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Welcome to MAST10005 – Calculus 1

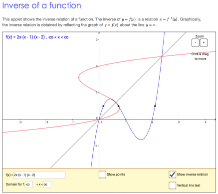
- **Coordinator and Lecturer:** Dr. John Banks, G41, Peter Hall Building, john.banks@unimelb.edu.au
- Three lectures each week:
  1. Monday 9AM,
  2. Thursday 11AM,
  3. Friday 11AM,

  all in Carillo Gantner Theatre, Asia Centre.
- One Workshop and one Practice class (AKA Tutorial) each week, starting week 1.
- Several consultation hours (AKA office hours) each week.
  - Come to any to get individual help with MAST10005 material.
  - See LMS for times.

Online Resources

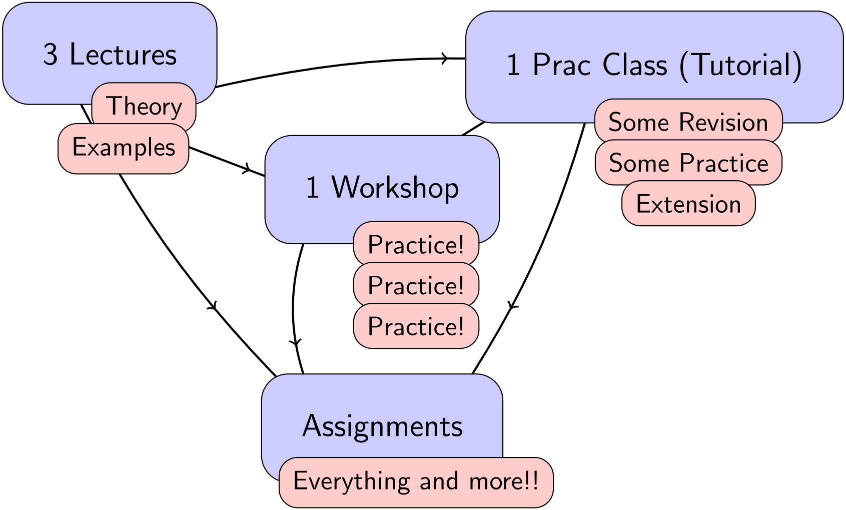
- The LMS Site contains information about assignments, consultation hours, and extra resources for additional help.
- Our interactive Geogebra applets are intended to help you understand mathematical concepts that arise in our subjects.
  - Those most useful for MAST10005 can be accessed at:

    http://www.melbapplets.ms.unimelb.edu.au/



by choosing the MAST10005 tab.

Our working week



Printed materials available from bookshop

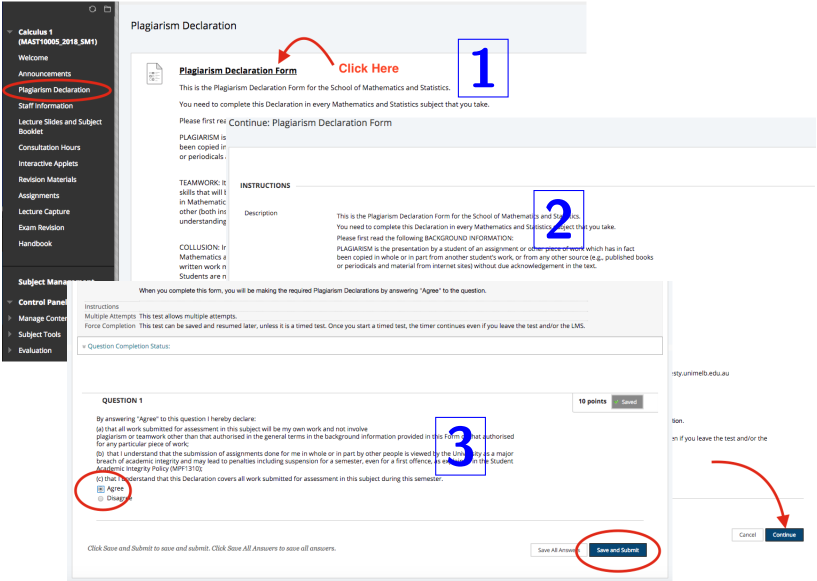
- These lecture notes contain an outline for each lecture, with spaces to fill in the examples as we work through them.
- Completed notes with solutions will not be made available!
- The subject handbook (AKA problem booklet) has:
  - all the information about the organisation of this subject, assessment details, etc.
  - a list of useful formulas, which will be given to you in the exam.
  - problem sheets (aim to work through these topic by topic), with answers at the back of the handbook.
- Recommended textbook:
  - Hass, Weir and Thomas - *University Calculus Early Transcendentals*, 3rd edition, Pearson, 2016.

