Chapter I

Structure and Spaces

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1 Set-Theoretic Digression: Sets

[1´1] Sets and Elements

[1´2] Equality of Sets

[1´3] The Empty Set

[1´4] Basic Set of Numbers

[1´5] Describing a Set By Listing Its Elements

1.1. *What is $\{\varnothing\}$? How many elements does it contain?*

Answer. This is a set made up of the empty set, it contains one element. □

1.2. *Which of the following formulas are correct:*

1. $\varnothing \in \{\varnothing, \{\varnothing\}\}$ *is correct*
2. $\{\varnothing\} \in \{\{\varnothing\}\}$ *is correct*
3. $\varnothing \in \{\{\varnothing\}\}$ *is incorrect*

Primary author: Willie Kaufman

1.3. *Yes, $\{\{\varnothing\}\}$ contains just one element.*
Primary author: Willie Kaufman

1.4. *How many elements do the following sets contains?*

1. $\{1, 2, 1\}$; 2 elements

2. $\{1, 2, \{1, 2\}\}$; 3 elements
3. $\{\{2\}\}$; 1 element
4. $\{\{1\}, 1\}$; 2 elements
5. $\{1, \varnothing\}$; 2 elements
6. $\{\{\varnothing\}, \varnothing\}$; 2 elements
7. $\{\{\varnothing\}, \{\}\}$; 1 element
8. $\{x, 3x - 1\}$for some $x \in \mathbb{R}$; 1 element if $x = \frac{1}{2}$, and 2 elements for all for all values of x

Primary author: Reilly Noonan Grant

1.5. *(a) :* $\{0, 1, 2, 3, 4\}$
(b) : $\varnothing$
(c) : $\{-1, -2, -3, \cdots\}$

[1′6] Subsets

1.A. *Let a set A have a elements, and let a set B have b elements. Prove that if $A \subset B$, then $a \leq b$.*

Proof. Suppose A is a set made up of a elements, and B is a set made up of b elements. Suppose $a > b$; then there must be some element $x \in A$ that satisfies $x \notin B$. This implies that $A \not\subset B$, as x is not a member of B. By the contrapositive, we obtain the desired result. □

[1′7] Properties of Inclusion

1.B (Reflexivity of Inclusion)**.** *Any set includes itself: $A \subset A$ holds true for any A.*

Proof. We first let A be an arbitrary Set. Suppose that A is the empty set. We would then see that as A doesn't have any elements, that it is vacuously true that every element of A is in A, and thus $A \subset A$. Now suppose that A is non empty. Let $a \in A$ be arbitrary. We see that $a \in A$, and as a was arbitrary, we know that this is true for all elements of A, and thus $A \subset A$. As A was arbitrary, and $A \subset A$ for all cases, we see that $A \subset A$ holds true for any A. □

Primary author: Reilly Noonan Grant

1.C. ***The Empty Set Is Everywhere.*** *The inclusion $\varnothing \subset A$ holds true for any A. In other words, the empty set is present in each set as a subset.*

Proof. Assume that $\varnothing \subset A$ does not hold, by defintion of inclusion, there exist at least one element $a \in \varnothing$ such that $a \notin A$. Since $\varnothing$ does not contain any element, we have a contradiction, and the statement above is true. □

Primary author: Jimin Tan

1.D. *If A , B, and C are sets, $A \subset B$, and $B \subset C$, then $A \subset C$.*

Proof. Let A, B, and C be arbitrary sets such that $A \subset B$ and $B \subset C$. If A is the empty set, it is true that $A \subset C$. If A is nonempty, choose an arbitrary element $a \in A$. Because $A \subset B$, we know that $a \in B$, and similarly because $B \subset C$, $a \in C$. Since a was arbitrary, $A \subset C$.

□

Primary author: Willie Kaufman

[1′8] To Prove Equality of Sets, Prove Two Inclusions

1.E (Criterion of Equality for Sets.)**.** *$A = B$ if and only if $A \subset B$ and $B \subset A$.*

Proof. Let A and B be arbitrary sets. Let us first suppose that $A = B$. We show that $A \subset B$ and $B \subset A$. Let $x \in A$ be arbitrary. Since $A = B$, they have the same elements; since x is an element of A, x is an element of B. Hence, $A \subset B$. A similar argument establishes that $B \subset A$.

We now establish the converse of the previous claim via contraposition. Suppose that it is not the case that $A \subset B$ and $B \subset A$; without loss of generality, we shall assume that $A \not\subset B$. Then there necessarily exists an element $x \in A$ satisfying $x \notin B$. As such, it is not the case that A and B contain the same elements, since B does not contain x. Hence, $A \neq B$. By the contrapositive, this establishes the converse. □

[1′9] Inclusion Versus Belonging

1.F. *$x \in A$ if and only if $\{x\} \subset A$.*

Proof. We will first show that if $\{x\} \subset A$, then $x \in A$. We can see that as $\{x\}$ is a set described by listing all of its elements, that $x \in \{x\}$. We also see that as $\{x\} \subset A$, that all of the elements of $\{x\}$ are also elements of A, and thus $x \in A$. We thus know that if $\{x\} \subset A$, then $x \in A$.

