WELCOME TO MAST10005 – CALCULUS 1

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Course Organisation

Three lectures every week:

- **Tuesday 2:15**, Public Lecture Theatre (Old Arts).
- **Thursday 11:00**, Kathleen Fitzpatrick Theatre (Arts West).
- **Friday 11:00**, Copland Theatre (The Spot).

You will also have 1 practice class every week. Consult your timetable for your time and location.

Practice classes start in week one in this subject!

LMS Site

You should check the LMS Site regularly.

It contains information about assignments, consultation hours, and extra resources for additional help.

Consultation Hours

Each lecturer, and some tutors, will have several consultation hours each week.

You can come to any of these sessions to get individual help with problems. Check the LMS for the consultation schedule.

Notes and text

- These lecture notes contain an outline of the theory for each lecture. Spaces are provided for you to fill in the solutions to the examples in the lectures as we work through them. If you miss a lecture, you should work through the theory yourself and try to work through the examples.

  Complete lecture notes will not be made available at any time during the semester!

- Recommended textbook, available from the bookshop:

  Hass, Weir and Thomas - *University Calculus Early Transcendentals*, 2nd edition, Pearson, 2012.

The Calculus 1 Handbook

The blue subject handbook (also referred to as the problem booklet) comes bundled with the lecture notes. In it you will find:

- all the information about the organisation of this subject, assessment details, etc.
- a list of useful formulas, which will be given to you in the exam.
- problem sheets (aim to work through these topic by topic), with answers at the back of the handbook.

Assessment

- There will be 10 assignments worth a total of 20%.
- A 3-hour exam at the end of semester worth 80%.

Assignment 1 due on Monday August 8 at 11:00AM is revision and is intended to help you identify any material you need to review.

The remaining assignments are due weekly on Fridays at 3 pm sharp, starting from Friday 12 August.

The assignment problems will be handed out during lectures and also posted on the LMS.

Note that **no calculators** are permitted in the final exam.

Prerequisites

The prerequisites for Calculus 1 are VCE Mathematical Methods 3/4 (or equivalent), or MAST10012 Introduction to Mathematics.

If you are from a non-VCE background, it is your responsibility to ensure that you know all the material in the VCE MM 3/4 course.

Future subjects

After successfully completing **MAST10005 Calculus 1**, you may continue your study of mathematics by taking **MAST10006 Calculus 2** and/or **MAST10007 Linear Algebra**.

You may also study statistics by taking **MAST10010 Data Analysis 1**.

Overview and Aims

This subject will give you a solid grounding in differential and integral calculus, as well as introducing the topic of differential equations. These core calculus topics have a wide range of applications in the real world, such as in biological and physical systems, in engineering and in economics. We also study functions, vectors and complex numbers, which include key concepts for further studies in mathematics, including further studies in calculus.

Topics covered

Topic 1: Trigonometric functions

Topic 2: Vectors

Topic 3: Complex Numbers

Topic 4: Differential Calculus

Topic 5: Integral Calculus

Topic 6: Differential Equations

Topic 1: Trigonometric Functions

In this topic we look mainly at trigonometric functions, extending your knowledge from school. We define the reciprocal trigonometric functions and inverse trigonometric functions, and derive several trigonometric identities.

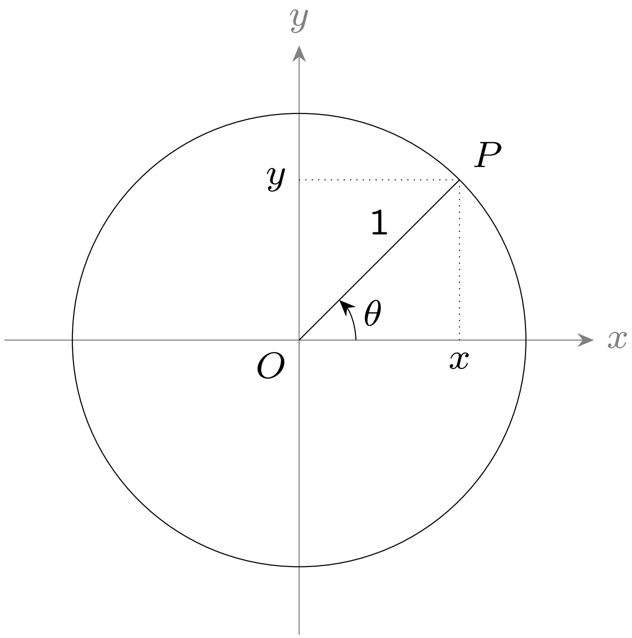
1.1 Revision of Trigonometric Functions [Chapter 1.3]

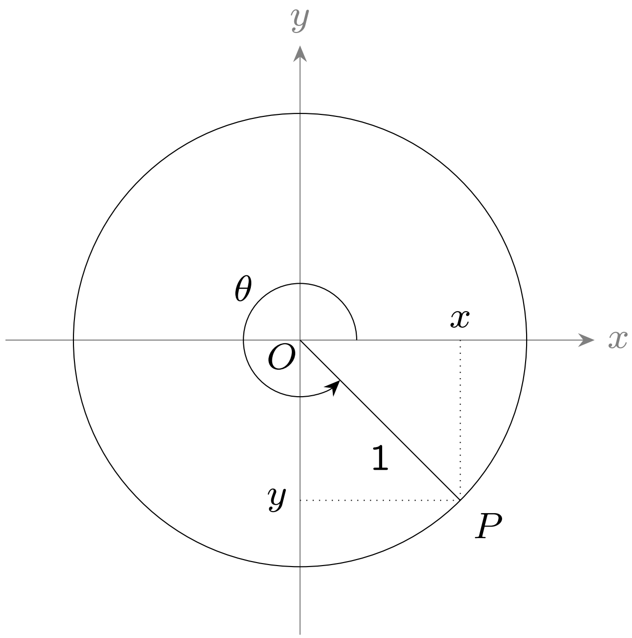
The basic concepts in trigonometry will have been covered in high school at years 10, 11 and 12. They are assumed knowledge. Please work through the exercises to refresh your memories.

Trigonometry is vital for many topics in this course (and beyond) so you need to have a good understanding of these concepts. If you are experiencing difficulty, you should work through the appropriate section in the text (or your old school text).

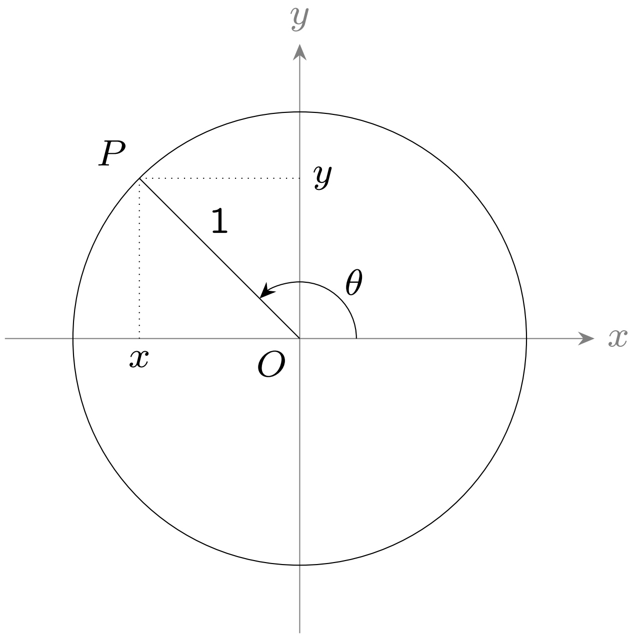
1.1.1 The unit circle

The **unit circle** is the circle of radius 1 centred at the origin. Any angle $\theta \in \mathbb{R}$ can be represented on the unit circle as the anticlockwise angle from the positive x-axis to the ray OP. For example:





To define the functions **sine**, **cosine** and **tangent**, let P be the point on the unit circle corresponding to the angle θ. Then

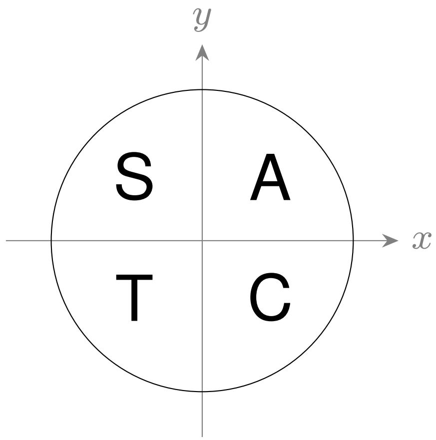


$$\sin(\theta) = y$$

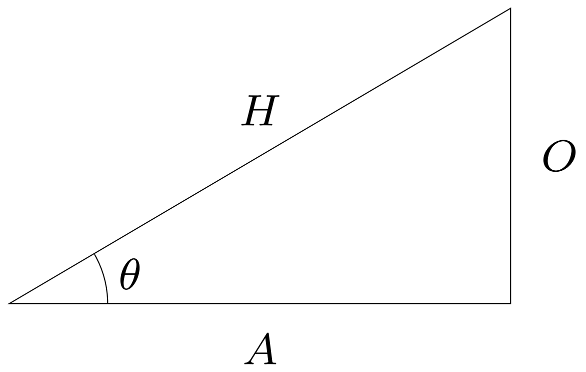
$$\cos(\theta) = x$$

$$\tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)} = \frac{y}{x}$$

The ‘CAST’ diagram shows which trigonometric function is positive for each of the four quadrants in the plane:



If θ happens to satisfy $0 < \theta < \pi/2$, we can evaluate the trigonometric functions as ratios in a right-angled triangle:



$$\sin(\theta) = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{O}{H}$$

$$\cos(\theta) = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{A}{H}$$

$$\tan(\theta) = \frac{\text{opposite}}{\text{adjacent}} = \frac{O}{A}$$

1.1.2 Degrees and Radians

Angles can be measured in either degrees or radians.

The angle around a full circle is 360° or 2π radians.

Hence to convert from degrees to radians we multiply by $\frac{2\pi}{360} = \frac{\pi}{180}$,

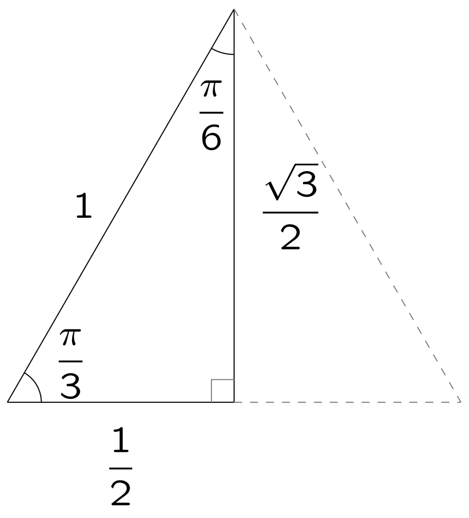
and to convert from radians to degrees we multiply by $\frac{360}{2\pi} = \frac{180}{\pi}$.

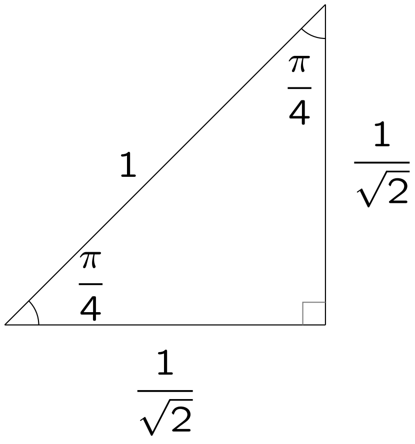
Homework: Complete the following table:

degrees	30			90	120		180		
radians		$\frac{\pi}{4}$	$\frac{\pi}{3}$			$\frac{3\pi}{4}$		$\frac{3\pi}{2}$	2π

We generally use radians to measure angles in our university courses.

1.1.3 Standard Triangles





Homework: Determine the following:

$$\sin\left(\frac{\pi}{6}\right) = \qquad \sin\left(\frac{\pi}{3}\right) = \qquad \sin\left(\frac{\pi}{4}\right) =$$

$$\cos\left(\frac{\pi}{6}\right) = \qquad \cos\left(\frac{\pi}{3}\right) = \qquad \cos\left(\frac{\pi}{4}\right) =$$

$$\tan\left(\frac{\pi}{6}\right) = \qquad \tan\left(\frac{\pi}{3}\right) = \qquad \tan\left(\frac{\pi}{4}\right) =$$

For angles in the 2nd, 3rd and 4th quadrants, we can use these standard triangles and the unit circle to determine the correct trigonometric ratios.

Homework: Find the following:

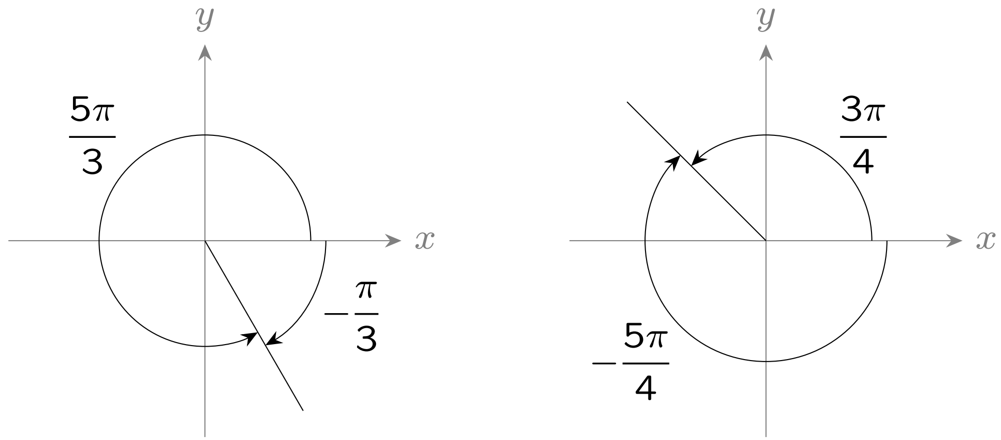
(a) $\cos\left(\frac{5\pi}{6}\right)$ (b) $\sin\left(\frac{7\pi}{4}\right)$ (c) $\tan\left(\frac{4\pi}{3}\right)$

1.1.4 Negative Angles

While positive angles are measured *anticlockwise* around the unit circle, negative angles are measured *clockwise* around the unit circle.

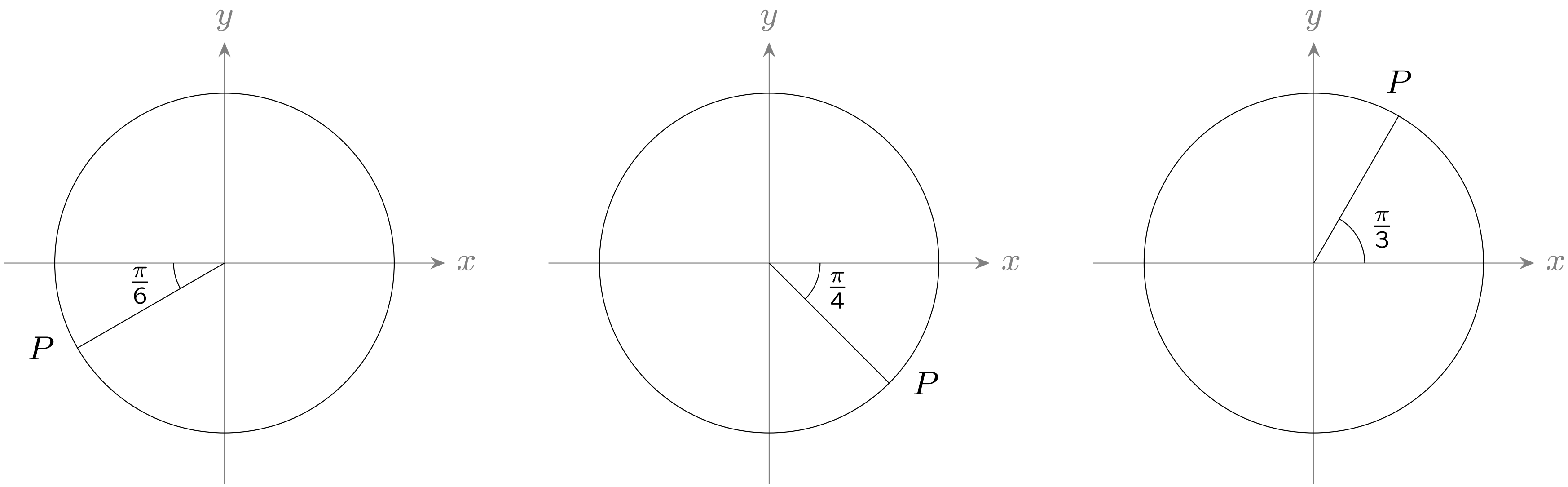
The same angle on the unit circle may be considered either positive or negative, since adding or subtracting multiples of 2π does not change its position on the unit circle.

For example:



$$-\frac{\pi}{3} \text{ is the same as } \frac{5\pi}{3}, \quad \text{and} \quad \frac{3\pi}{4} \text{ is the same as } -\frac{5\pi}{4}.$$

Homework: Express the angles corresponding to the points P as both a positive angle in the range $[0, 2\pi]$ and a negative angle in the range $[-2\pi, 0]$.



Homework: Find the following:

(a) $\sin\left(-\frac{\pi}{3}\right)$

(b) $\cos\left(-\frac{3\pi}{4}\right)$

(c) $\tan\left(-\frac{2\pi}{3}\right)$

Homework: Draw angles in the relevant quadrant(s) to solve each of the following:

(a) $\sin\left(\frac{3\pi}{4}\right) = x$

(b) $\sin(\theta) = -\frac{1}{\sqrt{2}}$

(c) $\cos(\phi) = -\frac{1}{\sqrt{2}}$

(d) $\cos\left(-\frac{\pi}{6}\right) = y$

(e) $\tan\left(-\frac{3\pi}{4}\right) = z$

(f) $\tan(t) = -1$

1.1.5 Graphs of Trigonometric Functions

We now consider the trigonometric *functions*:

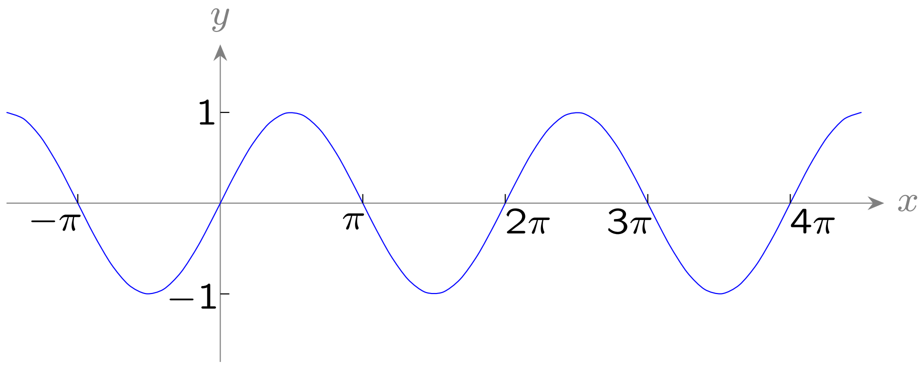
$$\mathsf{sin}(x), \qquad \mathsf{cos}(x), \qquad \mathsf{tan}(x).$$

Recall that the *domain* of a function is the set of “input” values of the function.

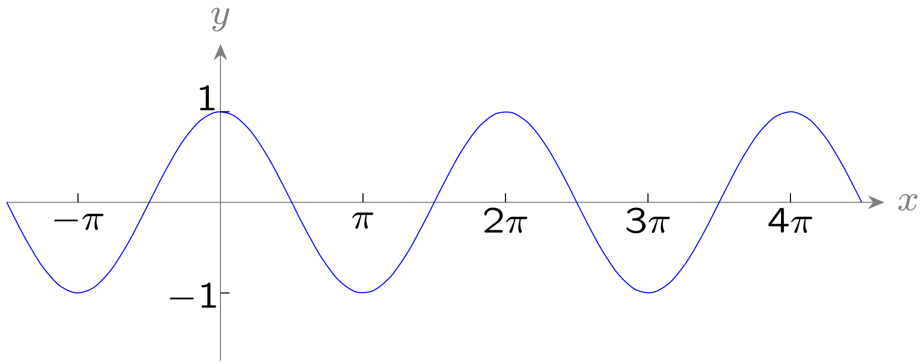
The sine and cosine functions have domain $\mathbb{R}$ and period 2π since $\mathsf{sin}(x+2\pi) = \mathsf{sin}(x)$ and $\mathsf{cos}(x+2\pi) = \mathsf{cos}(x)$ for all $x \in \mathbb{R}$.

The tangent function has domain $\mathbb{R} \setminus \left\{\frac{\pi}{2} + k\pi \mid k \in \mathbb{Z}\right\}$, since the tangent ratio is not defined when its denominator is 0. Its period is π since $\mathsf{tan}(x+\pi) = \mathsf{tan}(x)$ for all x in its domain.

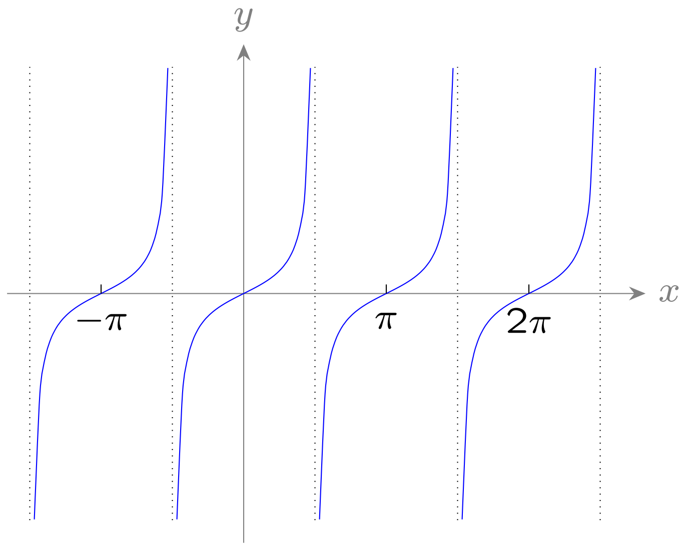
$\sin\colon \mathbb{R} \to [-1, 1]$



$\cos\colon \mathbb{R} \to [-1, 1]$



$\tan\colon \mathbb{R} \setminus \{\pi/2 + \pi\mathbb{Z}\} \to \mathbb{R}$



Homework: Sketch the following graphs:

(a) $\cos(3x)$ (b) $\tan\left(x - \frac{\pi}{4}\right)$ (c) $2\sin\left(x + \frac{\pi}{3}\right) + 2$

1.1.6 Solving Trigonometric Equations

By using the two standard triangles, the graphs of $\sin$, $\cos$ and $\tan$, and the unit circle, we can solve a range of trigonometric equations.

Homework: Solve the following equations for $x \in [0, 2\pi]$:

(a) $\sin(x) = \frac{1}{2}$ (b) $\cos(x) = -1$ (c) $\tan(x) = -\sqrt{3}$

(d) $\cos(x) = \sin(x)$ (e) $2\sin^2(x) + \sin(x) - 1 = 0$

It is essential that you have a good understanding of this section by next lecture.

You should attempt the homework questions and some additional questions after every lecture. This will help your understanding of the material, and highlight problems for which you may need to seek assistance.

Additional questions

You may now attempt a selection of exercises from chapter 1.3 in the textbook.

1.2 Reciprocal trigonometric functions [Chapter 1.3]

You are now familiar with the three trigonometric functions *sine*, *cosine* and *tangent* and their graphs. Today we define the reciprocals of these functions and sketch their graphs.

1.2.1 The cosecant function

The reciprocal of the sine function is called the **cosecant** function. It is abbreviated to “cosec” and is defined as:

$$\boxed{\operatorname{cosec}(x) = \frac{1}{\sin(x)} \qquad \text{provided } \sin(x) \neq 0.}$$

Since $\sin(x) = 0$ exactly when $x = n\pi$ for some $n \in \mathbb{Z}$, the domain of cosec is $\mathbb{R} \setminus \{n\pi \mid n \in \mathbb{Z}\}$.

Graph of $y = \mathrm{cosec}(x)$

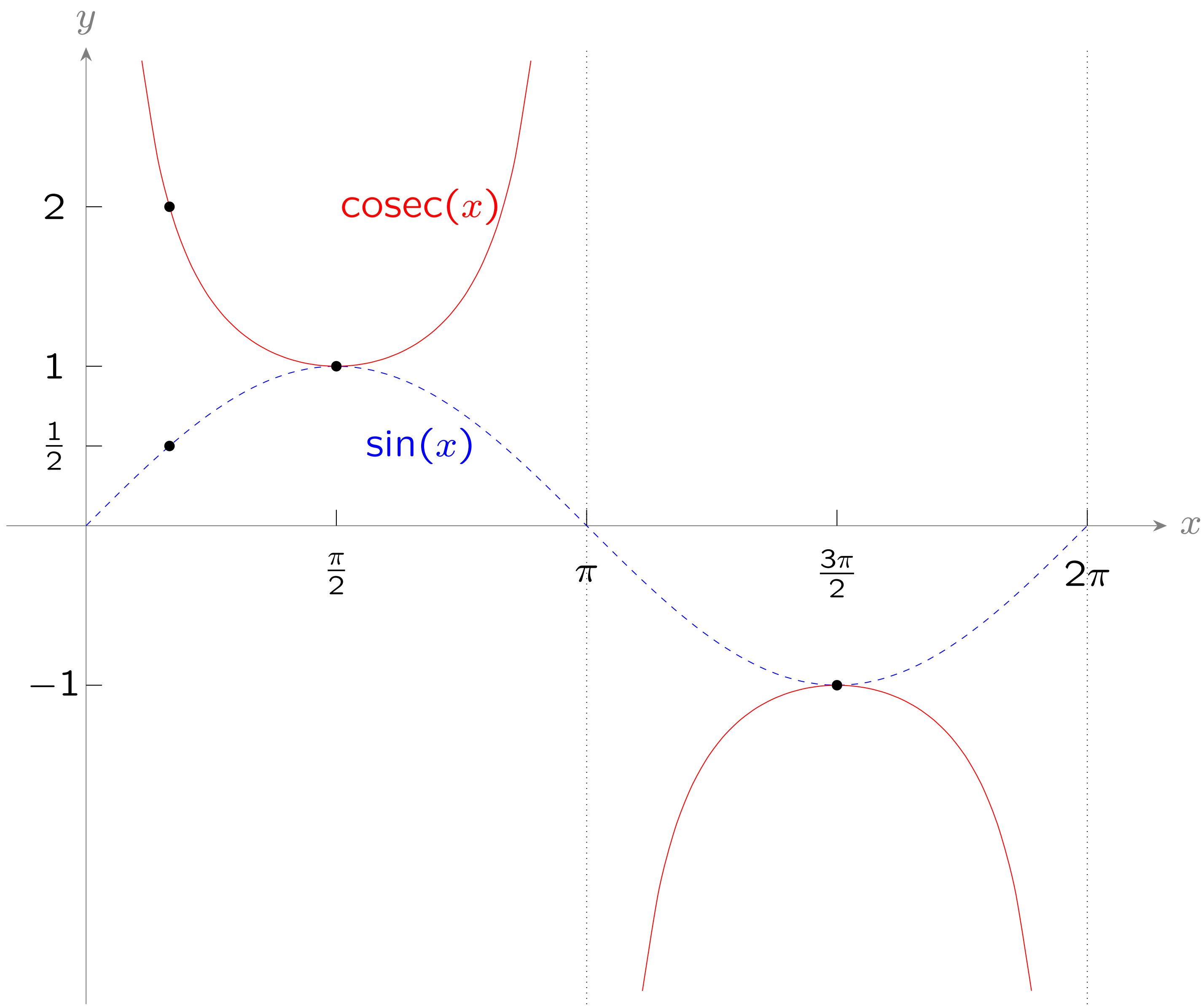
We derive the graph of $\mathrm{cosec}(x)$ from the graph of $\sin(x)$.

First we need to determine the domain and range of cosec. We already know that the **domain** of cosec is $\mathbb{R} \setminus \{n\pi \mid n \in \mathbb{Z}\}$.

Recall that the *range* of a function is the set of “output” values of the function. The range of sin is $[-1, 1]$ and since we are taking the reciprocal, the **range** of cosec is $\mathbb{R} \setminus (-1, 1)$.

The function sin has turning points at $\left\{\frac{\pi}{2} + n\pi \,\middle|\, n \in \mathbb{Z}\right\}$, and therefore so does cosec.

The values of x for which cosec is not defined give rise to vertical asymptotes.



y
2
1
$\frac{1}{2}$
-1
$\mathrm{cosec}(x)$
$\sin(x)$
x
$\frac{\pi}{2}$
π
$\frac{3\pi}{2}$
2π

Example: Sketch the graph of $\operatorname{cosec}(2x)$ over the interval $[0, 2\pi]$.

Example continued:

Homework: Sketch the graph of $\operatorname{cosec}\left(x - \frac{\pi}{4}\right)$ over the interval $[0, 2\pi]$.

1.2.2 The cotangent function

The reciprocal of the tangent function is called the *cotangent* function. It is abbreviated to "cot" and is defined as:

$$\boxed{\cot(x) = \frac{1}{\tan(x)} = \frac{\cos(x)}{\sin(x)} \qquad \text{provided } \sin(x) \neq 0.}$$

Earlier we saw that $\sin(x) = 0$ exactly when $x = n\pi$ for some $n \in \mathbb{Z}$ so the domain of cot is $\mathbb{R} \setminus \{n\pi \mid n \in \mathbb{Z}\}$.

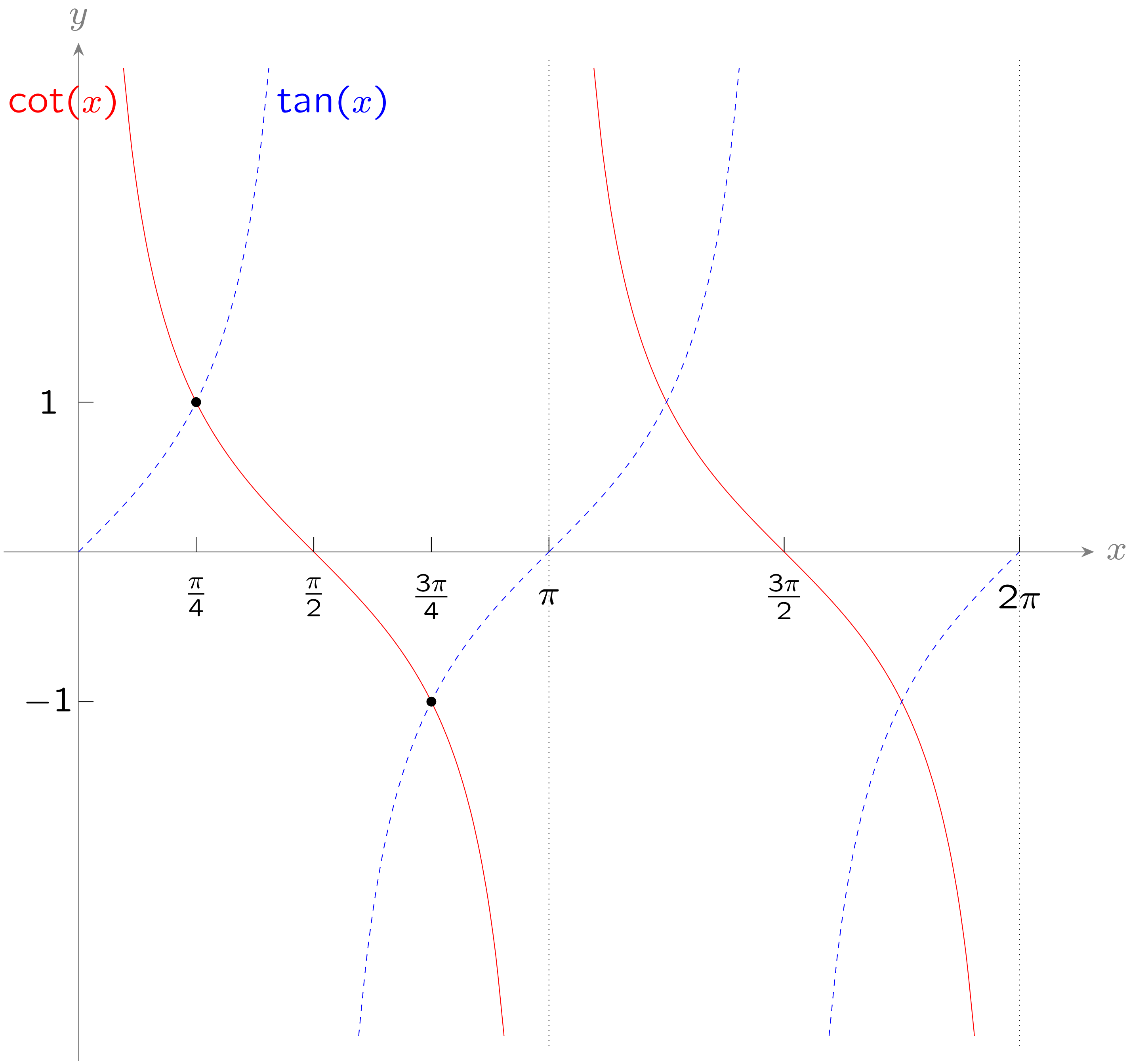
Graph of $y = \texttt{cot}(x)$

We derive the graph of $\texttt{cot}(x)$ from the graph of $\texttt{tan}(x)$.

The **domain** of $\texttt{cot}$ is $\mathbb{R} \setminus \{n\pi \mid n \in \mathbb{Z}\}$.

The range of $\texttt{tan}$ is $\mathbb{R}$ and since we are taking the reciprocal, the **range** of $\texttt{cot}$ is $\mathbb{R}$.

The values of x for which $\texttt{tan}$ is zero give rise to asymptotes for $\texttt{cot}$. The zeros of $\texttt{cot}$ occur at the zeros of $\texttt{cos}$, and coincide with the asymptotes for $\texttt{tan}$.


y
$\cot(x)$
$\tan(x)$
1
-1
x
$\frac{\pi}{4}$
$\frac{\pi}{2}$
$\frac{3\pi}{4}$
π
$\frac{3\pi}{2}$
2π

Example: Sketch the graph of $\cot\left(x+\dfrac{\pi}{4}\right)$ over the interval $[0, 2\pi]$.

Example continued:

Homework: Sketch the graph of $3\cot(x)$ over the interval $[0, 2\pi]$.

1.2.3 The secant function

The reciprocal of the cosine function is called the *secant* function. It is abbreviated to "sec" and is defined as:

$$\boxed{\sec(x) = \frac{1}{\cos(x)} \qquad \text{provided } \cos(x) \neq 0.}$$

Homework: Using the work on cosecant as a guide:

(a) find the domain and range of sec;

(b) sketch $\cos(x)$ and $\sec(x)$ for $0 \leq x \leq 2\pi$ on the same graph.

Homework: Sketch the graph of $2\sec(x)$ over the interval $[0, 2\pi]$.

Homework: Sketch the graph of $-\sec(x)$ over the interval $[0, 2\pi]$.

Harder example: Sketch the graph of $2\,\text{cosec}\left(x - \frac{\pi}{4}\right) + 1$ over the interval $[-2\pi, 2\pi]$.

Example continued:

Example continued:

Example continued:

Summary

$$\operatorname{cosec}(x) = \frac{1}{\sin(x)}, \qquad \sin(x) \neq 0$$

$$\sec(x) = \frac{1}{\cos(x)}, \qquad \cos(x) \neq 0$$

$$\cot(x) = \frac{1}{\tan(x)} = \frac{\cos(x)}{\sin(x)}, \qquad \sin(x) \neq 0.$$

Example/Homework: Evaluate the following:

(a) $\operatorname{cosec}\left(\frac{\pi}{6}\right)$ (b) $\sec\left(\frac{\pi}{6}\right)$ (c) $\cot\left(\frac{\pi}{6}\right)$

(d) $\operatorname{cosec}\left(\frac{3\pi}{4}\right)$ (e) $\sec\left(\frac{3\pi}{4}\right)$ (f) $\cot\left(\frac{3\pi}{4}\right)$

Additional questions

You can now attempt problems 1–4 from Topic 1 in the handbook.

You may also attempt a selection of problems from Exercise set 1.3 in the textbook.

1.3 Trigonometric Formulae [Chapter 1.3]

An identity is a statement written in terms of a variable, which is true for all values of the variable for which the identity is valid. A familiar example of a trigonometric identity is:

$$\boxed{\sin^2(\theta) + \cos^2(\theta) = 1.}$$

This is called the *Pythagorean identity*. It is true for all values of $\theta \in \mathbb{R}$. We will now look at some other trigonometric formulae, in particular:

- trigonometric identities;
- compound and double angle formulae.

1.3.1 Trigonometric identities

We start with the Pythagorean identity

$$\sin^2(\theta) + \cos^2(\theta) = 1. \tag{1}$$

From this identity, we can derive identities involving reciprocal trigonometric functions. If we divide both sides of equation (1) by $\cos^2(\theta)$, we obtain:

$$\frac{\sin^2(\theta)}{\cos^2(\theta)} + 1 = \frac{1}{\cos^2(\theta)}, \qquad \text{provided } \cos(\theta) \neq 0,$$

which is the same as

$$\boxed{\tan^2(\theta) + 1 = \sec^2(\theta), \qquad \text{for } \cos(\theta) \neq 0.}$$

Alternatively, we can divide equation (1) by $\sin^2(\theta)$ to obtain:

$$1 + \frac{\cos^2(\theta)}{\sin^2(\theta)} = \frac{1}{\sin^2(\theta)}, \qquad \text{provided } \sin(\theta) \neq 0,$$

which is the same as

$$\boxed{1 + \cot^2(\theta) = \operatorname{cosec}^2(\theta), \qquad \text{for } \sin(\theta) \neq 0.}$$

These identities can be used to evaluate or to simplify trigonometric expressions.

Example: Simplify the expression $\sin^2(x)\left(1+\cot^2(x)\right)$.

Example: If $x \in \left[\frac{\pi}{2}, \pi\right]$, and $\sin(x) = \frac{4}{5}$, find:

(a) $\cos(x)$ (b) $\cot(x)$.

Example: If $x \in \left[\frac{3\pi}{2}, 2\pi\right]$, and $\sec(x) = \frac{3}{2}$, find:

(a) $\tan(x)$ (b) $\operatorname{cosec}(x)$.

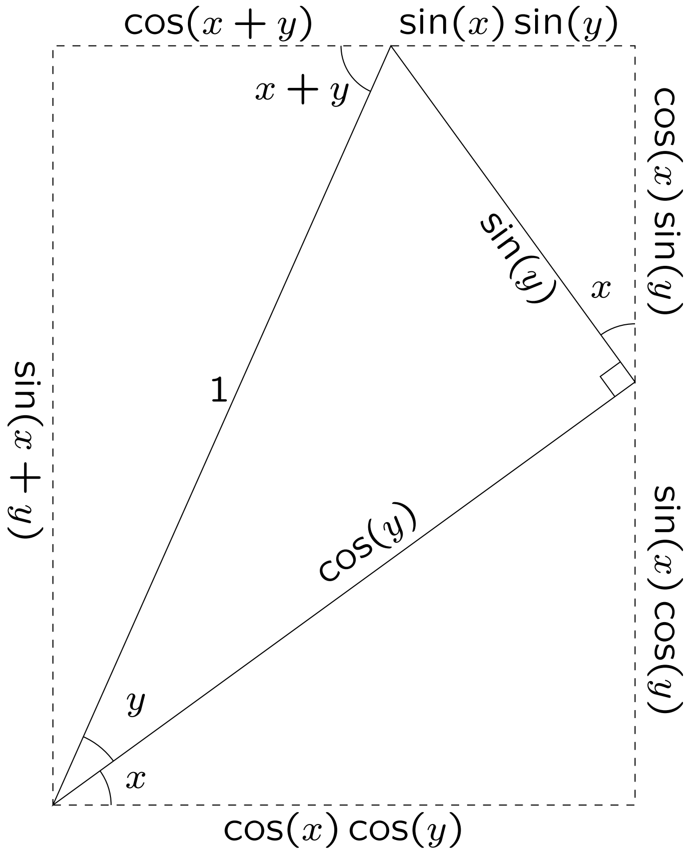
Example continued:

1.3.2 Compound angle formulae

The standard triangles make it possible to evaluate trigonometric functions at specific standard angles. But what about a non-standard angle?

If the angle can be written as some combination of standard angles (for example: $\frac{5\pi}{12} = \frac{\pi}{6} + \frac{\pi}{4}$), then we can use the standard triangles and *compound angle formulae* to evaluate the trigonometric function.

The compound angle formulae for $\sin$ and $\cos$ are summarised (and proved) in the following diagram:



vertical sides of the rectangle:

$$\boxed{\sin(x+y) = \sin(x)\cos(y) + \cos(x)\sin(y)}$$

horizontal sides of the rectangle:

$$\boxed{\cos(x+y) = \cos(x)\cos(y) - \sin(x)\sin(y)}$$

By replacing the angle y with $-y$, and recalling that

$$\cos(-y) = \cos(y) \qquad \text{and} \qquad \sin(-y) = -\sin(y),$$

we can also deduce:

$$\begin{aligned} \sin(x-y) &= \sin(x+(-y)) \\ &= \sin(x)\cos(-y) + \cos(x)\sin(-y) \\ &= \sin(x)\cos(y) - \cos(x)\sin(y) \end{aligned}$$

$$\boxed{\sin(x-y) = \sin(x)\cos(y) - \cos(x)\sin(y)}$$

and

$$\begin{aligned} \cos(x-y) &= \cos(x+(-y)) \\ &= \cos(x)\cos(-y) - \sin(x)\sin(-y) \\ &= \cos(x)\cos(y) + \sin(x)\sin(y) \end{aligned}$$

$$\boxed{\cos(x-y) = \cos(x)\cos(y) + \sin(x)\sin(y).}$$

We can use these identities to deduce compound angle formulae for the tangent function:

$$\tan(x+y) = \frac{\sin(x+y)}{\cos(x+y)}, \quad \cos(x+y) \neq 0$$

$$= \frac{\sin(x)\cos(y) + \cos(x)\sin(y)}{\cos(x)\cos(y) - \sin(x)\sin(y)}$$

Dividing the numerator and denominator by $\cos(x)\cos(y)$, where $\cos x \neq 0,\ \cos y \neq 0$, this expression simplifies to:

$$\boxed{\tan(x+y) = \frac{\tan(x) + \tan(y)}{1 - \tan(x)\tan(y)}.}$$

Similarly:

$$\boxed{\tan(x-y) = \frac{\tan(x) - \tan(y)}{1 + \tan(x)\tan(y)}.}$$

Example: Express $\cos\left(x + \frac{\pi}{6}\right)$ in terms of $\cos(x)$ and $\sin(x)$.

Example: Simplify the expression $\operatorname{cosec}\left(x - \frac{\pi}{2}\right)$.

Example: Find the exact value of $\tan\left(\frac{7\pi}{12}\right)$.

Homework: Find the exact value of $\cos\left(\frac{5\pi}{12}\right)$.

1.3.3 Double angle formulae

We now deduce the *double angle formulae* for $\sin$, $\cos$ and $\tan$ of a double angle $2x$. These can be found by starting with the compound angle formulae.

Recall for $\sin$ we have:

$$\sin(x+y) = \sin(x)\cos(y) + \cos(x)\sin(y).$$

If we let $y = x$ then we have:

$$\begin{aligned}\sin(2x) &= \sin(x+x)\\ &= \sin(x)\cos(x) + \cos(x)\sin(x)\\ &= 2\sin(x)\cos(x).\end{aligned}$$

So the double angle formula for the sine function is:

$$\boxed{\sin(2x) = 2\sin(x)\cos(x).}$$

Similarly for cos we have:

$$\begin{aligned}\cos(2x) &= \cos(x+x)\\ &= \cos(x)\cos(x) - \sin(x)\sin(x)\\ &= \cos^2(x) - \sin^2(x).\end{aligned}$$

So:

$$\boxed{\cos(2x) = \cos^2(x) - \sin^2(x).}$$

We can obtain two alternative versions of this formula by recalling that $\sin^2(x) + \cos^2(x) = 1$, so:

$$\begin{aligned}\cos(2x) &= \cos^2(x) - \sin^2(x)\\ &= (1 - \sin^2(x)) - \sin^2(x)\\ &= 1 - 2\sin^2(x),\\ \\ &= 1 - 2(1 - \cos^2(x))\\ &= 2\cos^2(x) - 1.\end{aligned}$$

Homework: Use the compound angle formula for $\tan(x+y)$ to show that

$$\boxed{\tan(2x) = \frac{2\tan(x)}{1-\tan^2(x)}}$$

Homework: Simplify the expression $\dfrac{\tan\left(\frac{x}{2}\right)}{1-\tan^2\left(\frac{x}{2}\right)}$.