Assessment

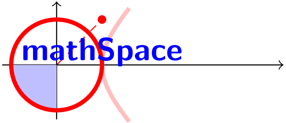
- 10 assignments worth a total of 20%, due weekly.
- One 3-hour exam at the end of semester worth 80%.
- The assignment problems will be handed out during lectures and also posted on the LMS.
- Note that no calculators are permitted in the final exam.
- There are no hurdle requirements.

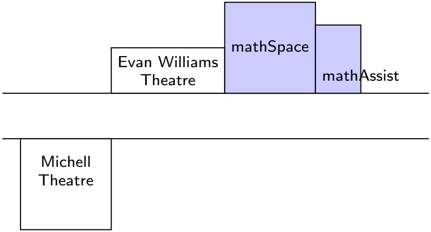
Online Plagiarism Declaration - easy as 1, 2 ,3!

- Please complete on LMS before submitting Assignment 1:



mathSpace and *mathAssist*





- Learning Assistant service available weekdays 12:00 – 2:00.
- Located next to mathSpace in Peter Hall Building.
- Learning Assistants can help with basic skills like:
  - Basic algebra, fractions and index laws.
  - Trigonometric functions, logarithms and exponentials.
  - Equations of circles, ellipses and hyperbolae.
  - Basic differentiation and integration.
- But not with current assignment questions or topics taught in MAST10005!

Topic 1 - Numbers and Sets

Everything you always wanted to know about sets, but were afraid to ask.

In this topic, we learn the names and the special personalities of some of the best known sets of numbers and we learn standard notations for describing them. Since the concept of a set underpins every field of modern mathematics, a good grasp of the concepts and notations will be essential to your progress in all future mathematical studies.

After reviewing the familiar real number system and learning about sets in general, we will introduce an exciting set of numbers you may not have met before. The set of complex numbers contains all of the familiar real numbers, but also contains something that may seem a bit strange if this is your first meeting - square roots of the negative numbers!

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The real numbers

Unless you have taken some advanced studies, all of the calculus you have studied up until now has been about a particular class of functions - often called functions of one variable - that take a real number as input and give a real number as output. In MAST10005, we will deepen our knowledge of the calculus of these functions and also start to extend our calculus studies a little beyond functions of one variable.

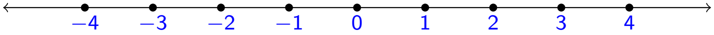
However, even in this wider context, where we learn to differentiate parametric functions, all of the calculus we learn is underpinned by the real numbers. We now review the nature of the real numbers, the most important subsets of the real numbers and the notations we use to describe these sets.

A sense of wholeness

- $\mathbb{N}$ is the set of positive integers or counting numbers:

$$\mathbb{N} = \{1, 2, 3, 4, \dots\}$$

  sometimes called the natural numbers.
- Note that at UoM we do not inclue 0 in $\mathbb{N}$.
- $\mathbb{Z}$ is the set of integers or whole numbers. It includes the positive and negative whole numbers as well as zero.



We try so hard to be rational . . .

- $\mathbb{Q}$ is the set of rational numbers - those that can be written as fractions $\frac{m}{n}$ where m and n are integers, for example:

$$\frac{1}{2}, \frac{78}{23}, \frac{-3}{4}, \frac{0}{5} = 0, \frac{-6}{1} = -6, 1 = \frac{1}{1} = \frac{2}{2} = \frac{3}{3}, \ldots$$

- There are infinitely many rational numbers each of which we can write in infinitely many different ways.
- Every integer is a rational number. Do you see why?

. . . but sometimes its impossible!

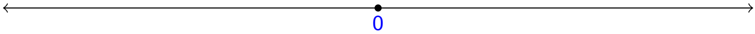
- Some numbers cannot be written as fractions! We say they are irrational (and we mean it in a nice way!).
- A few examples:

$$\pi, e, \sqrt{2}, \sqrt[3]{2}, \log_2(5), \log_5(2), \ldots$$

- In order to convince ourselves that irrational numbers actually exist, we need to give a proof by contradiction.
- We will explore this in a Practice class.