We will now show that if $x \in A$, then $\{x\} \subset A$. We can see that as $\{x\}$ is a set described by listing all of its elements, that x is the only element in $\{x\}$. We also see that as $x \in A$, that all elements of $\{x\}$ belong to A, and thus $\{x\} \subset A$. We now can see that $x \in A$ if and only if $\{x\} \subset A$. □

Primary author: Reilly Noonan Grant

1.G. ***Non-Reflexivity of Belonging.*** *Construct a set A such that $A \notin A$. The example, $\{1\} \notin \{1\}$ shows the statement above. The set that contains $\{1\}$ is $\{\{1\}\}$. Primary author: Jimin Tan*

1.H. ***Non-Transitivity of Belonging.*** *Construct three sets A, B, and C such that $A \in B$ and $B \in C$, but $A \notin C$.*
$A = \{1\}$

$B = \{\{1\}, 2\}$
$C = \{\{\{1\}, 2\}, 3\}$

Primary author: Willie Kaufman

[1′10] Defining a Set by a Condition (Set-Builder Notation)

[1′11] Intersection and Union

1.I (Commutativity of $\cap$ and $\cup$)**.** *For any two sets A and B, we have*

$$A \cap B = B \cap A \quad \textit{and} \quad A \cup B = B \cup A.$$

Proof. For our proof we rely on the commutativity of logical operators, which can be verified via truth tables. Namely, we have

$$\alpha \text{ and } \beta = \beta \text{ and } \alpha \quad \text{and} \quad \alpha \text{ or } \beta = \beta \text{ or } \alpha,$$

where α and β are arbitrary statements. We will show that the statements about intersections and unions reduce to statements with "and" and "or" operators, respectively.

Let A and B be arbitrary sets. Then $x \in A \cap B$ if and only if $x \in A$ and $x \in B$, which holds if and only if $x \in B$ and $x \in A$, which holds if and only if $x \in B \cap A$. This establishes via double-containment that $A \cap B = B \cap A$.

Similarly, $x \in A \cup B$ if and only if $x \in A$ or $x \in B$, which holds if and only if $x \in B$ or $x \in A$, which holds if and only if $x \in B \cup x \in A$. This establishes via double-containment that $A \cup B = B \cup A$. □

Primary author: David Kraemer.

1.6. *Prove that for any set A we have $A \cap A = A$, $A \cup A = A$, $A \cup \varnothing = A$, and $A \cap \varnothing = \varnothing$.*

Proof. Let A be an arbitrary set. If A is the empty set, each of these is true. So we consider when A is not the emptyset. Choose an arbitrary element $a \in A$. This element is in $A \cap A$ by the definition of intersection, as it belongs to both A and A. This element is also in $A \cup A$ by the definition of union, as it belongs to at least one of A and A. This element is in $A \cup \varnothing$, as it belongs to at least one of A and $\varnothing$. This element is not in $A \cap \varnothing$, as it does not belong to both A and $\varnothing$. Since a was arbitrary, each element of A will be in $A \cup A$, $A \cap A$, and $A \cup \varnothing$, and no elements of A will be in $A \cap \varnothing$. No elements that do not belong to A could be in any of these sets, so combining these two facts requires that $A \cup A$, $A \cap A$, and $A \cup \varnothing$ each equal A and $A \cup \varnothing = \varnothing$. □

Primary author: Willie Kaufman

1.7. *Prove that for any sets A and B we have*

$$A \subset B, \quad \text{iff} \quad A \cap B = A, \quad \text{iff} \quad A \cup B = B$$

Proof. We will break this chain if and only if statement into two parts and then prove them separately.

To begin with, we want to show that $A \subset B$, if and only if $A \cap B = A$. For if and only if statement, we need to prove it in both directions. For the forward direction, assume that $A \subset B$ and let $x \in A$, since $A \subset B$, we know that $x \in B$ by definition of inclusion. We have $x \in A$ and $x \in B$, so $x \in A \cap B$. Since $A \cap B \subset A$ by definition of intersection, we have

$$A \cap B = A$$

Then, we consider the backward direction, assume that $A = A \cap B$ and let $x \in A$, since $A \subset A \cap B$, we have $x \in A$ and $x \in B$. Hence, we have

$$A \subset B$$

Now we want to prove the second if and only if statement which is $A \cap B = A$ iff $A \cup B = B$.

We start with the forward. Assume that $A \cap B = A$ and let $x \in A \cup B$, by definition, we know $x \in A$ or $x \in B$. If $x \in A$, since $A \subset A \cap B$, $x \in B$. Since $B \subset A \cup B$, we have:

$$A \cup B = B$$

Backward direction: Let $x \in A$, since $A \cup B \subset B$, $x \in B$. We have whenever $x \in A$, $x \in B$, so $x \in A \cap B$ and $A \subset A \cap B$. Since $A \cap B \subset A$, we have

$$A \cap B = A$$

□

Primary author: Jimin Tan

1.J. *Associativity of $\cap$ and $\cup$. For any sets A, B, and C, we have*

$$(A \cap B) \cap C = A \cap (B \cap C)$$

and

$$(A \cup B) \cup C = A \cup (B \cup C)$$

Proof. Let A, B, and C be arbitrary sets. First, consider the associativity of $\cap$. In the case of intersection, if any of A, B, or C are $\varnothing$, the top claim is true, as both evaluate to the emptyset. So consider the case where none of A, B, or C are the emptyset. Choose an arbitrary element $a \in A \cup B \cup C$. Only elements in $A \cup B \cup C$ will be in the intersection of these sets, so we can ignore other elements; if the two sets we are considering contain exactly the same elements from these sets, they will be the same. $a \in (A \cap B) \cap C$ iff $a \in A \cap B$ and $a \in C$ by the definition of intersection. $a \in A \cap B$ iff $a \in A$ and $a \in B$ by the definition