Summary

Compound Angle Formulae

1. $\sin(x+y) = \sin(x)\cos(y) + \cos(x)\sin(y)$
2. $\sin(x-y) = \sin(x)\cos(y) - \cos(x)\sin(y)$

3. $\cos(x+y) = \cos(x)\cos(y) - \sin(x)\sin(y)$
4. $\cos(x-y) = \cos(x)\cos(y) + \sin(x)\sin(y)$

5. $\tan(x+y) = \dfrac{\tan(x)+\tan(y)}{1-\tan(x)\tan(y)}$
6. $\tan(x-y) = \dfrac{\tan(x)-\tan(y)}{1+\tan(x)\tan(y)}$

Double Angle Formulae

1. $\sin(2x) = 2\sin(x)\cos(x)$

2. $$\begin{aligned}\cos(2x) &= \cos^2(x) - \sin^2(x) \\ &= 1 - 2\sin^2(x) \\ &= 2\cos^2(x) - 1\end{aligned}$$

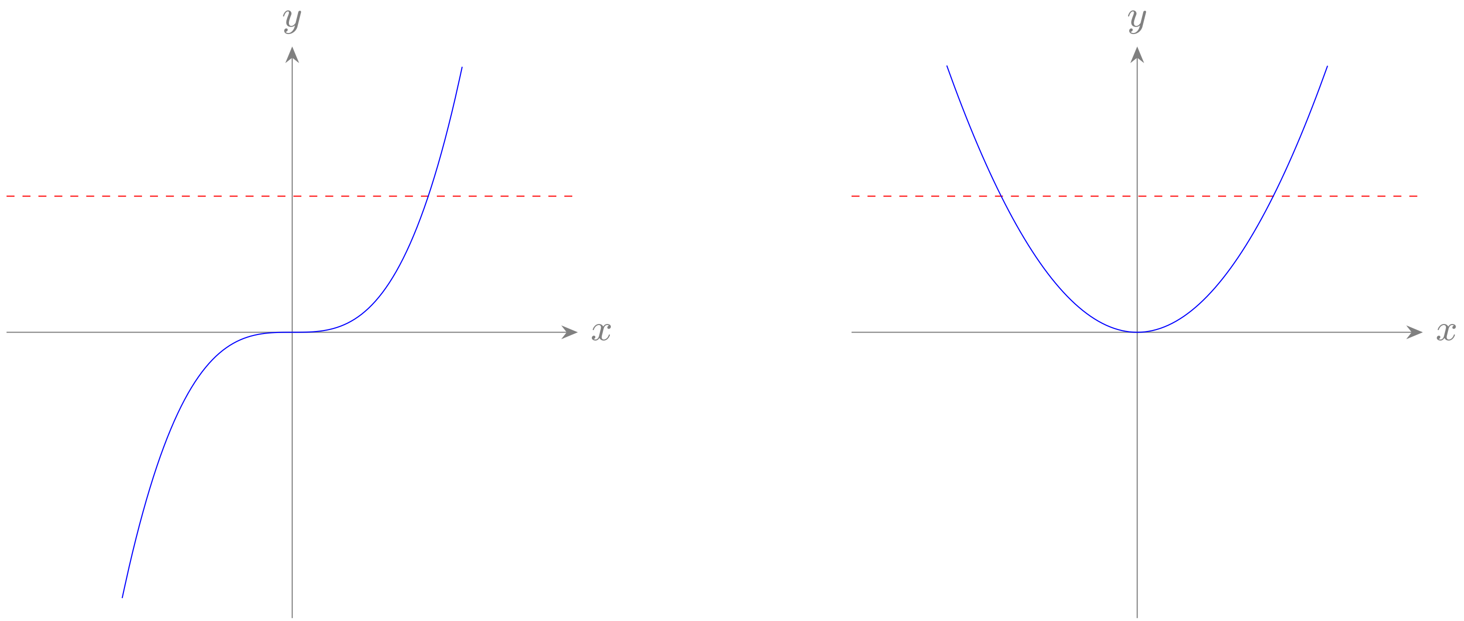
3. $\tan(2x) = \dfrac{2\tan(x)}{1 - \tan^2(x)}$

Additional questions

You can attempt problems 5–13 from Topic 1 of the handbook.

You may also try a selection of problems from Exercise set 1.3 in the textbook.

Recall that a function f is **one-to-one** if for each element y in the codomain of f there is *at most one* x in the domain of f such that $f(x) = y$. An easy way to determine this is to draw a horizontal line through the graph of f.



The function is one-to-one if there is no horizontal line that cuts the graph in more than one point. Here the graph on the left is one-to-one, but the graph on the right is not.

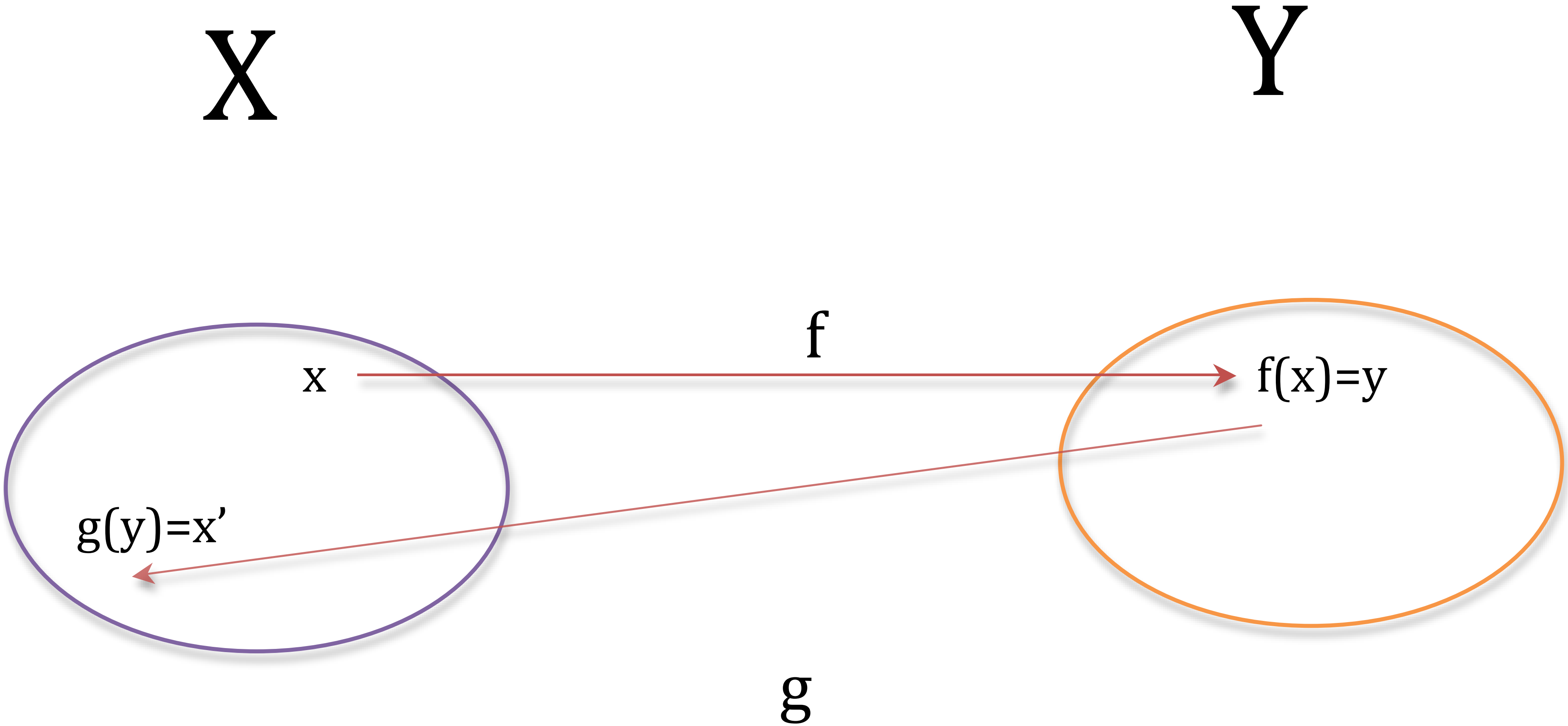
Clearly, the functions $\mathtt{sin}$, $\mathtt{cos}$ and $\mathtt{tan}$ are not one-to-one because they are periodic. However we can restrict the domains of these functions to obtain one-to-one functions. For these restricted functions we can then define inverse functions.

Recall the definition of an inverse function:

If $f\colon X \to Y$ is a function, then an **inverse function** of f is a function $g\colon Y \to X$ such that

$$\begin{aligned} g(f(x)) &= x \qquad \text{for all } x \in X \\ \text{and} \quad f(g(y)) &= y \qquad \text{for all } y \in Y. \end{aligned}$$

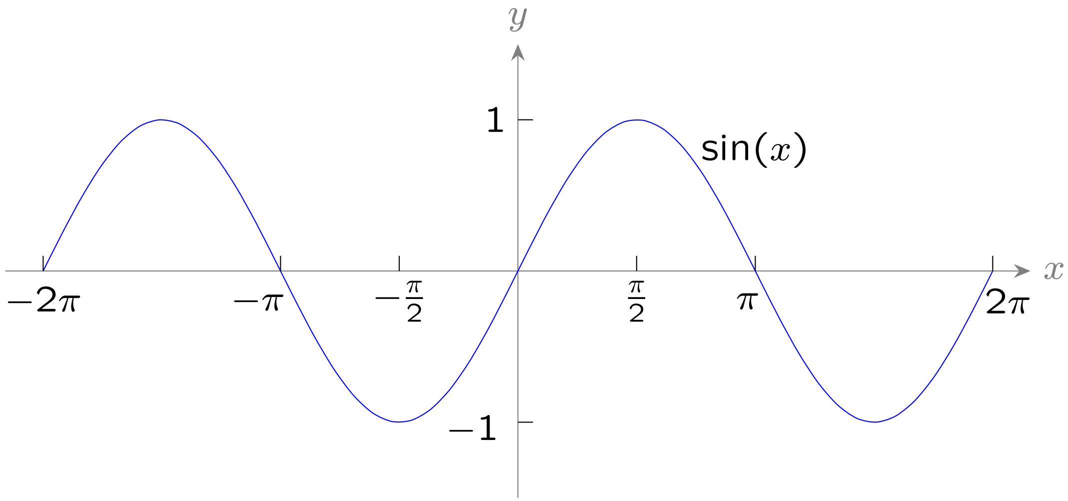
Not every function has an inverse function. However, if $f\colon X \to Y$ is one-to-one and has range Y, then it always has an inverse function and this inverse is unique.


X
Y
f
x
f(x)=y
g(y)=x’
g

Example: Consider the function $f : \mathbb{R} \to [0, \infty)$ given by $f(x) = x^2$.
Is this function invertible? If not, how can it be made invertible?

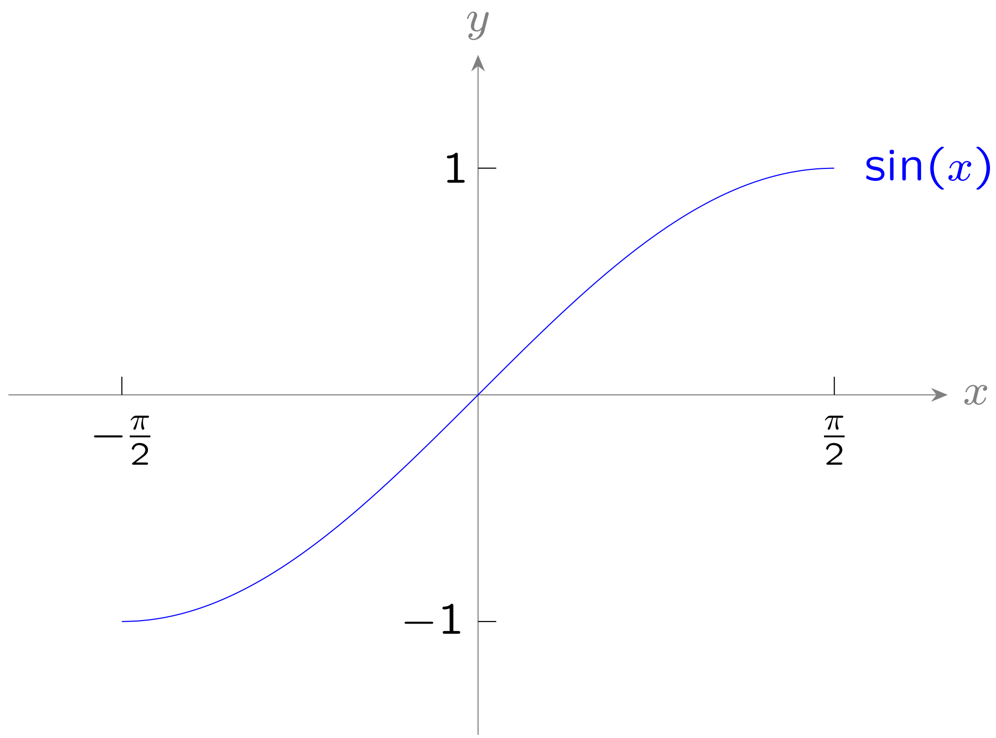
1.4.1 The inverse of the sine function

Recall the graph of sine:



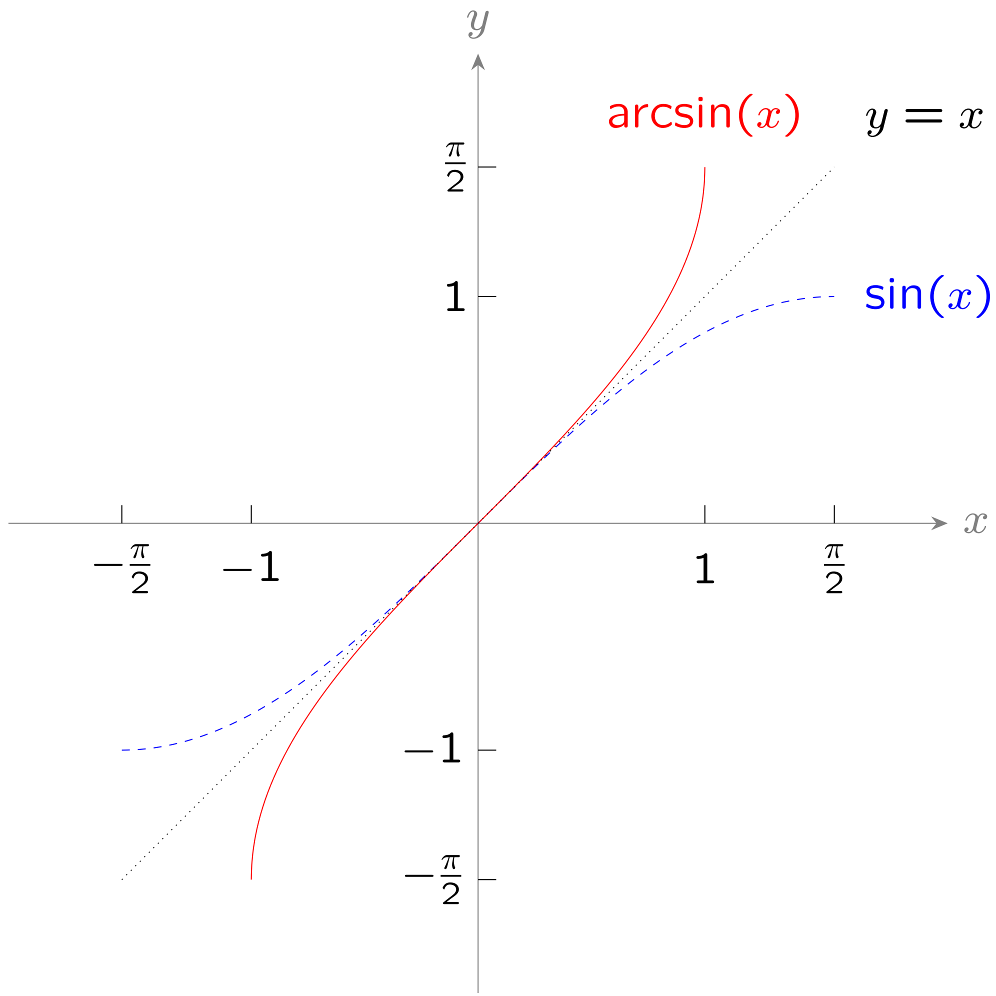
To define the inverse, we need to restrict the domain of the sine function so that the new function is one-to-one on this restricted domain, but the range of the function is not restricted.

Clearly many choices are possible. By convention, the restricted domain is chosen to be $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

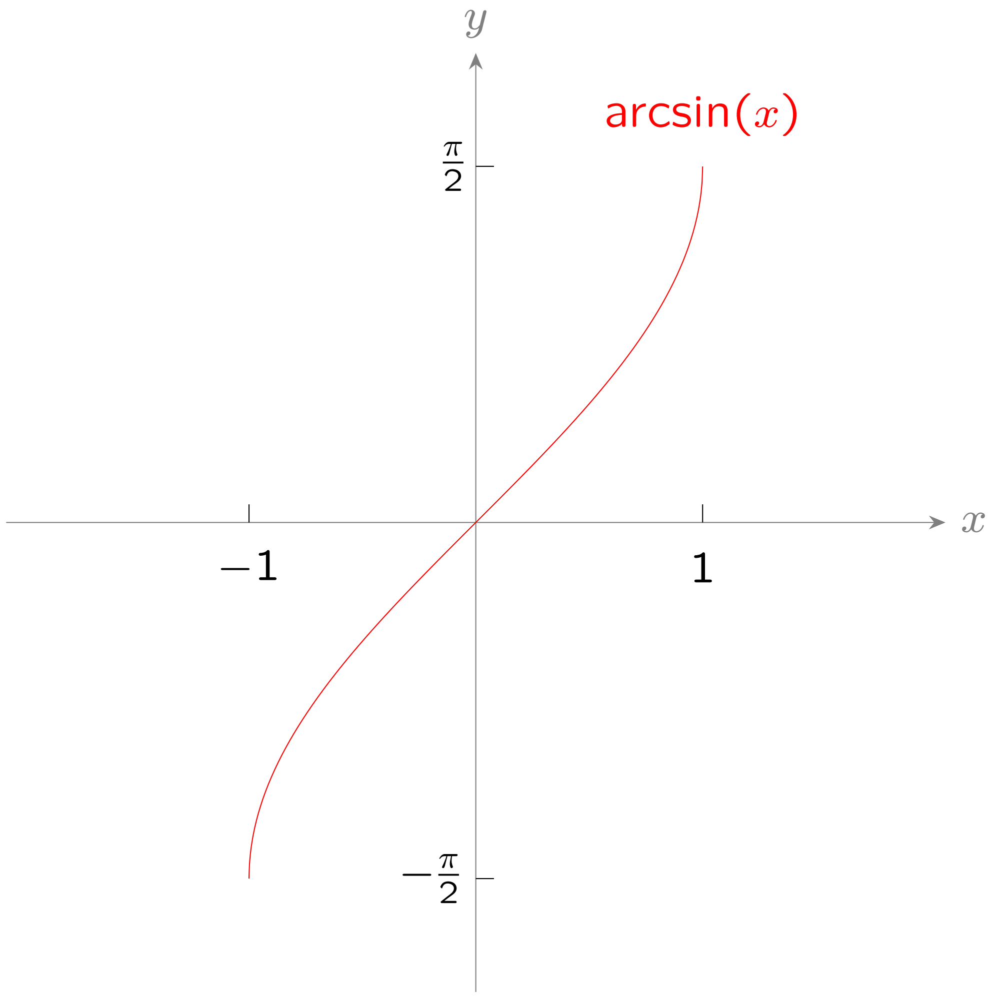


Notice that the range of this restricted function is still $[-1, 1]$.

Now we can obtain the inverse function by reflecting the graph through the line $y = x$.



We call this function *arcsine* (denoted $\arcsin$).
Its domain is $[-1, 1]$ and its range is $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.



This function is also referred to in some texts as Sin^{-1} where the capital S denotes the restricted domain and the superscript -1 means inverse function rather than reciprocal.

In this subject we will only use the arcsin **notation.**

This avoids potential confusion between $\mathsf{Sin}^{-1}(x)$ and $\frac{1}{\sin(x)}$.

The function $\operatorname{arcsin}\colon [-1,1] \to \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ satisfies:

$$\boxed{\theta = \operatorname{arcsin}(x) \qquad \Rightarrow \quad \sin(\theta) = x.}$$

Note that the reverse implication holds only if θ is in the range of arcsin, i.e. $\theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

Using properties of inverse functions, we can also say that, for any $x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$,

$$\operatorname{arcsin}(\sin(x)) = x$$

and, for any $y \in [-1,1]$,

$$\sin(\operatorname{arcsin}(y)) = y.$$

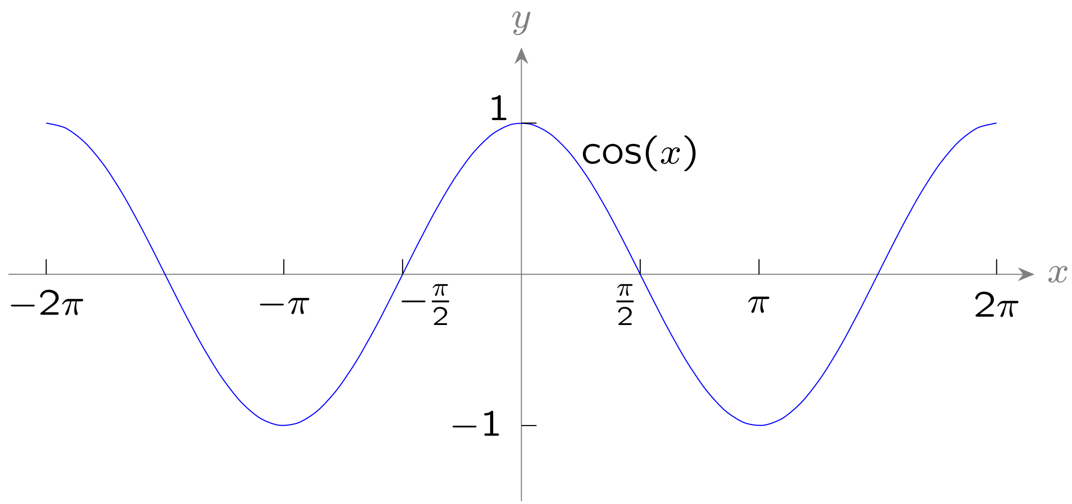
Example: Evaluate $\arcsin\left(\frac{1}{2}\right)$.

Example: Simplify $\arcsin\left(\sin\left(\frac{\pi}{4}\right)\right)$.

Homework: Evaluate $\arcsin\left(\sin\left(\frac{2\pi}{3}\right)\right)$.

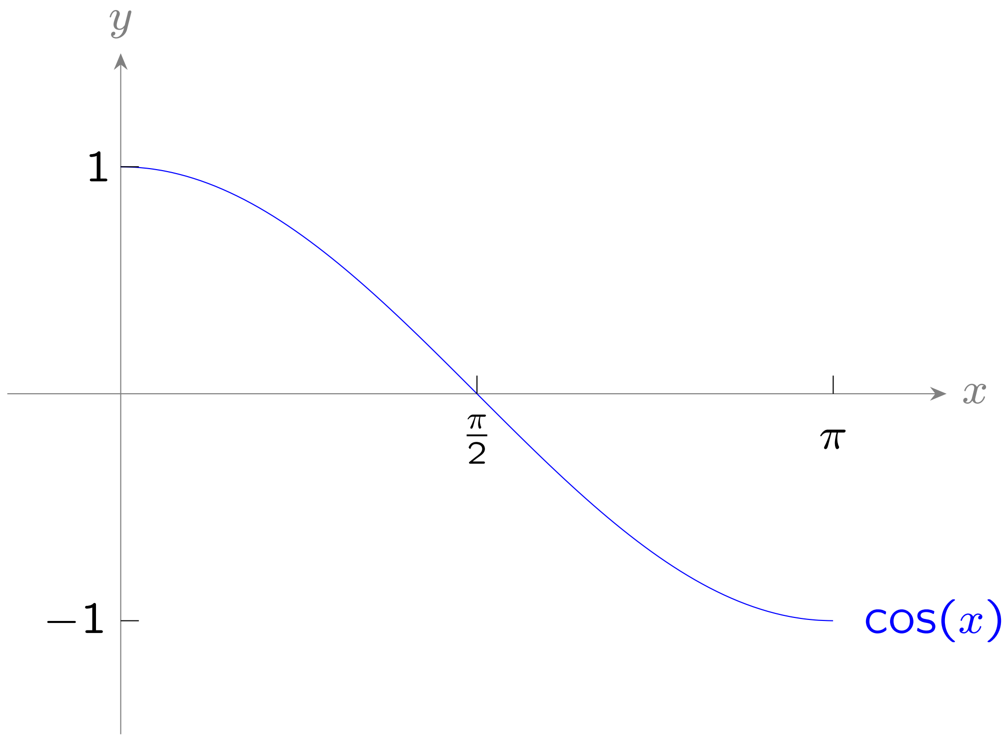
1.4.2 The inverse of the cosine function

Recall the graph of cosine:



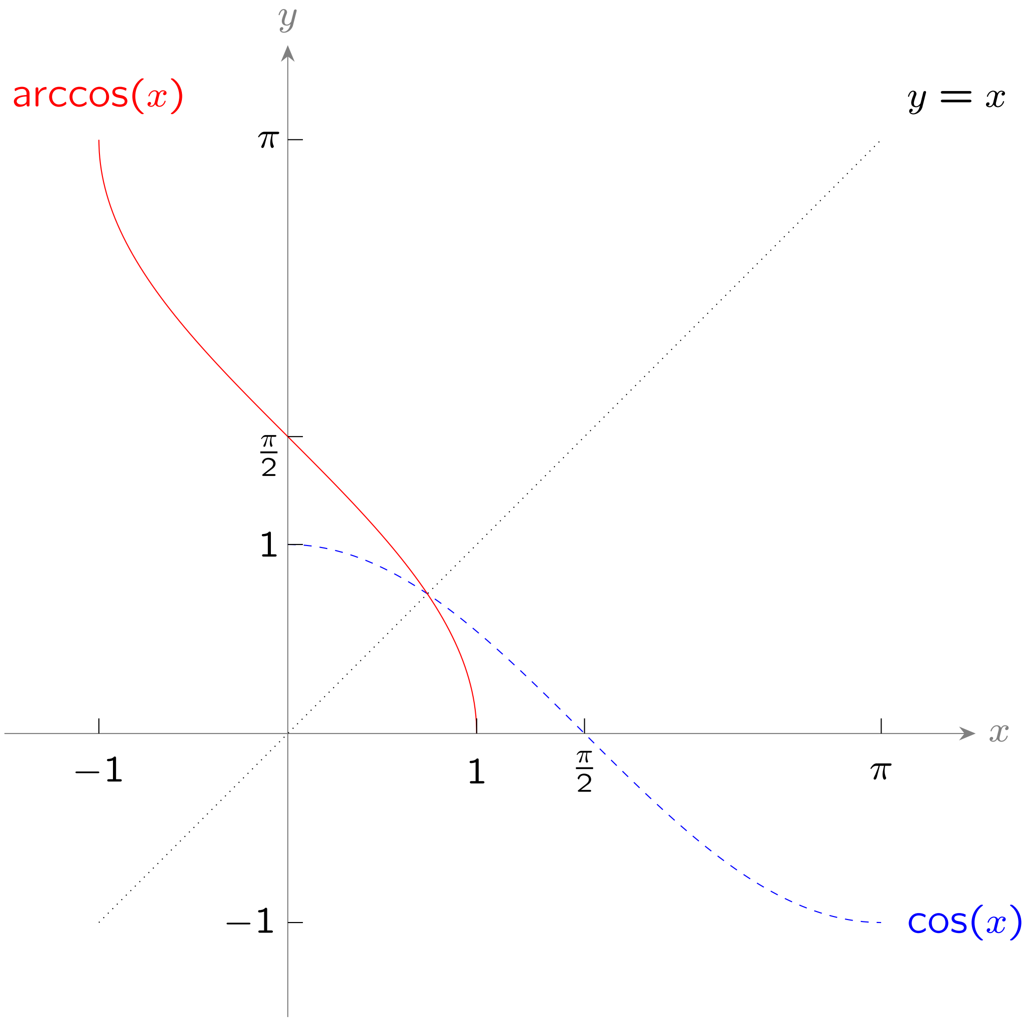
In a similar way we can restrict the domain of the cosine function to obtain a new one-to-one function.

In this case the restricted domain is chosen to be $[0, \pi]$.



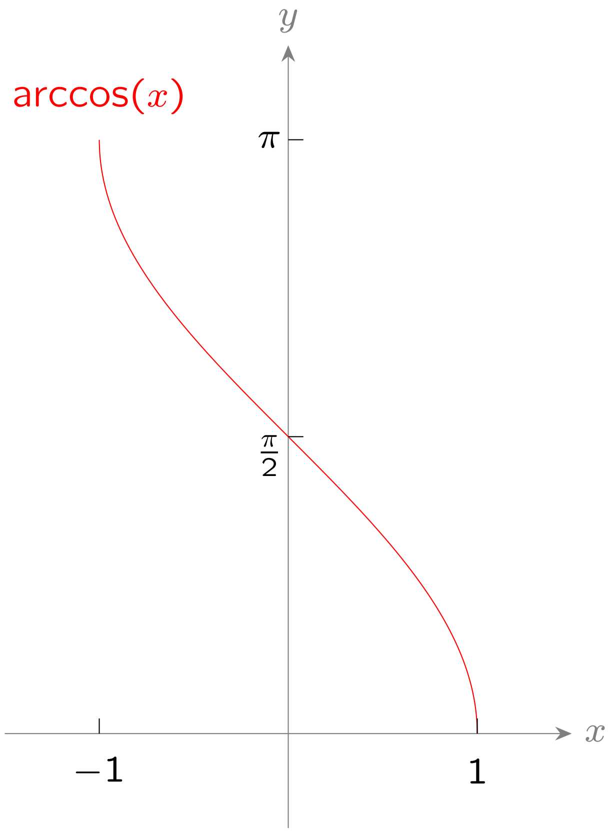
The range of this restricted function is still $[-1, 1]$.

We can find the inverse function by reflecting the graph through the line $y = x$.



We call this function *arccosine* (denoted $\arccos$).

Its domain is $[-1, 1]$ and its range is $[0, \pi]$.



The function $\mathsf{arccos}\colon [-1,1] \to [0,\pi]$ satisfies:

$$\boxed{\theta = \mathsf{arccos}(x) \qquad \Rightarrow \quad \cos(\theta) = x.}$$

Note that the reverse implication holds only if θ is in the range of arccos, i.e. $\theta \in [0,\pi]$.

For any $x \in [0,\pi]$,

$$\mathsf{arccos}(\cos(x)) = x$$

and for any $y \in [-1,1]$,

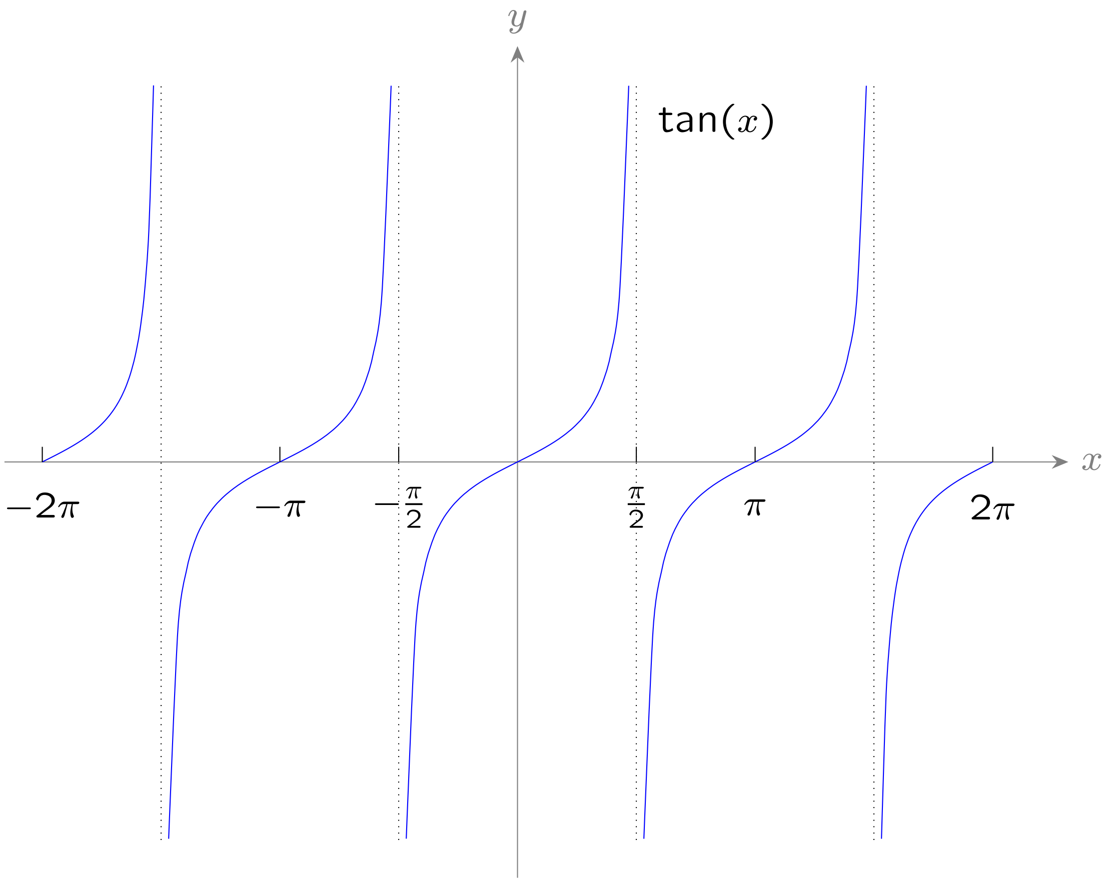
$$\cos(\mathsf{arccos}(y)) = y.$$

Example: Evaluate $\arccos(-1)$.

Example: Evaluate $\arccos\left(\sin\left(-\frac{\pi}{3}\right)\right)$.

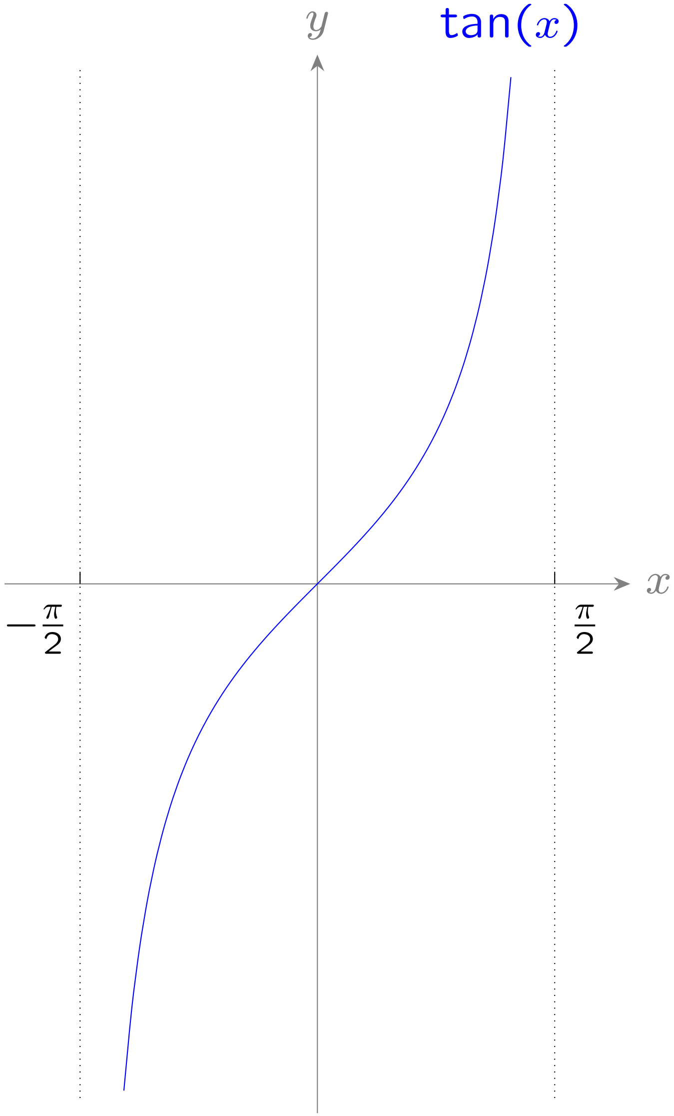
Homework: Evaluate $\arccos\left(\cos\left(-\frac{\pi}{3}\right)\right)$.

1.4.3 The inverse of the tangent function



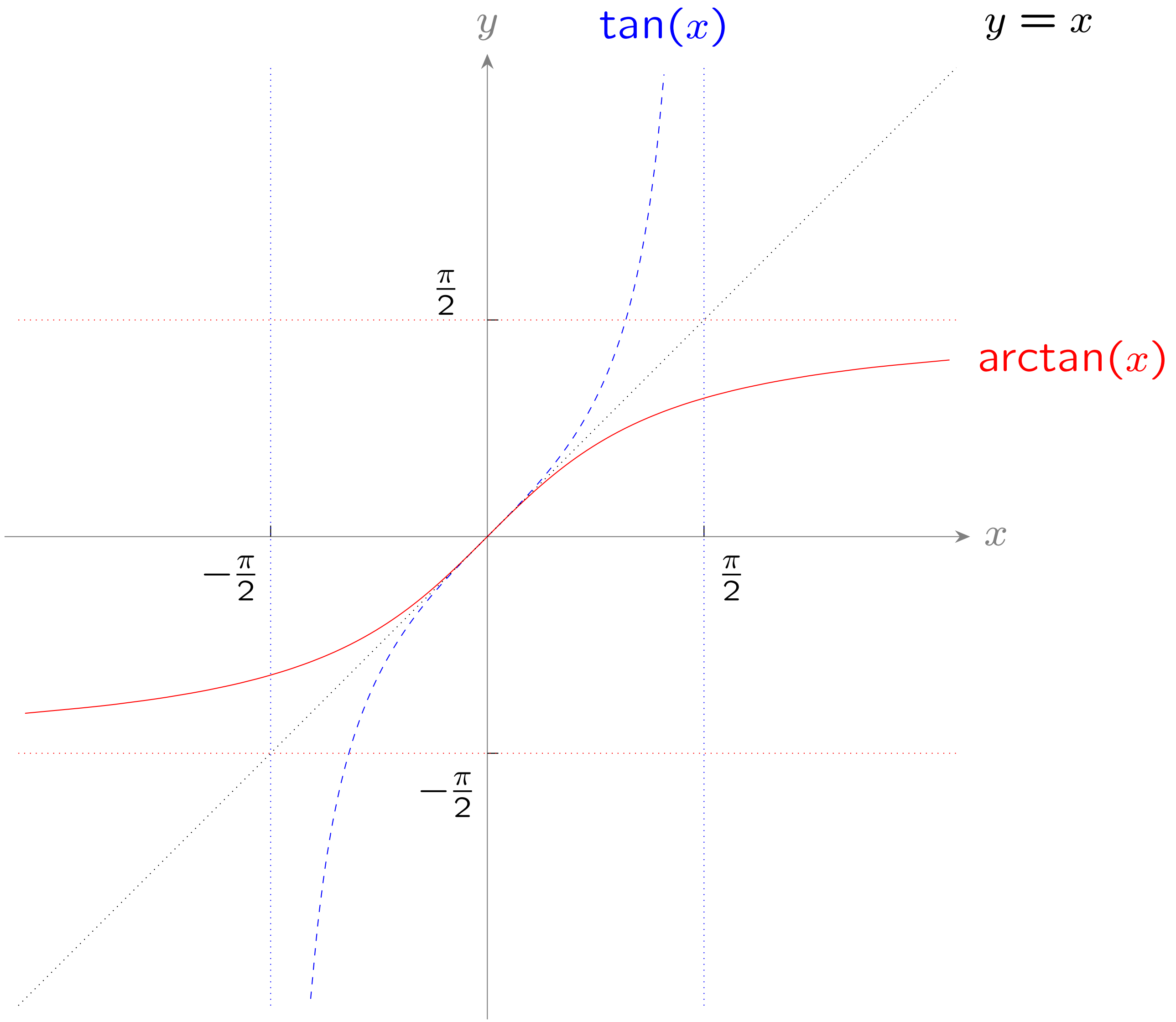
We can restrict the domain of the tangent function to $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ to obtain a new one-to-one function.

Notice that the endpoints are not included in this case. Why?

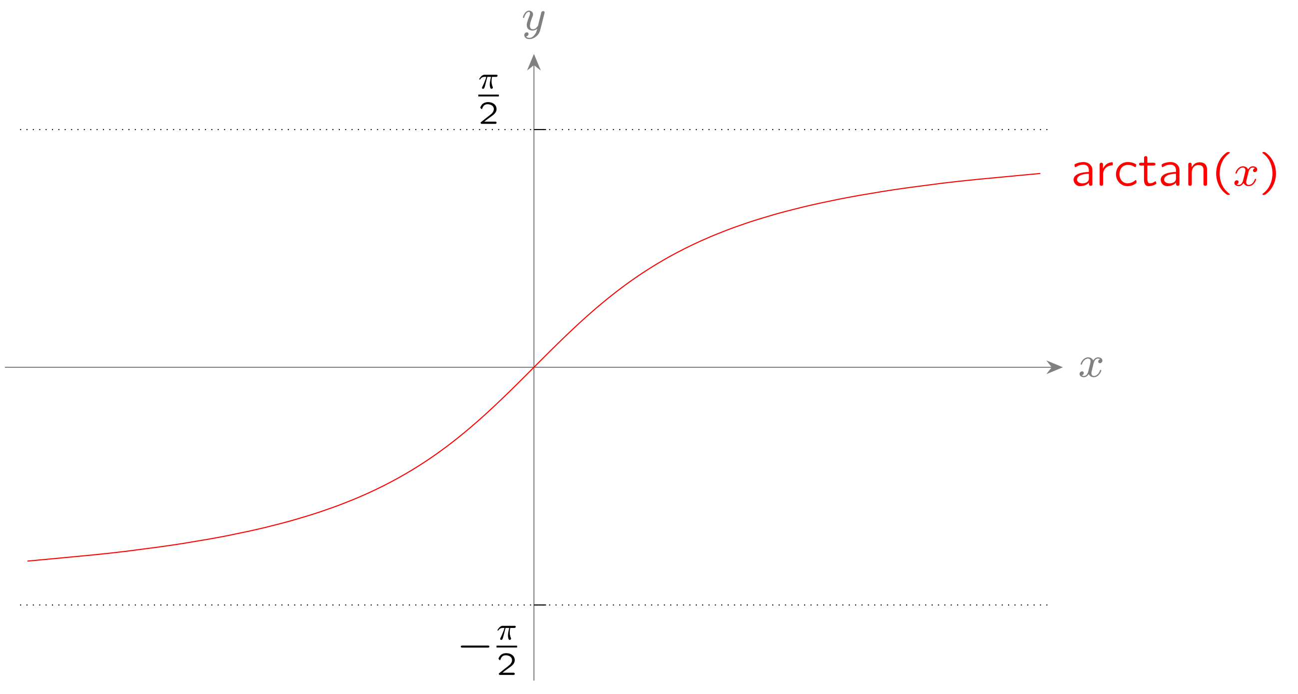


The range of this restricted function is still $\mathbb{R}$.

