A \cap V'$ is an element of the collection of sets whose finite intersection defines B. As Ω is a topological space, we know that $V \in \Omega$, as Ω is closed under finite intersection. We thus have that $A \cap V \in \Omega_A$, we will now show that $B = A \cap V$, and thus that $B \in \Omega_A$. Let $b \in B$ be arbitrary. We see that by the definition of intersection, that $b \in A$, and that for every V' such that $A \cap V'$ the collection of sets whose finite intersection defines B, we have that $a \in V'$. We thus have that by the definition of V, $b \in V$, and thus $b \in A \cap V$. As b was arbitarary, we know that $B \subset A \cap V$. We now let $a \in A \cap V$ be arbitrary. We see that for any $A \cap V'$ which defines B, that $a \in A$, and by definition of V, $a \in V'$. We thus have that by definition of intersection, that $a \in B$, and as a was arbitrary, and $B \subset A \cap V$, we have that $B = A \cap V$. □

5.B. *The canonical topology on $\mathbb{R}^1$ coincides with the topology induced on $\mathbb{R}^1$ as a subspace of $\mathbb{R}^2$.*

Proof. Let Ω be the canonical topology on $\mathbb{R}^1$. Let Σ be the canonical topology on $\mathbb{R}^2$ and let $\sigma \in \Sigma$ be arbitrary. Let R be the set $x|x \in \mathbb{R}^2 and x = (R, 0) for some R \in \mathbb{R}$. Consider the set $S = R \cap \sigma$. Each ball that forms σ intersects R for some open interval, so S will be the union of open intervals, and so will be in the canonical topology on $\mathbb{R}^1$. We can also reverse engineer a σ that generates an arbitrary $\omega \in \Omega$ by choosing an appropriate ball for each open interval, the union of which form ω, so there are no elements $\omega \in \Omega$ that are not induced by considering $\mathbb{R}^1$ as a subspace of $\mathbb{R}^2$, and so the sets Ω and Σ_R are equivalent. □

Primary author: Willie Kaufman

5.1 (Riddle)**.** *How to construct a base for at topology induced on A by using a base for the topology on X.*

If Σ is a base for the topology on X, the set $A \cap \sigma | \sigma \in \Sigma$ forms a base for A. Primary author: Willie Kaufman

5.2. *Describe the topological structures induced (1) on the set $\mathbb{N}$ of positive integers by the topology of the real line: (2) on $\mathbb{N}$ by the topology of the arrow (3) on the two-element set 1, 2 by the topology of $\mathbb{R}_T$ (4) on the same set by the topology of the arrow*

Answer. (1) We see that for any $n \in \mathbb{N}$, that $(n-1/2, n+1/2)$ is in the topology of the real line, and thus that $\{n\}$ is in the induced topology, and thus as n was arbitrary every element in $\mathbb{N}$ would have in a set in the topology which contains only it, and thus this topology would be the discrete topology on $\mathbb{N}$.

(2) This topology would be the collection of all set in $\mathbb{N}$ such of the form $[n, \infty)$ where n is a natural number greater than 0. We can see this, by seeing that for any $n > 0$, $(n - 1/2, \infty)$ is in the topology of the real line, and that if n is in a set, by the definition of the arrow, all numbers greater than it must be in the set.

(3) We see that this would be the discrete topology, as $\varnothing, \{1\}, \{2\}, \{1, 2\}$ could all individually be included by considering the finite complement of 1, and 2, 2, 1, and 0 respectively.

(4) We see that this would be the topology $\varnothing, \{2\}, \{1, 2\}$, as if 1 is in the set, any set in the topology of the arrow would also have 2 in it. □

5.3. *Is the half open interval $[0, 1)$ open in the segment $[0, 2]$, regarded as a subspace of the real line?*

5.C. *A set F is closed in a subspace $A \subset X$ iff F is the intersection of A and a closed subset of X.*

Proof. □

5.4. *If a subset of a subspace is open (respectively, closed) in the ambient space, then it also open (respectively, closed) in the subspace.*

Proof. Let B be an arbitrary set in a subspace (A, Ω_A) of the space (X, Ω). We will show that $B$is an open in (X, Ω), that it is open in (A, Ω_A). We first see that as B is in (A, Ω_A), that $B \subset A$. As B is open, we know that $B \in \Omega$, and as $B \subset A$, we know that $A \cap B = B$. We now see by the definition of of subspace, that $A \cap B \in \Omega_A$, and thus is an open set, and thus B is an open set. □

[5′2] Relativity of Openness and Closedness

5.D. *The unique open set in $\mathbb{R}^1$ which is also open in $\mathbb{R}^2$ is $\varnothing$.*