of intersection. Combining these logically, $a \in (A \cap B) \cap C$ iff it is an element of A, B, and C. Now considering the right side of the equation, $a \in A \cap (B \cap C)$ iff it is in both A and $B \cap C$ by the definition of intersection. a is in $B \cap C$ iff it is in B and C by the definition of intersection. Combining these logically, we have that $a \in A \cap (B \cap C)$ iff $a \in A$, $a \in B$ and $a \in C$. We know that $a \in (A \cap B) \cap C$ iff $a \in A$ and $a \in B$ and $a \in C$ iff $a \in A \cap (B \cap C)$, or $a \in (A \cap B) \cap C$ iff $a \in A \cap (B \cap C)$. Since a was arbitrary, these sets must be the same.
Second, consider the associativity of $\cup$. If all of A, B, and C are $\varnothing$, both the left and right hand sides of the equation evaluate to $\varnothing$, and so the bottom claim is true. So consider the case where $A \cup B \cup C$ is nonempty. Choose an arbitrary element $a \in A \cup B \cup C$. Only elements in $A \cup B \cup C$ will be in the union of these sets, so we can ignore other elements; if the two sets we are considering contain exactly the same elements from these sets, they will be the same. $a \in (A \cup B) \cup C$ iff $a \in (A \cup B)$ or $a \in C$ by the definition of union. $a \in (A \cup B)$ iff $a \in A$ or $a \in B$ by the definition of union. Combining these logically, $a \in (A \cup B) \cup C$ iff $a \in A$, $a \in B$ or $a \in C$. Now consider the right hand side of the equation. $a \in A \cup (B \cup C)$ iff $a \in A$ or $a \in B \cup C$ by the definition of union. $a \in B \cup C$ iff $a \in B$ or $a \in C$ by the definition of union. Combining these logically, $a \in A \cup (B \cup C)$ iff $a \in A$, $a \in B$, or $a \in C$. We know that $a \in (A \cup B) \cup C$ iff $a \in A$, $a \in B$, or $a \in C$ iff $a \in (A \cup B) \cup C$. Since a was arbitrary, these sets must be the same. ☐

Primary author: Willie Kaufman

1.K. *The notions of intersection and union of an arbitrary collection of sets generalize the notions of intersection and union of two sets: for* $\Gamma = \{A, B\}$*, we have*

$$\bigcap_{C \in \Gamma} C = A \cap B \text{ and } \bigcup_{C \in \Gamma} C = A \cup B$$

Proof. We will discuss the intersection of multiple sets first.

Let $x \in \bigcap_{C \in \Gamma}$ and assume we can list all element like $C_0, C_1 \cdots, C_n \cdots$, we have $x \in C_0 \cap C_1 \cdots \cap C_n$ by definition. Since there are only two sets in Γ, which are A, B, $x \in A \cap B$. We have $\bigcap_{C \in \Gamma} \subset A \cap B$. Let $y \in A \cap B$. Since A and B are the only two elements in Γ, we have $y \in \bigcap_{C \in \Gamma}$, and $A \cap B \subset \bigcap_{C \in \Gamma}$. Then we have

$$A \cap B = \bigcap_{C \in \Gamma}$$

Union:

Let $x \in \bigcup_{C \in \Gamma}$ and assume we can list all element like $C_0, C_1 \cdots, C_n \cdots$, we have $x \in C_0 \cup C_1 \cdots \cup C_n$. Since A and B are the only two sets in Γ, we have $x \in A \cup B$ and $\bigcup_{C \in \Gamma} \subset A \cup B$. Let $y \in A \cup B$, since A and B are the only two sets in Γ, we have $y \in \bigcup_{C \in \Gamma}$, and $A \cup B \subset \bigcup_{C \in \Gamma}$. Then we have

$$A \cup B = \bigcup_{C \in \Gamma}$$

☐

Primary author: Jimin Tan

1.8 (Riddle)**.** *How are the notions of system of equations and intersection of sets related to each other?*

Answer. If $E_1, E_2, \dots, E_n$ are a system of equations and $S_1, S_2, \dots, S_n$ are the solution sets corresponding to each equation, then the set

$$S = \bigcap_{i=1}^{n} S_i$$

is the solution to the system of equations, as any solution $s \in S$ solves each equation E_i simultaneously. □

Primary author: David Kraemer

1.L (Two Distributitivites)**.** *For any sets A, B, and C, we have*

$$(A \cap B) \cup C = (A \cup C) \cap (B \cup C)$$
$$(A \cup B) \cap C = (A \cap C) \cup (B \cap C)$$

Proof. These properties follow from unpacking the definitions of set union and intersection, as well as from recalling the distributive properties of logical operators:

$$(\alpha \text{ and } \beta) \text{ or } \gamma \Longleftrightarrow (\alpha \text{ or } \gamma) \text{ and } (\beta \text{ or } \gamma)$$
$$(\alpha \text{ or } \beta) \text{ and } \gamma \Longleftrightarrow (\alpha \text{ and } \gamma) \text{ or } (\beta \text{ and } \gamma).$$