We obtain the inverse function by reflecting the graph through the line $y = x$.



We call this function *arctangent* (denoted $\mathtt{arctan}$).
Its domain is $\mathbb{R}$ and its range is $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.



The function $\mathsf{arctan}: \mathbb{R} \to \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ satisfies:

$$\boxed{\theta = \mathsf{arctan}(x) \qquad \Rightarrow \qquad \mathsf{tan}(\theta) = x.}$$

The reverse implication holds only if θ is in the range of arctan, i.e. $\theta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

For any $x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$,

$$\mathsf{arctan}(\mathsf{tan}(x)) = x$$

and for any $y \in \mathbb{R}$,

$$\mathsf{tan}(\mathsf{arctan}(y)) = y.$$

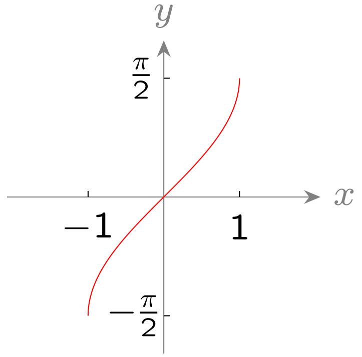
Beware! $\mathsf{arctan} \neq \frac{\mathsf{arcsin}}{\mathsf{arccos}}$

Example: Evaluate `arctan(1)`.

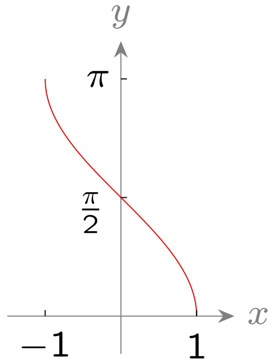
Example: Evaluate $\sin(\arctan(-\sqrt{3}))$.

Summary

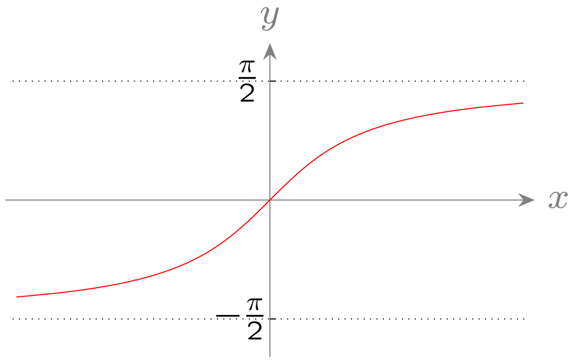
$$\arcsin\colon [-1,1] \to \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$$



$$\arccos\colon [-1,1] \to [0,\pi]$$



$$\arctan\colon \mathbb{R} \to \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$$



Homework: Evaluate the following:

(a) $\cos\left(\arcsin\left(-\frac{1}{\sqrt{2}}\right)\right)$

(b) $\arccos\left(\tan\left(-\frac{\pi}{4}\right)\right)$

(c) $\arctan\left(\sin\left(\pi\right)\right)$

(d) $\arcsin\left(\sin\left(\frac{2\pi}{3}\right)\right)$

Additional questions

You can attempt problems 14–16 from Topic 1 in the handbook.

You may also attempt a selection of problems from Exercise set 1.5 of the textbook.

1.5 Implied domain and range [Chapter 1.5]

Recall that a function f is a mapping that takes us from a set X, called the **domain** of f, to a set Y, called the **codomain** of f, according to some given rule. For example,

$$f\colon \mathbb{R} \to \mathbb{R}, \quad \text{where} \quad f(x) = x - 2 \quad \text{for each } x \in \mathbb{R},$$

is a function with domain $\mathbb{R}$ and codomain $\mathbb{R}$.

However,

$$g\colon (0, \infty) \to (-2, \infty), \quad \text{where} \quad g(x) = x - 2 \quad \text{for each } x \in (0, \infty),$$

is a function different from f because although the rules for f and g are the same, the domains and codomains are different.

The shorthand notations for the domain and codomain of f are $\mathrm{dom}(f)$ and $\mathrm{codom}(f)$.

The **range** of a function f, denoted $\mathsf{ran}(f)$, is the set of all values $f(x)$ such that x is in the domain of f.

Notice that $\mathsf{ran}(f) \subseteq \mathsf{codom}(f)$ but the two sets are not necessarily equal.

Sometimes we get sloppy and write down a function only by giving the rule and not the domain or codomain. So if I write a function

$$h(x) = x - 2$$

how can you tell what I mean? Is this the same as f, g or something different? In such a case, we assume that the domain of h is the biggest possible set for which h is defined; that is,

The **implied domain** of a function f is the set of all possible x values such that $f(x)$ is defined.

For example, consider the function $f(x) = \sqrt{x}$.

• The implied domain of f is $[0, \infty)$, since we can't take the square root of a negative number.

• The range of f is $[0, \infty)$, since for any $x \geq 0$, $f(x) \geq 0$ (that is, $\sqrt{x}$ is never negative).

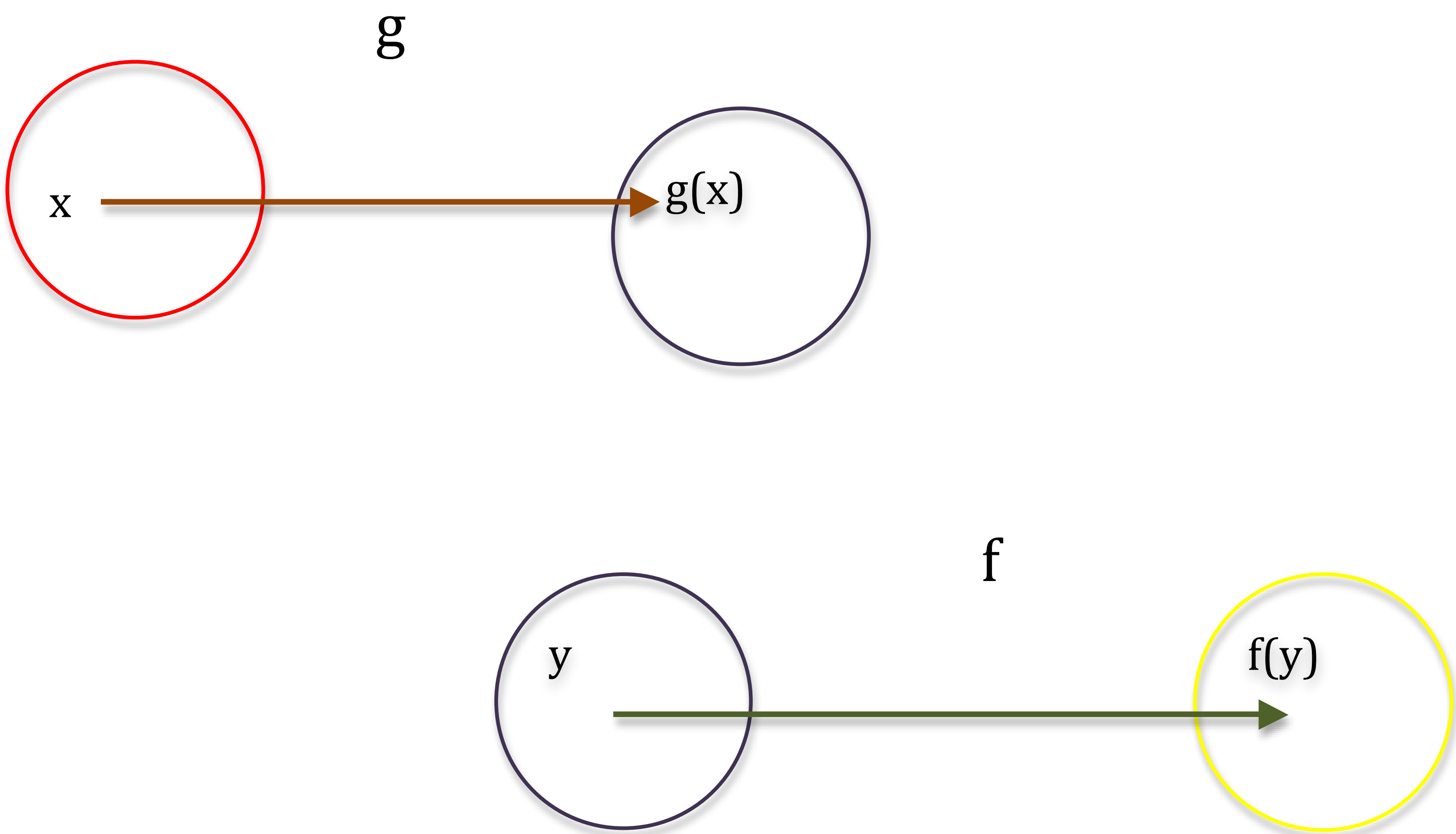
• There are infinitely many codomains for f. Any set $Y \subseteq \mathbb{R}$ with $[0, \infty) \subseteq Y$ is a codomain of f.

• For example: $\mathbb{R}$, $[-100, \infty)$, and $[0, \infty)$ are all codomains of f.

1.5.1 Composite functions

A **composite function** is a function of the form $f(g(x))$, that is, a function of a function. Sometimes the notation $f \circ g$ is used to denote this composite function.

For the composite function $f \circ g$ to be defined, the range of g must intersect with the domain of f. Hence the composite function $f \circ g$ is not defined for all possible pairs of functions f and g.


g
x
g(x)
f
y
f(y)

Example: Let $f : \mathbb{R} \to \mathbb{R},$ be the function $f(y) = y - 2$ and $g : \mathbb{R} \to \mathbb{R},$ the function $g(x) = 2x + 1$.
Is the composite function $f \circ g$ defined?

Example: Let $f : \mathbb{R} \to (-\infty, 0],$ be the function $f(y) = -y^2$ and $g : (0, \infty) \to \mathbb{R},$ the function $g(x) = 2x + 1.$
Is the composite function $g \circ f$ defined?

Composite functions require great care when finding their implied domains and ranges. In Example 1, the range of g and the domain of f were equal. In such a case, the implied domain of $f \circ g$ is the domain of g and the range of $f \circ g$ is the range of f.

However if the range of g is not equal to the domain of f, but the two sets intersect, the composition can still be defined but perhaps on a restricted set. This restricted set will be the implied domain of the composite function.

The **implied domain of** $f \circ g$ is the set of all x in the domain of g such that $(f \circ g)(x) = f(g(x))$ is defined. This can be expressed as:

> ***all x in the domain of g for which $g(x)$ is in the domain of f.***

Note that the range of $f \circ g$ may not be the same as the range of f.

The **range of** $f \circ g$ is the set of all y in the range of f such that $y = (f \circ g)(x)$ for some $g(x) \in \mathsf{ran}(g) \cap \mathsf{dom}(f)$. That is:

> ***all*** y ***in the range of*** f ***for which*** $y = (f \circ g)(x)$ ***for some*** x ***in the domain of*** g***.***

Example: Find the implied domain of $h(x) = \sqrt{\log(x)}$.

This is a composite function with inner function $\log$ and outer function the square root function. So:

- What is the range of the inner function $\log$?

- What is the domain of the outer function $\sqrt{}$?

- What is the intersection between the range of $\mathtt{log}$ and the domain of $\sqrt{\ }$?

Since this set is not equal to the range of $\mathtt{log}$, we must restrict the domain of $\mathtt{log}$ so that the range of the restricted function is $[0, \infty)$.

- What is the domain of the inner function $\mathtt{log}$?

• Which part of the domain of $\log$ is mapped to $[0, \infty)$?

• So the implied domain is:

In this case $\mathrm{ran}(g) \cap \mathrm{dom}(f) = \mathrm{dom}(f)$ so that $\mathrm{ran}(f \circ g) = \mathrm{ran}(f)$. That is, the range of $f \circ g$ is $[0, \infty)$.

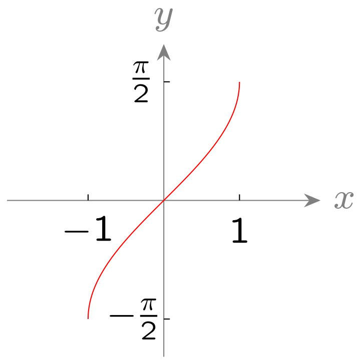
Example: Find the implied domain of $f \circ g$, where $f(x) = \log(x)$ and

$$g\colon \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \to [-1, 1] \quad \text{is given by} \quad g(x) = \sin(x).$$

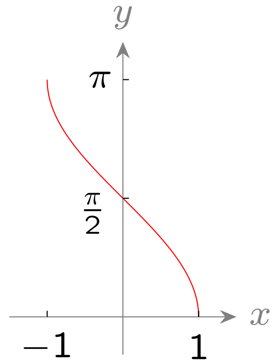
1.5.2 Implied domain and range involving inverse trigonometric functions

Recall the inverse trigonometric functions:

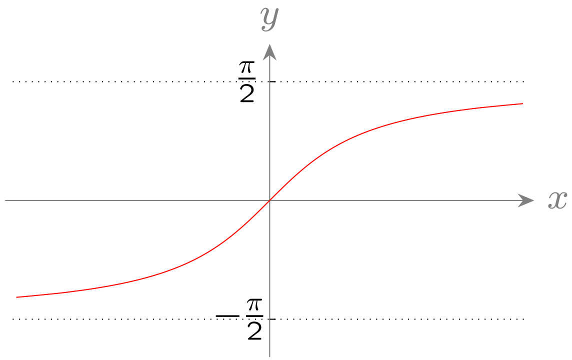
$\arcsin\colon [-1,1] \to \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$



$\arccos\colon [-1,1] \to [0,\pi]$



$\arctan\colon \mathbb{R} \to \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$



Example: Find the implied domain and the range of $f(x) = \arccos(2x - 3)$.

Example continued:

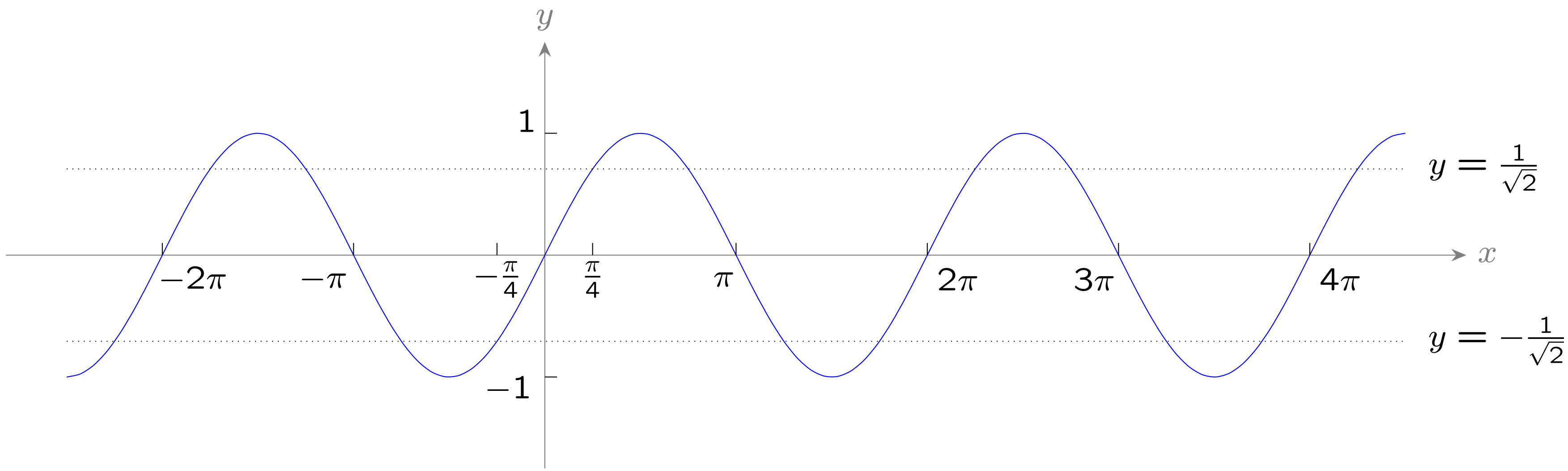
Example: Find the implied domain and the range of $f(x) = \arcsin\left(\sqrt{2}\,\sin(x)\right)$.

The implied domain here is the set of *all* x values such that

$$-\frac{1}{\sqrt{2}} \leq \sin(x) \leq \frac{1}{\sqrt{2}}.$$

This is the union of all intervals of the form

$$\left[-\frac{\pi}{4} + k\pi, \frac{\pi}{4} + k\pi\right] \qquad \text{for } k \in \mathbb{Z}.$$



Additional questions

You can now attempt the remaining problems from Topic 1 in the handbook.

You may also attempt a selection of problems from Chapter 1.5 of the textbook.

Topic 2: Vectors [Chapter 10.2]

2.1 Vector notation

2.2 Vector algebra

2.3 Rectangular coordinates

2.4 Length of a vector

2.5 Standard unit vectors

2.6 Dot product

2.7 Using vectors in problem solving

2.8 Scalar and vector projections

2.9 Graphs of circles, ellipses and hyperbolae

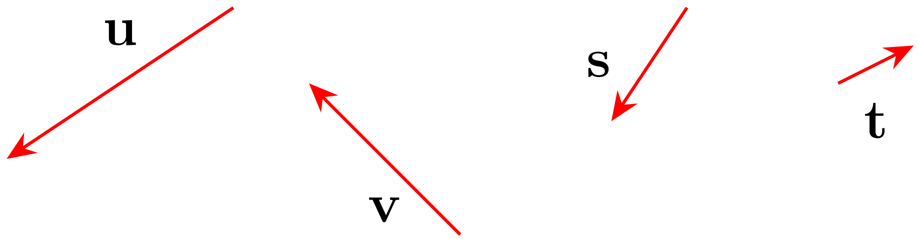
2.10 Parametric curves

There are many different types of quantifiable objects, for example, time, length, speed and height. We can quantify these things with a single (real) number which we call a *scalar*. Scalar quantities have *magnitude* only.

Other quantities, however, require more than just a scalar magnitude to be completely specified, for example: displacement, velocity and force. These quantities also have a *direction*. We call these quantities **vectors**. Vectors have magnitude and direction.

2.1 Vector notation

We can represent a vector diagramatically as a directed line segment:



The length of the line represents the magnitude of the vector and the orientation of the line represents its direction.

There are several different ways of writing a vector. $\overrightarrow{AB}$ means the vector starting at the point A and ending at the point B. The notation $\mathbf{u}$ (this is a boldface u) is also used to indicate that the quantity you are using is a vector rather than a scalar. When we are writing a vector by hand we often use a symbol called a *tilde*: $\underset{\sim}{u}$.

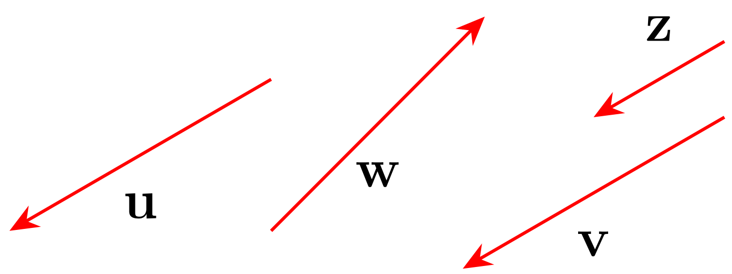
2.2 Vector algebra

Since we are no longer dealing with scalars, we need to define what we mean by equality, addition, subtraction and so on, for vectors.

2.2.1 Equality of vectors

Since vectors have both magnitude and direction, to say that two vectors are equal means that they have the same direction **and** the same magnitude.

Example:



In the figure above we can make the following statements:

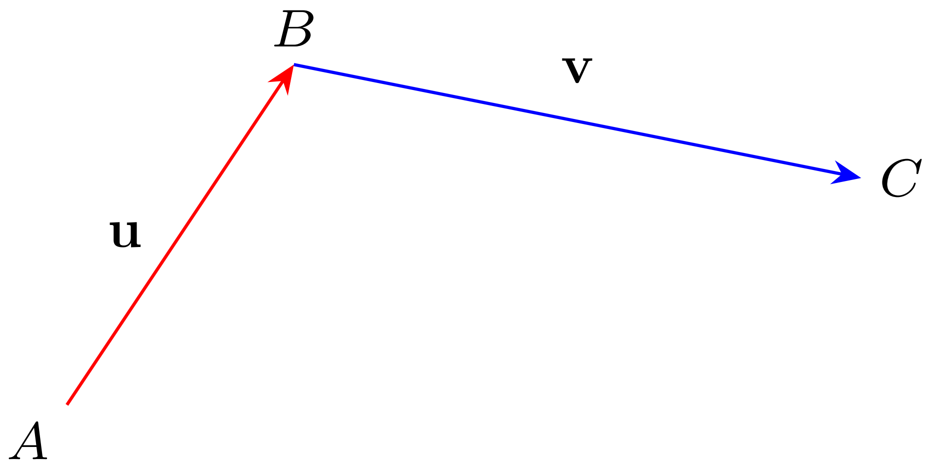
$$\mathbf{u} = \mathbf{v}$$

$$\mathbf{u} \neq \mathbf{w} \quad \text{(since they have different direction)}$$

$$\mathbf{u} \neq \mathbf{z} \quad \text{(since they have different magnitude).}$$

2.2.2 Addition of vectors

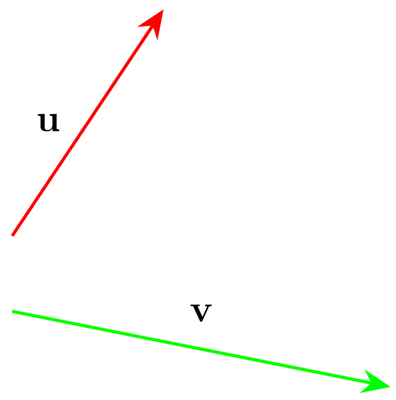
Suppose we have two vectors: $\mathbf{u} = \overrightarrow{AB}$ and $\mathbf{v} = \overrightarrow{BC}$.



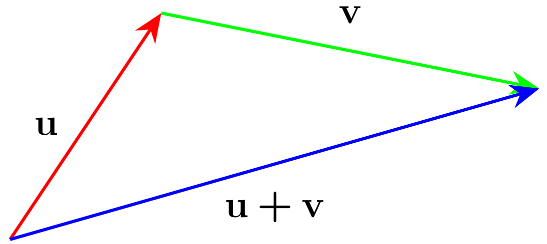
The overall effect can be represented by a new vector $\mathbf{w}$ which starts at the point A and ends at the point C. We say that

$$\mathbf{w} = \mathbf{u} + \mathbf{v}.$$

In general, to add two vectors $\mathbf{u}$ and $\mathbf{v}$, join the tail of $\mathbf{v}$ to the head of $\mathbf{u}$.



The resultant vector starts at the tail of $\mathbf{u}$ and ends at the head of $\mathbf{v}$:

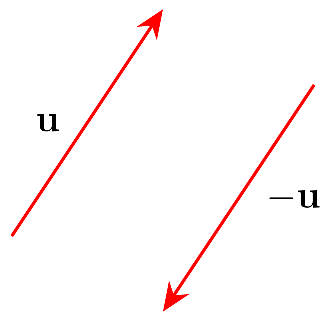


We call this resultant vector the **sum** of $\mathbf{u}$ and $\mathbf{v}$.

2.2.3 The zero vector

The zero vector is the vector that has zero length and no direction. It is denoted by $\mathbf{0}$ (boldface).

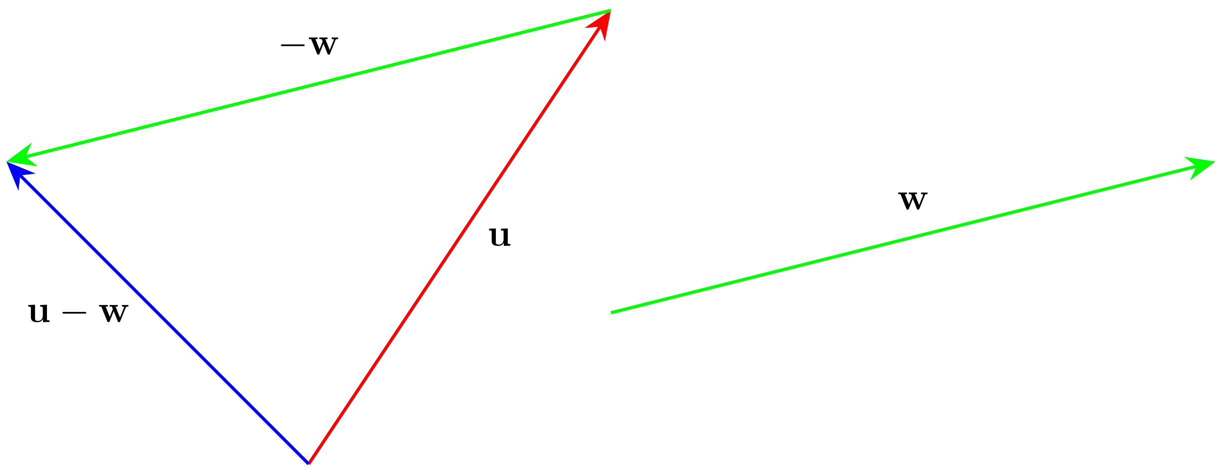
2.2.4 The negative of a vector



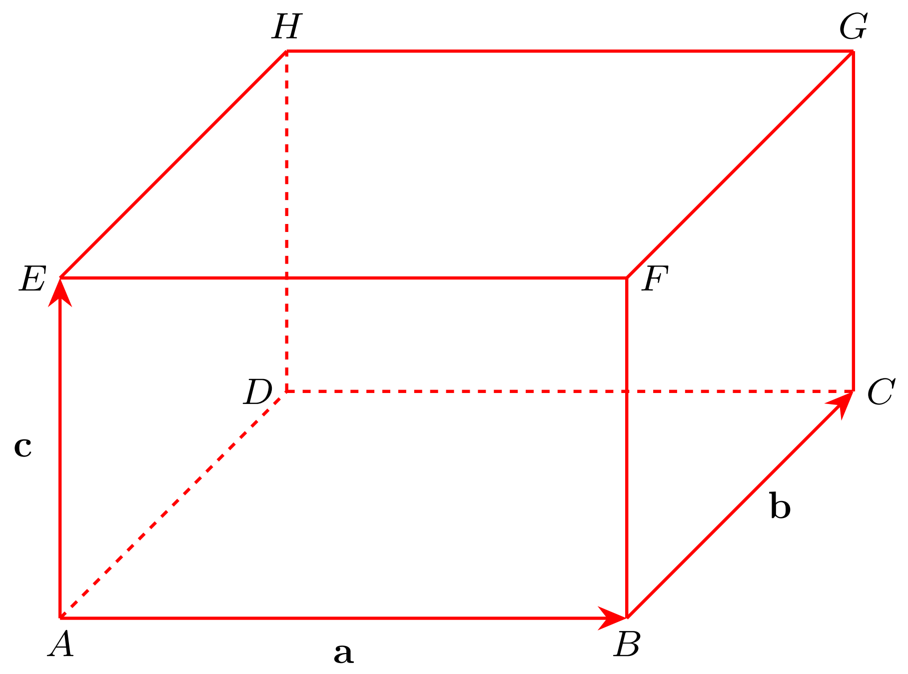
If $\mathbf{u}$ is the vector from A to B then $-\mathbf{u}$ is simply the vector from B to A: the negative of a vector has the same magnitude but the opposite direction of the original vector.

We subtract the vector $\mathbf{w}$ from the vector $\mathbf{u}$ by adding the negative of $\mathbf{w}$ to $\mathbf{u}$:

$$\mathbf{u} - \mathbf{w} = \mathbf{u} + (-\mathbf{w}).$$



Example: Consider the rectangular prism below, where $\overrightarrow{AB} = \mathbf{a}$, $\overrightarrow{BC} = \mathbf{b}$ and $\overrightarrow{AE} = \mathbf{c}$:



Express the following in terms of $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$:

(a) $\overrightarrow{EF}$

(b) $\overrightarrow{FB}$

(c) $\overrightarrow{AC}$

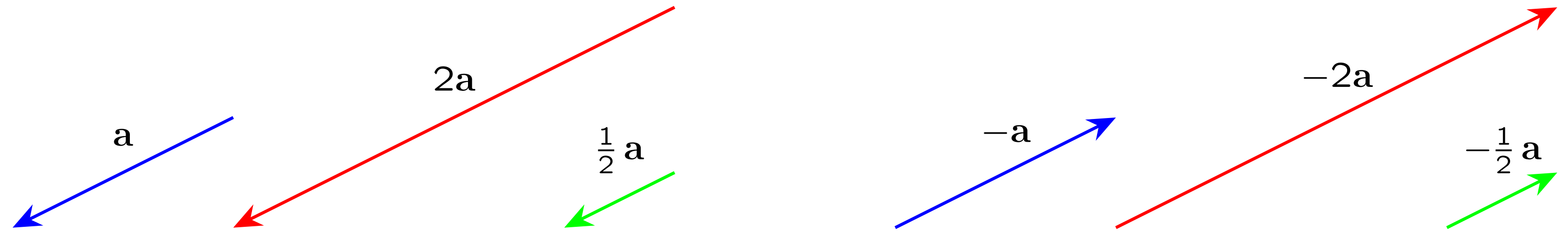
(d) $\overrightarrow{AG}$

(e) $\overrightarrow{DB}$

(f) $\overrightarrow{BH}$

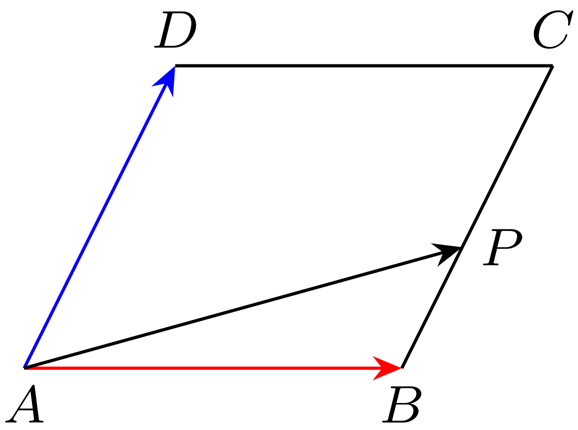
2.2.5 Multiplying a vector by a scalar

We can multiply any vector by a scalar (number). If the scalar is positive, then only the magnitude will be affected; that is, the direction will not change.



If the scalar is negative, the magnitude will change and the direction will be reversed.

Example: Let $ABCD$ be a parallelogram and let P divide the segment BC in the ratio $2\colon 3$. Express $\overrightarrow{AP}$ in terms of the vectors $\overrightarrow{AB}$ and $\overrightarrow{AD}$.



Example continued:

2.2.6 Parallel vectors

Two non-zero vectors $\mathbf{u}$ and $\mathbf{v}$ are said to be **parallel** if there is a non-zero scalar $k \in \mathbb{R}$ such that $\mathbf{u} = k\mathbf{v}$.

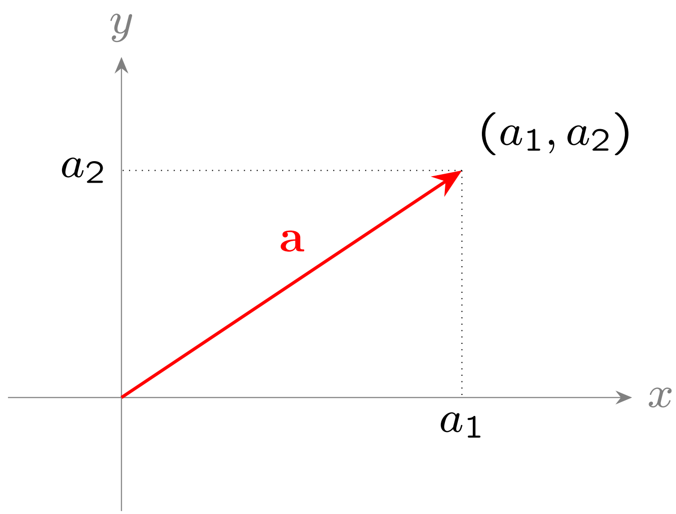
2.2.7 Properties of vectors

1. $\mathbf{a} + \mathbf{b} = \mathbf{b} + \mathbf{a}$ (vector addition is commutative)
2. $(\mathbf{a} + \mathbf{b}) + \mathbf{c} = \mathbf{a} + (\mathbf{b} + \mathbf{c})$ (vector addition is associative)
3. $\mathbf{a} + \mathbf{0} = \mathbf{a}$ (property of the zero vector)
4. $\mathbf{a} + (-\mathbf{a}) = \mathbf{0}$ (property of the negative vector)
5. $m(\mathbf{a} + \mathbf{b}) = m\mathbf{a} + m\mathbf{b}$ (scalar multiplication is distributive)

2.3 Rectangular coordinates

2.3.1 Rectangular coordinates in $\mathbb{R}^2$

To simplify problems involving vectors, we can introduce a rectangular coordinate system. Let $\mathbf{a}$ be a vector in the plane that has its starting point at the origin of a rectangular coordinate system. Its tip is at the point (a_1, a_2) as pictured below.



The coordinates (a_1, a_2) are called the **components** of $\mathbf{a}$, and we say that

$$\mathbf{a} = (a_1, a_2).$$

If $\mathbf{a} = (a_1, a_2)$ and $\mathbf{b} = (b_1, b_2)$ are both vectors with initial points at the origin, they are equal exactly when their terminal points coincide. In terms of components, this means that

$$\mathbf{a} = \mathbf{b} \qquad \text{if and only if} \qquad a_1 = b_1 \;\text{ and }\; a_2 = b_2.$$

The algebraic operations that we have already defined for vectors are easy to perform when we write vectors in component form. We simply apply the operations *component-wise*.

Examples: Let $\mathbf{v} = (3, -2)$ and $\mathbf{w} = (1, 5)$. Calculate

(a) $\mathbf{v} + \mathbf{w}$

(b) $\mathbf{v} - \mathbf{w}$

(c) $\mathbf{w} - \mathbf{v}$

(d) $3\mathbf{v}$

Homework: Let $\mathbf{v} = (1, -3)$ and $\mathbf{w} = (-1, -4)$. Repeat (a)–(d) above.

2.3.2 Rectangular coordinates in $\mathbb{R}^n$

We can extend the definition of a vector in component form to any number of components we like. So $\mathbf{v} = (1, 4, 7)$ is a vector in $\mathbb{R}^3$ whilst $\mathbf{w} = (1, 0, 2, 4, 7)$ is a vector in $\mathbb{R}^5$. The same algebraic operations apply in any $\mathbb{R}^n$.

Be careful, however, because you can only perform operations between vectors of the same dimension (that is, between vectors with the same number of components).

So:

$$(1, 4, 7) + (2, 0, 1) =$$

but $\quad (1, 4, 7) + (2, 0) =$

The properties of vectors in section 2.2.7 can be easily proved using rectangular coordinates in $\mathbb{R}^n$.

Example: Prove that $\mathbf{a} + \mathbf{b} = \mathbf{b} + \mathbf{a}$ for vectors $\mathbf{a}, \mathbf{b} \in \mathbb{R}^3$.

Let $\mathbf{a} = (a_1, a_2, a_3)$ and $\mathbf{b} = (b_1, b_2, b_3)$ be vectors in $\mathbb{R}^3$. Then

$$
\begin{aligned}
\mathbf{a} + \mathbf{b} &= (a_1, a_2, a_3) + (b_1, b_2, b_3) \\
&= (a_1 + b_1, a_2 + b_2, a_3 + b_3) \quad && \text{(by definition of vector addition)} \\
&= (b_1 + a_1, b_2 + a_2, b_3 + a_3) \quad && \text{(addition of real numbers is commutative)} \\
&= (b_1, b_2, b_3) + (a_1, a_2, a_3) \quad && \text{(by definition of vector addition)} \\
&= \mathbf{b} + \mathbf{a}.
\end{aligned}
$$

2.3.3 Vectors with initial points not at the origin

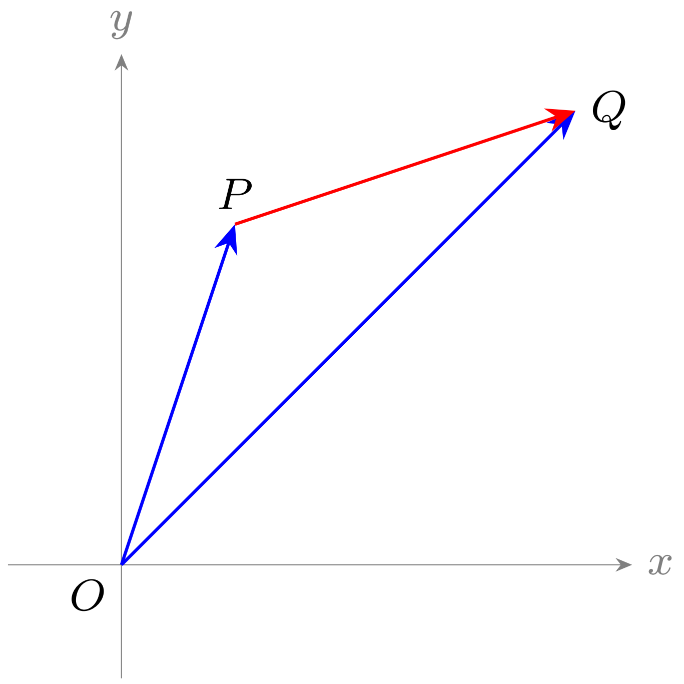
Sometimes a vector is positioned so that its initial point is not at the origin. Suppose that the vector $\overrightarrow{PQ}$ has initial point $P(x, y, z)$ and terminal point $Q(a, b, c)$.

We can work out the components of $\overrightarrow{PQ}$ as follows:

$$\begin{aligned}\overrightarrow{PQ} &= \overrightarrow{PO} + \overrightarrow{OQ} \\ &= \overrightarrow{OQ} - \overrightarrow{OP} \\ &= (a, b, c) - (x, y, z) \\ &= (a - x, b - y, c - z).\end{aligned}$$

So:

$$\boxed{\overrightarrow{PQ} = (a - x, b - y, c - z).}$$



An easy way to remember this is

$$\boxed{\begin{aligned}\overrightarrow{PQ} &= \text{endpoint} - \text{initial point}\\ &= \overrightarrow{OQ} - \overrightarrow{OP}.\end{aligned}}$$

Example: If P_1 has coordinates $(1, 5, -2)$ and P_2 has coordinates $(2, -1, 0)$, find $\overrightarrow{P_1P_2}$.

Homework: If A has coordinates $(2, -3, -7)$ and B has coordinates $(1, -3, 2)$, find $\overrightarrow{AB}$.

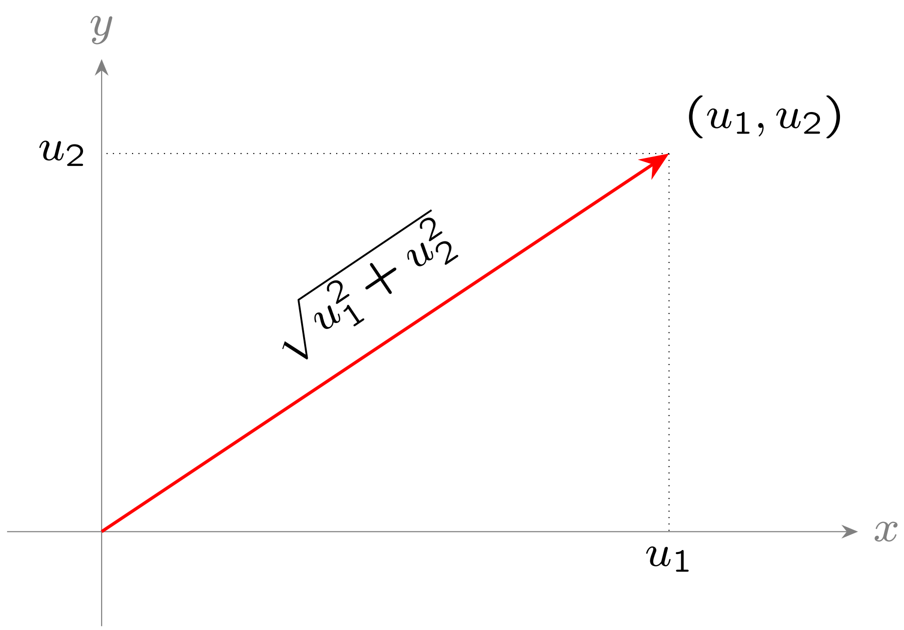
Additional questions

You can now attempt problems 1–4 from Topic 2 in the handbook.

You may also attempt a selection of exercises 1–12 from Exercise set 10.2 in the textbook.

2.4 Length of a vector [Chapter 10.2]

As already discussed, every vector has a magnitude or *length*. We can find the length of a vector in component form by using Pythagoras's Theorem.



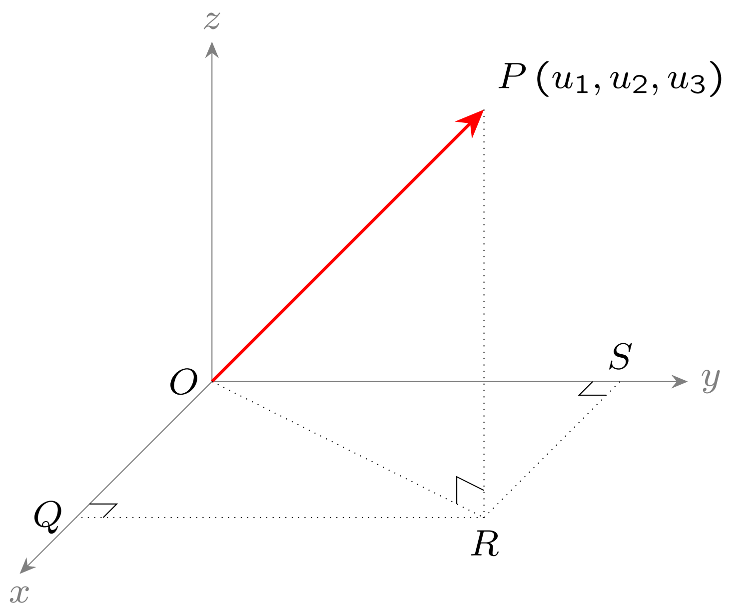
The **length** of $\mathbf{u}$ (also called the **norm** of $\mathbf{u}$), denoted $\|\mathbf{u}\|$, is given by:

$$\|\mathbf{u}\| = \sqrt{u_1^2 + u_2^2}$$

We can extend this definition to a vector $\mathbf{u}$ in $\mathbb{R}^n$ (for any $n \in \mathbb{N}$):

$$\|\mathbf{u}\| = \sqrt{u_1^2 + u_2^2 + \cdots + u_n^2}$$

To see the $\mathbb{R}^3$ case, first note that the length of the line segment OR below is $\sqrt{u_1^2 + u_2^2}$.



Therefore

$$
\begin{aligned}
\left\|\overrightarrow{OP}\right\| &= \sqrt{\left\|\overrightarrow{OR}\right\|^2 + \left\|\overrightarrow{RP}\right\|^2} \\
&= \sqrt{\left(\sqrt{u_1^2 + u_2^2}\right)^2 + u_3^2} \\
&= \sqrt{u_1^2 + u_2^2 + u_3^2} \qquad \text{as claimed.}
\end{aligned}
$$

Example: Find the length of the vectors (a) $(1, 2)$ (b) $(-1, 3, -2)$.

Homework: Find the length of the vectors (a) $(-2, -1)$ (b) $(-3, 6, -2)$.

2.4.1 Properties of the norm

1. for any vector $\mathbf{u}$, $\|\mathbf{u}\| \geq 0$; $\|\mathbf{u}\| = 0$ if and only if $\mathbf{u} = \mathbf{0}$.
2. $\|k\mathbf{u}\| = |k|\|\mathbf{u}\|$ for any $k \in \mathbb{R}$ and any vector $\mathbf{u}$.
3. $\|\mathbf{u} + \mathbf{v}\| \leq \|\mathbf{u}\| + \|\mathbf{v}\|$ for any vectors $\mathbf{u}$ and $\mathbf{v}$.