Proof. In order for a set $X \in \mathbb{R}^2$ to be open, it must be the union of open balls. Any open ball in $\mathbb{R}^2$ will contain some values that do not fit the form $(x, 0)$ for some $x \in \mathbb{R}$, so it is not possible for a union of open balls to be exlusively values in $\mathbb{R}$. $\varnothing$ is trivially open in both $\mathbb{R}$ and $\mathbb{R}^2$, so it is the unique set that is open in both. □

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5.E. *An open set of an open subspace is open in the ambient space, i.e., if $A \in \Omega$, then $\Omega_A \in \Omega$.*

Proof. Let $A \in \Omega$, and $B \in \Omega_A$ be arbitrary. We see by definition of Ω_A, that $B = A \cap V$ for some $V \in \Omega$. As $A \in \Omega$, and topological spaces are closed under finite intersection, we have that $B \in \Omega$. As A,B was arbitrary, we see that this is true for any open subspace. □

5.F. *Closed sets of a closed subspace are closed in the ambient space.*

Proof. Let A be an arbitrary closed subspace, and B be an arbitrary closed set in that subspace. We see that for some $V \in \Omega$, that $B = A \setminus (A \cap V)$ where $V \in \Omega$. As V is open, we see that $X \setminus V$ is closed, and as closed sets are closed under intersection, we see that $A \cap (X \setminus V)$ is also closed. We will now show that $B = A \cap (X \setminus V)$. Let $b \in B$ be arbitrary. We see that $b \in A$, but that $b \notin A \cap V$, and thus $b \notin V$. We now can see that as $A \subset X$, that $b \in X$, and thus that $b \in X$ and as $b \notin V$ $b \in (X \setminus V)$. As $b \in A$, we have that $b \in A \cap (X \setminus V)$. As b was arbitrary, we have that $B \subset A \cap (X \setminus V)$. We now let $a \in A \cap (X \setminus V)$ be arbitrary. By definition of intersection, we have that $a \in A$, and $a \in (X \setminus V)$. By the definition of the difference of sets, we have that $a \notin V$, and thus $a \notin (A \cap V)$. We now can see that $a \in A \setminus (A \cap V) = B$, and thus as a was arbitrary, and $B \subset A \cap (X \setminus V)$, we have that $B \subset A \cap (X \setminus V)$, and thus that B is closed. As B and A were arbitrary, we have that closed sets of a closed subspace are closed in the ambient space. □

5.5. *Prove that a set U is open in X iff each point in U has a neighborhood V in X such that $U \cap B$ is open in V.*

5.6. *Show that the property of being closed is not local.*

5.G. *Let (X, Ω) be a topological space, $X \supset A \supset B$. Then $(\Omega_A)_B = \Omega_B$, i.e., the topology induced on B by the relative topology of A coincides with the topology induced on B directly from X.*

Proof. We will show $(\Omega_A)_B = \Omega_B$ by double containment. We first let $C \in (\Omega_A)_B$ be arbitrary. We see that by the for some $V \in \Omega_A$, that $C = B \cap V$. We also see that as $V \in \Omega_A$, that $V = A \cap V'$, and thus $C = B \cap A \cap V'$. We see as $B \subset A$, that $B \cap A \cap V' = B \cap V'$, and thus $C = B \cap V'$. As $V' \in \Omega$, we know that $C \in \Omega_B$ by definition. As C was arbitrary, we have that $(\Omega_A)_B \subset \Omega_B$.

We now let $D \in \Omega_B$ be arbitrary. We see that for some $V \in \Omega$, $D = B \cap V$. As $B \subset A$, we have that $B \cap V = A \cap B \cap V = A \cap (B \cap V)$. We now can see by definition of Ω_A that $(B \cap V) \in \Omega_A$. We now see by the definition of $(\Omega_A)_B$ that $D \in (\Omega_A)_B$. As D was arbitrary, and $(\Omega_A)_B \subset \Omega_B$, we have $(\Omega_A)_B = \Omega_B$, and thus the topology induced on B by the relative topology of A coincides with the topology induced on B directly from X.

□

5.7. *Let (X, ρ) be a metric space, $A \subset X$. Then the topology on A generated by the induced metric $\rho_{|AxA}$ coincides with the relative topology induced on A by the metric topology on X.*

Proof. The topology on A generated by the induced metric $\rho_{|AxA}$ is equivalent to the portion of each open ball contained in A. Let Ω equal the set of all open balls in X. The relative topology induced on A by the metric topology on X is equivalent to $\Omega_A = \omega \cap A | \omega \in \Omega$. Given a portion of a ball centered at b with radius r in the topology generated by the induced metric, that same ball - as defined by its center and radius - in Ω intersected with A will be the same portion of ball. Similarly, given a ball in Ω intersected with A with radius b_2 and radius r_2, that same ball - as defined by its center and radius - in the topology generated by the induced metric will be equivalent. Since any element in either topology has a mirror in the other, the topologies are equivalent.

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5.8 (Riddle)**.** *The statement in 5.7 is equivalent to a pair of inclusions. Which of these is less obvious?*

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