We first show $(A \cap B) \cup C = (A \cup C) \cap (B \cup C)$. We have that

$$x \in (A \cap B) \cup C$$

if and only if either

$$x \in A \cap B \text{ or } x \in C,$$

which holds if and only if

$$(x \in A \text{ and } x \in B) \text{ or } x \in C,$$

which holds if and only if

$$(x \in A \text{ and } x \in C) \text{ or } (x \in B \text{ and } x \in C),$$

which holds if and only if

$$(x \in A \cap C) \text{ or } (x \in B \cap C),$$

which holds if and only if

$$x \in (A \cap C) \cup (B \cap C).$$

We now show $(A \cup B) \cap C = (A \cap C) \cup (B \cap C)$. We have that

$$x \in (A \cup B) \cap C$$

if and only if either

$$x \in A \cup B \text{ and } x \in C,$$

which holds if and only if

$$(x \in A \text{ or } x \in B) \text{ and } x \in C,$$

which holds if and only if

$$(x \in A \text{ or } x \in C) \text{ and } (x \in B \text{ or } x \in C),$$

which holds if and only if

$$(x \in A \cup C) \text{ and } (x \in B \cup C),$$

which holds if and only if

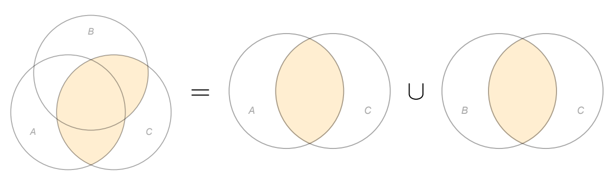
$$x \in (A \cup C) \cap (B \cup C).$$

These equivalencies establish the claim. □

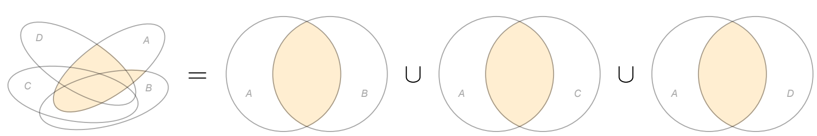
Primary author: David Kraemer

1.M. *Draw a Venn diagram illustrating (2). Prove (1) and (2) by tracing all details of the proofs in the Venn diagrams. Draw Venn diagrams illustrating all formulas below in this section.*

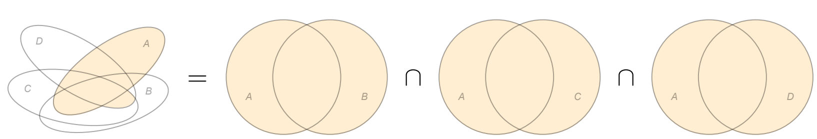
Answer. Demonstration of $(A \cup B) \cap C = (A \cap C) \cup (B \cap C)$:



Demonstration of $A \cap \bigcup_{B \in \Gamma} B = \bigcup_{B \in \Gamma}(A \cap B)$, with $|\Gamma| = 3$.



Demonstration of $A \cup \bigcap_{B \in \Gamma} B = \bigcap_{B \in \Gamma}(A \cup B)$, with $|\Gamma| = 3$.



□

Primary author: David Kraemer.

1.9 (Riddle)**.** *Generalize Theorem 1.L to the case of arbitrary collections of sets.*

Proof. (See 1.N) Let A be a set and let Γ be a set consisting of sets, then we have

$$A \cap \bigcup_{B \in \Gamma} B = \bigcup_{B \in \Gamma} (A \cap B) \text{ and } A \cup \bigcap_{B \in \Gamma} B = \bigcap_{B \in \Gamma} (A \cup B)$$

□

Primary author: Reilly Noonan Grant

1.N (Yet Another Pair of Distributivities)**.** *Let A be a set and let Γ be a set consisting of sets, then we have*

$$A \cap \bigcup_{B \in \Gamma} B = \bigcup_{B \in \Gamma} (A \cap B) \textit{ and } A \cup \bigcap_{B \in \Gamma} B = \bigcap_{B \in \Gamma} (A \cup B)$$

Proof. We will first show that $A \cap \bigcup_{B \in \Gamma} B = \bigcup_{B \in \Gamma} (A \cap B)$ using double containment.

First let $x \in A \cap \bigcup_{B \in \Gamma} B$ be arbitrary. We see that

$$x \in A \text{ and } x \in B$$

for some $B \in \Gamma$. We thus know that for some B,

$$x \in (A \cap B)$$

We now can see that as

$$\bigcup_{B \in \Gamma} (A \cap B)$$

contains every $(A \cap B)$, we know that

$$x \in \bigcup_{B \in \Gamma} (A \cap B)$$

As x was arbitrary, we know that

$$A \cap \bigcup_{B \in \Gamma} B \subseteq \bigcup_{B \in \Gamma} (A \cap B).$$

We now let $x \in \bigcup (A \cap B)$ be arbitrary. We see that

$$x \in A \text{ and } x \in B$$

for some $B \in \Gamma$. We also see that as $B \in \Gamma$, that

$$B \subset \bigcup_{B \in \Gamma} B$$

and thus $x \in A$ and $x \in \bigcup_{B\in\Gamma}$. We now can see that this only holds if

$$x \in A \cap \bigcup_{B\in\Gamma}$$

and thus as x was arbitrary

$$A \cap \bigcup_{B\in\Gamma} B \supseteq \bigcup_{B\in\Gamma} (A \cap B).$$

We now can see by double containment, that $A \cap \bigcup_{B\in\Gamma} B = \bigcup_{B\in\Gamma}(A \cap B)$.

We will now show that

$$A \cup \bigcap_{B\in\Gamma} B = \bigcap_{B\in\Gamma} (A \cup B)$$

by double containment.