Example: We prove property 2 for $\mathbf{u} = (u_1, u_2, u_3) \in \mathbb{R}^3$.

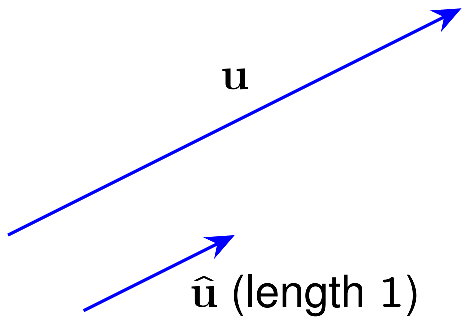
$$
\begin{aligned}
\|k\mathbf{u}\| &= \|k\,(u_1, u_2, u_3)\,\| \\
&= \|\,(ku_1, ku_2, ku_3)\,\| \\
&= \sqrt{(ku_1)^2 + (ku_2)^2 + (ku_3)^2} \\
&= \sqrt{k^2\left(u_1^2 + u_2^2 + u_3^2\right)} \\
&= |k|\sqrt{u_1^2 + u_2^2 + u_3^2} \\
&= |k|\|\mathbf{u}\|.
\end{aligned}
$$

2.4.2 Unit vectors

A vector of length 1 is called a **unit vector**. If $\mathbf{u}$ is any nonzero vector, we can construct a unit vector in the same direction as $\mathbf{u}$ by simply multiplying the vector $\mathbf{u}$ by the reciprocal of its length $\|\mathbf{u}\|$. (We will refer to this as *dividing* $\mathbf{u}$ *by its length*.)

We write $\hat{\mathbf{u}}$ to denote the unit vector in the direction of $\mathbf{u}$:

$$\boxed{\hat{\mathbf{u}} = \frac{\mathbf{u}}{\|\mathbf{u}\|}}$$



We can check that $\hat{\mathbf{u}}$ has length 1 as follows:

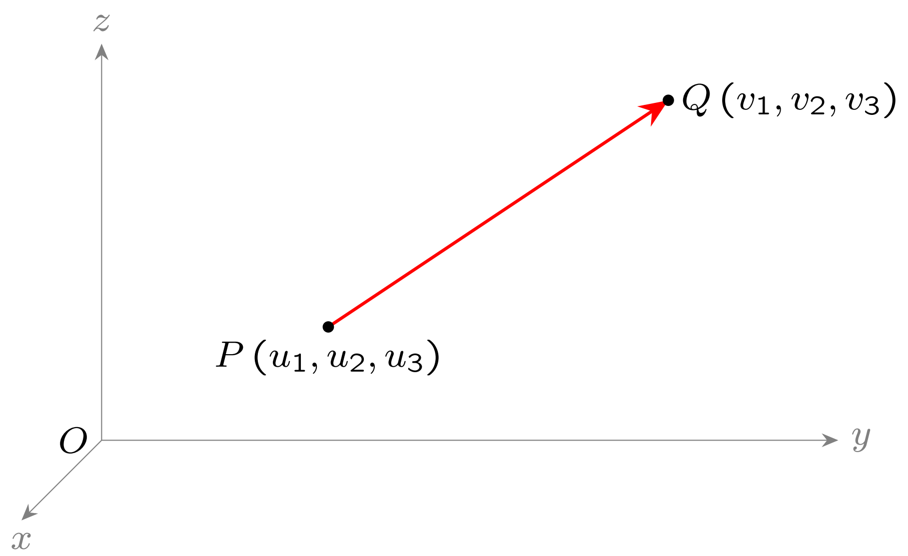
$$\|\hat{\mathbf{u}}\| = \left\|\frac{\mathbf{u}}{\|\mathbf{u}\|}\right\| = \left\|\frac{1}{\|\mathbf{u}\|}\mathbf{u}\right\| = \frac{1}{\|\mathbf{u}\|}\|\mathbf{u}\| = 1.$$

Example: Is the vector $\mathbf{v} = (1, -2, 1)$ a unit vector? If it is not, find a unit vector in the same direction as $\mathbf{v}$.

Example: Let $\mathbf{v}$ be the vector from $A(2, 0, -1)$ to $B(1, 2, -3)$. Find two unit vectors parallel to $\mathbf{v}$.

2.4.3 Distance between two points

If $P\,(u_1, u_2, u_3)$ and $Q\,(v_1, v_2, v_3)$ are two points in $\mathbb{R}^3$, then the distance between them is the length of the vector $\overrightarrow{PQ}$:



We know that $\overrightarrow{PQ} = (v_1 - u_1, v_2 - u_2, v_3 - u_3)$, so the distance between P and Q is

$$\text{dist}(P, Q) = \left\|\overrightarrow{PQ}\right\| = \sqrt{(v_1 - u_1)^2 + (v_2 - u_2)^2 + (v_3 - u_3)^2}.$$

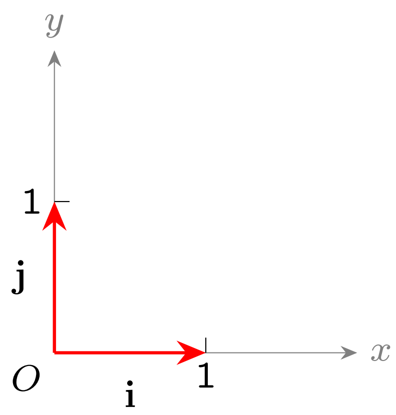
Clearly we can extend this definition to $\mathbb{R}^n$ in the obvious way.

Example: Find the distance between the points $P(-2, 1, 0)$ and $Q(3, -1, 1)$.

Homework: Find the distance between the points $A(-3, 1, -1)$ and $B(2, 0, -1)$.

2.5 Standard unit vectors

We now introduce some special unit vectors. Let $\mathbf{i} = (1, 0)$ and $\mathbf{j} = (0, 1)$. It is easy to see that both of these vectors have length 1. It is also easy to see that the vectors lie along the x-axis and y-axis respectively.



They are called the **standard unit vectors** in $\mathbb{R}^2$.

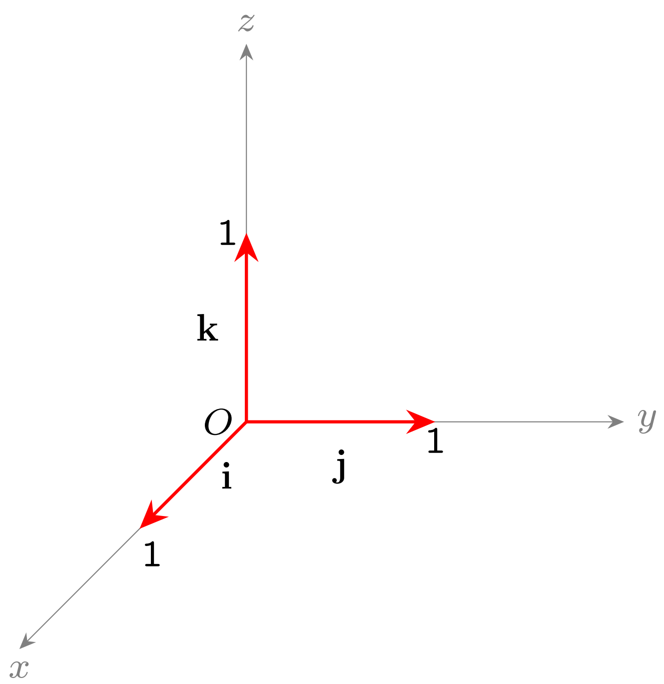
Every vector in $\mathbb{R}^2$ can be expressed in terms of $\mathbf{i}$ and $\mathbf{j}$.

Example: Express the vectors $\mathbf{v} = (1, 2)$ and $\mathbf{u} = (-1, 3)$ in terms of $\mathbf{i}$ and $\mathbf{j}$.

We can extend the definition of standard unit vectors to $\mathbb{R}^3$ by viewing $\mathbf{i}$ and $\mathbf{j}$ as vectors in $\mathbb{R}^3$ and introducing another unit vector $\mathbf{k}$ in the z-direction.

The standard unit vectors in $\mathbb{R}^3$ are therefore:

$$\mathbf{i} = (1, 0, 0),\ \mathbf{j} = (0, 1, 0) \text{ and } \mathbf{k} = (0, 0, 1).$$



Every vector in $\mathbb{R}^3$ can be expressed in terms of $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$.

Example: Express the vectors $\mathbf{v} = (-3, 1, 2)$ and $\mathbf{u} = (-1, 1, -2)$ in terms of $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$.

In general, if $\mathbf{u} = (u_1, u_2) \in \mathbb{R}^2$, then

$$\mathbf{u} = u_1\,\mathbf{i} + u_2\,\mathbf{j}.$$

Similarly, if $\mathbf{v} = (v_1, v_2, v_3) \in \mathbb{R}^3$, then

$$\mathbf{v} = v_1\,\mathbf{i} + v_2\,\mathbf{j} + v_3\,\mathbf{k}.$$

The two notations (rectangular coordinates or $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$) may be used interchangeably, and the same algebraic operations apply.

Warning! It is important to specify whether a vector of the form $\mathbf{u} = u_1\,\mathbf{i} + u_2\,\mathbf{j}$ is in $\mathbb{R}^2$ or $\mathbb{R}^3$. If $\mathbf{u}$ is in $\mathbb{R}^3$ we can write $\mathbf{u} = u_1\,\mathbf{i} + u_2\,\mathbf{j} + 0\,\mathbf{k}$ to be clear.

Example: If $\mathbf{u} = \mathbf{i} + 3\mathbf{j} - 2\mathbf{k}$ and $\mathbf{v} = 2\mathbf{i} - \mathbf{j} + 3\mathbf{k}$, find

1. $\mathbf{u} + \mathbf{v}$

2. $\mathbf{v} - \mathbf{u}$

3. $2\mathbf{u} + 3\mathbf{v}$

Additional questions

You can now attempt problems 5–8 from Topic 2 in the handbook.

You may also attempt a selection of exercises 9 – 34 from Exercise set 10.2 in the textbook.

2.6 The dot product [Chapter 10.3]

We have seen how to multiply a vector by a scalar but we have not yet discussed the possibility of multiplying a vector by another vector. In vector algebra there are several concepts of multiplication of vectors. One is called the **dot product**, which we discuss now; another, the *cross product*, is specific to vectors in $\mathbb{R}^3$ and will be introduced in Calculus 2.

2.6.1 Definition of the dot product

Let $\mathbf{u}$ and $\mathbf{v}$ be vectors in $\mathbb{R}^n$ (for some $n \in \mathbb{N}$), with components $\mathbf{u} = (u_1, u_2, \ldots, u_n)$ and $\mathbf{v} = (v_1, v_2, \ldots, v_n)$.

The **dot product** (also known as **scalar product**) of $\mathbf{u}$ and $\mathbf{v}$ is defined as:

$$\boxed{\mathbf{u} \cdot \mathbf{v} = u_1 v_1 + u_2 v_2 + \ldots + u_n v_n.}$$

Note that the dot product is a sum of real numbers, therefore it is a scalar quantity (not a vector)!

For instance, in $\mathbb{R}^3$ we have $\mathbf{u} = (u_1, u_2, u_3)$, $\mathbf{v} = (v_1, v_2, v_3)$ and

$$\mathbf{u} \cdot \mathbf{v} = u_1 v_1 + u_2 v_2 + u_3 v_3.$$

Example: Let $\mathbf{u} = (2, 3, -1)$ and $\mathbf{v} = (4, 5, 0)$. Calculate $\mathbf{u} \cdot \mathbf{v}$.

Homework: Calculate $\mathbf{u} \cdot \mathbf{v}$ if $\mathbf{u} = (3, 2, 1)$ and $\mathbf{v} = (1, 1, 3)$.

2.6.2 Properties of the dot product

Let $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ be vectors in $\mathbb{R}^n$ and let $\lambda \in \mathbb{R}$.

1. The dot product of two vectors is a scalar.
2. The dot product is *commutative*; that is
$$\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}.$$
3. The dot product is *distributive*; that is
$$\mathbf{v} \cdot (\mathbf{u} + \mathbf{w}) = \mathbf{v} \cdot \mathbf{u} + \mathbf{v} \cdot \mathbf{w}.$$
4. $\lambda(\mathbf{u} \cdot \mathbf{v}) = (\lambda \mathbf{u}) \cdot \mathbf{v} = \mathbf{u} \cdot (\lambda \mathbf{v})$.
5. If $\mathbf{u}$ and $\mathbf{v}$ are perpendicular, then $\mathbf{u} \cdot \mathbf{v} = 0$. (We'll see why this is true in section 2.6.4.)
6. The dot product of a vector $\mathbf{u}$ with itself satisfies $\mathbf{u} \cdot \mathbf{u} = \|\mathbf{u}\|^2$.
7. $\mathbf{u} \cdot \mathbf{u} \geq 0$, and $\mathbf{u} \cdot \mathbf{u} = 0$ if and only if $\mathbf{u} = \mathbf{0}$.

Some questions

1. Can you take the dot product of three vectors?

2. Why is property 6 true?

3. What if $\mathbf{u}$ and $\mathbf{v}$ are parallel?

2.6.3 A couple of proofs

We can use the definition of the dot product to prove some general facts about the dot product. For example:

1. Let $\mathbf{u} = (u_1, u_2, \ldots, u_n)$ and $\mathbf{v} = (v_1, v_2, \ldots, v_n)$ be two vectors in $\mathbb{R}^n$ and let $\lambda \in \mathbb{R}$. Then

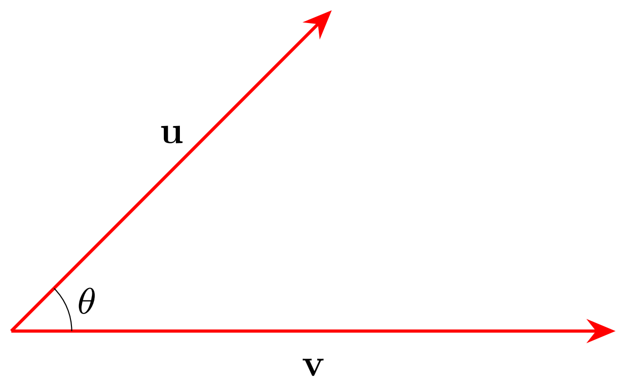
$$\begin{aligned}\lambda(\mathbf{u} \cdot \mathbf{v}) &= \lambda\,(u_1v_1 + u_2v_2 + \cdots + u_nv_n)\\ &= \lambda u_1v_1 + \lambda u_2v_2 + \cdots + \lambda u_nv_n\\ &= (\lambda u_1)v_1 + (\lambda u_2)v_2 + \cdots + (\lambda u_n)v_n\\ &= (\lambda\mathbf{u}) \cdot \mathbf{v}.\end{aligned}$$

2. If $\mathbf{0}$ is the zero vector, we have:

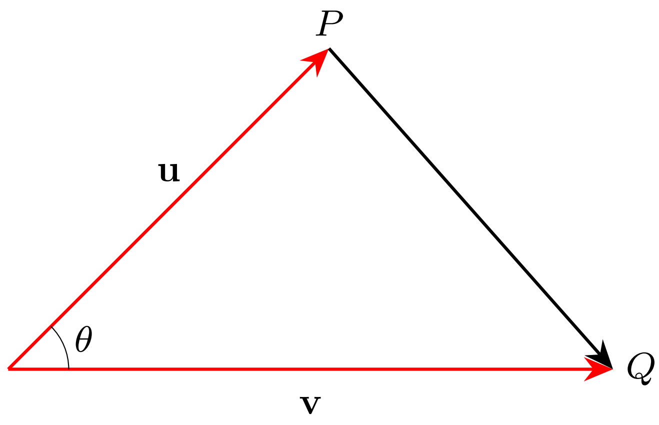
$$\begin{aligned}\mathbf{u} \cdot \mathbf{0} &= (u_1, u_2, \ldots, u_n) \cdot (0, 0, \ldots, 0)\\ &= u_1 \times 0 + u_2 \times 0 + \cdots + u_n \times 0\\ &= 0.\end{aligned}$$

2.6.4 Dot product and the angle between vectors

Let $\mathbf{u}$ and $\mathbf{v}$ be non-zero vectors in $\mathbb{R}^n$ and assume that they are positioned so that their tails meet. We define the **angle between** $\mathbf{u}$ and $\mathbf{v}$ to be the angle θ such that $0 \leq \theta \leq \pi$ as pictured below.



The notion of angle between $\mathbf{u}$ and $\mathbf{v}$ is closely related to the dot product $\mathbf{u} \cdot \mathbf{v}$. To see this, we need the Law of Cosines.



The Law of Cosines says that

$$\|\overrightarrow{PQ}\|^2 = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 - 2\|\mathbf{u}\|\|\mathbf{v}\|\cos(\theta).$$

Since $\overrightarrow{PQ} = -\mathbf{u} + \mathbf{v} = \mathbf{v} - \mathbf{u}$, this can be rearranged to give:

$$\cos(\theta) = \frac{\|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 - \|\mathbf{v} - \mathbf{u}\|^2}{2\|\mathbf{u}\|\|\mathbf{v}\|}.$$

We can simplify this expression by working on the numerator.

Using the properties of the dot product, we have

$$\begin{aligned}\|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 - \|\mathbf{v}-\mathbf{u}\|^2 &= \mathbf{u}\cdot\mathbf{u} + \mathbf{v}\cdot\mathbf{v} - (\mathbf{v}-\mathbf{u})\cdot(\mathbf{v}-\mathbf{u}) \\ &= \mathbf{u}\cdot\mathbf{u} + \mathbf{v}\cdot\mathbf{v} - (\mathbf{v}\cdot\mathbf{v} - 2\mathbf{u}\cdot\mathbf{v} + \mathbf{u}\cdot\mathbf{u}) \\ &= 2\mathbf{u}\cdot\mathbf{v}.\end{aligned}$$

We conclude that

$$\boxed{\cos(\theta) = \frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{u}\|\|\mathbf{v}\|}.}$$

In particular, this gives us a simple explanation of property 5 of the dot product: if two vectors $\mathbf{u}$ and $\mathbf{v}$ are perpendicular, then the angle between them is $\theta = \frac{\pi}{2}$, so $\cos(\theta) = 0$ and the above formula tells us that $\mathbf{u}\cdot\mathbf{v} = 0$.

Example: Find the angle between $\mathbf{u} = (1, -2, 0)$ and $\mathbf{v} = (3, 1, -2)$.

Homework: Find the angle between $\mathbf{u} = (-3, 0, 1)$ and $\mathbf{v} = (2, -2, 1)$.

2.6.5 Acute, obtuse, and right angles

Suppose we have non-zero, non-parallel vectors $\mathbf{u}$ and $\mathbf{v}$, with angle θ between them. If θ is acute, i.e. in the 1st quadrant, then $\cos(\theta)$ is positive. If θ is obtuse, i.e. in the 2nd quadrant, then $\cos(\theta)$ is negative. If θ is a right angle, then $\cos(\theta) = 0$.

We also note that the sign of the dot product is the same as the sign of $\cos(\theta)$, since $\mathbf{u} \cdot \mathbf{v} = \|\mathbf{u}\|\|\mathbf{v}\| \cos(\theta)$, and $\|\mathbf{u}\|$ and $\|\mathbf{v}\|$ are positive. This means that we can tell from the dot product whether the angle θ is acute or obtuse.

> If $\mathbf{u}$ and $\mathbf{v}$ are non-zero, non-parallel vectors in $\mathbb{R}^n$ and θ is the angle between them, then
>
> | | | |
> |---|---|---|
> | θ is acute | if and only if | $\mathbf{u} \cdot \mathbf{v} > 0$; |
> | θ is obtuse | if and only if | $\mathbf{u} \cdot \mathbf{v} < 0$; |
> | $\theta = \dfrac{\pi}{2}$ | if and only if | $\mathbf{u} \cdot \mathbf{v} = 0$. |

Example: Decide whether the angle between the following pairs of vectors $\mathbf{u}$ and $\mathbf{v}$ is acute or obtuse:

(a) $\mathbf{u} = (1, -2, 0)$ and $\mathbf{v} = (3, 1, -2)$;

(b) $\mathbf{u} = (2, 1, 1)$ and $\mathbf{v} = (-3, -1, 2)$;

(c) $\mathbf{u} = (2, 0, 1)$ and $\mathbf{v} = (-1, -3, 2)$.

2.6.6 Standard unit vectors and the dot product

Let's see what happens when we take dot products of pairs of vectors from the list $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$.

As we saw previously, if we take the dot product between any vector and itself, the answer will be square of the length of the vector. So

$$\mathbf{i} \cdot \mathbf{i} = 1, \quad \mathbf{j} \cdot \mathbf{j} = 1, \quad \mathbf{k} \cdot \mathbf{k} = 1$$

since they are all unit vectors. However,

$$\mathbf{i} \cdot \mathbf{j} = 0, \quad \mathbf{j} \cdot \mathbf{k} = 0, \quad \mathbf{k} \cdot \mathbf{i} = 0$$

since they are pairwise perpendicular.

Additional questions

You can attempt problems 9–14 from Topic 2 in the handbook.

You may also attempt a selection of problems from 1–15 in Exercise set 10.3 from the textbook.

2.7 Using vectors in problem solving

The ability to manipulate vectors for the purposes of problem solving is a skill that requires practice. It will be particularly important for students studying physics, as many physical situations involving velocity, force and so on, can be represented using vectors.

We will spend some time solving vector problems to begin developing these skills.

Example 1: Let A, B and C be points defined respectively by the position vectors

$$\mathbf{a} = -2\mathbf{i} + \mathbf{j} + 5\mathbf{k}, \quad \mathbf{b} = 2\mathbf{j} + 3\mathbf{k}, \quad \mathbf{c} = -2\mathbf{i} + 4\mathbf{j} + 5\mathbf{k}.$$

1. Show that ABC forms an isosceles triangle.
2. Find $\overrightarrow{OM}$ where M is the midpoint of BC.
3. Find $\overrightarrow{AM}$.
4. Find the area of the triangle ABC.

Example 1 (continued):

Example 2: Let $\mathbf{i}$ be a unit vector in the east direction and $\mathbf{j}$ be a unit vector in the north direction. A runner heads off in a direction which is $\frac{\pi}{3}$ west of north.

1. Find a unit vector in this direction.
2. The runner covers 3 km. Find the position of the runner with respect to her starting point.
3. The runner now changes direction and heads directly north for 5 km. Find the position of the runner with respect to the starting point.
4. Find the distance of the runner from the starting point.

Example 2 (continued):

Example 3: A hang-glider jumps from a 50 m cliff.

1. Give a position vector which describes his position A relative to a point O on the ground directly beneath him.

2. After some time the hang-glider has drifted to position B which is given as $-80\mathbf{i} + 20\mathbf{j} + 40\mathbf{k}$ relative to O.

   (a) Find the vector $\overrightarrow{AB}$.

   (b) Find the magnitude of $\overrightarrow{AB}$.

3. The hang-glider then moves 600 m in the $\mathbf{j}$ direction and 60 m in the $\mathbf{k}$ direction. Give the new position vector of the hang-glider.

Example 3 (continued):

Example 4: A ship leaves port and sails in a north easterly direction for 100 km to a point P. Assuming that $\mathbf{i}$ and $\mathbf{j}$ are unit vectors in the directions east and north respectively.

1. Find the position vector of the point P.
2. Let B be a point on the shore with position vector $\overrightarrow{OB} = 100\mathbf{i}$. Determine:
   (a) the vector $\overrightarrow{BP}$;
   (b) the bearing of P from B.

Example 4 (continued):

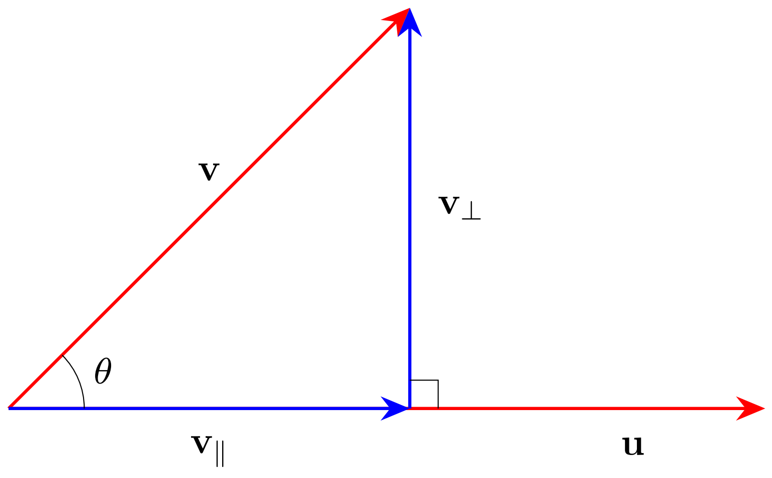
Additional questions

You can attempt problems 15–22 from Topic 2 in the handbook.

You may also attempt a selection of problems from Exercise set 10.2 from the textbook.

2.8 Scalar and vector projections [Chapter 10.3]

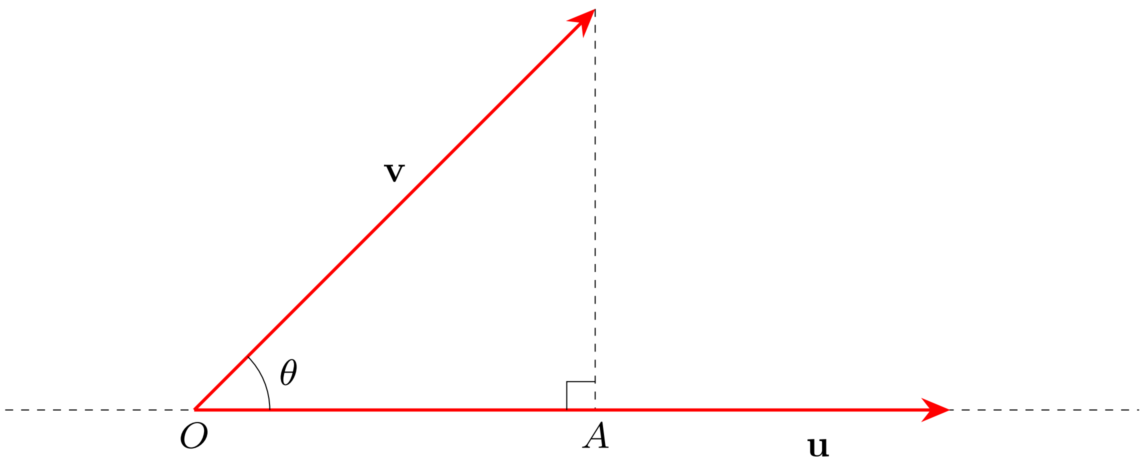
Consider the vectors $\mathbf{u}$ and $\mathbf{v}$ below, with angle θ between them. In this section, we look at how $\mathbf{v}$ can be decomposed into the sum of a vector $\mathbf{v}_{\|}$ parallel to $\mathbf{u}$ and another vector $\mathbf{v}_{\perp}$ perpendicular to $\mathbf{u}$.



2.8.1 The scalar projection

First we obtain the component of $\mathbf{v}$ in the direction of $\mathbf{u}$ by the following construction.

Drop a perpendicular from the tip of $\mathbf{v}$ to the line through $\mathbf{u}$. This line is perpendicular to $\mathbf{u}$ and intersects the line through $\mathbf{u}$ at the point A.



Next we find the length of the line segment OA. From trigonometry we see that

$$\cos(\theta) = \frac{|OA|}{\|\mathbf{v}\|}.$$

But from the relation between angle and dot product, we have

$$\cos(\theta) = \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\|\|\mathbf{v}\|}.$$

Combining these gives

$$\frac{|OA|}{\|\mathbf{v}\|} = \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\|\|\mathbf{v}\|}.$$

Then cancelling the $\|\mathbf{v}\|$'s on each side gives

$$|OA| = \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\|} = \left(\frac{\mathbf{u}}{\|\mathbf{u}\|}\right) \cdot \mathbf{v}.$$

But we know that $\dfrac{\mathbf{u}}{\|\mathbf{u}\|} = \hat{\mathbf{u}}$, the unit vector in the direction of $\mathbf{u}$, so

$$|OA| = \hat{\mathbf{u}} \cdot \mathbf{v}.$$

The length of this line segment OA is called the **scalar projection** of $\mathbf{v}$ onto $\mathbf{u}$.

Note: The vector we're projecting *onto* is the one with the hat (the unit vector).

Example: Let $\mathbf{u} = (2, -1, 3)$ and $\mathbf{v} = (4, -1, 2)$. Find the scalar projection of $\mathbf{v}$ onto $\mathbf{u}$ and indicate this in a sketch.

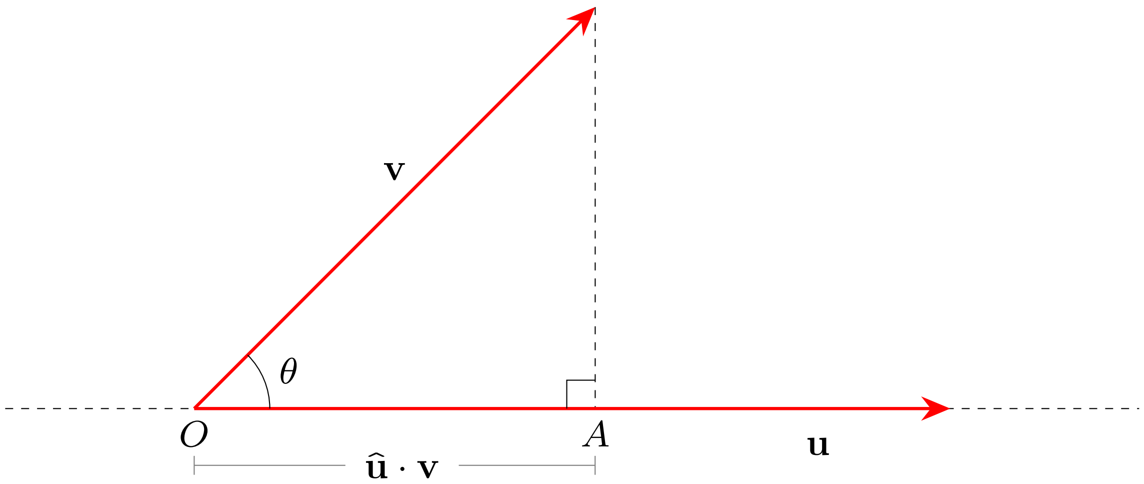
We can also calculate the scalar projection of $\mathbf{u}$ onto $\mathbf{v}$. Note that the two quantities are most likely not equal.

Example: For the same vectors, now find the scalar projection of $\mathbf{u}$ onto $\mathbf{v}$. Illustrate this in a sketch.

Homework: Let $\mathbf{u} = (1, 6, -2)$ and $\mathbf{v} = (-2, 0, -3)$. Find the scalar projection of $\mathbf{v}$ onto $\mathbf{u}$ and the scalar projection of $\mathbf{u}$ onto $\mathbf{v}$.

2.8.2 Vector projections

Let's go back to our diagram

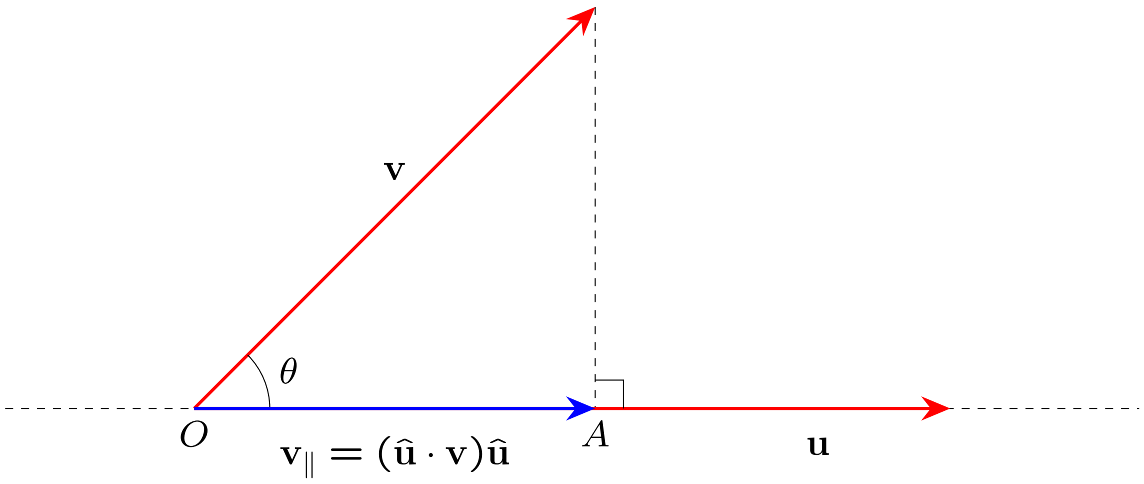


We know that the length of the line segment OA is just the scalar projection $\hat{\mathbf{u}} \cdot \mathbf{v}$. If we think of the vector $\overrightarrow{OA}$, its direction is the same as the direction of $\mathbf{u}$. To construct a vector in the direction of $\mathbf{u}$ with the same length as OA we simply multiply $|OA|$ by a unit vector in the direction of $\mathbf{u}$.

This gives:

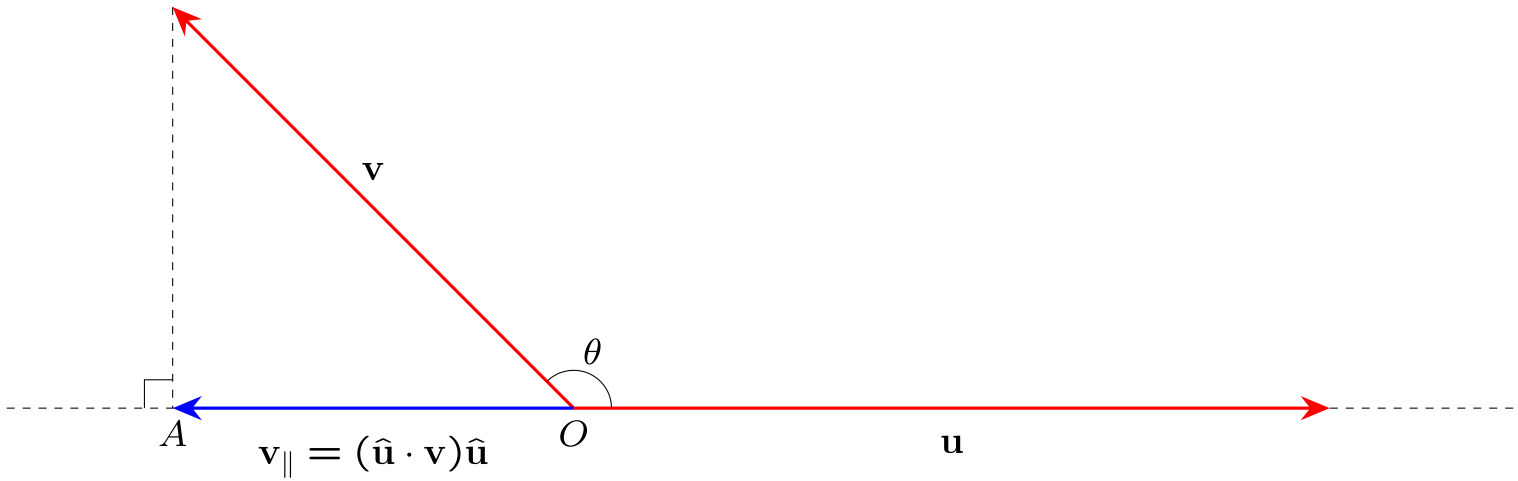
$$\boxed{\mathbf{v}_{\parallel} = \text{proj}_{\mathbf{u}}\,\mathbf{v} = (\hat{\mathbf{u}} \cdot \mathbf{v})\hat{\mathbf{u}}.}$$

We call this vector the **projection of $\mathbf{v}$ onto $\mathbf{u}$**. We denote it by either $\text{proj}_{\mathbf{u}}\,\mathbf{v}$ or $\mathbf{v}_{\parallel}$.



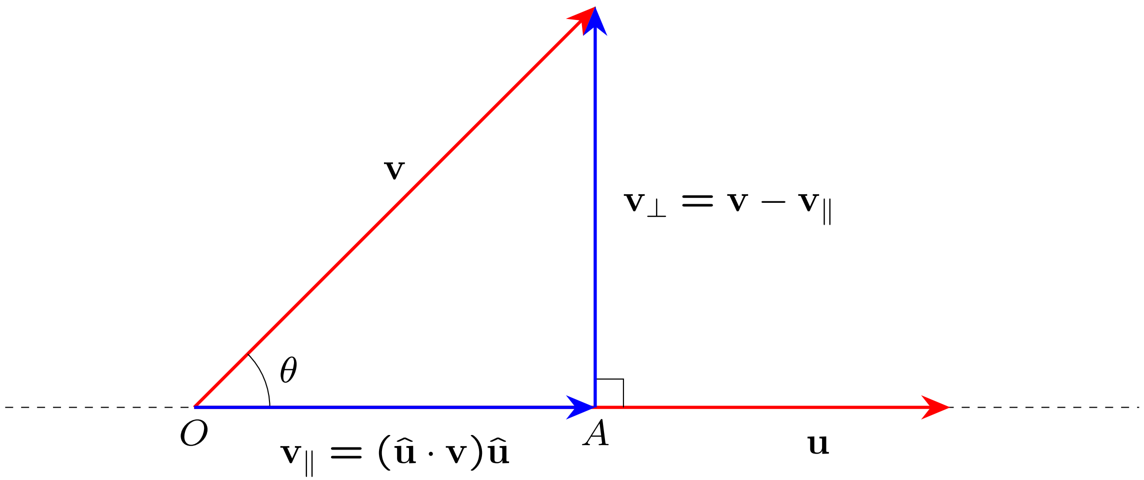
Example: Let $\mathbf{u} = (3, 1, -2)$ and $\mathbf{v} = (1, 0, 5)$. Find the scalar projection of $\mathbf{v}$ onto $\mathbf{u}$ and the vector projection of $\mathbf{v}$ onto $\mathbf{u}$.

Note that we have obtained a negative scalar projection. This simply means that the vector $\overrightarrow{OA}$ is in the opposite direction to $\mathbf{u}$, which occurs when the angle between $\mathbf{u}$ and $\mathbf{v}$ is obtuse.



Homework: Let $\mathbf{u} = (1, 1, 0)$ and $\mathbf{v} = (1, -2, 3)$. Find the scalar projection of $\mathbf{v}$ onto $\mathbf{u}$ and the vector projection of $\mathbf{v}$ onto $\mathbf{u}$.

Back to the diagram:



Clearly

$$\mathbf{v} = \mathbf{v}_\parallel + \mathbf{v}_\perp, \qquad \text{so} \qquad \mathbf{v}_\perp = \mathbf{v} - \mathbf{v}_\parallel.$$

The vector $\mathbf{v}_\perp$ is called the **vector component of $\mathbf{v}$ perpendicular to $\mathbf{u}$**, or the **projection of $\mathbf{v}$ orthogonal to $\mathbf{u}$.**

Example: Let $\mathbf{u} = (3, 1, -2)$ and $\mathbf{v} = (1, 0, 5)$, as in the previous example. Find the projection of $\mathbf{v}$ orthogonal to $\mathbf{u}$.

Homework: Let $\mathbf{u} = (1, 1, 0)$ and $\mathbf{v} = (1, -2, 3)$, as in the previous homework question. Find the projection of $\mathbf{v}$ orthogonal to $\mathbf{u}$.

Example: Let $\mathbf{u} = (-4, 0, 2)$ and $\mathbf{v} = (1, 2, -7)$. Find the following, and illustrate with a sketch:

(a) the scalar projection of $\mathbf{u}$ onto $\mathbf{v}$;

(b) the vector projection of $\mathbf{u}$ onto $\mathbf{v}$;

(c) the projection of $\mathbf{u}$ orthogonal to $\mathbf{v}$.

Example continued:

Example continued:

Summary

1. The **scalar projection** of $\mathbf{v}$ onto $\mathbf{u}$ is:

$$\hat{\mathbf{u}} \cdot \mathbf{v}.$$

2. The **vector projection** of $\mathbf{v}$ onto $\mathbf{u}$ is:

$$\mathbf{v}_{\parallel} = (\hat{\mathbf{u}} \cdot \mathbf{v})\hat{\mathbf{u}}.$$

3. The **vector component of** $\mathbf{v}$ **perpendicular to** $\mathbf{u}$ or **projection of** $\mathbf{v}$ **orthogonal to** $\mathbf{u}$ is:

$$\mathbf{v}_{\perp} = \mathbf{v} - \mathbf{v}_{\parallel} = \mathbf{v} - (\hat{\mathbf{u}} \cdot \mathbf{v})\hat{\mathbf{u}}.$$

Additional questions

You can now attempt problems 23–27 from Topic 2 in the handbook.

You may also attempt a selection of exercises from Exercise set 10.3 in the textbook.

2.9 Graphs of circles, ellipses and hyperbolae [Chapter 9.4]

In this section, we consider circles, ellipses and hyperbolae; we discuss how to express them algebraically and how to sketch them.

The treatment of graph sketching in the textbook is more advanced than the one we will give here. The material on foci is interesting, but is not part of the course.

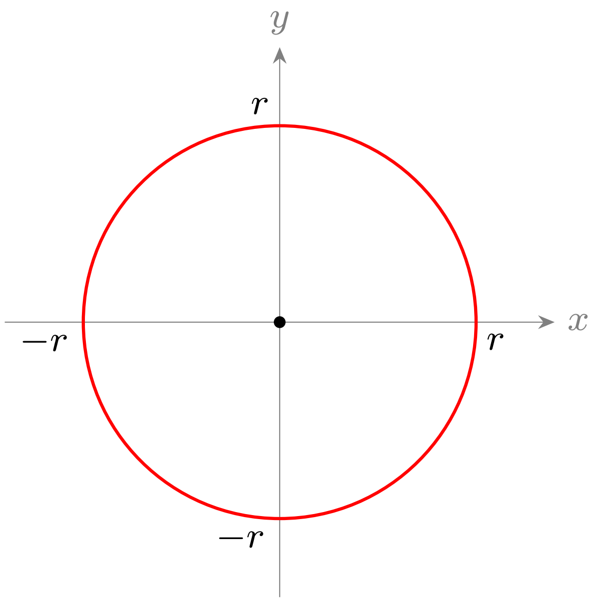
2.9.1 Graphs of circles

A **circle** is the set of points in the plane whose distance from a specified point is constant.

The basic equation of a circle is

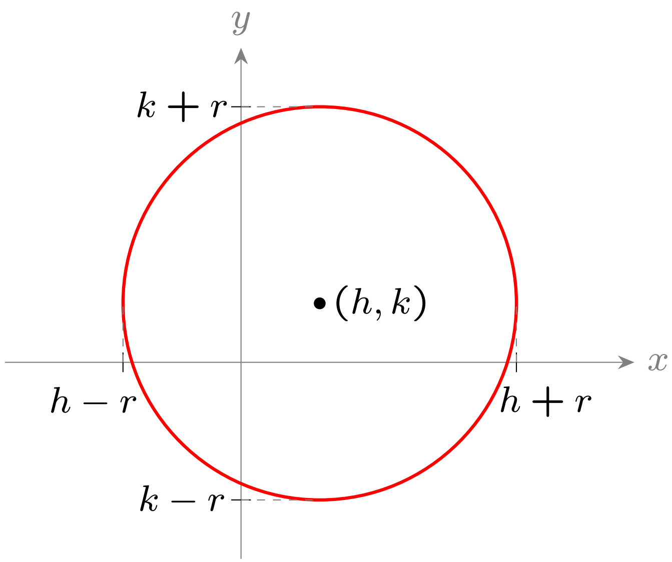
$$x^2 + y^2 = r^2.$$

This circle has centre $(0,0)$ and radius r.
By setting $y = 0$ we find x-intercepts at $(r,0)$ and $(-r,0)$.
By setting $x = 0$ we find y-intercepts at $(0,r)$ and $(0,-r)$.