The most important set we know

- The set of irrational numbers joined together with the set of rational numbers gives the set of real numbers, which we write as $\mathbb{R}$.



- There are infinitely many irrational numbers. In a sense, “most” real numbers are irrational.
- Soon we will meet an even bigger set of numbers - the set $\mathbb{C}$ of complex numbers.

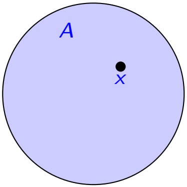
Set notations and operations

The concept of set membership is the foundation on which all of set theory is built. A set is defined precisely by its members or - to put it another way - two sets are equal precisely if they have the same members (also called elements). Here we review the notation for set membership and use it to develop the key ideas of elementary set theory - subsets, union, intersection, complement and Cartesian product.

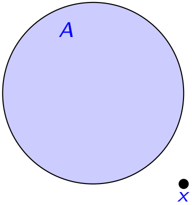
We consider how to define a set having members with some specific property and use this to give definitions of various special subsets of the real numbers. In day-to-day mathematics, this is one of the most important ideas in set theory. It is used, for example, when writing domains of functions and solution sets of equations.

Membership has its rewards

- If A is a set, saying $x \in A$ means x is in the set A.
- We read this as:
  - "x is an element of A" or
  - "x is a member of A" or
  - "x is in A" or
  - "A contains the point x".



- We express the fact that $x \in A$ is false by writing $x \notin A$.
- We read this as:
  - "x is not an element of A" or
  - "x is not a member of A" or
  - "x is not in A" or
  - "A does not contain the point x".



- Venn diagrams (like those above) help us to visualise set concepts.

Membership examples

Membership Examples:

- Saying $n \in \mathbb{Z}$ means n is an integer (positive, negative or zero).
- Saying $x \in \mathbb{Q}$ means x can be written as a fraction, so $x = \frac{m}{n}$ where $m \in \mathbb{Z}, n \in \mathbb{Z}$ and $n \neq 0$.
- Saying $y \in \mathbb{R}$ means y is some real number.

Not Membership Examples:

- $\frac{3}{2} \notin \mathbb{Z}$ means $\frac{3}{2}$ is not an integer.
- $\log_2(5) \notin \mathbb{Q}$ means $\log_2(5)$ is not rational (or irrational).
- $0 \notin \mathbb{N}$ means 0 is not a natural number.

Description is the key

- To define a set, we must say which elements it contains.
- Descriptive notation (AKA set builder notation) does this by stating a property its elements (and only its elements) have. For example, writing

$$A = \{x \in \mathbb{R} \mid x^2 + 1 > 37\}$$

defines the set A to be:

"the set of all real numbers x such that $x^2 + 1 > 37$ is true."

- We can read this notation part by part:

$$\begin{array}{cccccc} A & = & \{ & x \in \mathbb{R} & \mid & x^2 + 1 > 37 \ \} \\ & & \uparrow & \uparrow & \uparrow & \uparrow \\ & & \text{The set of} & \text{all real numbers} & \text{such that} & \text{this statement is true.} \end{array}$$

- The symbol $:$ is often used instead of the symbol $\mid$.

Example 1.1

1. Express *the set of real numbers whose natural (base e) logarithm is positive* in descriptive notation.
2. Express *the set of integers whose square is even* in descriptive notation.
3. Describe the set $\{n \in \mathbb{N} \mid \sin(n) > 0\}$ in words.

⚠ We denote the natural logarithm $\log_e(x)$ by $\log(x)$ at UoM. We do not not use the notation $\ln(x)$.

Keeping it brief

- We often abbreviate descriptive set notation.
  For example, writing $A = \{x \in \mathbb{R} \mid \sin(x) = 0\}$ in the form

$$\{k\pi \mid k \in \mathbb{Z}\}$$

  indicates that the set A is the set of numbers of the form $k\pi$ where k is an integer.
- This is really just an abbreviated form of

$$\{x \in \mathbb{R} \mid x = k\pi \text{ for some } k \in \mathbb{Z}\}$$

- Again, we can read the abbreviated notation part by part:

$$\begin{array}{cccccc} A & = & \{ & k\pi & | & k \in \mathbb{Z}\ \} \\ & & \uparrow & \uparrow & \uparrow & \uparrow \\ & & \text{The set of} & \text{real numbers} & \text{such that} & \text{this statement} \\ & & & \text{of this form} & & \text{is true.} \end{array}$$

Example 1.2

1. Express the set of odd integers in abbreviated set notation.
2. Express the set $\{x \in \mathbb{R} \mid \cos(x) = 0\}$ in abbreviated set notation.
3. Express the set $\{x \in \mathbb{R} \mid \sin(x) = -1\}$ in abbreviated set notation.