We first let $x \in A \cup \bigcap_{B\in\Gamma} B$ be arbitrary. We see that

$$x \in A \text{ or } x \in \bigcap_{B\in\Gamma} B$$

We now suppose that $x \in A$. We would then see that $x \in A \cup B$ for any B, and thus

$$x \in \bigcap_{B\in\Gamma} (A \cap B)$$

Now suppose that $x \notin A$. We would then see that

$$x \in \bigcap_{B\in\Gamma} B$$

and if x is in some B, we know that it would also be in $A \cup B$, so thus, we would know that

$$x \in \bigcap_{B\in\Gamma} (A \cup B)$$

As x was arbitrary, we know that

$$A \cup \bigcap_{B\in\Gamma} B \subseteq \bigcap_{B\in\Gamma} (A \cup B)$$

We now let $x \in \bigcap_{B\in\Gamma}(A \cup B)$ be arbitrary. We can see that $x \in A$ or $x \in B$ for every $B \in \Gamma$. Suppose $x \in A$. It would then be the case that $x \in A \cup \bigcap_{B\in\Gamma} B$ as $x \in A$. Now, suppose that $x \notin A$ we would then have that $x \in \bigcap_{B\in\Gamma} B$ as $x \in \bigcap_{B\in\Gamma}(A \cap B)$, and $x \notin A$. We thus see that as x was arbitrary, that

$$A \cup \bigcap_{B\in\Gamma} B \supseteq \bigcap_{B\in\Gamma} (A \cup B).$$

and thus

$$A \cup \bigcap_{B \in \Gamma} B = \bigcap_{B \in \Gamma} (A \cup B)$$

We can now see that

$$A \cap \bigcup_{B \in \Gamma} B = \bigcup_{B \in \Gamma} (A \cap B) \text{ and } A \cup \bigcap_{B \in \Gamma} B = \bigcap_{B \in \Gamma} (A \cup B)$$

□

Primary author: Reilly Noonan Grant

[1′12] Different Differences

1.10. *Prove that for any two sets A and B their union $A \cup B$ is the union of the following three sets: $A \setminus B$, $B \setminus A$, and $A \cap B$, which are pairwise disjoint.*

Proof. Let A, B and C be arbitrary sets. For an arbitrary value a, $a \in A \cup B$ iff $a \in A$ or $a \in B$. This is equivalent to $a \in A \cup B$ iff one of the three following conditions are met; $a \in A$ and $a \notin B$, $a \notin A$ and $a \in B$, or $a \in A$ and $a \in B$. $a \in A \setminus B$ iff $a \in A$ and $a \notin B$, $a \in B \setminus A$ iff $a \notin A$ and $a \in B$, and $a \in A \cap B$ iff $a \in A$ and $a \in B$. So a is in the union of these sets iff it meets one of those criteria by the definition of union. These criteria are the same as those enumerated about for $A \cup B$. Since a was arbitrary, these sets must be the same. □

Primary author: Willie Kaufman

1.11. *Prove that $A \setminus (A \setminus B) = A \cap B$ for any sets A and B.*

Proof. Let $x \in A \setminus (A \setminus B)$, by definition of set difference, we have $x \in A$ and $x \notin A \setminus B$. By basic set operation, $x \notin A \setminus B$ is the same as $x \in (A \setminus B)^c$ which is equal to $B \cup A^c$. By distribution rule, $(A \cap (A^c \cup B)) = (A \cap A^c) \cup (A \cap B) = A \cap B$, so $x \in A \cap B$ and we have $A \setminus (A \setminus B) \subset A \cap B$. Since this process is reversible, we have $A \cap B \subset A \setminus (A \setminus B)$, and we have:

$$A \setminus (A \setminus B) = A \cap B$$

□

Primary author: Jimin Tan

1.12. *Prove that $A \subset B$ if and only if $A \setminus B = \varnothing$.*

Proof. We have $A \subset B$ if and only if there does not exist an $x \in A$ with $x \notin B$; which holds if and only if it is not the case that $A \setminus B \neq \varnothing$, which holds if and only if $A \setminus B = \varnothing$. □

1.13. *Prove that $A \cap (B \setminus C) = A \cap B \setminus A \cap C$ for any sets A,B, and C.*

Proof. We will show this by using double containment. Let $x \in (A \cap (B \setminus C)$ be arbitrary. We see that x is in A, and in B, but not in C and thus $x \in (A \cap B)$, and as x is not in C, that $x \notin (A \cap C)$. We thus see that $x \in (A \cap B) \setminus (A \cap C)$. We thus see that as x is arbitrary, we know that $A \cap (B \setminus C) \subseteq A \cap B \setminus A \cap C$. We now let $x \in (A \cap B) \setminus (A \cap C)$. We see because of this, that x is in A and B, but that x is not in A and C.We thus know that $x \in A$, and because x is in A, and x is in B, but x is not in A and C, that x is not in B and C, and thus $x \in (B \setminus C)$. We thus see that $x \in A \cap (B \setminus C)$

□

Primary author: Reilly Noonan Grant

1.14. *Prove that for any sets A and B we have*

$$A \vartriangle B = (A \cup B) \setminus (A \cap B)$$

Proof. Let A and B be arbitrary sets. $A \vartriangle B$ denotes the set of values a for which it is true either that $a \in A$ and $a \notin B$ or $a \notin A$ and $a \in B$. $(A \cup B) \setminus (A \cap B)$ denotes the set of values b for which $b \in A \cup B$ and $b \notin A \cap B)$. We then know $(A \cup B) \setminus (A \cap B)$ denotes the set of values b for which $b \in A$ or $b \in B$ and $b \notin A \cap B$. This is the same as the set of values b for which b belongs to exactly one of A or B, i.e. the set of values for which it is true either that $b \in A$ and $b \notin B$ or $b \notin A$ and $b \in B$. We then have the exact same characterizations of $A \vartriangle b$ and $(A \cup B) \setminus (A \cap B)$, so the sets are the same. □

1.15 (Associativity of Symmetric Difference.)**.** *Prove that for any sets A, B and C we have*

$$(A \vartriangle B) \vartriangle C = A \vartriangle (B \vartriangle C)$$

Proof. To prove this equation we need to reinterpret the formula.