If the circle is shifted h units to the right and k units up, with the radius unchanged, then the centre moves to (h,k) and its equation becomes

$$(x-h)^2 + (y-k)^2 = r^2. \tag{2}$$



Equation (2) remains valid if one or both of h and k are negative.

Homework: Sketch the curve with equation $(x-3)^2 + (y+1)^2 = 4$.

Example continued:

2.9.2 Graphs of ellipses

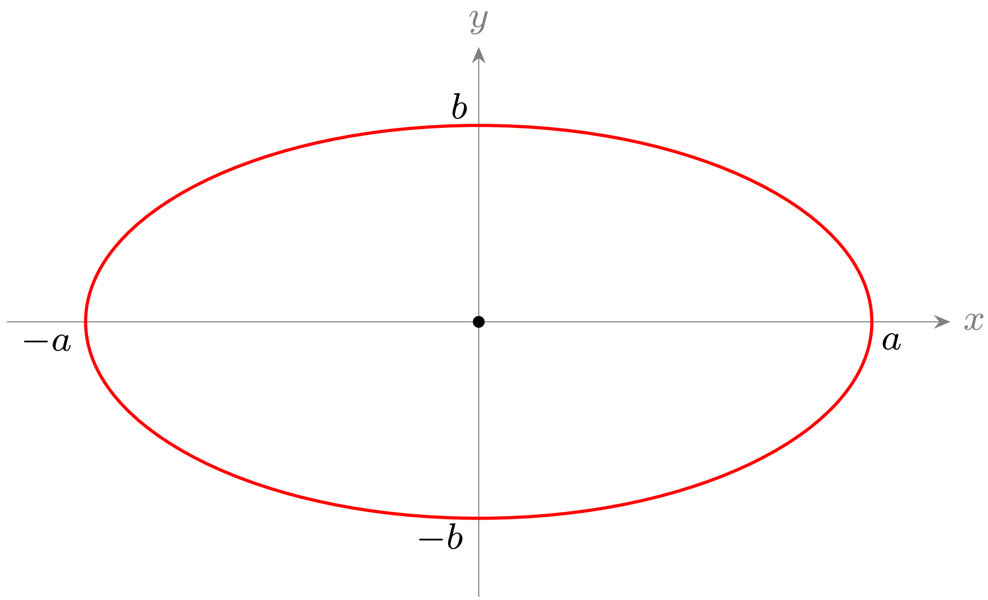
The basic equation of an ellipse is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

This ellipse has centre $(0, 0)$.

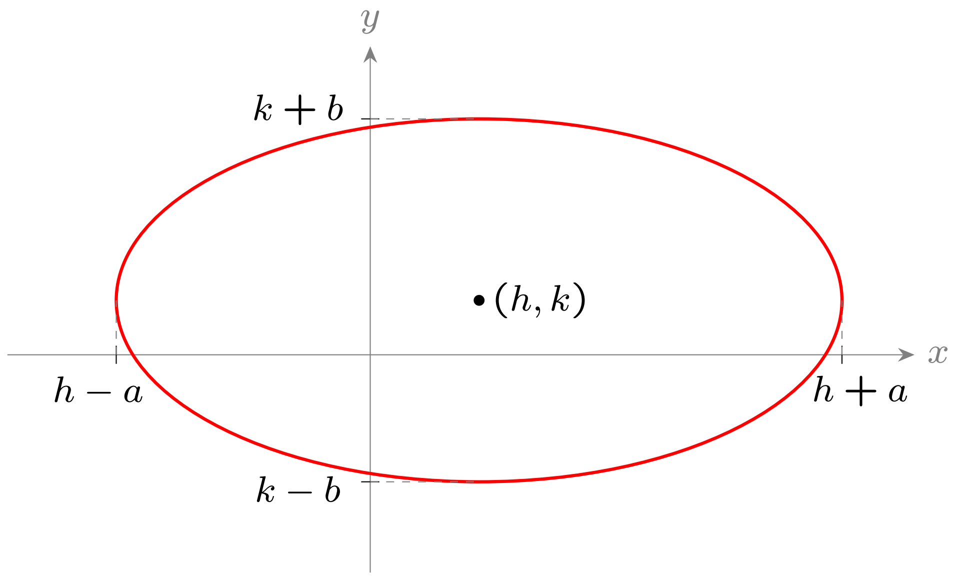
By setting $y = 0$ we find x-intercepts at $(a, 0)$ and $(-a, 0)$.

By setting $x = 0$ we find y-intercepts at $(0, b)$ and $(0, -b)$.



If the ellipse is shifted h units to the right and k units up, then the centre moves to (h, k) and its equation becomes

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1. \tag{3}$$



Of course, equation (3) remains valid if one or both of h and k are negative.

If $a = b$ then equation (3) can be rearranged as

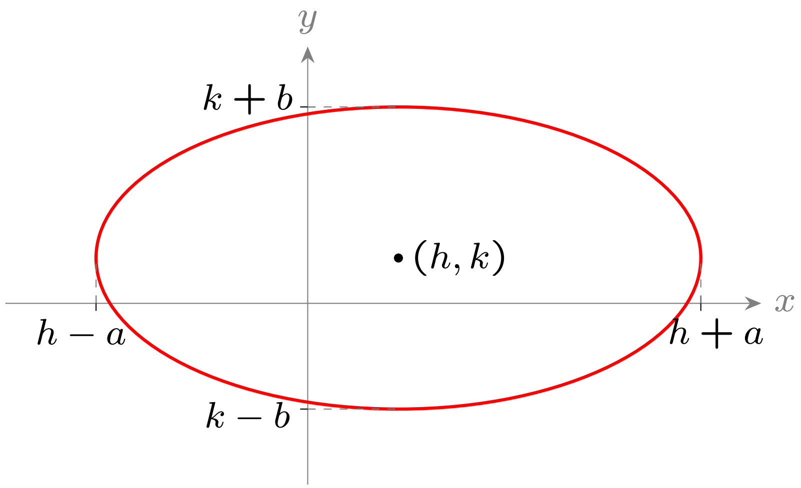
$$(x-h)^2 + (y-k)^2 = a^2,$$

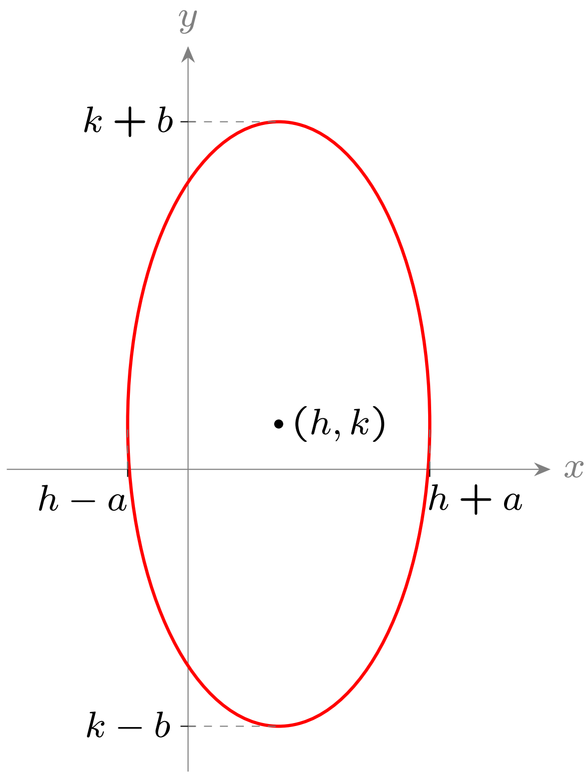
which is just the equation of a circle with centre (h, k) and radius a.

An ellipse does not have a radius. It has a **major axis** and a **minor axis**.

If $a > b$ then the major axis is parallel to the x-axis and has length $2a$ whilst the minor axis is parallel to the y-axis and has length $2b$.

If $a < b$ then the major axis is parallel to the y-axis and has length $2b$ whilst the minor axis is parallel to the x-axis and has length $2a$.





Example: Sketch the curve with equation $\frac{(x+2)^2}{9} + \frac{(y-4)^2}{64} = 1$.

Example continued:

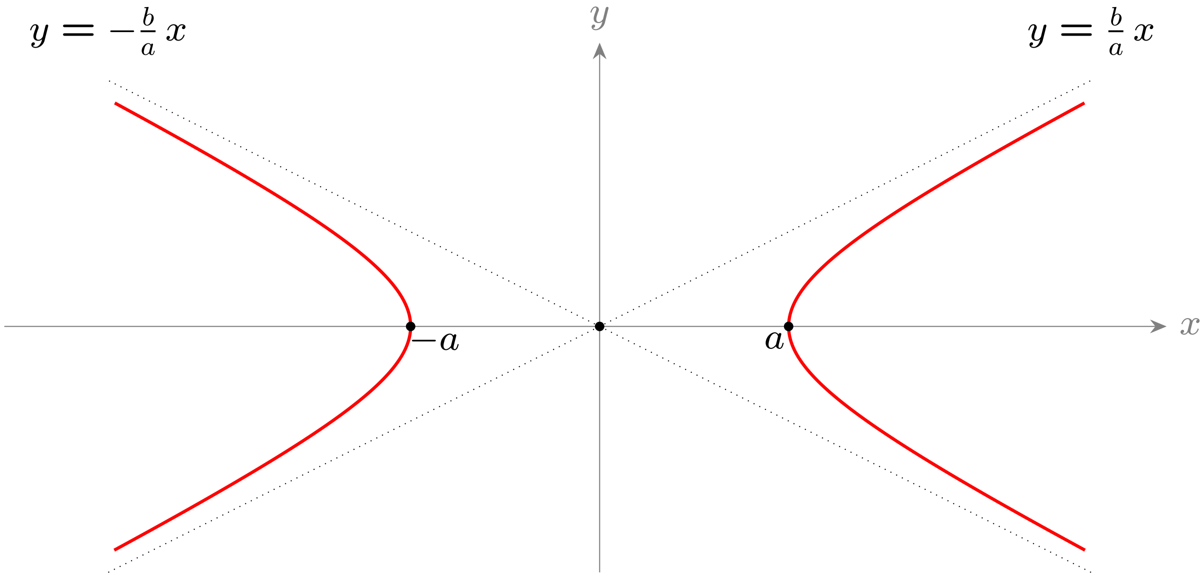
Homework: Sketch the curve with equation $\frac{(x+2)^2}{16} + \frac{(y-4)^2}{9} = 1$.

2.9.3 Graphs of hyperbolae

The basic equation of a hyperbola is

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1.$$

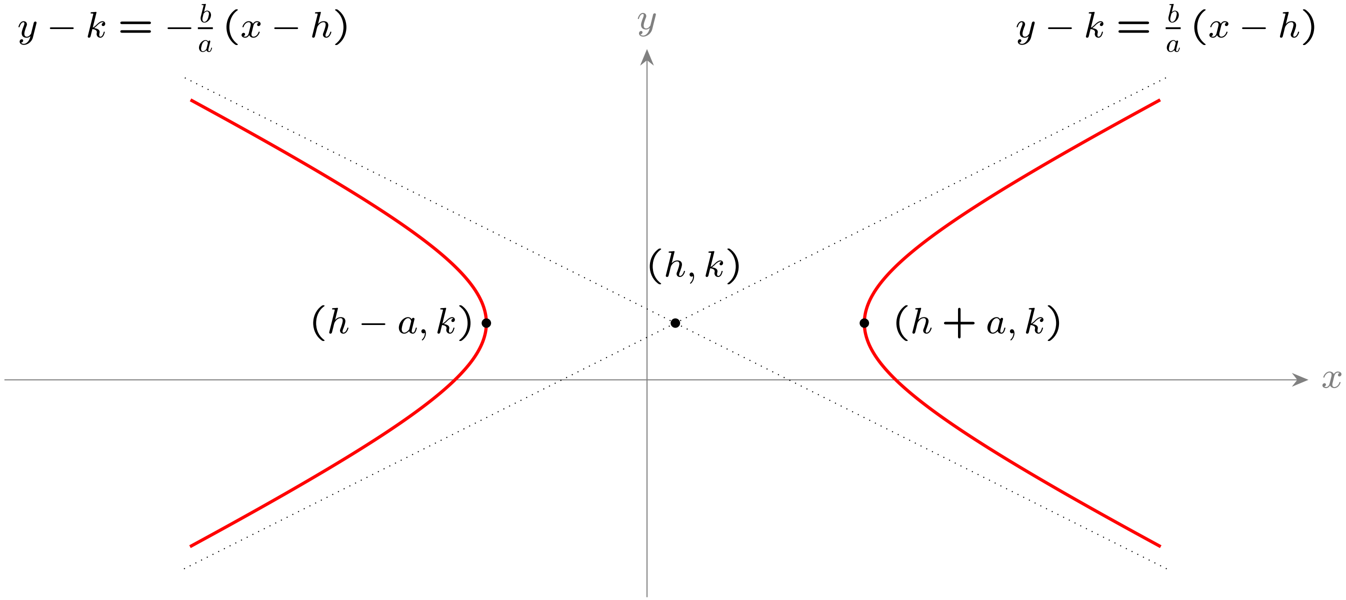
This hyperbola has centre $(0, 0)$.
By setting $y = 0$ we find x-intercepts (the **vertices**) at $(-a, 0)$ and $(a, 0)$.
If we set $x = 0$, there are no solutions for y, so no y-intercepts.



There are two (oblique) asymptotes, with equations $y = \frac{b}{a}x$ and $y = -\frac{b}{a}x$.

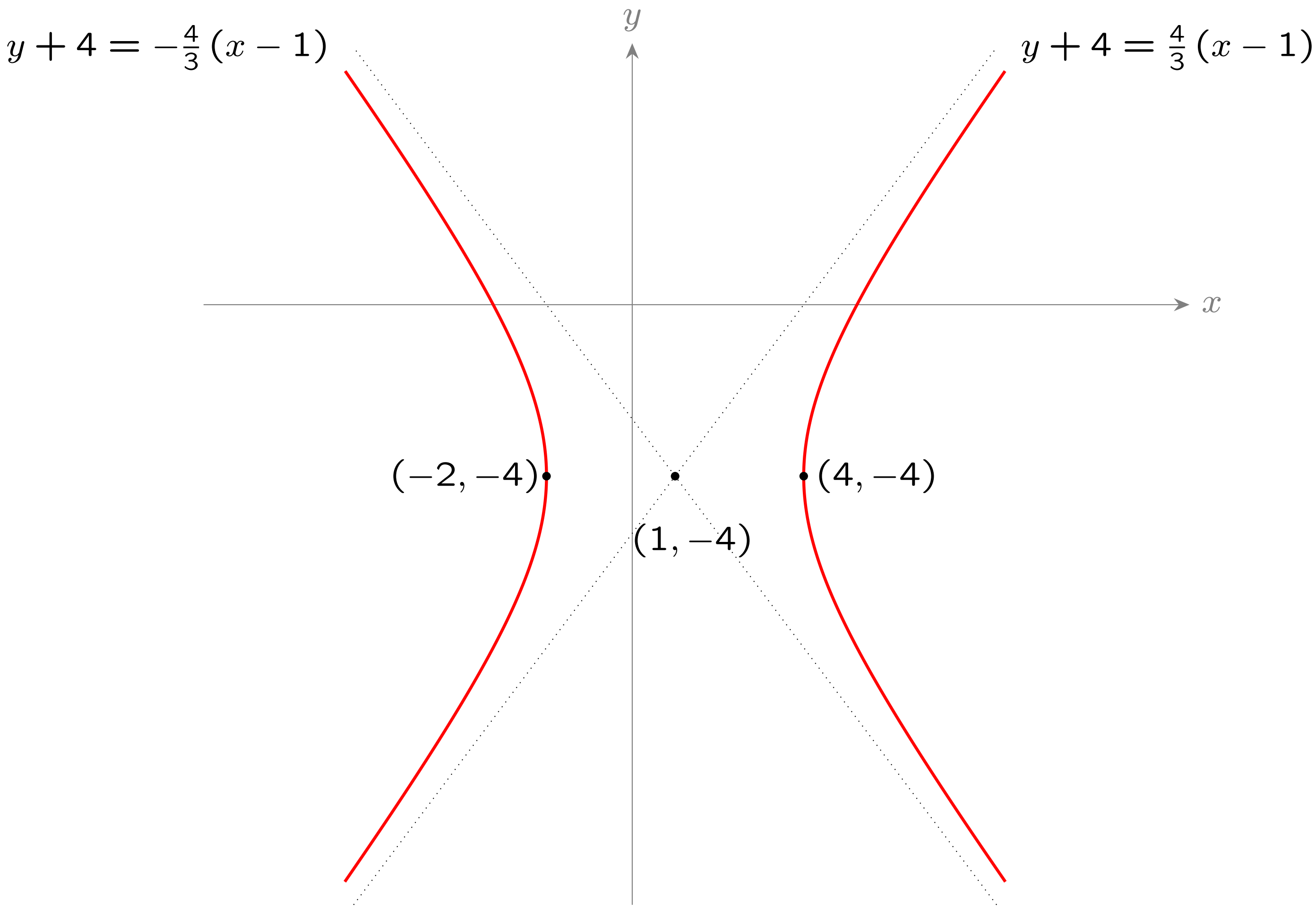
If the hyperbola is shifted h units to the right and k units up, its equation becomes

$$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1.$$



Example: Sketch the curve with equation

$$16(x-1)^2 - 9(y+4)^2 = 144.$$


$y+4=-\frac{4}{3}(x-1)$
y
$y+4=\frac{4}{3}(x-1)$
x
$(-2,-4)$
$(4,-4)$
$(1,-4)$

Homework: Sketch the curve with equation $\frac{(x-3)^2}{9} - \frac{(y+4)^2}{25} = 1$.

Example: Sketch the curve with equation $y^2 - x^2 = 1$.

Example continued:

Additional questions

You can now attempt problems 28–30 from Topic 2 in the handbook.

You may also attempt a selection of problems from Exercise set 9.4 from the textbook (ignore the parts about foci).

2.9 Parametric curves [Chapter 9.6]

2.9.1 Introduction to parametric curves

Up to this point we have seen vectors of the form $\mathbf{u} = x\mathbf{i} + y\mathbf{j}$, where the components of the vectors have been constant. Suppose now that these components are functions of time; that is,

$$\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j}, \qquad \text{where } t \in \mathbb{R}.$$

Here $\mathbf{r}$ is a function $\mathbf{r}\colon \mathbb{R} \to \mathbb{R}^2$, since for each $t \in \mathbb{R}$, $\mathbf{r}(t)$ is a vector in $\mathbb{R}^2$. This is called a **vector-valued** function of a real variable. The functions $x(t)$ and $y(t)$ are called **parametric equations**, since they depend on the parameter t. The resulting curve that $\mathbf{r}(t)$ traces out in $\mathbb{R}^2$ is called a **parametric curve**.

Such vector-valued functions are particularly useful in applications, for example, in describing the motion of a particle at any time t.

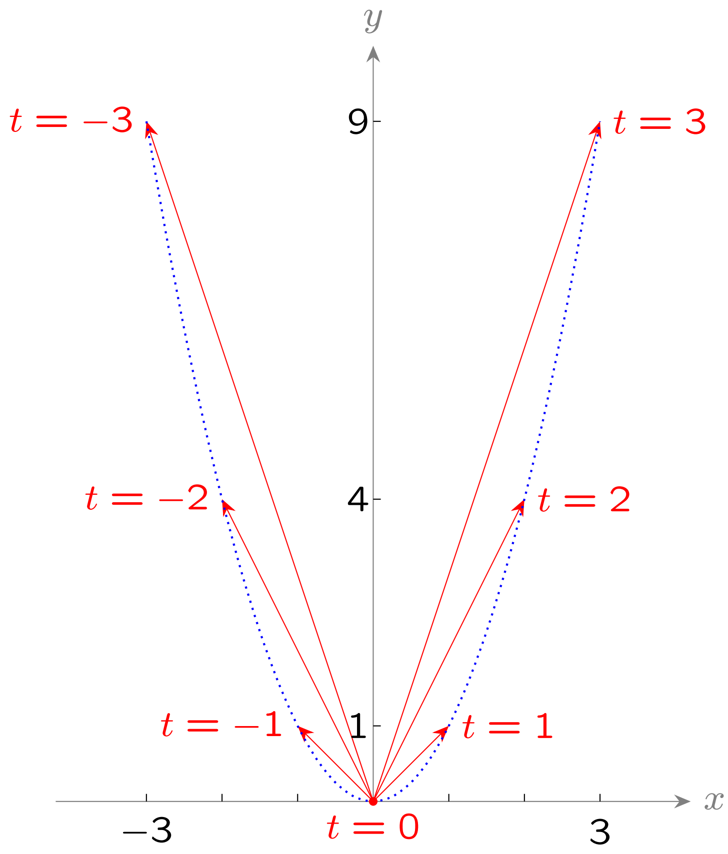
Let's look at a simple example. Let

$$\mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j}, \qquad \text{for } t \in \mathbb{R}.$$

This means that $x(t) = t$ and $y(t) = t^2$. To sketch the graph of $\mathbf{r}(t)$, we need to know the values of x and y for different values of t. A simple way to do this is to construct a table to give us a rough picture of what the curve looks like.

t	$x(t) = t$	$y(t) = t^2$	$\mathbf{r}$
-3	-3	9	$-3\mathbf{i} + 9\mathbf{j}$
-2	-2	4	$-2\mathbf{i} + 4\mathbf{j}$
-1	-1	1	$-\mathbf{i} + \mathbf{j}$
0	0	0	$\mathbf{0}$
1	1	1	$\mathbf{i} + \mathbf{j}$
2	2	4	$2\mathbf{i} + 4\mathbf{j}$
3	3	9	$3\mathbf{i} + 9\mathbf{j}$

So for each point t we get a vector $\mathbf{r}(t)$.



We can see the *path* of the function by following the curve traced out by the heads of the vectors (indicated by the dotted line). Our job is to find the equation of this path.

2.9.2 Finding the equation of a path

To find the equation of a path defined by parametric equations, we need to solve the equations simultaneously. The aim is to eliminate the parameter t, so as to obtain a relationship between x and y.

Consider our example $\mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j}$. The component in the x direction is t and the component in the y direction is t^2. We put

$$x = t \quad \text{and} \quad y = t^2.$$

To eliminate the parameter we can simply substitute the first equation into the second. Thus:

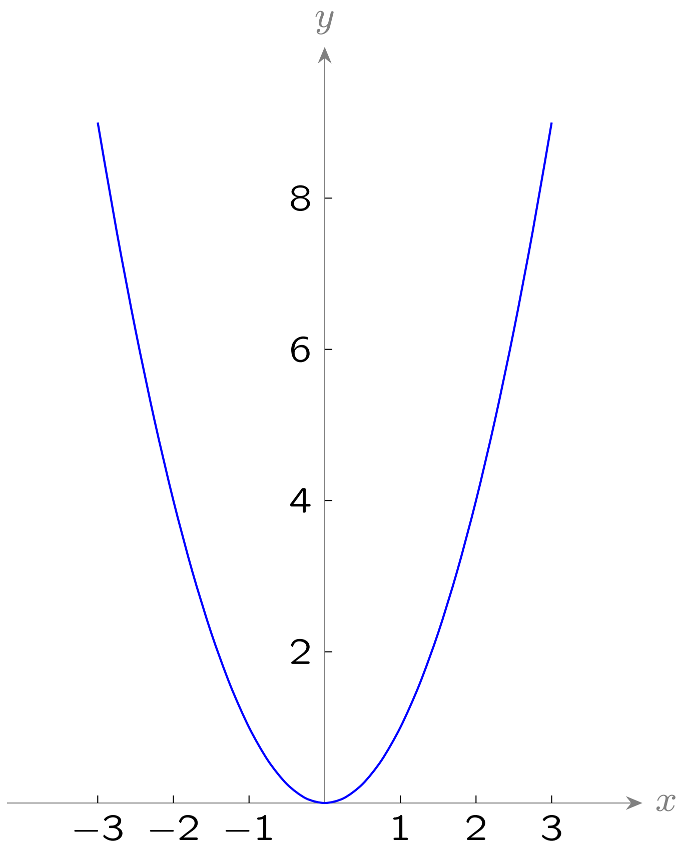
$$\begin{aligned} y &= t^2 \\ &= x^2 \end{aligned}$$

$$\text{i.e.} \quad y = x^2.$$

So the equation of the path defined by the parametric equations is

$$y = x^2.$$

Since $t \in \mathbb{R}$ and $x = t$, the domain of this function is $\mathbb{R}$.



Example: Find the equation of the path of a particle whose position is given by

$$\mathbf{r}(t) = (t^2 - t)\mathbf{i} + 3t\,\mathbf{j}, \qquad \text{for } t \geq 0.$$

Sketch the graph of the path, indicating the direction of increasing t.

Example continued:

Homework: Find the equation of the path of a particle whose position is given by $\mathbf{r}(t) = t\mathbf{i} + (t^2 + 1)\mathbf{j}$, for $t \geq 0$.

Example: Find the equation of the path of a particle whose position is given by

$$\mathbf{r}(t) = \cos(t)\mathbf{i} + \sin(t)\mathbf{j}, \qquad \text{for } t \in \mathbb{R}.$$

Sketch the graph of the path.

Example continued:

Homework: Find the equation of the path of a particle whose position is given by $\mathbf{r}(t) = \cos(t)\mathbf{i} + 2\sin(t)\mathbf{j}$, for $t \in \mathbb{R}$.

In the last example, the path described is a circle, which we could have simply sketched from the cartesian equation $x^2 + y^2 = 1$. However, the parametric equations give us more information about the motion of the particle:

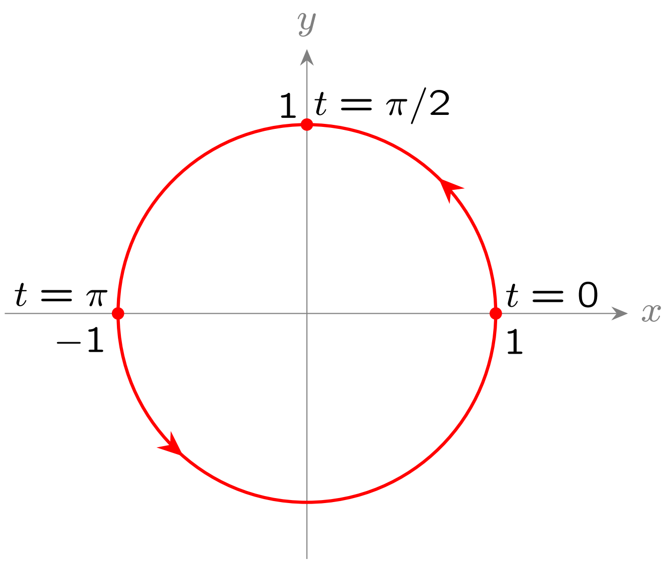
- they tell us that at time $t = 0$ the particle is at the point $(1, 0)$.
- they tell us that the direction of motion is anticlockwise.
  This can be seen in two ways:
  - As t increases from 0, $x = \cos(t)$ decreases, while $y = \sin(t)$ increases (consider the graphs of $\cos$ and $\sin$).
  - The point given by $\mathbf{r}(t) = \cos(t)\mathbf{i} + \sin(t)\mathbf{j}$ is at angle t around the unit circle, so as t increases, it moves in an anticlockwise direction.
- they tell us that it takes the particle time 2π to travel around the circle, i.e. its **period of motion** is 2π.
  This is because $\mathbf{r}(t)$ first returns to its original position when $t = 2\pi$, since $\cos(t)$ and $\sin(t)$ both have period 2π.

Example: Find the equation of the path of a particle whose position is given by

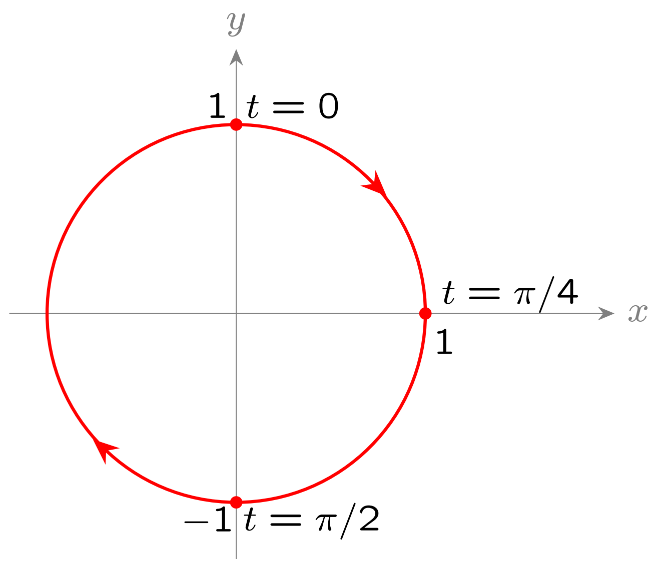
$$\mathbf{r}(t) = \sin(2t)\mathbf{i} + \cos(2t)\mathbf{j}, \qquad \text{for } t \in \mathbb{R}.$$

Sketch the graph of the path.

Here we have described the same path as in the previous example, but for any given t we may not be at the same point on the path. For instance, when $t = 0$, in the first example we are at the point $(1, 0)$, whereas in the second example we are at the point $(0, 1)$.



Graph of $\cos(t)\mathbf{i} + \sin(t)\mathbf{j}$



Graph of $\sin(2t)\mathbf{i} + \cos(2t)\mathbf{j}$

In the first example, the particle moves anticlockwise with a period of 2π; in the second example, the particle moves clockwise with a period of π.

Example: The motion of a particle is described by

$$\mathbf{r}(t) = (3 - 5\cos(2t))\mathbf{i} + (1 - 2\sin(2t))\mathbf{j}, \qquad \text{for } t \geq 0.$$

Determine:

(a) the cartesian equation of the path;

(b) the position of the particle at time

(i) $t = 0$ (ii) $t = \frac{\pi}{4}$ (iii) $t = \frac{\pi}{2}$;

(c) the time taken by the particle to return to its original position;

(d) the direction of motion.

Example continued:

Example continued:

Example: The motion of two particles is given by the equations

$$\mathbf{r}_1(t) = (t+1)\mathbf{i} + (t^2 - 4t)\mathbf{j} \qquad \text{and} \qquad \mathbf{r}_2(t) = (2t)\mathbf{i} + (6t-9)\mathbf{j}.$$

Determine:

(a) the point(s) at which the particles collide;

(b) the distance between the particles when $t = 2$.

Example continued:

Example continued:

Note: Since the particles collide at the point $(2, -3)$ (when $t = 1$), the two paths must cross at this point. However, there may be other points where the paths cross, but where the particles do not collide since they are not there at the same time. (Imagine two people walking around a room. Their paths may cross, but they will only bump into each other if they are at the same point *at the same time*.)

Homework: Find the equations of the two paths in this example and hence find all points where the paths cross.

Additional questions

You can attempt the remaining problems from Topic 2 in the handbook.

You may also attempt a selection of problems from 1–12 in Exercise set 9.6 in the textbook.

Topic 3: Complex Numbers

To complement our knowledge of functions of real variables, we now introduce basic properties and applications of complex numbers. These form an enlargement of the real number system and are used extensively in physics and engineering, in areas such as electromagnetic waves and electric circuits; together with calculus, they form the mathematical study of complex analysis.

3.1 Introduction to Complex Numbers

3.2 Arithmetic of Complex Numbers

3.3 Modulus and Argument

3.4 Polar Form and Complex Exponential

3.5 Sketching Regions in the Complex Plane

3.6 Powers of Complex Numbers

3.7 Roots of Complex Numbers

3.8 Roots of Polynomials and the Fundamental Theorem of Algebra

3.1 Introduction to Complex Numbers

We know that many polynomial equations have real solutions, for example:

$$\begin{aligned} x^2 - 1 &= 0 \\ \Rightarrow \qquad x^2 &= 1 \\ \Rightarrow \qquad x &= \pm 1. \end{aligned}$$

However, there are also many polynomial equations that do not have real solutions, for example:

$$\begin{aligned} x^2 + 1 &= 0 \\ \Rightarrow \qquad x^2 &= -1. \end{aligned} \tag{4}$$

Since no real number can be squared to give -1, this equation has no real solutions.

However, if we define a new number, denoted by i, such that

$$i^2 = -1,$$

then the solutions to (4) become

$$\begin{aligned} x &= \pm\sqrt{-1} \\ &= \pm\sqrt{i^2} \\ &= \pm\, i \end{aligned}$$

So this new number i allows us to solve equations that we could previously not solve over the reals!

Definition:

A **complex number** (generally denoted z) is a quantity consisting of a real number added to a real multiple of the number i. That is:

$$z = x + iy, \qquad \text{where } x, y \in \mathbb{R} \text{ and } i^2 = -1.$$

x is called the **real part** of z and is denoted $\mathrm{Re}(z)$;
y is called the **imaginary part** of z and is denoted $\mathrm{Im}(z)$.
The set of all complex numbers is denoted $\mathbb{C}$.

A complex number written in the form $z = x + iy$ is said to be in **cartesian form**.

Note that the set of real numbers $\mathbb{R}$ is a subset of the set of complex numbers $\mathbb{C}$. (Why?)

Example:

(a) Using the number i, write down an expression for $\sqrt{-25}$.

(b) Simplify i^7.

(c) For the complex number $z = 2 - 3i$, write down:

(i) $\text{Re}(z) =$

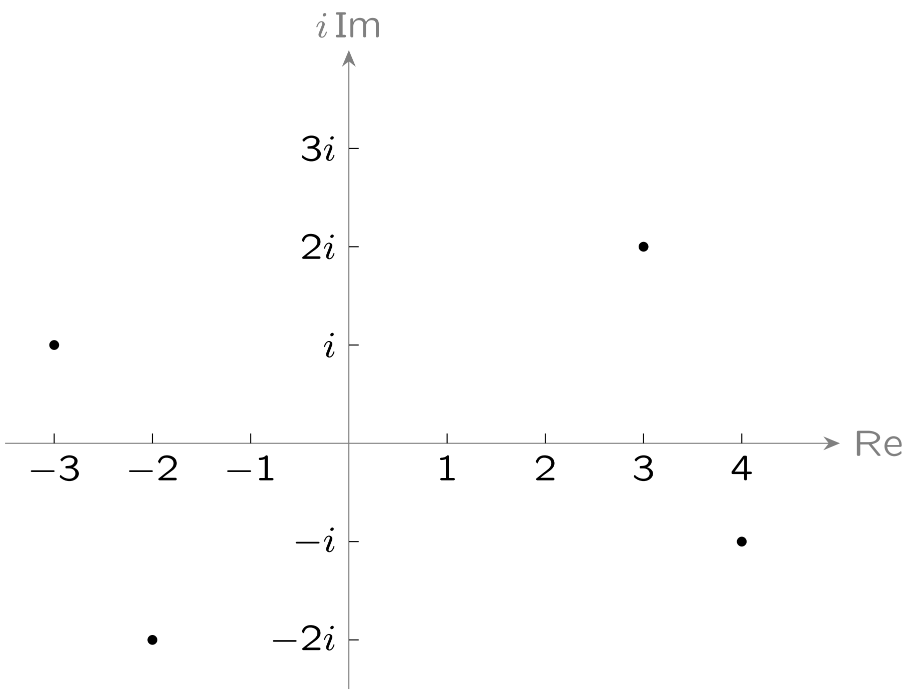
(ii) $\text{Im}(z) =$

(iii) $\text{Re}(z) - \text{Im}(z) =$

The complex plane

Complex numbers can be represented graphically in the **complex plane**. We regard the complex number $z = x + iy$ as corresponding to a point in the plane, where the horizontal axis is now called the **real axis** and corresponds to the real part of z, while the vertical axis is the **imaginary axis** and corresponds to i times the imaginary part of z.

The complex plane is sometimes called the **Argand plane**, or the **Argand diagram**.



Example: Sketch the complex numbers $3 - i$, $-2 + 2i$, -4, and $3i$ in the complex plane.

3.2 Arithmetic of Complex Numbers

Let $z_1 = a + ib$ and $z_2 = c + id$ be complex numbers.

Equality: The complex numbers z_1 and z_2 are equal if and only if

$$\mathrm{Re}(z_1) = \mathrm{Re}(z_2) \qquad \text{and} \qquad \mathrm{Im}(z_1) = \mathrm{Im}(z_2).$$

Addition: We add z_1 and z_2 as follows:

$$z_1 + z_2 = (a + ib) + (c + id) = (a + c) + i(b + d).$$

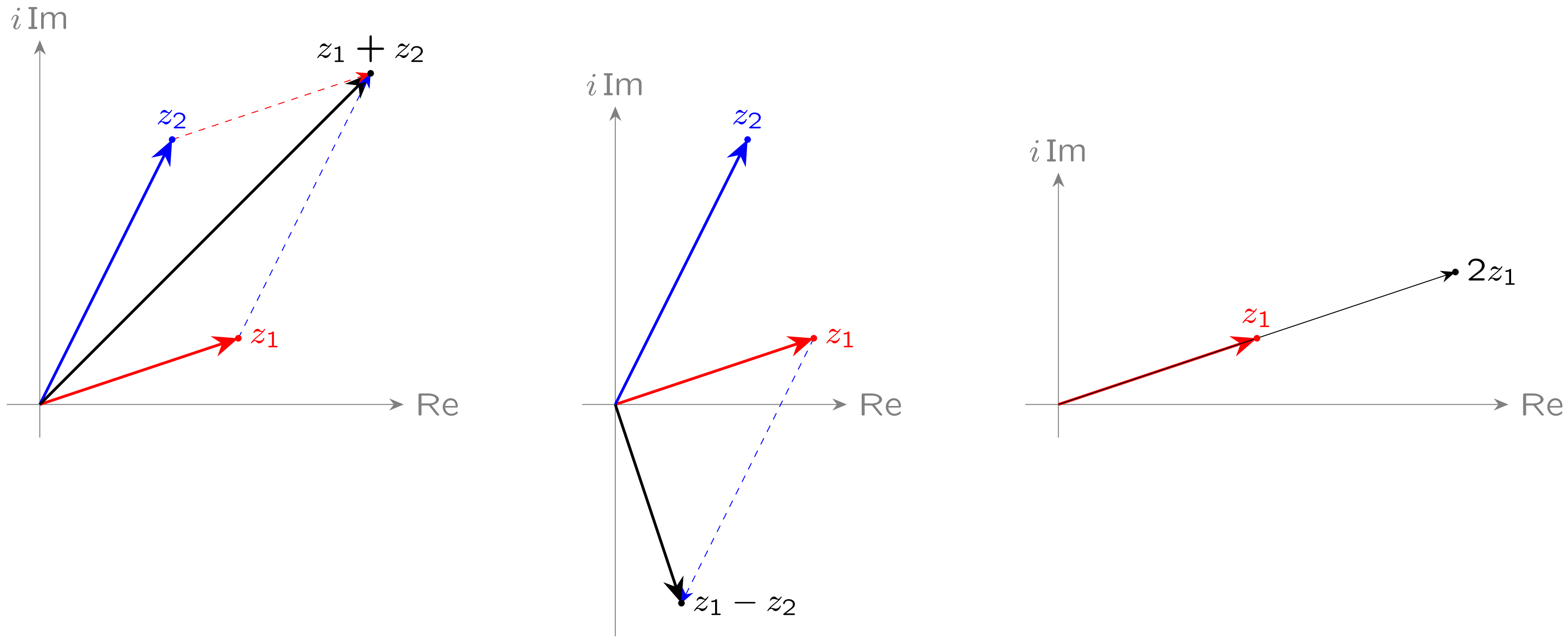
Subtraction: We subtract z_2 from z_1 as follows:

$$z_1 - z_2 = (a + ib) - (c + id) = (a - c) + i(b - d).$$

Multiplication by $k \in \mathbb{R}$: We multiply z_1 by $k \in \mathbb{R}$ as follows:

$$kz_1 = k(a + ib) = (ka) + i(kb).$$

Complex addition, subtraction, and multiplication by a real number can be represented geometrically in the complex plane, and should remind you of vector arithmetic.



Multiplication of complex numbers

We multiply complex numbers by simply 'expanding the brackets', remembering that $i^2 = -1$:

$$z_1 z_2 = (a + ib)(c + id)$$

The Complex Conjugate

In order to divide complex numbers, we need to define the complex conjugate of a complex number.

Definition: If $z = a + ib$ is a complex number, then the **complex conjugate** of z is denoted $\bar{z}$ ("z bar"), and is defined to be

$$\bar{z} = a - ib.$$

That is, the real part stays the same and the imaginary part changes sign.

Example: Write down the complex conjugate of:

(i) $-3 + 7i$ (ii) $2 - 5i$ (iii) $3i$ (iv) 4

Example: If $z_1 = 1 + i$ and $z_2 = -3 - 2i$, plot z_1, z_2, $\bar{z}_1$ and $\bar{z}_2$ in the complex plane. What is the geometric relationship between z and $\bar{z}$?

Homework: Plot $z_1 = 2i$, $z_2 = 3$, $\bar{z}_1$ and $\bar{z}_2$ in the complex plane.

Example: Find the solutions of $z^2 - 6z + 10 = 0$. What is the relationship between the solutions? Plot the solutions in the complex plane.

Example: Let z and w be complex numbers. Prove the following properties of the complex conjugate.

(i) $z + \bar{z}$ is real;

(ii) $z - \bar{z}$ is imaginary;

(iii) $z\bar{z}$ is real;

(iv) $\overline{z + w} = \bar{z} + \bar{w}$;

(v) $\overline{zw} = \bar{z}\bar{w}$.

Example continued:

Example continued:

Division of complex numbers

Suppose we want to divide two complex numbers:

$$\frac{a+ib}{c+id}. \tag{5}$$

In order to get an answer in the form $x+iy$ we need to make the denominator real. To do so, we can make use of property (iii) of the previous example, which says that a complex number multiplied by its conjugate is real. So, we simply multiply top and bottom of (5) by the complex conjugate of the denominator:

$$\frac{a+ib}{c+id} = \frac{a+ib}{c+id} \times \frac{c-id}{c-id} =$$

Example: Express the following in cartesian form $x + iy$:

$$\frac{1 + 2i}{-1 + 3i}$$

Example: Find $\mathrm{Re}\left(\dfrac{1+5i}{2-2i}\right)$ and $\mathrm{Im}\left(\dfrac{1+5i}{2-2i}\right)$.

Homework: Let $z = 1 - 5i$ and $w = -2 + i$. Express the following complex numbers in cartesian form $a + ib$ where a and b are real.

(i) $w^2 z$

(ii) $\dfrac{w}{w + 2z}$

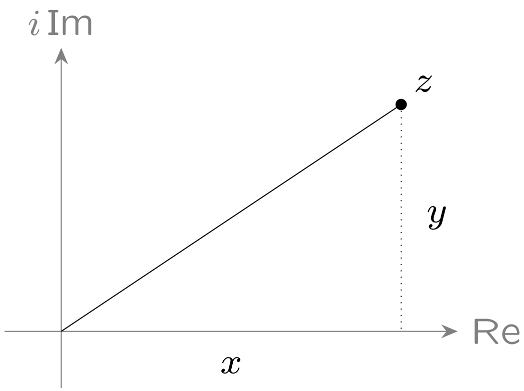
Additional questions

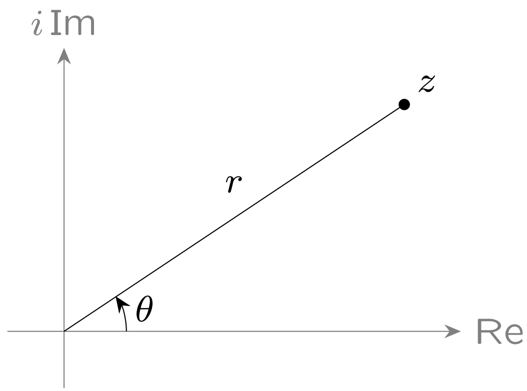
You can now attempt problems 1–12 from Topic 3 in the handbook.