Homework 1

1. *Express the set of integers whose square is no greater than their cube in descriptive notation.*
2. *Describe the set* $\{x \in \mathbb{R} \mid x^2 > x^3\}$ *in words.*
3. *Express the set* $\{x \in \mathbb{R} \mid \cos(x) = 1\}$ *in abbreviated set notation.*
4. *Express the set* $\{x \in \mathbb{R} \mid \cos(x) = -1\}$ *in abbreviated set notation.*

An open and closed case

For $a, b \in \mathbb{R}$ with $a \leq b$:

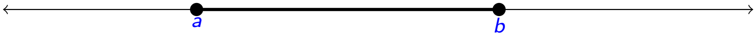
- (a, b) means the interval of real numbers between a and b (exclusive) so

$$(a, b) = \{x \in \mathbb{R} \mid a < x < b\}.$$

  This is called an open interval.

- $[a, b]$ means the interval of all real numbers from a to b (inclusive) so

$$[a, b] = \{x \in \mathbb{R} \mid a \leq x \leq b\}.$$



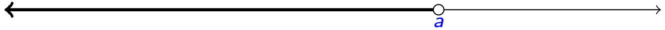
  This is called a closed interval.

Example 1.3

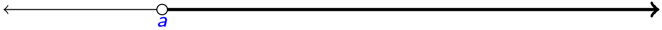
Give definitions of $[a, b)$ and $(a, b]$, with diagrams.

Unbounded intervals

- For any $a \in \mathbb{R}$
  - $(-\infty, a)$ is the interval of real numbers strictly less than a so
$$(-\infty, a) = \{x \in \mathbb{R} \mid x < a\}.$$



  - (a, ∞) is the interval of real numbers strictly greater than a so
$$(a, \infty) = \{x \in \mathbb{R} \mid x > a\}.$$



Homework 2

Give definitions of $(-\infty, a]$ *and* $[a, \infty)$*, with diagrams.*

Small sets

- We can describe a set with only finitely many elements in "list of elements" form.
- **Some Examples:**
  - $\{x \in \mathbb{R} \mid x^2 - 1 = 0\} = \{-1, 1\}$.
  - $\{x \in \mathbb{Z} \mid x^2 - 1 < 0\} = \{0\}$.
  - $\{x \in \mathbb{Z} \mid x^3 - x = 0\} = \{-1, 0, 1\}$.

Example 1.4

Write the set of prime numbers less than 20 in descriptive and "list of elements" form.

Smallish sets

- We can use a kind of "list of elements" notation to denote some infinite sets, by adding some dots.
  Example: $\{(2k+1)\pi \mid k \in \mathbb{Z}\} = \{\ldots, -3\pi, -\pi, \pi, 3\pi, \ldots\}$.
- Where we can write the elements as an infinite list, this often makes it easier to visualise the set.

Example 1.5

Express the set $\{k\pi \mid k \in \mathbb{Z}\}$ in list of elements form and hence sketch it.

The smallest set of all

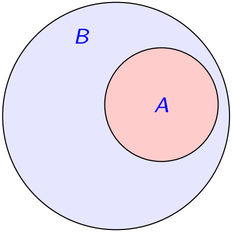
- There is only one set with no elements. It is called the empty set and is usually written as $\varnothing$.
- We can also describe the set with no elements in "list of elements" form as $\{\ \}$.
- This notation is often useful when expressing solutions of equations and inequalities.
- **Some Examples:**
  - $\{x \in \mathbb{R} \mid x^2 + 1 = 0\} = \varnothing$.
  - $\{x \in \mathbb{R} \mid \cos(x) > 1\} = \varnothing$.
- There is a diagram of $\varnothing$ at the bottom of this slide.

Subsets

- If A and B are sets, then saying

$$A \subseteq B$$

means that every element of A is also an element of B.