LHS =

$$\begin{aligned}
&(A \vartriangle B) \vartriangle C \\
=&((A \cup B) \setminus (A \cap B)) \cup C \setminus ((A \cup B) \setminus ((A \cap B)) \cap C) \\
=&A \cup B \cup C \setminus (A \cap B) \setminus (((A \cup B) \cap C) \setminus A \cap B \cap C) \\
=&A \cup B \cup C \setminus (A \cap B) \setminus (((A \cap C) \cup (B \cap C) \setminus A \cap B \cap C) \\
=&A \cup B \cup C \setminus ((A \cap B) \cup (A \cap C) \cup (B \cap C) \setminus A \cap B \cap C))
\end{aligned}$$

RHS:

$$\begin{aligned}
=&(A \vartriangle (B \vartriangle C) \\
=&A \vartriangle (B \cup C \setminus B \cap C) \\
=&A \cup (B \cup C \setminus B \cap C) \setminus A \cap (B \cup C \setminus B \cap C) \\
=&(A \cup B \cup C \setminus B \cap C) \setminus (A \cap (B \cup C)) \setminus (A \cap B \cap C) \\
=&A \cup B \cup C \setminus (((B \cap C) \cup (A \cap B) \cup (A \cap C)) \setminus (A \cap B \cap C)) = LHS
\end{aligned}$$

We have:

$$(A \vartriangle B) \vartriangle C = A \vartriangle (B \vartriangle C)$$

□

Primary author: Jimin Tan

1.16. *Riddle. Find a symmetric definition of the symmetric difference* $(A \vartriangle B) \vartriangle C$ *of three sets and generalize it to arbitrary finite collections of sets.*

Proof. We will start with the definition of symetric difference for three sets. By the definition of symmetric difference of two sets, we can see that the symmetric difference between two sets is the result of removing their intersection from their union. We can then see that by iteratively applying this definition to a set C, that we have that all the elements of A, B, and C that don't belong another set are are included, that all the elements which belong to 2 sets are excluded, and all the elements which belong to both A, B, and C are included. From this, we can see that for 3 sets, the symmetric difference is composed of all elements which belong to an odd number of sets. By continuing to apply the symmetric diffence opperator, we would see that this remains to be the pattern, and thus the general pattern for the definition of symmetric difference will be the collection of elements of all sets which belong to an odd number of sets. □

Primary author: Reilly Noonan Grant

1.17 (Distributivity)**.** *Prove that* $(A \vartriangle B) \cap C = (A \cap C) \vartriangle (B \cap C)$ *for any sets A,B ,and C*

Proof. Let A, B, and C be arbitrary sets. We will show $(A \vartriangle B) \cap C = (A \cap C) \vartriangle (B \cap C)$ by double containment.

We first let $x \in (A \vartriangle B) \cap C$ be arbitrary. We can see that $x \in C$ and $x \in (A \vartriangle B)$ by the properties of intersection. Because $x \in (A \vartriangle B)$, we know that $x \in A$ but $x \notin B$ or $x \in A$ but $x \notin B$. We thus know that $x \in C$ and $x \in A$, but $x \notin B$ or $x \in C$ and $x \in B$, but $x \notin A$. We can see that this is true if and only if $x \in (C \cap A) \setminus B \cup (C \cap B) \setminus A$, and thus $x \in (A \cap C) \vartriangle (B \cap C)$. As x was arbitrary, we can see that $(A \vartriangle B) \cap C \subseteq (A \cap C) \vartriangle (B \cap C)$.

We now let $x \in (A \cap C) \vartriangle (B \cap C)$ be arbitrary. We can see that $x \in (A \cap C)$, but $x \notin (B \cap C)$ or $x \in (B \cap C)$ but $x \notin (A \cap C)$. We thus can see that equivalently, $x \in A$ and $x \in C$ but $x \notin (B \cap C)$ or $x \in B$ and $x \in C$ but $x \notin (A \cap C)$. We can see that in every case that $x \in C$, and thus $x \in C$ and $x \in A$ but $x \notin B$, or $x \in B$ but $x \notin A$. We can see that this is equivilent to $x \in C \cap ((A \setminus B) \cup (B \setminus A))$, and by the definition of symmetric difference, we can see that $x \in (A \vartriangle B) \cap C$. As x was arbitrary, we can now see that $(A \vartriangle B) \cap C \supseteq (A \cap C) \vartriangle (B \cap C)$, and thus $(A \vartriangle B) \cap C = (A \cap C) \vartriangle (B \cap C)$ □