3.3 Modulus and Argument

The position of a complex number z in the complex plane can be specified in two different ways:

- by its real and imaginary parts x and y, such that $z = x + iy$;
- by its distance r from the origin, and the angle θ from the positive real axis.

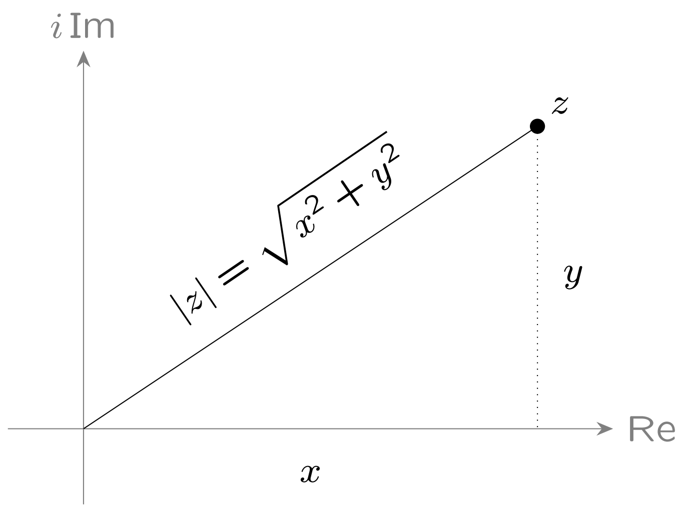




Definition: The **modulus** of z, denoted $|z|$, is the distance r from z to the origin in the complex plane.

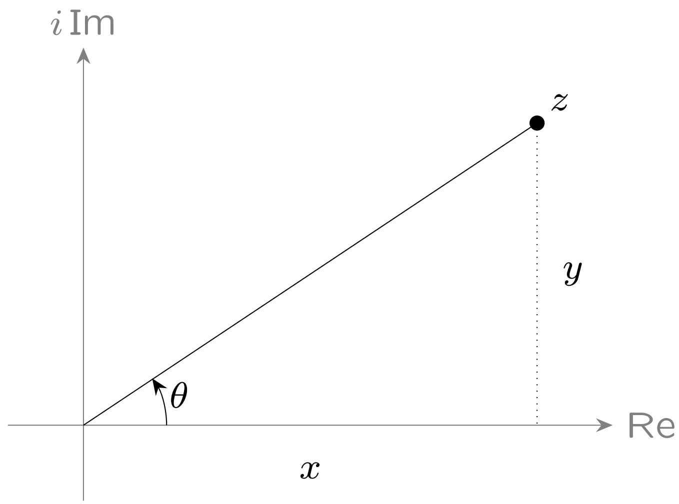
Writing $z = x + iy$, we can find $|z|$ by Pythagoras:

$$|z| = \sqrt{x^2 + y^2}.$$



Definition: The **argument** of z, denoted $\mathsf{arg}(z)$, is the angle θ that z makes with the positive real axis in the complex plane.

To determine the argument of a complex number $z = x + iy$, draw z in the complex plane and use standard triangles if possible to determine $\theta = \mathsf{arg}(z)$.



If θ is not a standard angle, we may note that

$$\mathsf{tan}(\theta) = \frac{y}{x}.$$

If θ is in the first or fourth quadrant we may conclude $\theta = \mathsf{arctan}(\frac{y}{x})$.
(Recall that the range of arctan is $(-\frac{\pi}{2}, \frac{\pi}{2})$.)

Caution:

The argument of z is not unique, since adding multiples of 2π does not change the position of z in the complex plane.

However, there is only one value of the argument that satisfies

$$-\pi < \theta \leq \pi.$$

This is called the **principal argument** of z and is sometimes denoted $\mathsf{Arg}(z)$ with a capital A.

In this subject, when we refer to the argument of z, we will always mean the principal argument.

Example: Find the modulus and argument of

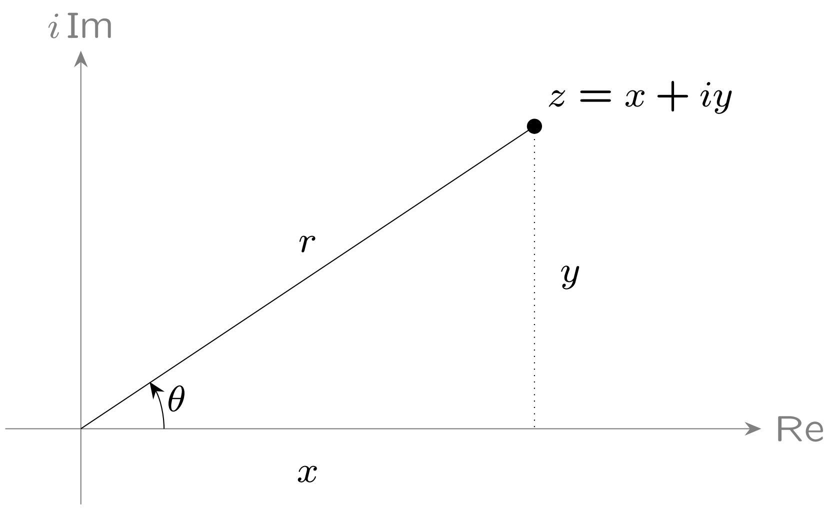
(i) $1+\sqrt{3}i$ (ii) $-3-3i$ (iii) $3+4i$ (iv) -7

Example continued:

Homework: Find the modulus and argument of $z = 2 - 2i$.

3.4 Polar Form and Complex Exponential

Recall that any complex number $z = x + iy$ can also be specified by its modulus r and argument θ.



From the diagram above, we see that

$$\cos\theta = \frac{x}{r} \qquad \text{and} \qquad \sin\theta = \frac{y}{r}$$

$$\Rightarrow \qquad x = r\cos\theta \qquad \text{and} \qquad y = r\sin\theta.$$

Substituting these back into z we obtain

$$\begin{aligned} z &= x + iy \\ &= r\cos\theta + i\,r\sin\theta. \end{aligned}$$

This is called the **polar form** of the complex number z:

$$\boxed{z = r(\cos\theta + i\sin\theta).}$$

Example: Express the following complex numbers in polar form.

(i) $z = \sqrt{3} + i$

(ii) $z = -1 - i$

The Complex Exponential

There is a very useful of writing the polar form of a complex number.

Definition: The **complex exponential** $e^{i\theta}$ is defined to be

$$\boxed{e^{i\theta} = \cos\theta + i\sin\theta.}$$

We can now write a complex number z in **exponential polar form**

$$\boxed{z = re^{i\theta},}$$

where $r = |z|$ and $\theta = \arg(z)$.

Example: Express $z = -2 + 2\sqrt{3}i$ in exponential polar form $re^{i\theta}$.

Homework: Express $z = 1 - i$ in exponential polar form.

Example: Express $z = 5e^{i\frac{3\pi}{4}}$ in cartesian form $x + iy$.

Homework: Express $z = 4e^{-i\frac{\pi}{3}}$ in cartesian form.

Properties of the complex exponential

Using the definition of the complex exponential, we will soon obtain the following:

1. $e^{i0} = 1$
2. $e^{i\theta_1}e^{i\theta_2} = e^{i(\theta_1+\theta_2)}$
3. $\dfrac{e^{i\theta_1}}{e^{i\theta_2}} = e^{i(\theta_1-\theta_2)}$

Notice that these properties are consistent with the usual index laws for working with exponentials (and so are easy to remember!).

We generally work with the exponential polar form $re^{i\theta}$ instead of the trigonometric polar form $r(\cos\theta + i\sin\theta)$.

Property 2: Multiplication in polar form

Recall the compound angle formulas for $\cos$ and $\sin$:

$$\cos(\theta_1 + \theta_2) = \cos\theta_1 \cos\theta_2 - \sin\theta_1 \sin\theta_2$$
$$\sin(\theta_1 + \theta_2) = \sin\theta_1 \cos\theta_2 + \cos\theta_1 \sin\theta_2.$$

From this and the definition of the complex exponential, we find

$$\begin{aligned} e^{i\theta_1} e^{i\theta_2} &= (\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2) \\ &= (\cos\theta_1 \cos\theta_2 - \sin\theta_1 \sin\theta_2) + i(\cos\theta_1 \sin\theta_2 + \cos\theta_2 \sin\theta_1) \\ &= \cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2) \\ &= e^{i(\theta_1 + \theta_2)}. \end{aligned}$$

If we have two complex numbers

$$z_1 = r_1 e^{i\theta_1} \qquad \text{and} \qquad z_2 = r_2 e^{i\theta_2},$$

we conclude that their product is given by

$$\boxed{z_1 z_2 = r_1 r_2 e^{i(\theta_1+\theta_2)}.}$$

Interpreting this geometrically shows that the product of two complex numbers in polar form is obtained by multiplying their moduli ($r_1 r_2$) and adding their arguments ($\theta_1 + \theta_2$):

$$\boxed{\begin{aligned} |z_1 z_2| &= |z_1||z_2| \\ \arg(z_1 z_2) &= \arg(z_1) + \arg(z_2). \end{aligned}}$$

Example: Describe geometrically what happens when a complex number z is multiplied by $w = i$.

Homework: Describe geometrically what happens when a complex number is multiplied by $w = -i$.

Homework: Property 3 – Division in polar form

Consider two complex numbers written in polar form:

$$z_1 = r_1 e^{i\theta_1} \qquad \text{and} \qquad z_2 = r_2 e^{i\theta_2} \neq 0.$$

Show that

$$\boxed{\frac{z_1}{z_2} = \frac{r_1}{r_2}\, e^{i(\theta_1 - \theta_2)}.}$$

The geometric interpretation is that the quotient of two complex numbers in polar form is obtained by dividing their moduli ($\frac{r_1}{r_2}$) and subtracting their arguments ($\theta_1 - \theta_2$):

$$\boxed{\begin{aligned} \left|\frac{z_1}{z_2}\right| &= \frac{|z_1|}{|z_2|} \\ \arg\left(\frac{z_1}{z_2}\right) &= \arg(z_1) - \arg(z_2) \end{aligned}}$$

Example: Describe geometrically what happens when a complex number z is divided by $w = \frac{1}{2} + \frac{\sqrt{3}}{2}i$.

Example: If $z = \sqrt{3} + i$ and $w = 1 - \sqrt{3}i$, use exponential polar form to find $\frac{1}{z}$ and zw.

Example continued:

Properties of the modulus and argument

We have proven the following properties of the modulus and argument of complex numbers z and w. These results are very useful for calculating moduli and arguments.

1. $|zw| = |z||w|$
2. $\left|\dfrac{z}{w}\right| = \dfrac{|z|}{|w|}$
3. $\arg(zw) = \arg(z) + \arg(w)$
4. $\arg\left(\dfrac{z}{w}\right) = \arg(z) - \arg(w)$.

Example: Using the above properties, evaluate

$$\left|\frac{-2(3-i)(5+2i)}{(1+3i)(7-i)}\right|$$

Example: Using the above properties, evaluate

(i) $\arg\Big((1+i)(-1+\sqrt{3}i)\Big)$

(ii) $\arg\left(\dfrac{-i}{-2+2i}\right)$

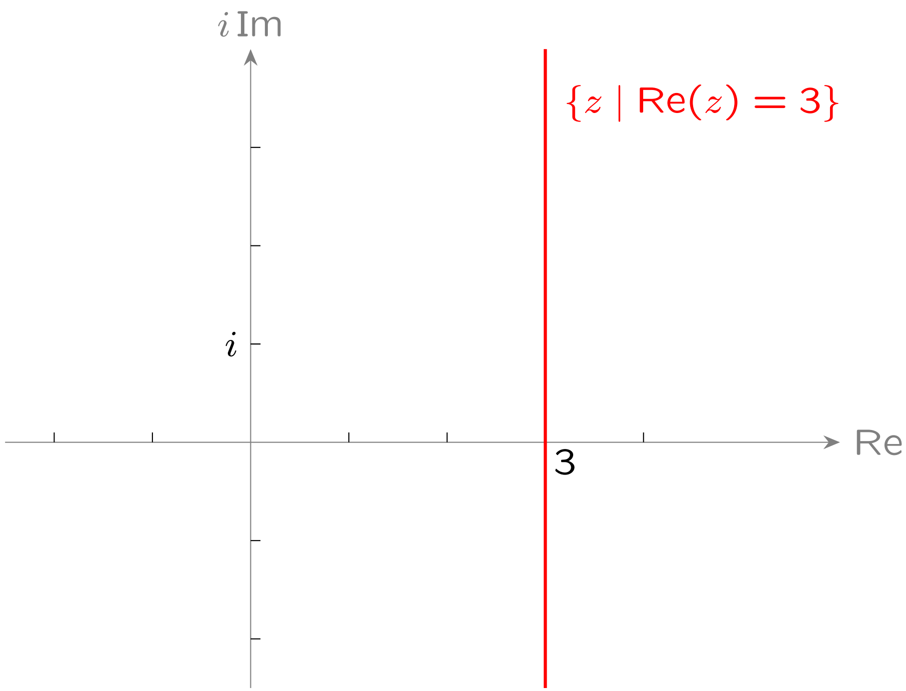
Additional questions

You can now attempt problems 13–14 from Topic 3 in the handbook.

3.5 Sketching Regions in the Complex Plane

By considering the set of complex numbers that satisfy certain conditions, we obtain a corresponding region in the complex plane.

For example: $\{z \mid \text{Re}(z) = 3\}$, the set of all complex numbers whose real part is 3, can be represented in the complex plane by the vertical line intersecting the real axis at 3.

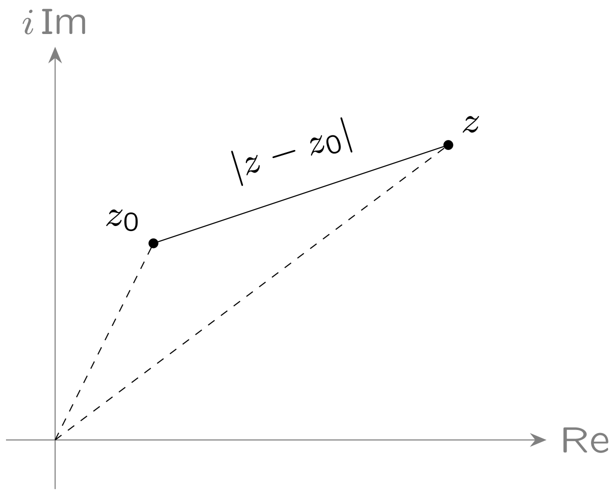


Example: Sketch the set of points in the complex plane satisfying $\mathrm{Im}(z) < 2$.

Example continued:

Example: Sketch the region in the complex plane given by $|z - i| = 2$.

Note that if z and z_0 are complex numbers, then $|z - z_0|$ equals the distance between z and z_0 in the complex plane.



So in our example we want all complex numbers z such that the distance between z and i is 2.

This is represented by a circle centred at i, and of radius 2.

Example continued: Now find the cartesian equation of this curve.

We can find the cartesian equation (i.e. an equation in terms of x and y), by substituting $z = x + iy$ and simplifying.

$$|z - i| = 2$$

So we have obtained the equation of the circle in cartesian form.

Example: Sketch the region given by $|z - 3 - 2i| \leq 3$.

Homework: Find the cartesian equation of this region by substituting $z = x + iy$.

Example: Sketch the region given by

$$\left\{ z \quad \middle| \quad 2 \leq |z| \leq 4 \quad \text{and} \quad \frac{\pi}{4} < \arg(z) < \frac{\pi}{2} \right\}.$$

Example: Sketch the region given by $|z+2| = |z+i|$.

Homework: Find the cartesian equation of the above curve.

Example: Sketch the region given by $z = i\bar{z}$.

Homework: Sketch the region given by $z\bar{z} = 16$.

Additional questions

You can now attempt problem 15 from Topic 3 in the handbook.

3.6 Powers of complex numbers

Let $z = re^{i\theta}$. **De Moivre's theorem** states that for any integer n:

$$\boxed{z^n = r^n e^{in\theta}.}$$

This formula is consistent with the usual exponent laws

$$z^n = (re^{i\theta})^n = r^n (e^{i\theta})^n = r^n e^{in\theta}.$$

De Moivre's theorem can be used to avoid expanding the brackets when finding large powers of complex numbers.

Example: Use de Moivre's theorem to find $(1 + i\sqrt{3})^8$ in both exponential and cartesian form.

Example continued:

Example: Find $\left(\dfrac{2}{1+i}\right)^{14}$ in exponential and cartesian form.

Example continued:

Trigonometric identities via the complex exponential

The properties of the complex exponential can be used to derive trigonometric identities.

First we recall the expansion of the binomial

$$(a+b)^n.$$

The tool we use is **Pascal's triangle**:

$$\begin{array}{ccccccccccc}
 & & & & & 1 & & & & & \\
 & & & & 1 & & 1 & & & & \\
 & & & 1 & & 2 & & 1 & & & \\
 & & 1 & & 3 & & 3 & & 1 & & \\
 & 1 & & 4 & & 6 & & 4 & & 1 & \\
1 & & 5 & & 10 & & 10 & & 5 & & 1 \\
 & & & & & \vdots & & & & &
\end{array}$$

Each entry is obtained by summing the two entries above it in the previous row.

Example: Expand $(a+b)^5$.

This is a special case of the **binomial formula**

$$(a+b)^n = \binom{n}{0}a^n + \binom{n}{1}a^{n-1}b + \binom{n}{2}a^{n-2}b^2 + \cdots + \binom{n}{n-2}a^2b^{n-2} + \binom{n}{n-1}ab^{n-1} + \binom{n}{n}b^n,$$

where $\binom{n}{j}$ is the $(j+1)$-st number in the $(n+1)$-st row of Pascal's triangle.

Note: To expand $(a-b)^n$, regard it as $(a+(-b))^n$.
This means that the signs of the terms will alternate $+, -, +, -, \ldots$

We also make the following observation regarding $\cos(n\theta)$ and $\sin(n\theta)$.

Since

$$e^{in\theta} = \cos(n\theta) + i\sin(n\theta),$$

we have

$$\boxed{\begin{aligned} \cos(n\theta) &= \mathrm{Re}(e^{in\theta}) \\ \sin(n\theta) &= \mathrm{Im}(e^{in\theta}). \end{aligned}}$$

Example: Use the complex exponential and de Moivre's theorem to express $\sin(3\theta)$ in terms of $\sin\theta$.

There is another way to express $\cos\theta$ and $\sin\theta$ in terms of complex exponentials.

Start with

$$e^{i\theta} = \cos\theta + i\sin\theta. \tag{6}$$

Since $\cos(-\theta) = \cos\theta$ and $\sin(-\theta) = -\sin\theta$, we have

$$e^{-i\theta} = \cos\theta - i\sin\theta. \tag{7}$$

Adding (6) and (7) gives

$$e^{i\theta} + e^{-i\theta} = 2\cos\theta$$

$$\boxed{\Rightarrow \qquad \cos\theta = \frac{1}{2}\left(e^{i\theta} + e^{-i\theta}\right).}$$

Similarly, subtracting (7) from (6) gives the formula

$$\boxed{\sin\theta = \frac{1}{2i}\left(e^{i\theta} - e^{-i\theta}\right).}$$

We can use these identities to find expressions for powers of $\cos\theta$ or $\sin\theta$.

Example: Express $\sin^4 \theta$ as a sum of sines or cosines of multiples of θ.

Example continued:

Example continued:

Additional questions

You can now attempt problems 16–17, 23–24 from Topic 3 in the handbook.

3.7 Roots of Complex Numbers

Suppose we wish to find the n-th roots of a complex number w. So we want to find z such that

$$z = w^{\frac{1}{n}}, \qquad \text{i.e.} \qquad z^n = w.$$

Let $z = re^{i\theta}$ and $w = se^{i\phi}$ be the respective exponential polar forms. The equation becomes

$$\begin{aligned} (re^{i\theta})^n &= se^{i\phi} \\ \Rightarrow \qquad r^n e^{in\theta} &= se^{i\phi} \end{aligned}$$

Equating the modulus and argument gives

$$r^n = s \qquad \text{and} \qquad e^{in\theta} = e^{i\phi}$$

$$\Rightarrow \quad r = s^{\frac{1}{n}} \qquad\qquad \Rightarrow \quad n\theta = \phi + 2k\pi, \qquad k \in \mathbb{Z}$$

$$\Rightarrow \quad \theta = \frac{1}{n}(\phi + 2k\pi), \qquad k \in \mathbb{Z}.$$

Therefore, the n-th roots of $w = se^{i\phi}$ are:

$$\boxed{w^{\frac{1}{n}} = s^{\frac{1}{n}}\, e^{i(\frac{1}{n}(\phi+2k\pi))} \qquad \text{for } k = 0, 1, \ldots, n-1.}$$

Note 1

Notice that $k = 0, 1, \ldots, n-1$ gives n different n-th roots of w. We stop at $n-1$ since for $k = n$ we would be adding a whole multiple of 2π to the argument, which would give the same complex number as $k = 0$.

Note 2

You do **not** need to memorise the formula in the box. It is easy to derive the n-th roots of a complex number w by starting with $z^n = w$, expressing w in exponential polar form and solving for z.

Example: Find the cube roots of 8, sketch them in the complex plane, and express them in cartesian form.

Recall that over the reals, there is just one cube root of 8, namely 2. However, over the complex numbers we expect there to be three cube roots of 8!

Sketch:

We see that the cube roots of 8 all have modulus 2, and are evenly spaced around a circle of radius 2 in the complex plane. This is a general property—the n-th roots of any complex number are evenly spaced around the origin in the complex plane. Can you see why?

Cartesian form:

Example: Find the 6-th roots of unity, and sketch these in the complex plane. You may keep your answers in exponential polar form.

Note: 'Unity' is just a fancy way of saying 1*!*

Example continued:

Example: Find the 4-th roots of $1 - i\sqrt{3}$.

Example continued:

Example continued:

Homework: Find the cube roots of $1 + i$.

Additional questions

You can now attempt problems 18–21 from Topic 3 in the handbook.

3.8 Roots of polynomials

Having looked at n-th roots of complex numbers, we now consider solving more general polynomial equations in a complex variable z.

Finding the roots of a polynomial $P(z)$ means solving $P(z) = 0$. One way to do this is to factorise $P(z)$ and set each of the factors equal to 0.

Recall that over the reals $\mathbb{R}$, some quadratics have roots because they can be factorised into linear factors, like

$$x^2 + 3x + 2 = (x + 2)(x + 1),$$

while others have no real roots because they cannot be factorised, such as

$$x^2 + 3x + 4.$$

Over the complex numbers $\mathbb{C}$, however, we have the following beautiful result.

The Fundamental Theorem of Algebra

Every polynomial $P(z)$ of degree n can be factorised into n linear factors over $\mathbb{C}$, that is

$$P(z) = a(z - \alpha_1)(z - \alpha_2) \ldots (z - \alpha_n),$$

where $a, \alpha_1, \alpha_2, \ldots, \alpha_n \in \mathbb{C}$.

This tells us that

every polynomial of degree n has exactly n roots over $\mathbb{C}$!

Another nice fact is the following:

If the coefficients of $P(z)$ are real, then any non-real roots of $P(z)$ occur in complex conjugate pairs z and $\bar{z}$.

Example: Consider the polynomial $P(z) = z^3 - 3iz^2 - 2z$.

(i) How many roots do you expect this polynomial to have?

(ii) Factorise $P(z)$.

(iii) Hence find the roots of $P(z)$.

(iv) Verify your answers by checking that $P(z) = 0$.

Example continued:

Example: Solve $z^4 + z^2 - 12 = 0$.

Example: Solve $z^4 - 2z^2 + 4 = 0$, expressing your answers in exponential polar form.

Example continued:

Homework: Express the above roots in cartesian form.

Example: Solve $z^3 + z^2 + z + 1 = 0$.

Example continued:

Additional questions

You can now attempt the remaining problems from Topic 3 in the handbook.

Topic 4: Differential Calculus

We now move on to our fourth topic, differential calculus. This is a fundamental topic in calculus, and is the foundation for our next topics of integral calculus and differential equations. All these topics have major real-world applications, in science, engineering, economics and the natural world.

4.1 Second and higher order derivatives

4.2 Implicit differentiation

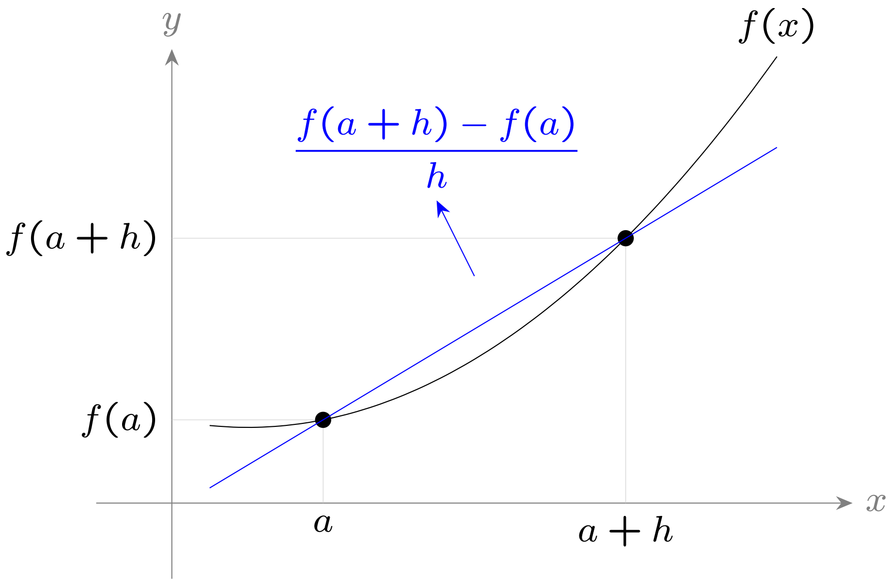
4.3 Derivatives of inverse trigonometric functions

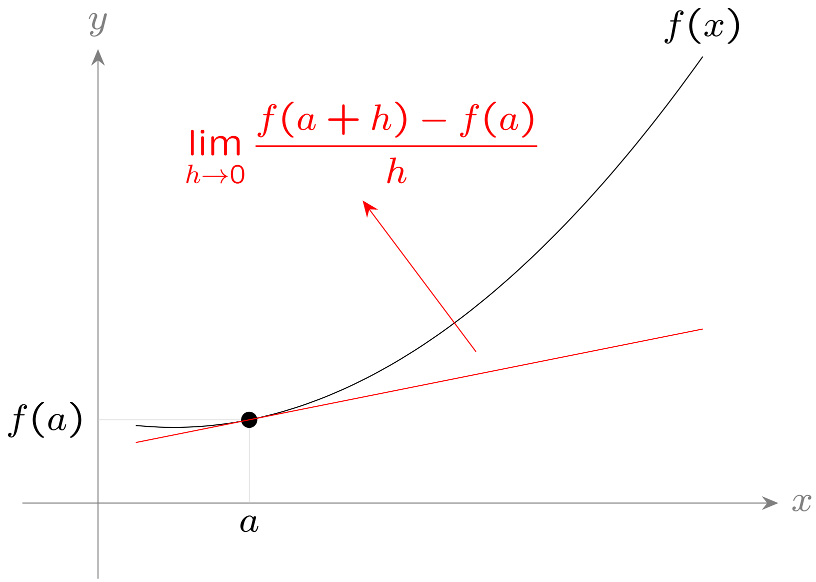
4.4 Graph sketching

4.5 Applications of differentiation

4.1 Second and higher order derivatives [Chapter 3.2]

4.1.1 Revision: First Derivative and Differentiation Rules





The **derivative of** f **at** a is defined as the slope of the tangent line to the graph of f at the point $(a, f(a))$:

$$f'(a) = \lim_{h\to 0}\frac{f(a+h)-f(a)}{h}.$$

If this limit exists, f is said to be **differentiable at** a.

In day-to-day work, we use the standard derivatives that you learnt at school, and the key rules of differentiation.

Some Standard Derivatives

$$\frac{d}{dx}\Big(x^n\Big) = nx^{n-1}$$

$$\frac{d}{dx}\Big(\sin(x)\Big) = \cos(x)$$

$$\frac{d}{dx}\Big(\cos(x)\Big) = -\sin(x)$$

$$\frac{d}{dx}\Big(e^x\Big) = e^x$$

$$\frac{d}{dx}\Big(\log(x)\Big) = \frac{1}{x}$$

Linearity of Derivatives

Differentiation is a linear operator:

$$\boxed{\begin{aligned} \frac{d}{dx}\Big(cf(x)\Big) &= c\frac{d}{dx}\Big(f(x)\Big)\,, \qquad c \in \mathbb{R} \\ \frac{d}{dx}\Big(f(x)+g(x)\Big) &= \frac{d}{dx}\Big(f(x)\Big) + \frac{d}{dx}\Big(g(x)\Big) \end{aligned}}$$

provided the functions f and g are both differentiable at x.

Example: If $f(x) = 2e^x + 3x^{-7}$, find $\frac{df}{dx}$.

Homework: If $f(x) = -9\cos(x) + 2\sqrt{x}$, find $\frac{df}{dx}$.

Product Rule

If f and g are differentiable at x, then so is the product fg, and:

$$\boxed{\frac{d}{dx}\Big(f(x)g(x)\Big) = \frac{d}{dx}\Big(f(x)\Big)g(x) + f(x)\frac{d}{dx}\Big(g(x)\Big).}$$

This is called the **product rule** and is often abbreviated to

$$\boxed{(uv)' = u'v + uv',}$$

where u' and v' denote the derivatives of u and v respectively.

Example: If $f : (0, \infty) \to \mathbb{R},\ f(x) = (x^4 - 3x)\log(x)$, find $f'(x)$.

Homework: If $f : \mathbb{R} \to \mathbb{R}$, $f(x) = e^x(2x^3 - x^2)$, find $f'(x)$.

Quotient Rule

If f and g are differentiable at x and if $g(x) \neq 0$, then $\frac{f}{g}$ is differentiable at x and:

$$\boxed{\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{\frac{d}{dx}\Big(f(x)\Big)g(x) - f(x)\frac{d}{dx}\Big(g(x)\Big)}{(g(x))^2}.}$$

This is called the **quotient rule** and is often abbreviated to

$$\boxed{\left(\frac{u}{v}\right)' = \frac{u'v - uv'}{v^2}.}$$

Example: If $y = \dfrac{x^3}{x^2 + 1}$ for all $x \in \mathbb{R}$, find $\dfrac{dy}{dx}$.

Homework: If $f(x) = \dfrac{e^x}{x^3}$, find $f'(x)$.

Chain Rule

If g is differentiable at x and f is differentiable at $g(x)$, then the composition $f \circ g$ is differentiable at x and:

$$\boxed{\frac{d}{dx}\big(f(g(x))\big) = f'(g(x))\, g'(x).}$$

This is called the **chain rule**.

If we let $y = f(u)$ where $u = g(x)$, then the chain rule can be written:

$$\boxed{\frac{dy}{dx} = \frac{dy}{du}\,\frac{du}{dx}.}$$

Example: Find $\dfrac{dy}{dx}$ if $y = \sin(3x^2 + 8)$.

Homework: Find $\dfrac{dy}{dx}$ if $y = \sqrt{e^x + 4}$.

With our differentiation skills and knowledge of trigonometric functions, we can now compute the derivatives of other trig functions.

Example: If $f(x) = \tan(x)$, find $f'(x)$.

So: $$\boxed{\frac{d}{dx}\Big(\tan(x)\Big) = \sec^2(x).}$$

Example: If $f(x) = \sec(x)$, find $f'(x)$.

Homework: If $f(x) = \mathrm{cosec}(x)$, find $f'(x)$.

4.1.2 Second order derivatives

If f is differentiable at x, then the derivative $f'(x)$ or $\frac{dy}{dx}$ may also be differentiable at x. We may then obtain the **second derivative** of $f(x)$, which is denoted $f''(x)$. There are several different notations which may also be used to denote the second derivative:

$$\boxed{f''(x) = \frac{d}{dx}\big(f'(x)\big) = \frac{d}{dx}\left(\frac{d}{dx}f(x)\right) = \frac{d^2}{dx^2}\big(f(x)\big).}$$

Example: If $f(x) = ax^2 + bx + c$, find $f'(x)$ and $f''(x)$.

Example: Let $f(x) = \frac{1}{2}(e^x + e^{-x})$ and $g(x) = \frac{1}{2}(e^x - e^{-x})$.
Find $f'(x)$, $f''(x)$, $g'(x)$ and $g''(x)$. What do you notice?

4.1.3 Higher order derivatives

Just as we defined the first derivative of f at x to be $f'(x)$, and the second derivative to be $f''(x)$, it is possible (for many functions f) to continue taking derivatives $f'''(x)$, $f^{(iv)}(x)$, and so on.

If $f^{(n-1)}(x)$ is differentiable at x, we define

$$\boxed{f^{(n)}(x) = \frac{d}{dx}\Big(f^{(n-1)}(x)\Big).}$$

This is called the n**-th derivative** of f.

Example: Write down the first five derivatives of $f(x) = \sin x$.

Example: Find a third degree polynomial $P(x)$ such that $P(1) = 1$, $P'(1) = 3$, $P''(1) = 6$ and $P'''(1) = 12$.

Example continued:

Example: If $f(x) = \dfrac{1}{x}$, compute f', f'', f''' and $f^{(4)}$ and use these to write down a formula for the n-th derivative $f^{(n)}(x)$.

Example continued:

Additional questions

You can now attempt problems 1–3 from Topic 4 in the handbook.

You may also attempt a selection of problems from 1–40 in Exercise set 3.2 from the textbook.

4.2 Implicit Differentiation [Chapter 3.6]

Usually, to find the derivative $\frac{dy}{dx}$ we are given or deduce y in terms of x and apply our knowledge of differentiation.

However, sometimes it is impossible to write down an expression for $y = f(x)$.

Example: Given $x^2 - xy + y^4 = 5$, what is y in terms of x?

Despite this, we can still find the derivative $\frac{dy}{dx}$, by using implicit differentiation.

To illustrate the technique, we start with a simpler example. Consider

$$x^2y = 1. \tag{8}$$

This equation describes an *implicit* dependence of y on x.

We differentiate both sides of (8) with respect to x:

$$\frac{d}{dx}\left(x^2y\right) = \frac{d}{dx}\left(1\right).$$

Applying the product rule to the left hand side gives

Finally we rearrange to isolate $\frac{dy}{dx}$:

$$\frac{dy}{dx} = -\frac{2y}{x}, \quad x \neq 0.$$

Note that the answer is given in terms of both x and y, and this is the best we can hope for in general.

Sometimes it is possible to express this entirely in terms of x. In our example,

$$x^2 y = 1 \quad \Rightarrow \quad y = \frac{1}{x^2},$$

so we can plug this into $-\frac{2y}{x}$ to get

$$\frac{dy}{dx} = -\frac{2}{x^3}.$$

General procedure for implicit differentiation

Start with an equation involving both x and y; so y implicitly depends on x.

1. Take the derivative of each side with respect to x.
2. Use the usual differentiation rules to simplify each side.
3. Rearrange to solve for $\frac{dy}{dx}$.

Example: Given that $y^3 = x$, find $\dfrac{dy}{dx}$ by implicit differentiation.

$$\frac{d}{dx}(y^3) = \frac{d}{dx}(x).$$

On the left hand side we need to differentiate y^3 with respect to x (not y). How can we do this?

Since y is implicitly dependent on x, we see that $y^3 = (y)^3$ is actually a composition of two functions: the 'cube' function is applied to y, which itself depends on x. So we must apply the chain rule:

$$\frac{d}{dx}(y^3) = \frac{d}{dy}(y^3) \cdot \frac{dy}{dx} = 3y^2 \cdot \frac{dy}{dx}.$$

The $3y^2$ term is the derivative of the outer function and the $\frac{dy}{dx}$ term is the derivative of the inner function.

Continuing with our calculation:

$$
\begin{aligned}
& \frac{d}{dx}(y^3) = \frac{d}{dx}(x) \\
\Rightarrow \qquad & 3y^2 \cdot \frac{dy}{dx} = 1 \\
& \frac{dy}{dx} = \frac{1}{3y^2}, \quad y \neq 0.
\end{aligned}
$$

Example: If y is implicitly dependent on x, find the following derivatives:

(a) $\frac{d}{dx}\left(y^5\right)$

(b) $\frac{d}{dx}\Big(\sin(y)\Big)$

(c) $\frac{d}{dx}\left(e^y\right)$

(d) $\frac{d}{dx}\Big(\log(y)\Big)$

Let's return to our original problem. Here we cannot solve for y in terms of x, so implicit differentiation is the only way to find $\frac{dy}{dx}$.

Example: If $x^2 - xy + y^4 = 5$, find $\dfrac{dy}{dx}$.

Example continued:

Homework: If $\cos(y) = x^2$, find $\dfrac{dy}{dx}$.

Example: If $\sin(\log(y)) = \frac{x}{y},\ y \neq 0$, find $\frac{dy}{dx}$.

Example continued:

Example: Find the equation of the tangent to the curve $y^4 - x^4 = 15$ at the point $(1, 2)$.

Example continued:

Homework: Find the equation of the tangent to the hyperbola $x^2 - y^2 = 9$ at the point $(5, 4)$.

Additional questions

You can now attempt problems 4–7 from Topic 4 in the handbook.

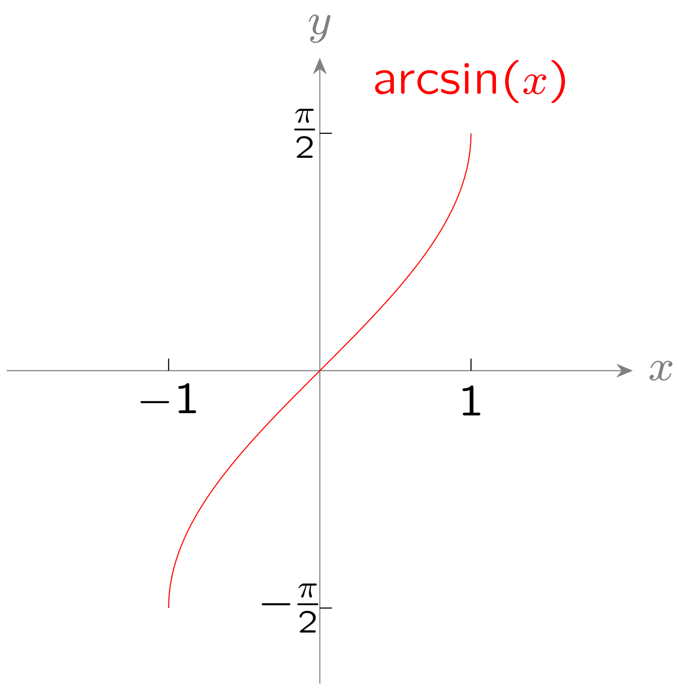
You may also attempt a selection of problems from 1–16 in Exercise set 3.6 from the textbook.

4.3 Derivatives of inverse trigonometric functions [Chapter 3.8]

Implicit differentiation can help us find the derivatives of inverse functions, for instance of the inverse trigonometric functions $\arcsin$, $\arccos$ and $\arctan$.

4.3.1 Derivative of $\arcsin$

Recall the function $\arcsin\colon [-1,1] \to \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ with graph



To find the derivative of $y = \text{arcsin}(x)$, we rewrite this relation as

$$\sin(y) = x.$$

By implicit differentiation with respect to x we get:

$$\Rightarrow \qquad \frac{dy}{dx} = \frac{1}{\cos(y)}, \qquad \cos(y) \neq 0. \tag{9}$$

The answer is in terms of y. To write it in terms of x, note that $\sin(y) = x$, so

$$\cos(y) = \sqrt{1 - \sin^2(y)} = \sqrt{1 - x^2}.$$

(The positive root is taken since $y = \text{arcsin}(x) \in [-\frac{\pi}{2}, \frac{\pi}{2}]$, so $\cos(y) \geq 0$.)

Substituting this in (9) gives

$$\frac{dy}{dx} = \frac{1}{\sqrt{1 - x^2}}.$$

So we have found the derivative of arcsin:

$$\boxed{\frac{d}{dx}\Big(\mathrm{arcsin}(x)\Big) = \frac{1}{\sqrt{1-x^2}}, \qquad \text{for } -1 < x < 1.}$$

Example: Find

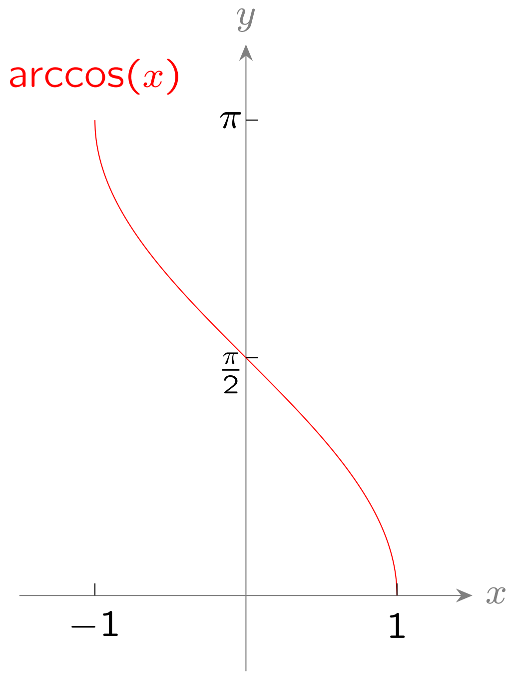
(a) $\frac{d}{dx}\Big(\mathrm{arcsin}(5x)\Big)$

(b) $\frac{d}{dx}\Big(\mathrm{arcsin}(x^3)\Big)$

Homework: Find $\dfrac{d}{dx}\Big(\mathrm{arcsin}(2x+1)\Big)$.

4.3.2 Derivative of arccos

Homework: Consider the function $\operatorname{arccos}\colon [-1,1] \to [0,\pi]$ with graph



Use implicit differentiation to show that

$$\boxed{\frac{d}{dx}\Big(\arccos(x)\Big) = \frac{-1}{\sqrt{1-x^2}}, \qquad \text{for } -1 < x < 1.}$$

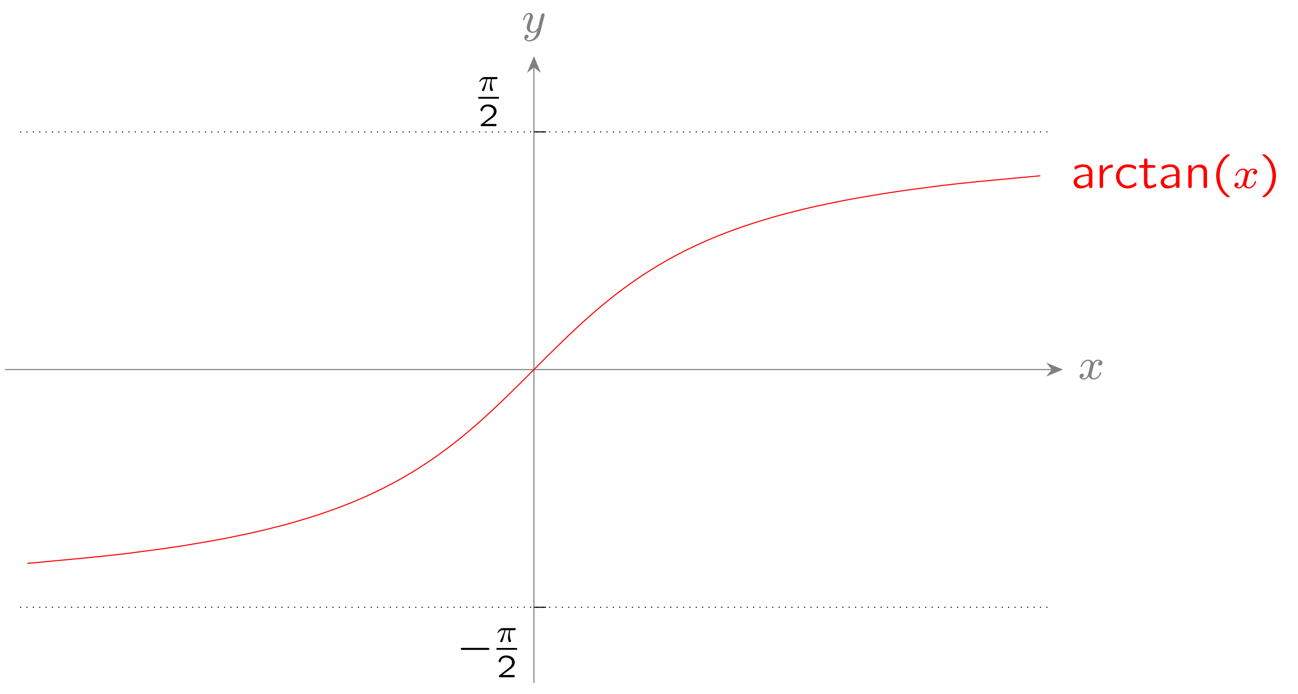
Example: If $y = \arccos\left(\frac{x}{3}\right)$, find $\frac{dy}{dx}$.

Example: If $y = \text{arccos}\left(\sqrt{x}\right)$, find $\dfrac{dy}{dx}$.

Homework: Find $\dfrac{d}{dx}\Big(\arccos(e^x)\Big)$.

4.3.3 Derivative of $\arctan$

Recall the function $\arctan\colon \mathbb{R} \to \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ with graph



We now work out its derivative.

$$y = \mathrm{arctan}(x)$$

$\Rightarrow$

So the derivative of arctan is:

$$\boxed{\frac{d}{dx}\Big(\mathsf{arctan}(x)\Big) = \frac{1}{1+x^2}\,, \qquad \text{for } x \in \mathbb{R}.}$$

Example: If $f(x) = \text{arctan}\left(\frac{2x-1}{5}\right)$, find $f'(x)$.

Homework: Find $\dfrac{d}{dx}\Big(\operatorname{arctan}(x^5)\Big)$.

Example: If $f(x) = \left(1 + x^2\right) \operatorname{arctan}(x)$, find $f'(x)$.

Example: If $f(x) = \sqrt{\operatorname{arcsin}(x)}$, find $f'(x)$.

Additional questions

You can now attempt problems 8–10 from Topic 4 in the handbook.

You may also attempt a selection of exercises from 1–12 and 21–42 in Exercise set 3.8 in the textbook.

4.4 Graph sketching [Chapter 4.1, 4.3 and 4.4]

Our differentiation skills allow us to analyse and sketch graphs of functions.

4.4.1 Increasing and decreasing functions [Chapter 4.3]

The intuitive idea of what it means for a function to be increasing or decreasing is formalised as follows:

Definition: Let f be defined on an interval I, and let x_1 and x_2 denote points in I.

1. f is **increasing** on I if $f(x_1) < f(x_2)$ whenever $x_1 < x_2$.
2. f is **decreasing** on I if $f(x_1) > f(x_2)$ whenever $x_1 < x_2$.

Example: Sketch a function on an interval $[a, b]$, that is (i) increasing; (ii) decreasing; (iii) neither increasing nor decreasing.

The first derivative detects whether a function is increasing or decreasing:

Theorem: Let f be continuous on $[a, b]$ and differentiable on (a, b).

1. If $f'(x) > 0$ for all $x \in (a, b)$ then f is increasing over $[a, b]$.

2. If $f'(x) < 0$ for all $x \in (a, b)$ then f is decreasing over $[a, b]$.

4.4.2 Local extrema and stationary points [Chapter 4.1]

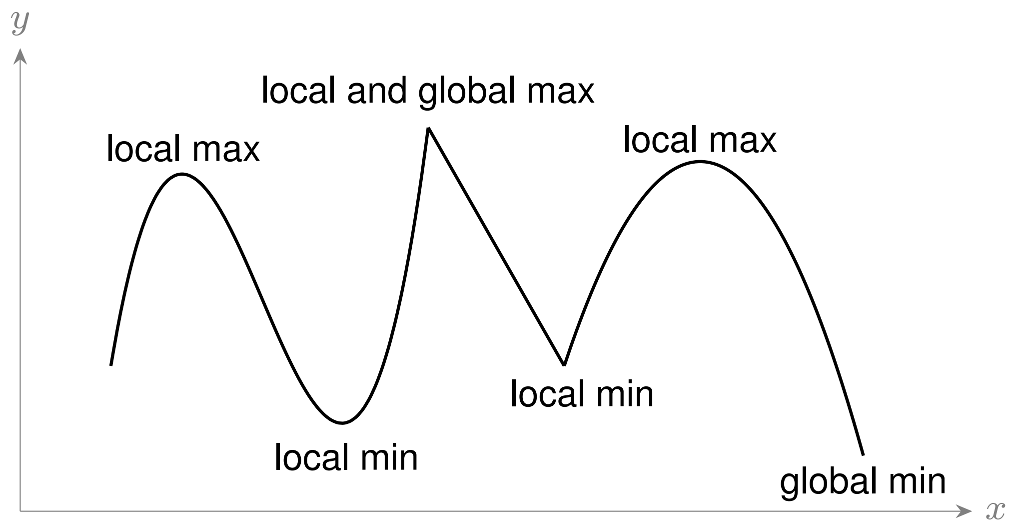
Definition:

A function f is said to have a **local maximum** at (x_0, y_0) if there is an open interval containing x_0 on which $y_0 = f(x_0)$ is the *largest* value.

A function f is said to have a **local minimum** at (x_0, y_0) if there is an open interval containing x_0 on which $y_0 = f(x_0)$ is the *smallest* value.

Local maxima and minima are also called **local extrema**.

The points where a function attains its *overall* largest and smallest values are called its **global extrema**.



We see that there are two types of local extrema:

1. Points where the graph of the function has a horizontal tangent.

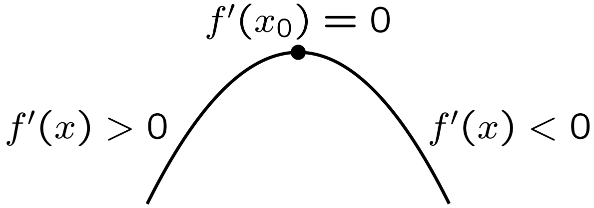
2. Points where the function is not differentiable (has a sharp point).

We will focus on the first type:

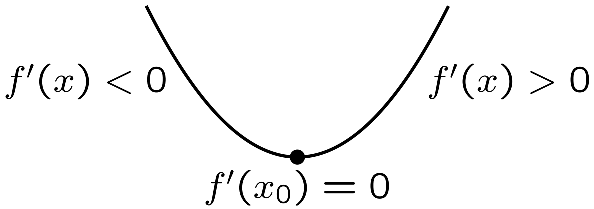
Definition: A **stationary point** of f is any point $(x, f(x))$ such that

$$f'(x) = 0.$$

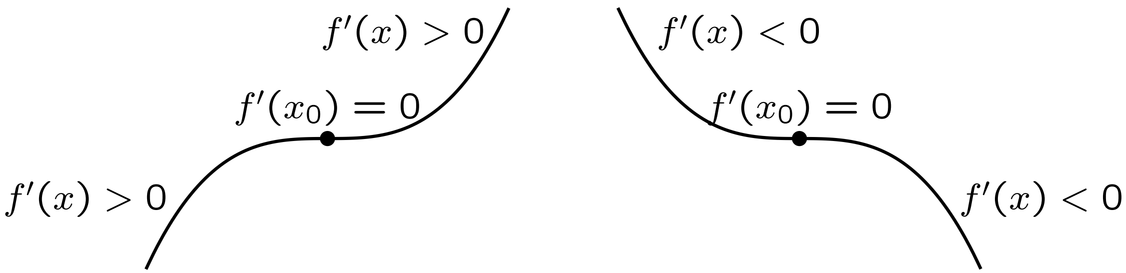
A stationary point can be a local maximum:



a local minimum:



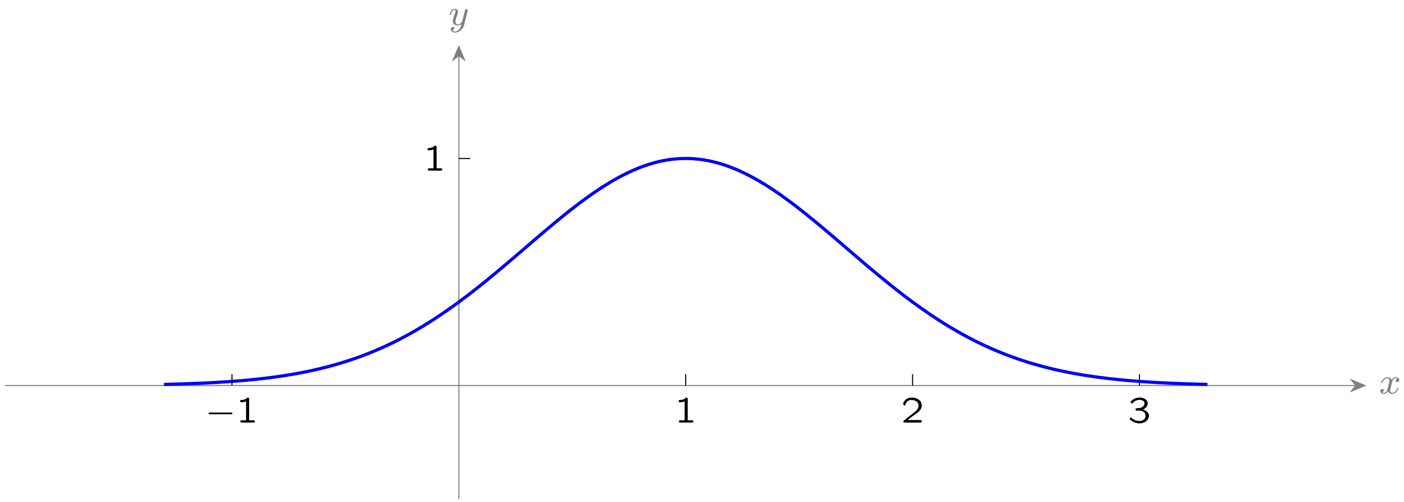
or neither:



Example: Find the stationary points of the function $f(x) = e^{-(x-1)^2}$. On what intervals is f increasing? Decreasing?

Example continued:

Our findings are verified by looking at the graph of f:

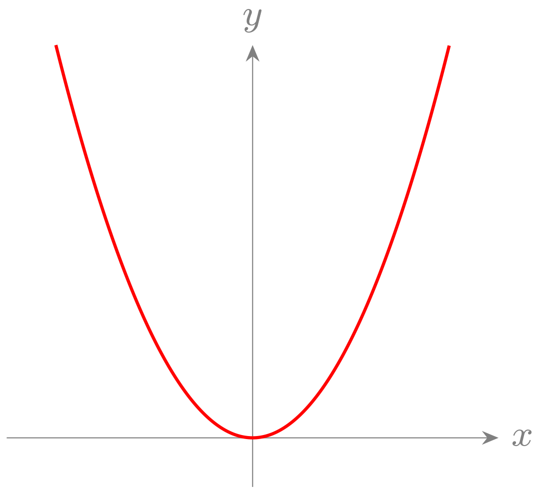


4.4.3 Concavity [Chapter 4.4]

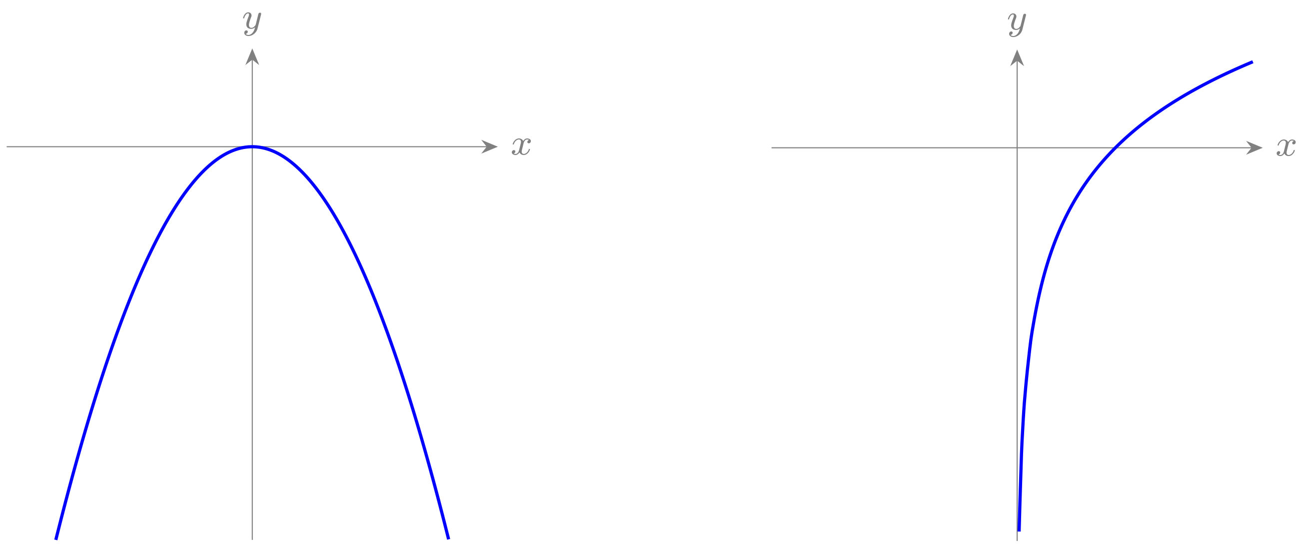
Concave up and concave down are geometric terms that describe the *curvature* of the graph of a function.

Definition: If f is differentiable on an interval I, then:

1. f is **concave up** on I if f' is increasing on I;
2. f is **concave down** on I if f' is decreasing on I.



Graphs of $y = x^2$ and $y = e^x$, both concave up



Graphs of $y = -x^2$ and $y = \log(x)$, both concave down

We can determine concavity using the second derivative:

Theorem: Suppose f and f' are differentiable on an interval I.

1. If $f''(x) > 0$ for all x in I, then f is concave up on I.
2. If $f''(x) < 0$ for all x in I, then f is concave down on I.

A point where f changes concavity has a special name:

Definition: A **point of inflection** is a point where f changes between being concave up and concave down.

Note: It is NOT correct to simply solve $f''(x) = 0$ to find points of inflection, because there are functions where $f''(x) = 0$ without a point of inflection (see the example on page 562).

Example: Sketch a function on an interval $[a, b]$ that is:

(i) increasing and concave down;

(ii) increasing and concave up;

(iii) decreasing and concave down;

(iv) decreasing and concave up.

Example: Sketch a function with a point of inflection where:

(i) f changes from concave down to concave up;

(ii) f changes from concave up to concave down.

Example: Find any points of inflection of:

(a) $f(x) = x^4$

(b) $f(x) = x^3$

Example continued:

Example continued:

Example: Consider the function:

$$f(x) = 3x^4 - 4x^3.$$

Find:

(i) the domain of f;

(ii) x- and y-intercepts of the graph $y = f(x)$;

(iii) stationary points of f;

(iv) intervals over which f is increasing;

(v) intervals over which f is decreasing;

(vi) local maxima/minima of f;

(vii) intervals over which f is concave up;

(viii) intervals over which f is concave down;

(ix) points of inflection of f.

Use this information to sketch the graph of $y = f(x)$.

Example continued:

Example continued:

Example continued:

Example continued:

Additional questions

You may now attempt a selection of problems from Exercise set 4.1, 4.3 and 4.4 from the textbook. Try do do these without using your graphics calculator. Remember that you won't have it in the exam!

4.4.4 Asymptotic behaviour of rational functions [Chapter 2.5]

An **asymptote** of a function f is a straight line to which the graph $y = f(x)$ becomes arbitrarily close either as $x \to x_0$ or as $x \to \pm\infty$.

The three types of straight lines in the plane give rise to:

- vertical asymptotes,
- horizontal asymptotes,
- oblique (slant) asymptotes.

Vertical asymptotes

We have already encountered vertical asymptotes when a ‘zero denominator’ occurs. That is, when $f(x) = \dfrac{g(x)}{h(x)}$ and $h(x) = 0$ for some values of x.

Vertical asymptotes may also arise in situations that do **not** involve quotients: for example, the function $f(x) = \log(x)$ does not involve divisions, but it has a vertical asymptote at $x = 0$.

Example: Recall and sketch the graph of $y = \mathtt{tan}\, x$.

Horizontal asymptotes

Horizontal asymptotes are associated with limiting behaviour as $x \to \pm\infty$. We have seen this kind of behaviour previously, with exponential functions.

Example: On the same figure sketch the graphs of $f(x) = e^x + 1$ and $g(x) = e^{-x} + 1$.

Example continued:

Oblique (slant) asymptotes

Sometimes the graph of a function f as $x \to \pm\infty$ approaches a slant line $y = \alpha x + \beta$ with $\alpha \in \mathbb{R} \setminus \{0\}$. We call this line an **oblique asymptote** and we write $f(x) \sim \alpha x + \beta$.

Oblique asymptotes may occur when considering *rational functions*, that is, functions of the form $f(x) = \dfrac{g(x)}{h(x)}$ where g and h are polynomials.

To find an oblique (or horizontal) asymptote of a rational function, we generally need to apply **polynomial long division**. This makes it easier to analyse the behaviour of the function as $x \to \pm\infty$.

Note: It is incorrect to think of an asymptote as "a line that may not be crossed"; this is only true of *vertical* asymptotes. A function may cross a *horizontal* or *oblique* asymptote many times.

Example: Sketch the graph of $f(x) = e^{-x}\sin(x)$.

Example: Find any asymptotes of $f(x) = \dfrac{2x^3 - 3x^2 + 2x - 2}{x^2 + 1}$.

Example continued:

Example: Find any asymptotes of $f(x) = \dfrac{2x^2 + 3x - 2}{x^2 - 1}$.

Example continued:

Example: Consider the function $f(x) = \dfrac{x^2}{x+1}$.

We will demonstrate how to analyse and sketch the graph of this function using a list of steps. Find:

(i) the domain of f;

(ii) the asymptotes of f;

(iii) the x- and y-intercepts of the graph $y = f(x)$;

(iv) the stationary points of f;

(v) the intervals where f is increasing;

(vi) the intervals where f is decreasing;

(vii) the local maxima/minima of f;

(viii) the intervals where f is concave up;

(ix) the intervals where f is concave down;

(x) the points of inflection of f.

Use all of this information to sketch the graph $y = f(x)$.

Example continued:

Example continued:

Example continued:

Example continued:

Example continued:

Example continued:

Additional questions

You can now attempt problems 11–12 from Topic 4 in the handbook.

You may also attempt a selection of problems from Exercise set 2.5 from the textbook.

4.5 Applications of Differentiation [Chapter 4.5]

Problems involving differentiation arise in a wide range of real world applications. In this section we look at several application problems which we can solve by using our differentiation skills.

Often these problems involve the optimisation (maximisation or minimisation) of a function subject to certain constraints. For such problems we can use the following general procedure.

General guidelines for solving max/min problems:

1. Draw a diagram with the appropriate quantities labelled.
2. Find a formula for the quantity to be maximised or minimised.
3. If this formula is in terms of more than one variable, use the conditions of the problem to reduce it to a function of a single variable.
4. Determine the domain of this function from the physical problem.
5. Determine maxima/minima by finding and analysing stationary points, and if relevant, the endpoints of the domain.

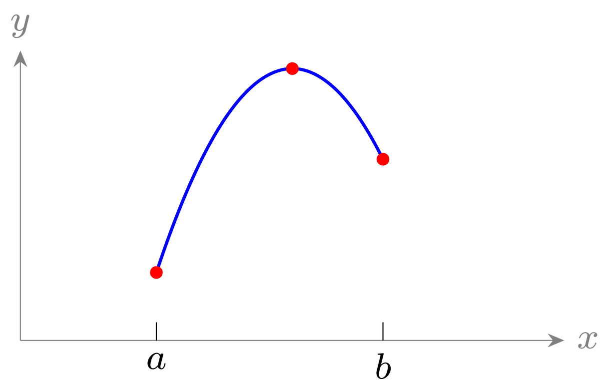
Note about Step 5:

For a function f defined on an interval $[a, b]$, it is important to consider *both*:

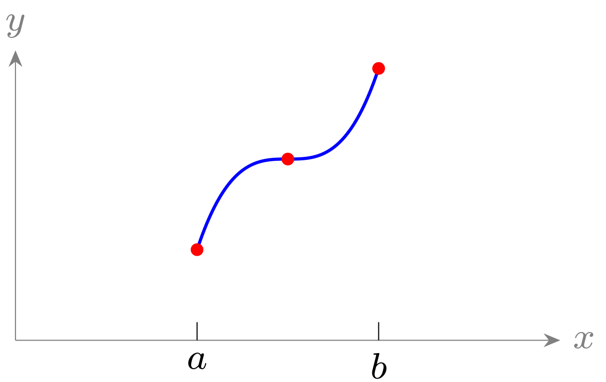
- the stationary points of f
- the endpoints of the domain $[a, b]$

because the global maximum/minimum could occur at either type of point.

The following graphs illustrate this:



Global max occurs at a stationary point inside $[a, b]$



Global max occurs at the endpoint b, not at the stationary point

Example: A farmer wishes to form a rectangular paddock. The paddock will be bounded on one side by a straight river, while the other three sides will be formed with 100m of fencing. What is the largest possible area of the paddock?

Example continued:

Example continued:

Example: A square sheet of cardboard measures 10cm by 10cm. Four equal squares are cut out of the corners and the sides are turned up to form an open rectangular box. What size should the squares be to obtain a box with the largest volume, and what is this largest volume?

Example continued:

Example continued:

Example: A pharmaceutical company can produce x batches of a certain drug at a cost (in dollars) of

$$C(x) = 28\,000 + 240x^2 - 2x^3.$$

If the company sells each batch for \$9600, and can produce up to 60 batches per month:

(a) Give an expression for the profit in producing x batches of the drug per month, over a suitable domain.

(b) Determine how many batches should be produced each month to maximise profit.

Example continued:

Example continued:

Additional questions

You can now attempt the remaining problems from Topic 4 in the handbook.

You may also attempt a selection of problems from Exercise set 4.5 in the textbook.

Topic 5: Integral Calculus

5.1 Introduction to Integration

5.2 Integration by Substitution

5.3 Integration using Trigonometric Identities

5.4 Integration using Partial Fractions

5.5 Definite Integrals and Areas

5.6 Application: Volumes of Solids of Revolution

5.1 Introduction to Integration [Chapter 4.8]

The two branches of calculus, differential calculus and integral calculus, originally developed as separate theories. Curiously, in most calculus courses differential calculus is taught before integral calculus, when in fact it was the latter theory that was developed first. Indeed the earliest mention of calculating volumes and areas (ideas that led to integral calculus) date back to the ancient Egyptians c. 1820 BC.

It was not until the 17th C that the relationship between differential and integral calculus was made clear. Newton's profound "Fundamental Theorem of Calculus" makes this relation precise and provides a standard way of calculating areas.

The definite integral

The **definite integral** of a function f from a to b, denoted

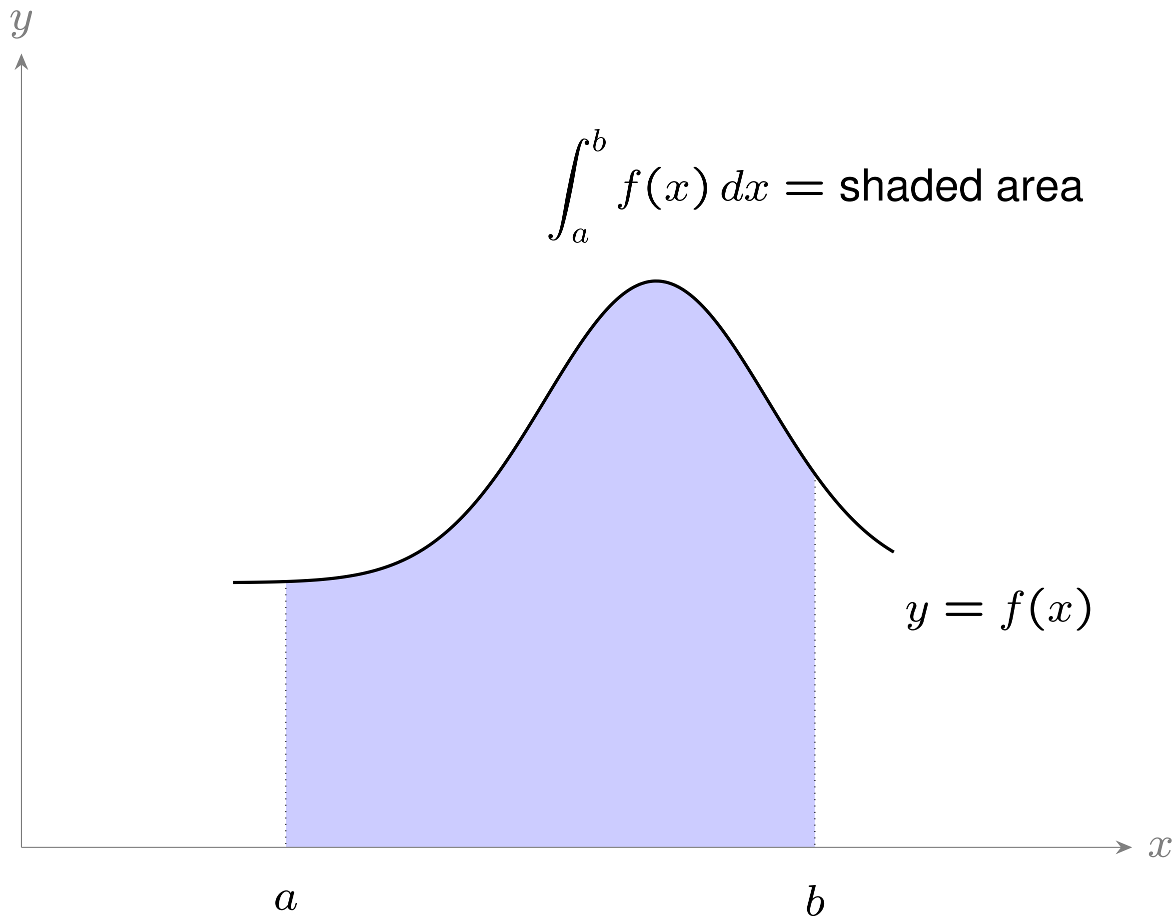
$$\int_a^b f(x)\ dx,$$

is defined to be the **signed area** bounded by:

- the curve $y = f(x)$,
- the x-axis,
- the vertical lines $x = a$ and $x = b$, where $a \leq b$.

Signed means that the parts of the graph above the x-axis are taken to have positive area, while the parts of the graph below the x-axis are taken to have negative area.

The values a and b are called the **terminals** of the integral.



y
$\int_a^b f(x)\,dx =$ shaded area
$y = f(x)$
x
a
b

Example: Evaluate the definite integral

$$\int_0^4 \sqrt{16 - x^2}\, dx$$

by interpreting it as the area of a known geometric figure.

Crucial to the development of calculus is the link between definite integrals and the concept of derivative.

Given a function f, we say that another function F is an **antiderivative of** f on an interval I if F is differentiable on I and

$$F'(x) = f(x) \qquad \text{for all } x \in I.$$

Fundamental Theorem of Calculus

If f is continuous on $[a, b]$ and F is an antiderivative of f on $[a, b]$, then

$$\int_a^b f(x)\, dx = F(b) - F(a).$$

It is therefore essential to be able to find antiderivatives. We will spend some time discussing various techniques of antidifferentiation, before returning to definite integrals in Section 5.5.

Notice that antiderivatives are not unique. For any F that satisfies

$$F'(x) = f(x),$$

the function $F(x) + C$ also has this property:

$$(F(x) + C)' = F'(x) = f(x).$$

We often write the operation of antidifferentiation using an integral sign without terminals, and refer to it as the **indefinite integral**:

$$\int f(x)\ dx = F(x) + C,$$

where F is some antiderivative of f.

For example,

$$\int x^2\,dx = \frac{1}{3}x^3 + C,$$

since

$$\frac{d}{dx}\left(\frac{1}{3}x^3 + C\right) = \frac{1}{3}\cdot 3x^2 = x^2.$$

Similarly,

$$\int \cos x\,dx = \sin x + C,$$

since

$$\frac{d}{dx}\big(\sin x + C\big) = \cos x.$$

Many integrals can be found directly in this way.

5.1.1 Standard Integrals

You should be familiar with the following integrals, where $k \in \mathbb{R}$:

$$\int x^n \, dx = \frac{x^{n+1}}{n+1} + C \qquad \text{for } n \neq -1$$

$$\int \frac{1}{x} \, dx = \log|x| + C$$

$$\int e^{kx} \, dx = \frac{1}{k} e^{kx} + C$$

$$\int \sin kx \, dx = -\frac{1}{k} \cos kx + C$$

$$\int \cos kx \, dx = \frac{1}{k} \sin kx + C$$

$$\int \sec^2 kx \, dx = \frac{1}{k} \tan kx + C$$

Since we have learnt how to differentiate inverse trigonometric functions, we also have:

$$\int \frac{1}{\sqrt{a^2 - x^2}}\, dx = \arcsin\left(\frac{x}{a}\right) + C$$

$$\int \frac{-1}{\sqrt{a^2 - x^2}}\, dx = \arccos\left(\frac{x}{a}\right) + C$$

$$\int \frac{1}{a^2 + x^2}\, dx = \frac{1}{a} \arctan\left(\frac{x}{a}\right) + C$$

where $a > 0$.

Homework: Check these by differentiating the right hand side using the chain rule.

Integrals, like derivatives, are **linear**. That is:

$$\int af(x)\,dx = a\int f(x)\,dx$$

$$\int (f(x)+g(x))\,dx = \int f(x)\,dx + \int g(x)\,dx$$

Example: Find $\int 5e^{2x} - \frac{3}{x}\,dx, x \neq 0$.

BEWARE of the following common errors, which are **NOT** properties of integrals:

$$\int f(x)g(x)\,dx \neq f(x)\int g(x)\,dx$$

$$\int f(x)g(x)\,dx \neq \int f(x)\,dx \int g(x)\,dx$$

$$\int \frac{f(x)}{g(x)}\,dx \neq \frac{\int f(x)\,dx}{\int g(x)\,dx}$$

Not all integrals are straightforward standard integrals like those we have considered so far. We need methods for solving more difficult integrals, just as we have the product, quotient and chain rules for finding more difficult derivatives.

We will consider three integration techniques:

- substitution,
- using trigonometric identities,
- partial fractions.

You will encounter other methods in Calculus 2 and further subjects.

5.2 Integration by Substitution [Chapter 5.5]

This method is a consequence of the chain rule. We start with an example.

According to the chain rule,

$$\frac{d}{dx}\left(\sin^5(x)\right) = 5\sin^4(x)\cdot\cos(x).$$

Therefore, by the Fundamental Theorem of Calculus we must have

$$\int 5\sin^4(x)\cdot\cos(x)\,dx = \sin^5(x) + C.$$

If we write

$$5\sin^4(x) = 5(\sin(x))^4 = g(h(x)) \quad \text{and} \quad h(x) = \sin(x) = u,$$

then

$$\begin{aligned}\int 5\sin^4(x)\cdot\cos(x)\,dx &= \int g(h(x))\cdot h'(x)\,dx\\ &= \int g(u)\frac{du}{dx}\,dx\\ &= \int g(u)\,du.\end{aligned}$$

We arrive at the substitution formula

$$\boxed{\int g\Big(u(x)\Big)\frac{du}{dx}\,dx = \int g(u)\,du.}$$

What does $\int g(u)\,du$ really mean?.

We can express (and prove) the integration by substitution formula in definite integral form:

$$\boxed{\int_a^b g\big(u(x)\big)u'(x)\,dx = \int_{u(a)}^{u(b)} g(x)\,dx.}$$

Proof:

So $\int g(u)\,du$ means “find $\int g(x)\,dx$ and plug in $u(x)$ values at the end”.

Idea of method:

- Find a composite function $g(h(x))$ as a factor in the integrand.
- Set $u = h(x)$ and try to express the remaining factor in the integrand as
$$\frac{du}{dx} = h'(x).$$
- If successful, this gives a new integral in the “variable” u.
- Find a formula for this new integral as a function of u.
- Use the relation $u = h(x)$ to express the result in terms of x.

Example: $\int 2x(x^2 - 5)^4 \, dx$

Example continued:

Example: $\int \cos(3x)\,\sqrt{\sin(3x)+4}\,dx$

The antiderivative of $\frac{1}{x}$

A classic example of substitution involves the $\log$ function. For $x > 0$ we know from the Fundamental Theorem of Calculus that

$$\int \frac{1}{x}\,dx = \log(x) + C.$$

But what happens when $x < 0$? The answer cannot be $\log(x)$ since $\log$ is not defined for $x < 0$.

For $x < 0$, consider

$$\int \frac{1}{x}\,dx = \int \frac{1}{-x}(-1)\,dx.$$

In this example, we can write $u = h(x) = -x$ and so $g(u) = \dfrac{1}{u}$ and $\dfrac{du}{dx} = -1$. Hence

$$\int \frac{1}{x}\,dx = \int \frac{1}{-x}(-1)\,dx = \int \frac{1}{u}\,du.$$

Notice that u is positive since $u = -x$ and x is negative. Therefore

$$\int \frac{1}{u}\,du = \log(u) + C = \log(-x) + C.$$

We have

$$\int \frac{1}{x}\,dx = \log(x) + C \quad \text{when} \quad x > 0,$$
$$\int \frac{1}{x}\,dx = \log(-x) + C \quad \text{when} \quad x < 0,$$

hence

$$\boxed{\int \frac{1}{x}\,dx = \log|x| + C\,, \qquad x \neq 0.}$$

Example: $\displaystyle\int \frac{6}{x^2+2x+5}\,dx$

Homework: $\displaystyle\int (x^4 + 1)\, e^{x^5+5x}\, dx$

Homework: $\int \frac{x^2+1}{\sqrt{x^3+3x}}\,dx$

Linear Substitutions

Sometimes, after substituting $u = h(x)$ and using $\dfrac{du}{dx} = h'(x)$, the integrand still contains the variable x. Completing the substitution would then require inverting the relation $u = h(x)$, which is often a bad sign. A special situation where this can be done successfully is when $h(x)$ is a linear function in x.

Example: $\displaystyle\int (2x+1)\sqrt{x-3}\,dx$

Example continued:

Example: $\displaystyle\int \frac{2x}{(x+1)^{10}}\,dx$

Homework: $\int 2x(x-5)^7\,dx$

Additional questions

You can now attempt problems 1–6 from Topic 5 in the handbook.

You may also attempt a selection of problems from Exercise sets 4.8 and 5.5 in the textbook.

5.3 Integration using Trigonometric Identities [Chapter 7.2]

An integral of the form

$$\int \sin^m(x)\cos^n(x)\,dx$$

(where m, n are nonnegative integers) can be solved using trigonometric identities. There are two cases to consider, depending on whether the powers m and n are even or odd.

5.3.1 Case 1: At least one of m, n is odd

In this case we can turn the integral into a simple substitution type integral. To do this, first split off a factor of the function with the odd power. This will become the derivative in the substitution, so we express everything else in terms of the *other* function, by using the identities

$$\sin^2(x) + \cos^2(x) = 1.$$

Example: $\int \sin^2(x)\cos^5(x)\,dx$

We split off a factor of the function with the odd power, $\cos(x)$, then express the other powers of $\cos$ in terms of $\sin$:

We can now substitute $u = \sin(x)$.

Example continued:

Example: $\int \sin^7(2x)\,dx$

Example continued:

5.3.2 Case 2: Both m and n are even

Here we use trigonometric identities that turn $\sin^2(x)$ and $\cos^2(x)$ into $\cos(2x)$.

The double angle formula for $\cos$ gives

$$\cos(2x) = 1 - 2\sin^2(x)$$
$$\Rightarrow \qquad \sin^2(x) = \frac{1}{2}\Big(1 - \cos(2x)\Big)$$

Homework: Derive the corresponding identity for $\cos^2(x)$:

$$\cos^2(x) = \frac{1}{2}\Big(1 + \cos(2x)\Big).$$

Example: $\int \cos^4(x)\,dx$

Example: $\displaystyle\int \sin^2(3x)\cos^2(3x)\,dx$

Homework: Solve the following:

(a) $\int \sin^2(x)\,dx$ **(b)** $\int \sin^3(x)\,dx$ **(c)** $\int \sin^4(x)\,dx$

Additional questions

You can now attempt problems 7–8 from Topic 5 in the handbook.

You may also attempt a selection of problems from Exercise set 7.2 in the textbook.

5.4 Integration using Partial Fractions [Chapter 7.4]

The method of **partial fractions** can be used to solve integrals of the form

$$\int \frac{f(x)}{ax^2 + bx + c}\, dx,$$

where $f(x)$ is a linear function (or constant), and the quadratic in the denominator can be factorised into linear factors.

The idea is to break up the integrand into smaller pieces that are easy to integrate. These pieces depend on the way the denominator factorises.

Case 1: The denominator factorises into *distinct* linear factors.

In this case we can rewrite the integrand as:

$$\boxed{\frac{f(x)}{(ax+b)(cx+d)} = \frac{A}{ax+b} + \frac{B}{cx+d}.}$$

Example: Decompose $\dfrac{9x+1}{(x-3)(x+1)}$ into partial fractions.

Example continued:

Example continued: Hence find $\displaystyle\int \frac{9x+1}{(x-3)(x+1)}\,dx$.

Example: $\displaystyle\int \frac{7}{x^2+3x-10}\,dx$

Example continued:

Case 2: The denominator factorises into *repeated* linear factors (i.e. is a perfect square).

In this case we rewrite the integrand as:

$$\boxed{\frac{f(x)}{(ax+b)^2} = \frac{A}{ax+b} + \frac{B}{(ax+b)^2}.}$$

Example: $\displaystyle\int \frac{3x+1}{x^2+4x+4}\,dx$

Example continued:

Homework: $\displaystyle\int \frac{2x-1}{x^2-6x+9}\,dx$

Notes about Partial Fractions

- If the numerator has degree greater than or equal to that of the denominator, we first need to apply **polynomial long division** to obtain an expression where the numerator has degree smaller than the denominator, before applying the partial fractions method.

- The above techniques generalise to cases where the denominator is a product of more than two linear factors. The partial fractions decompositions for the two extreme cases are:

$$\frac{f(x)}{(x-a_1)(x-a_2)\ldots(x-a_n)} = \frac{A_1}{x-a_1} + \frac{A_2}{x-a_2} + \cdots + \frac{A_n}{x-a_n}$$

$$\frac{f(x)}{(x-a)^n} = \frac{A_1}{x-a} + \frac{A_2}{(x-a)^2} + \cdots + \frac{A_n}{(x-a)^n}.$$

These cases are looked at further in Calculus 2.

Example: $\displaystyle\int \frac{2x^3 - 3x^2 - 8x + 24}{x^2 - 4}\,dx$

Example continued:

Example continued:

Homework: Practice your polynomial long division on these:

(a) $$\frac{2x^4 - 6x^3 + 14x^2 - 10x + 19}{x^2 - 3x + 5}$$

(b) $$\frac{5x^5 + 11x^4 - 3x^3 - 2x^2 - 2x + 1}{x^2 + 2x - 1}$$

Homework: Write down the partial fractions decomposition you would use for each of the following:

(a) $\dfrac{x+2}{x^2+4x-5}$

(b) $\dfrac{1-2x}{x^2+6x+9}$

(c) $\dfrac{3}{x^2-4}$

(d) $\dfrac{4x}{x^3-x^2}$

Additional questions

You can now attempt problems 9–11 from Topic 5 in the handbook.

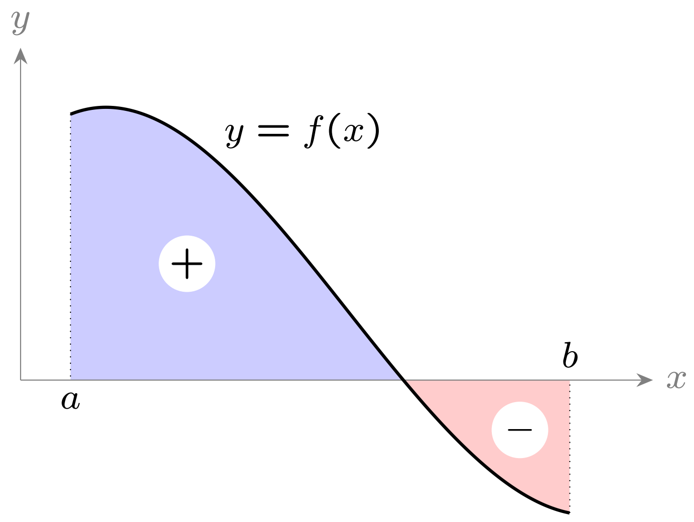
You may also attempt a selection of exercises 1–16 and 29–34 from Chapter 7.4 in the textbook.

5.5 Definite Integrals and Areas [Chapter 5.3, 5.4 and 5.6]

Recall that the **definite integral** of a function f from a to b, denoted

$$\int_a^b f(x)\ dx,$$

is defined to be the **signed area** between the graph of the function on $[a, b]$ and the x-axis.

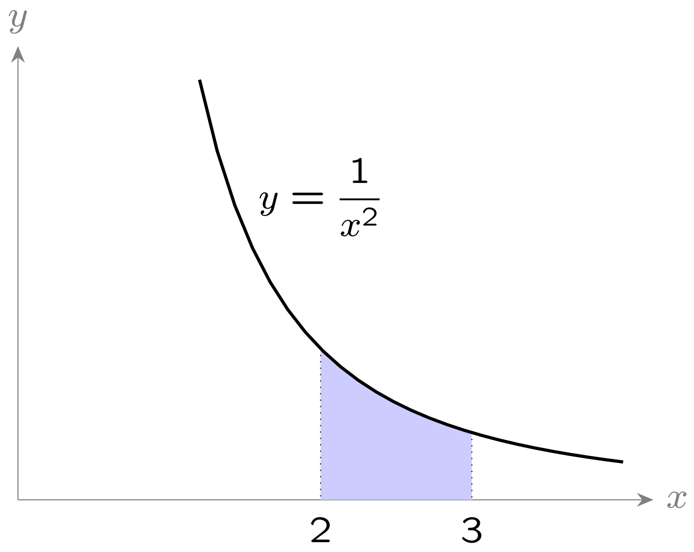


Computing its value can be done using the fundamental theorem of calculus:

$$\int_a^b f(x)\ dx = \Big[F(x)\Big]_a^b = F(b) - F(a),$$

where F is an antiderivative of f.

Example: Find the area enclosed by the curve $y = \frac{1}{x^2}$, the x-axis and the lines $x = 2$ and $x = 3$.



Example continued:

5.5.1 Properties of Definite Integrals

Definite integrals share the linearity properties of indefinite integrals:

1. $$\int_a^b f(x) + g(x)\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx;$$

2. $$\int_a^b k\,f(x)\,dx = k\int_a^b f(x)\,dx;$$

and also have the following properties:

3. $$\int_a^a f(x)\,dx = 0;$$

4. $$\int_a^c f(x)\,dx = \int_a^b f(x)\,dx + \int_b^c f(x)\,dx \qquad \text{for } a \le b \le c;$$

5. $$\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx.$$

Properties 1 to 4 can be motivated by considering areas, while Property 5 follows by setting $c = a$ in Property 4.

The next example indicates the importance of drawing a sketch whenever you are asked to find an area. Blindly evaluating a definite integral can lead to positive and negative areas (above/below the x-axis) cancelling out. Property 4 above can help in this case.

Example:

(a) Evaluate the definite integral $\displaystyle\int_0^{2\pi} \sin(2x)\, dx$.

(b) Find the area enclosed by the curve $y = \sin(2x)$ and the x-axis for $0 \leq x \leq 2\pi$.

Example continued:

5.5.2 Change of Variable with Definite Integrals

Recall that when using the substitution rule for a definite integral, the substitution is reflected in the terminals of the integral.