Primary author: Reilly Noonan Grant

1.18. *Does the following inequality hold true for any sets A, B, and C?*

$$(A \vartriangle B) \cup C = (A \cup C) \vartriangle (B \cup C)$$

Proof. For any sets A, B, C where $C \subset A \cap B$, $(A \triangle B) \cup C = (A \cup C) \triangle (B \cup C)$. □

Primary author: Willie Kaufman

2 Topology on a Set

[2′1] Definition of Topological Space

[2′2] Simplest Examples

2.A. *Check that the discrete topological space is a topological space, i.e., all axioms of topological structure hold true.*

Proof. Let the discrete topological space be given by (X, Ω) We will first show that Axiom (1) holds true. Let A be the union of an arbitrary collection of sets in Ω. If $A = \varnothing$, we know by 1.C, and the definition of the discrete topological space that A belongs to Ω. Now suppose that A is non empty. Let $a \in A$ be arbitrary. We see that because $a \in A$, that by properties of a union, that a is in at least one set in Ω, and every set in Ω is a subset of X, we know that $a \in X$. As a was arbitrary, we see that $A \subset X$, and thus A belongs to Ω. As A was arbitrary, we know that Axiom (1) holds.

We will now show that Axiom (2) holds true. Let A be an arbitrary intersection of a finite collection of sets that are elements of Ω. If $A = \varnothing$, we know by 1.C that $A \subset X$ and the definition of the discrete topological space that A belongs to Ω. Now suppose that A is non empty. Let $a \in A$ be arbitrary. We see by the definition of intersection, and the definition of the discrete topological space that a is an element of a subset of X, and thus $a \in X$. As $a \in X$, and a was arbitrary, we see that $A \subset X$, and thus A belongs to Ω. As A was arbitrary, we know that Axiom (2) holds.

By 1.B and 1.C we see that $\varnothing$ and X are subsets of X, and thus by the definition of the discrete topological space, we have that $\varnothing$ and X belong to Ω, and thus Axiom (3) holds true.

As all Axioms of topological structure hold true, we know that a discrete topological space is a topological space. □

Primary author: Reilly Noonan Grant

2.B. *The indiscrete topological space is a topological structure, is it not?*

Proof. We show that the indiscrete topology $\Omega_I = \{X, \varnothing\}$ is indeed a topological structure. We see immediately that $X \in \Omega_I$ and $\varnothing \in \Omega_I$, so Axiom 3 is satisfied.

To see that Ω_I is closed under arbitrary unions, let $\Omega_I' \subseteq \Omega_I$ be arbitrary. Then since $A \cup A = A$ and since

$$\bigcup_{A \in \Omega_I'} A$$

□

2.1. *Let X be the ray $[0, +\infty)$, and let Ω consist of $\varnothing$, X, and all rays $(a, +\infty)$ with $a \geq 0$. Prove that Ω is a topological space.*

2.2. *Let X be a plane. Let Σ consist of $\varnothing$, X, and all open disks centered at the origin. Is Σ a topological structure?*

Answer—NEEDS FLESHING OUT. Yes. We see immediately that Axiom 3 is satisfied, since $\varnothing, X \in \Sigma$. In general, we will represent D_r to indicate the open disk of radius $r > 0$ centered at the origin.

To see that Axiom 1 holds, let $\Sigma' \subseteq \Sigma$ be arbitrary. To proceed, we observe that $D_r \subseteq D_{r'}$ whenever $r \leq r'$ (and equality holding exactly when $r = r'$). Let $f : \Sigma' \to \mathbb{R}$ be defined by $f(D_r) = r$, and consider $f(\Sigma')$. Either $f(\Sigma')$ is bounded above, or it is unbounded. If it is bounded above, we have

$$s = \sup f(\Sigma').$$

In this case,

$$\bigcup_{D_r \in \Sigma'} D_r = D_s \in \Sigma.$$

Otherwise,

$$\bigcup_{D_r \in \Sigma'} D_r = X \in \Sigma.$$

In both cases, we see that the unions are elements of Σ.

To see that Axiom 2 holds, let $\Sigma' \subseteq \Sigma$ be an arbitrary finite subset of Σ, and consider

$$\bigcap_{D_r \in \Sigma'} D_r.$$

Now, since $D_r \subseteq D_{r'}$ whenever $r \leq r'$, we have that $D_r \cap D_{r'} = D_r$ whenever $r \leq r'$. In this case, $f(\Sigma')$ is finite, so by taking

$$m = \min(f(\Sigma')),$$

we have that

$$\bigcap_{D_r \in \Sigma'} D_r = D_m.$$

$\square$

2.3. *Let X consist of four elements: $X = \{a, b, c, d\}$. Which of the following collections of its subsets are topological structures in X, i.e., satisfy the axioms of topological structure:*

1. $\varnothing, X, \{a\}, \{b\}, \{a, c\}\{a, b, c\}, \{a, b\}$;
2. $\varnothing, X, \{a\}, \{b\}, \{a, b\}\{b, d\}$;
3. $\varnothing, X, \{a, c, d\}, \{b, c, d\}$?

We see that (1) is a topological structure, however (2) and (3) both fail to satisfy the axioms of topological structure. For (1), we see that any union of the elements of the set is also included, and that any intersection of the elements is also included. Because $\varnothing$ and X are also in (1), we know that (1) satisfies the axioms of topological struture, and thus is a topological structure.