For a simple substitution with the outer function expressed as a derivative, the rule becomes:

$$\int_a^b f'(g(x))g'(x)\,dx = \int_{g(a)}^{g(b)} f'(x)\,dx = \Big[f(x)\Big]_{g(a)}^{g(b)},$$

where we have made the substitution $u = g(x)$.

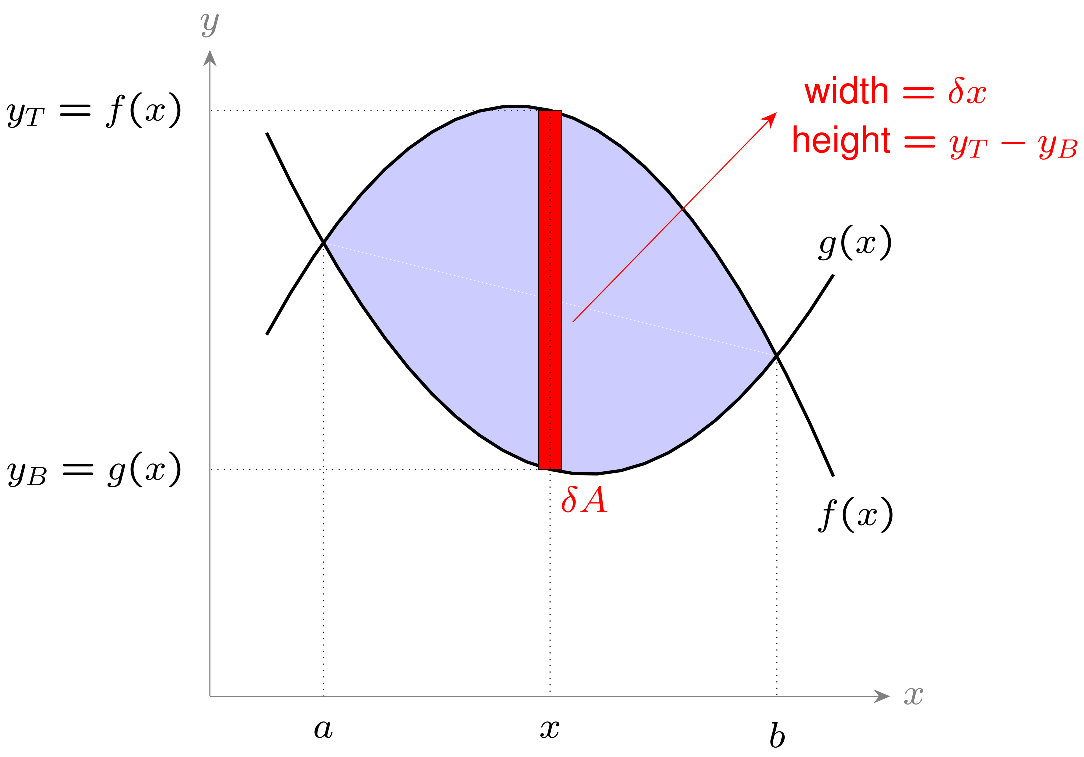
Example: Evaluate the definite integral $\int_0^{\frac{\pi}{2}} \cos(x)\sqrt{1+\sin(x)}\,dx$.

Example continued:

5.5.3 Areas between curves

Vertical strips

By dividing the area between two curves into thin vertical rectangles of width δx, we can find the area by **vertical strips**.



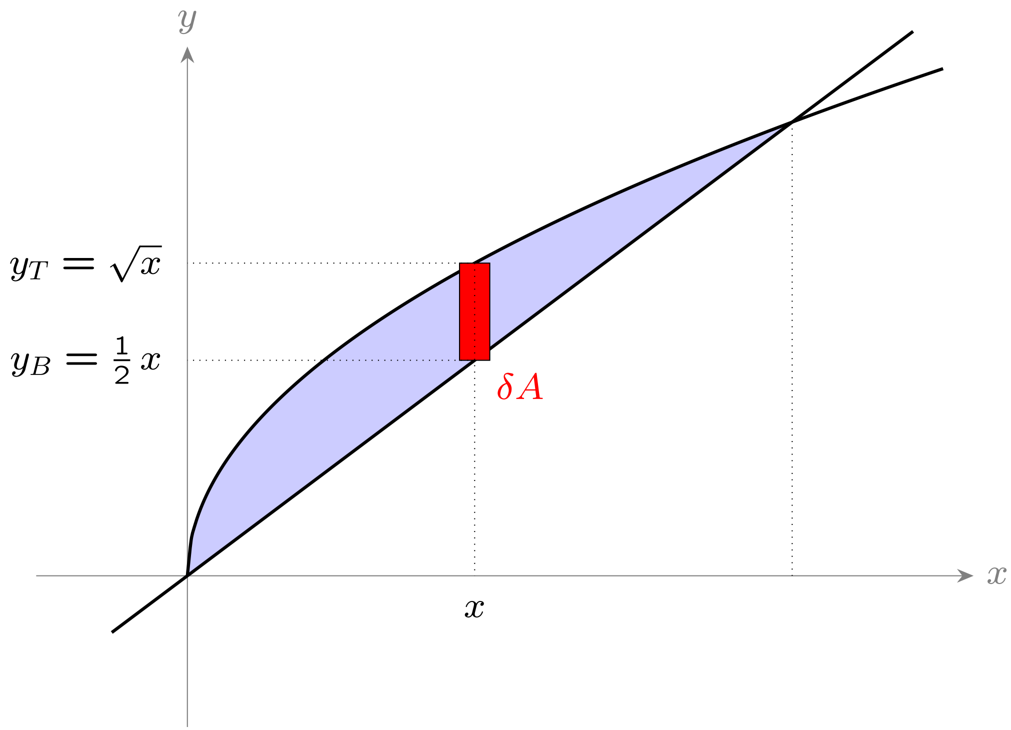
To find the area between curves using vertical strips:

- Sketch the curves.
- Identify **top curve** $y_T = f(x)$
  and **bottom curve** $y_B = g(x)$.
- Divide the region into thin vertical strips of width δx.
  These have area $(y_T - y_B)\delta x$.

Summing over these rectangles and taking the limit as the number of rectangles goes to infinity gives:

$$\text{Area between } f(x) \text{ and } g(x) = \int_a^b y_T - y_B \, dx.$$

Example: Use vertical strips to find the area of the region enclosed by $y = \sqrt{x}$ and $y = \frac{1}{2}x$.



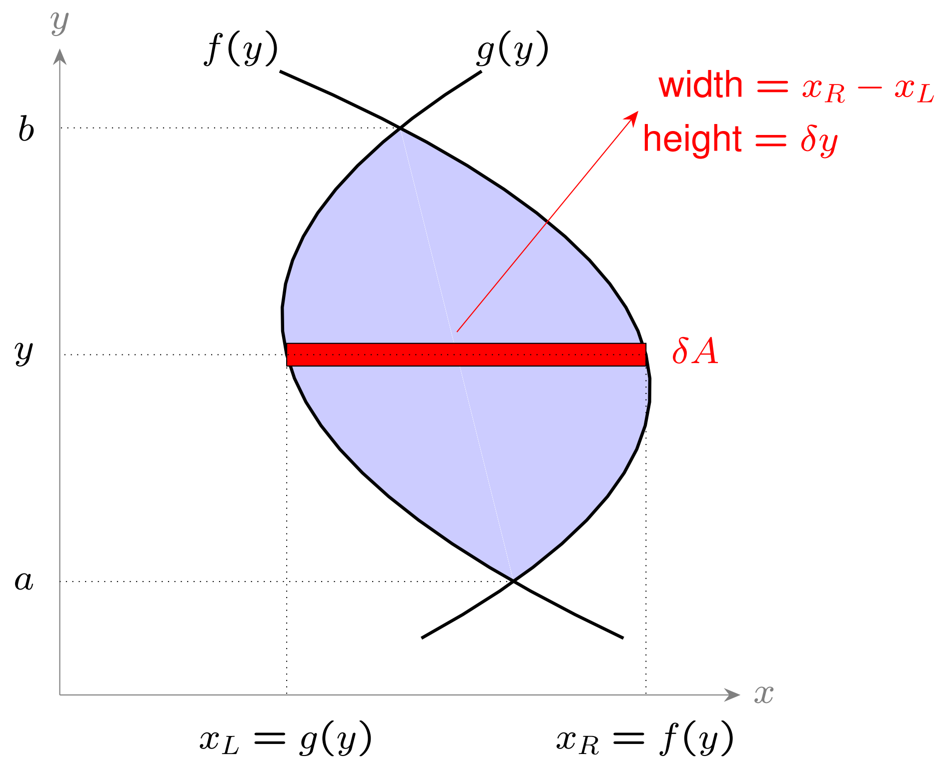
The curves intersect where:

Then,

Area =

Horizontal strips

By dividing the area between two curves into thin horizontal rectangles of width δy, we can find the area by **horizontal strips**.



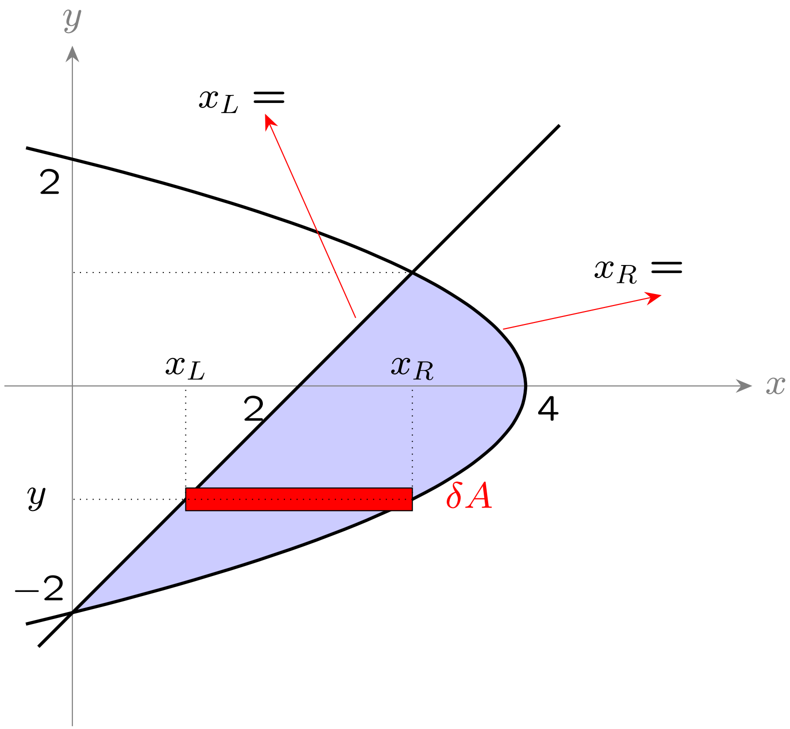
To find the area between curves using horizontal strips:

- Sketch the curves.
- Identify **right curve** $x_R = f(y)$
  and **left curve** $x_L = g(y)$.
- Divide the region into thin horizontal strips of width δy.
  These have area $(x_R - x_L)\delta y$.

Summing over these rectangles and taking the limit as the number of rectangles goes to infinity gives:

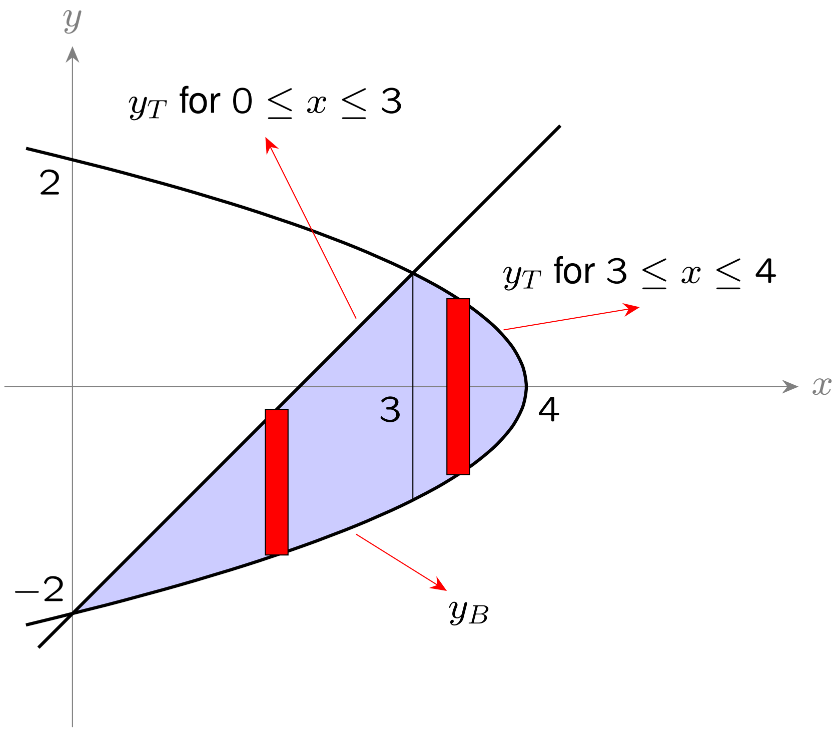
$$\boxed{\text{Area between } f(y) \text{ and } g(y) = \int_a^b x_R - x_L \, dy.}$$

Example: Find the area enclosed by the curves $y = x - 2$ and $x = 4 - y^2$.



Example continued:

Homework: Try to find this area using vertical strips – much harder!



Additional questions

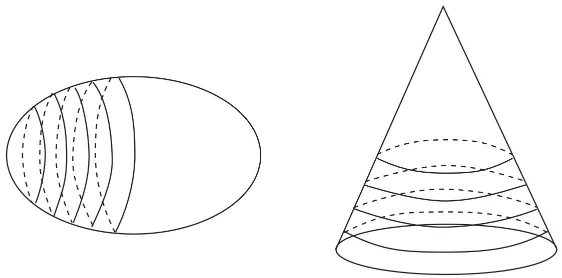
You can now attempt problems 12–18 from Topic 5 in the handbook.

You may also attempt a selection of exercises from Exercise sets 5.3 and 5.4 and some of problems 47–80 from Chapter 5.6 in the textbook.

5.6 Application: Volumes of Solids of Revolution [Chapter 6.1]

To find an area between curves, we have viewed the area as a sum of thin vertical or horizontal strips. Adding the areas of these strips and taking the limit as their width goes to zero, we obtain the exact area of the region as an integral.

We can apply the same principles to finding the volumes of certain solids in 3 dimensions. The idea is to divide the solid into thin 'slices', sum the volumes of these slices, and then take the limit as the thickness of these slices goes to zero.



The solids to which we can apply this technique are called **solids of revolution**, as they are formed by rotating regions in the xy-plane about axes.

5.6.1 Volume by Slices

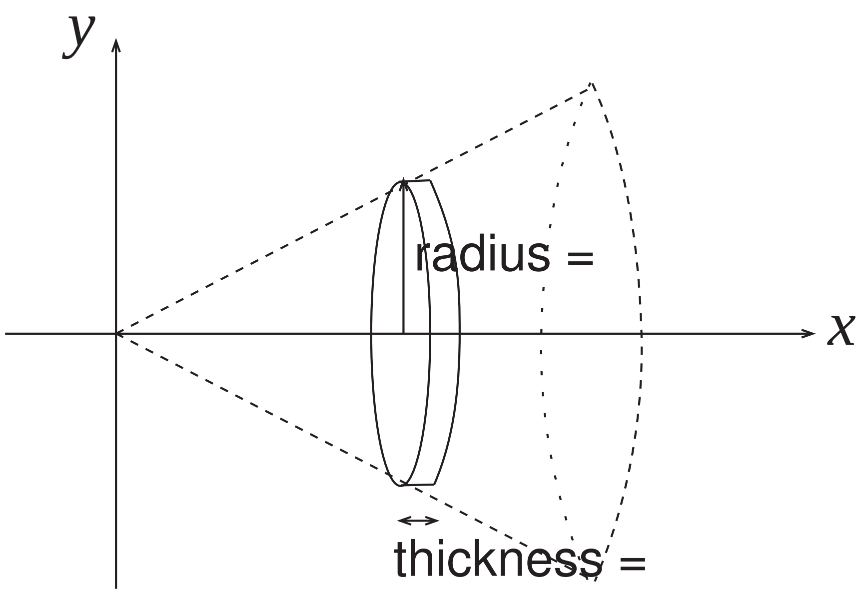
Rotation about the x-axis

Example: Find the volume of the solid formed when the region enclosed by the lines $y = \frac{1}{2}x$, $x = 6$ and the x-axis is rotated about the x-axis.

We start by drawing the region and the solid of revolution formed:
Then think of dividing the region into thin cross-sectional slices, perpendicular to the axis of rotation, i.e. perpendicular to the x-axis here.

We see that each cross-sectional slice has the volume of a thickened-up circle, namely:

$$V_{\text{slice}} = \pi(\text{radius})^2 \cdot \text{thickness} .$$

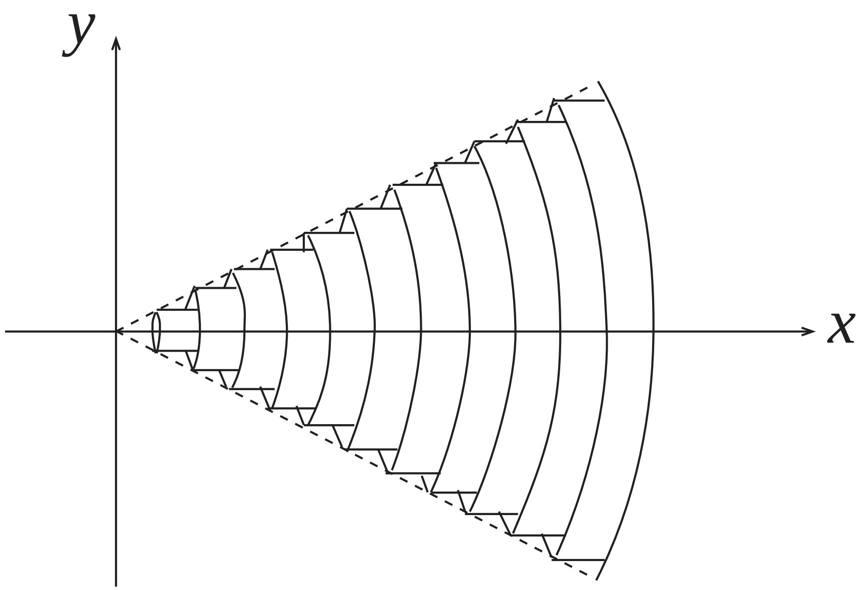


Here, we have: radius $=$

thickness $=$

So, $V_{\text{slice}} =$

Adding up the volumes of all the slices that make up the solid, we obtain an approximation to the volume of the entire solid:



$$
\begin{aligned}
V &\approx \sum V_{\text{slice}} \\
&= \sum
\end{aligned}
$$

In the limit as the thickness δx goes to zero, noting that the slices range from $x = 0$ to $x = 6$, we obtain the exact volume as the integral:

$$V =$$

To evaluate this integral, we need to express y in terms of x in the integrand. We can then find the volume of the solid:

Example: Consider the solid of revolution obtained when the semicircle $y = \sqrt{9 - x^2}$ is rotated about the x-axis.

(a) Draw the solid of revolution, and a typical 'slice' perpendicular to the axis of rotation.

(b) Write down an expression for V_{slice}, the volume of this typical slice.

(c) Hence find the volume of the solid. Check your answer against the known volume of a sphere, $V = \frac{4}{3}\pi r^3$.

Example continued:

Example continued:

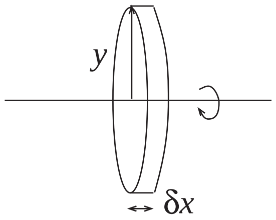
Rotation about the y-axis

Example: Consider the solid obtained by rotating the region enclosed by $y = x^2$, the line $y = 25$ and the y-axis, about the y-axis. Use the method of slices to find the volume of this solid, noting that a typical slice will now be perpendicular to the y-axis.

Example continued:

The previous two examples have illustrated that in general:

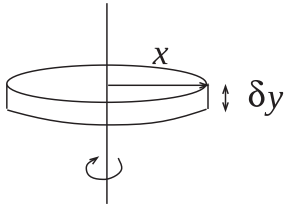
(1) For rotation about the x-axis, a slice has radius y and width δx:



which yields a volume:

$$V \approx \sum \pi y^2 \delta x = \int_{x=a}^{x=b} \pi y^2 \, dx$$

(2) For rotation about the y-axis, a slice has radius x and width δy:



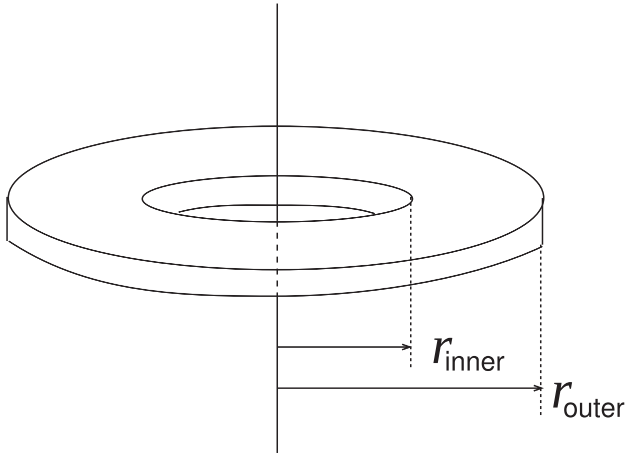
which yields a volume:

$$V \approx \sum \pi x^2 \delta y = \int_{y=a}^{y=b} \pi x^2 \, dy$$

It is best NOT to simply memorise these formulae, but to use the method of slices to determine the volume of solids of revolution.

5.6.2 Volume by Washers

A solid of revolution may have a cross-section that is shaped like an annulus rather than a disc. A thickened-up annulus then looks like a 'washer':



The volume of a washer is:

$$\begin{aligned} V_{\text{washer}} &= V_{\text{large disc}} - V_{\text{small disc}} \\ &= \pi \cdot r_{\text{outer}}^2 \cdot \text{thickness} - \pi \cdot r_{\text{inner}}^2 \cdot \text{thickness} \\ &= \pi \cdot \left(r_{\text{outer}}^2 - r_{\text{inner}}^2\right) \cdot \text{thickness} \end{aligned}$$

We can then use the same principles to find the volume of solids which have annular cross-sections.

Example: Sketch the region bounded by the graphs of $y = x^3$ and $y = 4x$ for $x \geq 0$, and consider the solid of revolution obtained when this region is rotated about the x-axis. Write down the volume V_{washer} of a typical slice of this solid, and hence find the volume of the entire solid.

Example continued:

Example continued:

Additional questions

You can now attempt the remaining problems from Topic 5 in the handbook.

You may also attempt a selection of exercises from Exercise set 6.1 in the textbook.

Topic 6: Differential Equations

In Topic 5 we looked at several techniques of integration. One of the key uses of integration is its application to solving differential equations. Differential equations (DEs) arise naturally in all sorts of real-world applications, and are used to model a huge variety of physical systems, some of which we consider in section 6.4. In this topic we look at:

6.1 Introduction to DEs and Verification of Solutions

6.2 Solving DEs by Direct Antidifferentiation

6.3 Separable DEs

6.4 Applications of DEs

6.1 Introduction to DEs and Verification of Solutions

An equation involving x, y, and the derivatives of y with respect to x:

$$x,\ y,\ \frac{dy}{dx},\ \frac{d^2y}{dx^2},\ \dots,\ \frac{d^ny}{dx^n}$$

is called an (ordinary) **differential equation**, or **DE**.

The order of the highest derivative in the DE (i.e. n above), is called the **order** of the DE.

Examples:

$$\frac{dy}{dx} = -5y \qquad \text{order} =$$

$$\frac{d^3y}{dt^3} + t\frac{dy}{dt} + (t^4 - 1)y = \sin t \qquad \text{order} =$$

$$e^x y' + y'' = y^3 + x \qquad \text{order} =$$

A **solution** of a differential equation is a function that satisfies the DE for some range of x values.

To verify if a given function y is a solution of a DE, simply substitute y and its derivatives into the DE, and check whether the left hand side equals the right hand side of the equation.

Example: Verify that $y = e^{3x}$ is a solution of:

$$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} - 15y = 0.$$

Example: Verify that $x = t^2 + \frac{2}{t}$ is a solution of

$$\frac{dx}{dt} + \frac{x}{t} = 3t.$$

Homework: Use implicit differentiation to verify that $\log y = xy^2 + C$ is a solution of the differential equation

$$\frac{dy}{dx} = \frac{y^3}{1 - 2xy^2}.$$

Homework: Find the constants a, b and c such that $y = a + bx + cx^2$ is a solution of the differential equation

$$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 4y = 4x^2.$$

Example: If the differential equation

$$x^2 \frac{d^2y}{dx^2} - 2x\frac{dy}{dx} - 10y = 0$$

has a solution $y = x^n$, find the possible values of n.

General Solutions and Particular Solutions

The **general solution** of a DE is a function that gives *all* possible solutions to the DE. The general solution of a 1st order DE will include one arbitrary constant. Similarly, the general solution of a 2nd order DE will include two arbitrary constants, and so on.

Example: Find the *general solution* of the DE

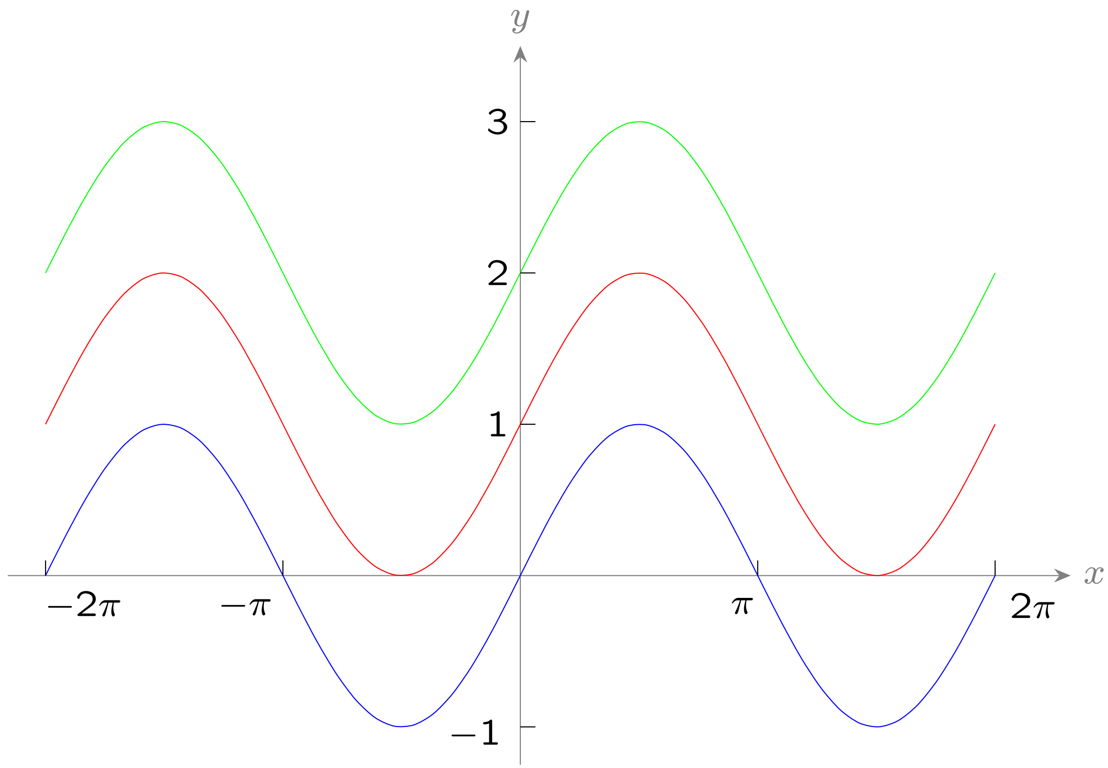
$$\frac{dy}{dx} = \cos(x).$$

This simple DE can be solved by integrating both sides:

Note the one arbitrary constant in the general solution of this first order DE. This gives an entire **family** of solutions which satisfy the DE (verify this!).

Example continued: Sketch the family of solution curves of the DE

$$\frac{dy}{dx} = \cos(x).$$



If more conditions are imposed, we are able to determine a **particular solution**.

Example continued: Now find the *particular solution* of the DE

$$\frac{dy}{dx} = \cos(x) \qquad \text{subject to} \qquad y = 3 \text{ when } x = \frac{\pi}{2},$$

and indicate this on your sketch.

The **initial condition** $y = 3$ when $x = \frac{\pi}{2}$ has determined the value of the constant in the general solution.

For an order n differential equation we need n such conditions to determine the n arbitrary constants in the general solution.

Example: If $y = Ae^{2x} + Bxe^{2x}$ is the general solution of a differential equation, what is the order of the differential equation? Find the particular solution satisfying the initial conditions $y = 1$ when $x = 0$ and $\frac{dy}{dx} = 5$ when $x = 0$.

Example continued:

Homework:

Verify that the general solution of $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} - 15y = 0$ is:

$$y = Ae^{3x} + Be^{-5x} \qquad \text{where } A \text{ and } B \text{ are arbitrary constants.}$$

Homework: Verify that the general solution of $\frac{dy}{dx} + \frac{y}{x} = 3x$ is:

$$y = x^2 + \frac{C}{x}$$ where C is an arbitrary constant.

6.2 Solving DEs by Direct Integration

Differential equations can be complicated and in many cases, we do not have methods which will allow us to find exact solutions. *Some* types of differential equations *can* be solved nicely.

In this section, we look at differential equations that can be solved by direct integration, of the form:

$$\frac{d^n y}{dx^n} = f(x)$$

Let's start by considering the case where $n = 1$ which is just a differential equation of the form

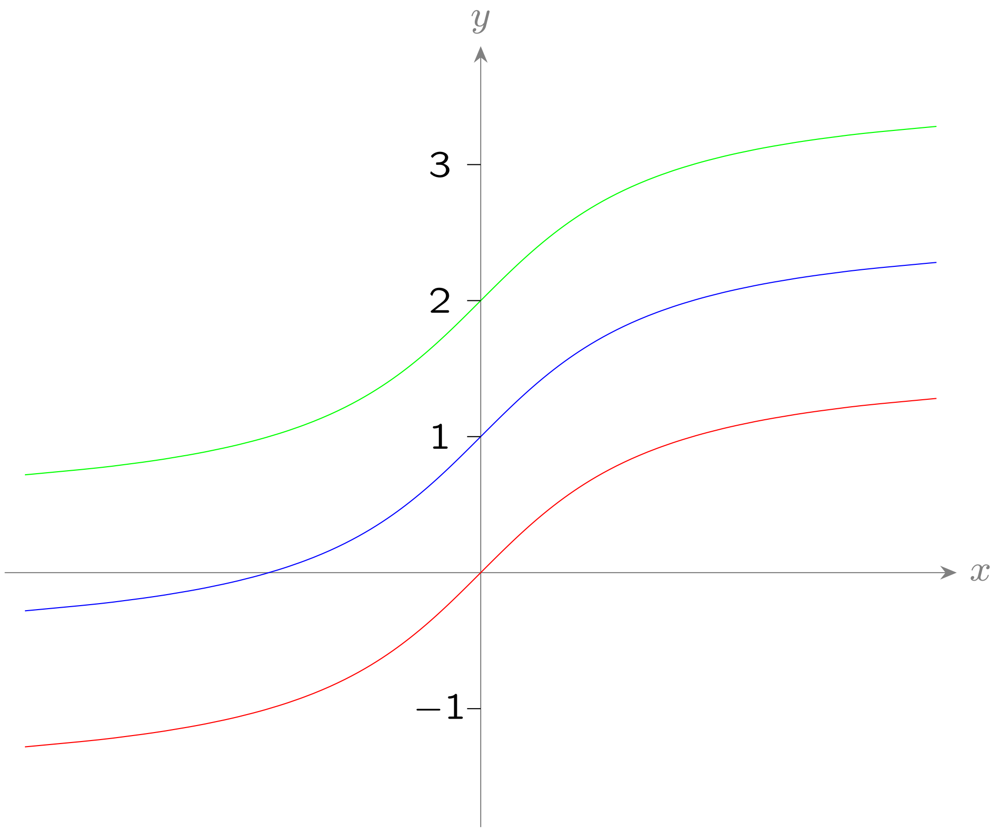
$$\frac{dy}{dx} = f(x).$$

This simply says that the derivative of y is $f(x)$. So y, the general solution of the DE, is simply the indefinite integral of $f(x)$:

$$y = \int f(x)\,dx.$$

Example: Find the general solution of the following differential equation, and sketch the family of solution curves.

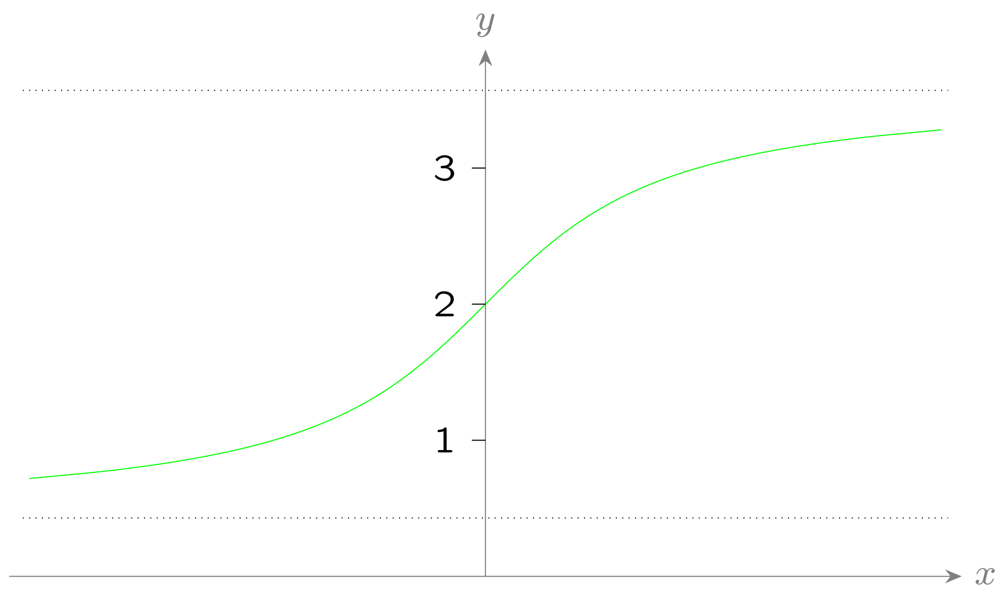
$$\frac{dy}{dx} = \frac{1}{1+x^2}.$$



Note there is one arbitrary constant in the solution of this first order DE.

Example continued: Now find the following *particular solution*, and indicate this on your sketch.

$$\frac{dy}{dx} = \frac{1}{1+x^2} \qquad \text{subject to} \qquad y = 2 \text{ when } x = 0.$$



This particular solution has picked out the solution curve satisfying $y = 2$ when $x = 0$, i.e. the curve that passes through the point $(0, 2)$.

To solve a DE of order n, we simply need to integrate n times.

Example: Find the general solution of the differential equation:

$$\frac{d^2y}{dx^2} = \frac{1}{\sqrt{x}}.$$

Note there are *two* arbitrary constants in the solution of this *second* order DE.

Additional questions

You can now attempt problems 1–7 from Topic 6 in the handbook.

6.3 Separable DEs

A differential equation of the form

$$\boxed{\frac{dy}{dx} = F(x)G(y)}$$

is called a **separable** differential equation. We can solve this type of DE by **separation of variables**, a method to be introduced shortly.

Note that the type of DE considered in Section 6.2 is a special case of separable DE, where $G(y) = 1$. Now let us consider another special case of separable DE, where $F(x) = 1$.

Consider a DE of the form:

$$\frac{dy}{dx} = G(y)$$

We cannot simply integrate both sides with respect to x, since the right-hand side is a function of y, not x. Instead, we divide both sides by $G(y)$ to obtain:

$$\frac{1}{G(y)}\frac{dy}{dx} = 1 \ , \qquad G(y) \neq 0.$$

We can now integrate both sides with respect to x:

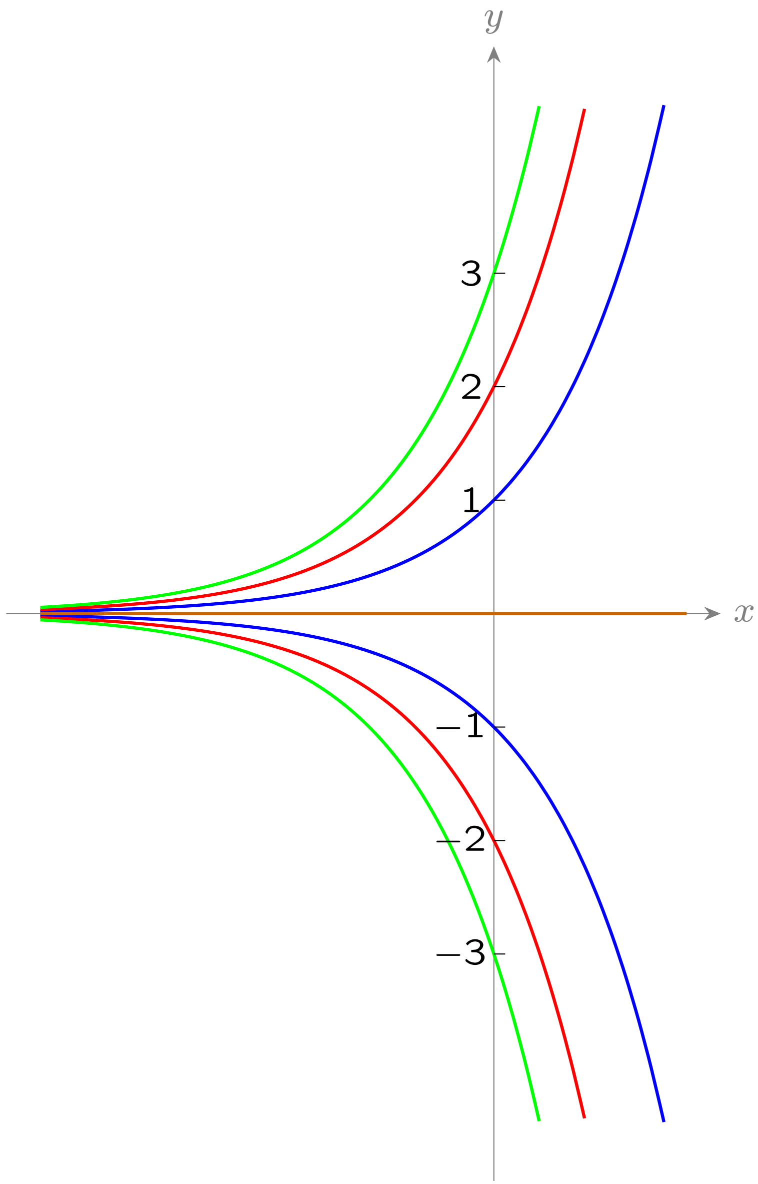
$$\int \frac{1}{G(y)} \frac{dy}{dx} \, dx = \int 1 \, dx$$

$$\Rightarrow \qquad \int \frac{1}{G(y)} \, dy = x$$

This determines x as a function of y, which we rearrange to find the solution y as a function of x (if possible). We then consider the case $G(y) = 0$ separately.

Example: Find the general solution of the differential equation

$$\frac{dy}{dx} = y.$$



y
x
3
2
1
−1
−2
−3

Example: Solve the initial value problem:

$$\frac{dy}{dx} = y \qquad \text{subject to} \qquad y = 5 \text{ when } x = 0.$$

Example: Solve the differential equation:

$$\frac{dy}{dx} = \sqrt{4 - y^2} \qquad \text{on } [-2, 2].$$

Separation of variables

We now consider the general form of a separable differential equation:

$$\frac{dy}{dx} = F(x)G(y).$$

This is called a **separable** DE because the right hand side can be written as the product of two functions, each in a different variable; this means we can separate these two variables to solve the DE.

Note: Some DEs may be separable but not look like it!

Example: Is the following DE separable

$$\frac{dy}{dx} = y^2 \sin(x) + y^2 - 4\sin(x) - 4?$$

Method of Solution: Separable Differential Equations

- Rearrange the DE to get it into separable form (if possible):

$$\frac{dy}{dx} = F(x)G(y).$$

- Separate the variables: $\dfrac{1}{G(y)}\dfrac{dy}{dx} = F(x), \quad G(y) \neq 0.$
- Integrate both sides with respect to x:

$$\int \frac{1}{G(y)}\frac{dy}{dx}\, dx = \int F(x)\, dx$$

$$\Rightarrow \qquad \int \frac{1}{G(y)}\, dy = \int F(x)\, dx$$

- Evaluate the integrals on each side. Don't forget to include an arbitrary constant of integration!
- Remember to consider the case $G(y) = 0$ if applicable.

Example: Solve $\dfrac{dy}{dx} = -6xy^2$.

Example: Solve $\dfrac{dy}{dx} = \dfrac{1+y}{x}$.

Example: Solve the initial value problem:

$$\frac{dy}{dx} = \frac{1}{2y\sqrt{1-x^2}} \qquad \text{given that} \quad y = 3 \text{ when } x = 0.$$

Example continued:

Example: Solve the initial value problem:

$$\frac{1}{3x^2y}\frac{dy}{dx} = \frac{1}{\sqrt{x^3 - 11}} \qquad \text{given that} \quad y = 1 \text{ when } x = 3.$$

Example continued:

Additional questions

You can now attempt problems 8–10 from Topic 6 in the handbook.

6.4 Applications of DEs

Mathematical modelling involves interpreting a worded question mathematically and determining the correct differential equation to model the situation. After solving the differential equation, the solution must be interpreted in terms of the physical problem.

Although most real-world populations have many complex factors affecting their growth, the underlying concepts of population growth can be modelled by 1st order DEs.

The simplest model for population growth assumes that the population is not constrained by environmental limitations. In this case, the number of births and the number of deaths are both proportional to the current population, and so the overall growth rate is also proportional to the current population.

Recall that if the rate of change of x with respect to t **is proportional to** some function f of x, we write:

$$\begin{aligned} & \frac{dx}{dt} \propto f(x) \\ \Rightarrow \qquad & \frac{dx}{dt} = kf(x) \end{aligned}$$

where k is a constant, called the **constant of proportionality**.

Example: The rate of growth of a certain mouse population is proportional to the current number of mice present, such that:

$$\frac{dM}{dt} = 0.2M,$$

where M is the number of mice t weeks after observation begins.

(a) Given that there are initially 50 mice, determine the number of mice $M(t)$ at any time.

(b) When will the mouse population reach 500?

Example continued:

Example: A tree is growing at a rate that is *inversely* proportional to its current height h. If the initial height of the tree is 2 metres and after 1 month it has grown to 3 metres, find the height of the tree after 1 year.

Example continued:

Example continued:

Example: A certain culture of bacteria is growing at a rate proportional to the current number B of bacteria present. If initially there are 1000 bacteria and after 1 hour this number has doubled:

(a) Find the number of bacteria $B(t)$ present at any time t.

(b) How many bacteria are present after $1\frac{1}{2}$ hours?

(c) When will the number of bacteria reach 1 million?

Example continued:

Example continued:

The heating or cooling of objects can also be modelled by 1st order DEs.

Newton's Law of Cooling:

The rate at which a body cools is proportional to the difference between its temperature T and the temperature T_s of its immediate surroundings. That is,

$$\frac{dT}{dt} = -k(T - T_s),$$

where k is a positive constant.

Example: A loaf of bread is placed in a freezer whose temperature is a constant -15°C. The bread obeys Newton's Law of Cooling. If the temperature T of the bread is initially 20°C and it takes 20 minutes for it to drop to 10°C, how long will it take for the bread to reach 0°C?

Example continued:

Example continued:

End of lectures

To finish the course, and to prepare for the exam, you should ensure that you have:

- completed all lecture examples, and worked through all homework problems;
- finished all problems in the handbook;
- finished all tutorial sheets.