We see that $\{a, b\} \cup \{a, d\} = \{a, b, d\}$. As $\{a, b\}$ and $\{a, d\}$ belong to (2), but $\{a, b, d\}$ does not we see that (2) does not satisfy Axiom 1

We see that $\{a, c, d\} \cap \{b, c, d\} = \{c, d\}$. As $\{a, c, d\}$ and $\{b, c, d\}$ belong to (3), but $\{c, d\}$ does not, we know that (3) does not satisfy Axiom 2

Primary Author: Reilly Noonan Grant

[2′3] The Most Important Example: Real Line

2.C. *Check whether Ω satisfies the axioms of topological structure*

[2′4] Additional Examples

2.4. *Let X be $\mathbb{R}$, and let Ω consist of the empty set and all infinte subsets of $\mathbb{R}$. Is Ω a topological structure*

2.5. *Let X be $\mathbb{R}$, and let Ω consist of the empty set and complements of all finite subsets of $\mathbb{R}$. Is Ω a topological structure*

2.6. *Let (X, Ω) be a topological space, Y the set obtained from X by adding a single element a. Is*

$$\Omega' = \{\{a\} \cup U : U \in \Omega\} \cup \{\varnothing\}$$

a topological structure in Y?

Answer. Yes. To see that Axiom 3 is satisfied, notice that since $X \in \Omega$, $\{a\} \cup X \in \Omega'$. We also have that $\varnothing \in \Omega'$. Thus, Axiom 3 holds.

A brief aside on the relationship of Ω and Ω'. Let $f : \Omega' \to \Omega$ be defined by $f(A) = A \setminus \{a\}$. We show that f is a bijection. To see that f is onto, let $U \in \Omega$ be arbitrary. Since $U \cup \{a\} \in \Omega'$ by definition, we have $f(U \cup \{a\}) = U$. Since U was arbitrary, we have that f is onto. To see that f is one-to-one, consider the function $g : \Omega \to \Omega'$ defined by $g(U) = U \cup \{a\}$, which is well-defined. We have

$$\begin{aligned}(f \circ g)(U) &= f(U \cup \{a\}) \\ &= U,\end{aligned}$$

while

$$\begin{aligned}(g \circ a)(A) &= g(A \setminus \{a\}) \\ &= A,\end{aligned}$$

showing that $g = f^{-1}$, which establishes that f is one-to-one. Thus, f is a bijection.

To see that Axiom 1 is satisfied, let $S \subseteq \Omega'$ be arbitrary, and consider $\bigcup_{A \in S} A$.

$$\begin{aligned}\bigcup_{A \in S} A &= \bigcup_{U \in f(S)} U \cup \{a\} \\ &= \{a\} \cup \left(\bigcup_{U \in f(S)} U \right).\end{aligned}$$

Since Ω is a topology, $\bigcup_{U \in f(S)} U = V$ for some $V \in \Omega$, so

$$\bigcup_{A \in S} A = \{a\} \cup V,$$

which is by definition an element of Ω'. This verifies Axiom 1.

To see that Axiom 2 is satisfied, let $S \subseteq \Omega'$ be an arbitrary finite set. Then

$$\begin{aligned}\bigcap_{A \in S} A &= \bigcap_{U \in f(S)} U \cup \{a\} \\ &= \{a\} \cup \left(\bigcup_{U \in f(S)} U \right).\end{aligned}$$

Since Ω is a topology, there is a $V \in \Omega$ such that

$$V = \bigcap_{U \in f(S)} U.$$

□

2.7. *Is the set $\{\varnothing, \{0\}, \{0,1\}\}$ a topological structure in $\{0,1\}$?*

The set $\{\varnothing, \{0\}, \{0,1\}\}$ a topological structure in $\{0,1\}$. We the union of any collection of elements in the topology will be either $\varnothing$, $\{0\}$ or $\{0,1\}$ and thus Axiom 1 is satisfied. We similarly see that an intersection of any collection of elements in the set will result in an element that already exists in the topology, and thus Axiom 2 is satisfied. Finally, we see that as $\varnothing$ and $\{0,1\}$ are in the set, that Axiom 3 is satisfied.

2.8. *List all topological structure in a two-element set, say, in $\{0,1\}$*

[2′5] Using New Words: Points, Open Sets, Closed Sets

2.D. *Reformulate the axioms of topological structure using the words open set wherever possible.*

[2′6] Set-Theoretic Digression: De Morgan Formulas

2.E. *Let Γ ba an arbitrary collection of subsets of a set X. Then*

$$X \setminus \bigcup_{A\in\Gamma} A = \bigcap_{A\in\Gamma} (X \setminus A) \tag{I.1}$$

$$X \setminus \bigcap_{A\in\Gamma} A = \bigcup_{A\in\Gamma} (X \setminus A) \tag{I.2}$$

[2′7] Properties of Closed Sets

[2′8] Being Open or Closed

2.F. *Find Examples of sets that are*

1. *both open and closed simultaneously*

2. *neither open, nor closed*

2.9.

2.G.

2.10. *Prove that the half-open interval $[0, 1)$ is neither open nor closed, but is both a union of closed sets and an instersection of open sets.*

Proof. □

2.11.

[2′9] Characterization of Topology in Terms of Closed Sets

[2′10] Neighborhoods

[2′11] Open Sets on Line

[2′12] Cantor Set

[2′13] Topology and Arithmetic Progressions

3 Bases

[3′1] Definition of Base

[3′2] When a Collection of Sets is a Base

[3′3] Bases for Plane

[3′4] Subbases

[3′5] Infiniteness of the Set of Prime Numbers

[3′6] Hierarchy of Topologies