- We read this as:
  - "A is a subset of B" or
  - "A is contained in B" or
  - "B contains A".
- The definition of $A \subseteq B$ may be written in symbols as

$$x \in A \Rightarrow x \in B$$

where $\Rightarrow$ means implies.

- We say "$x \in A$ implies $x \in B$" or "if $x \in A$ then $x \in B$".

Subset examples

- $\mathbb{N} \subseteq \mathbb{Z} \subseteq \mathbb{Q} \subseteq \mathbb{R}$. Soon we will add an even bigger set to this chain of inclusions. Excited?
- When proving $A \subseteq B$, our assumption is $x \in A$ and we must use valid mathematical reasoning to conclude that $x \in B$.
- To prove A is not a subset of B (sometimes written $A \not\subseteq B$) we must find an x in A that is not in B.

Theorem 1.6

The empty set $\varnothing$ is a subset of every set.

- This arises from the way we interpret "if $x \in A$ then $x \in B$" in cases where "$x \in A$" is always false.
- You will learn more about this in MAST20026 Real Analysis.

What is a proof?

- In order to be sure that a mathematical statement is true, we need to give a proof.
- A proof starts with a set of true assumptions and uses valid mathematical reasoning to reach a desired conclusion.
- Valid mathematical reasoning might include some well known algebra, geometry, etc.
- When we prove that $A \subseteq B$, there is only one assumption:

  $$x \in A$$

  and we must use valid mathematical reasoning to show that

  $$x \in B.$$

- In the case where A is infinite, it is never enough to just check inclusion for a few specific elements of A.

A subset proof

Example 1.7

Prove $A \subseteq B$ where $A = \{4n \mid n \in \mathbb{Z}\}$ and $B = \{2m + 2 \mid m \in \mathbb{Z}\}$.

- Every subset proof should start with "Let $x \in A$."
- Note that we do not need to "know" what all of the elements of A are to complete this proof.

Not a subset proof

- To prove a statement is false, we must find a counterexample - an example for which the statement does not hold.
- To prove A is not a subset of B (sometimes written $A \not\subseteq B$) we must find an x in A that is not in B.

Example 1.8

Prove that $A \not\subseteq B$ where $A = \{3n + 1 \mid n \in \mathbb{Z}\}$ and $B = \{6m + 1 \mid m \in \mathbb{Z}\}$.

- In this proof, we made use of an important technique called proof by contradiction.

Proof by contradiction

- **Q:** How can we prove that the equation
$$x^2 + x + 1 = 0$$
has no real solutions?
- **A:** Proof by contradiction – assume there is a solution $x \in \mathbb{R}$ and show that this leads to an absurd conclusion:
$$x^2 + x + 1 = 0 \Rightarrow (x + \frac{1}{2})^2 + \frac{3}{4} = 0 \Rightarrow (x + \frac{1}{2})^2 = -\frac{3}{4}$$
which this is absurd since the square of every real number is non-negative!
- **In general:** Assume the opposite of the statement you wish to prove. Show that this leads to absurd consequences.
- We will explore this idea further in a Practice Class.

Homework 3

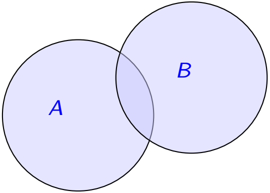
Prove that $\{x \in \mathbb{R} \mid \sin(x) = 0\} \subseteq \{x \in \mathbb{R} \mid \sin(2x) = 0\}$.

Homework 4

Prove that $\{x^3 \mid x \in \mathbb{R}\} \not\subseteq \{x^2 \mid x \in \mathbb{R}\}$.

The union makes us strong

- For sets A and B the union of A and B is the set of elements that are in at least one of A or B.
- The union of A and B is written $A \cup B$.



- In descriptive notation, the definition of union is

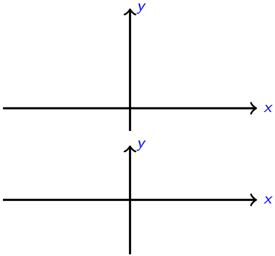
$$A \cup B = \{x \mid x \in A \text{ or } x \in B\}.$$

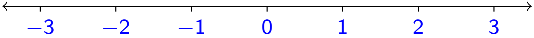
- The or in this expression is inclusive or - it is true when $x \in A$ and $x \in B$.

Example 1.9

Express each of the following sets as a union of intervals.
Drawing graphs may help for the first two.

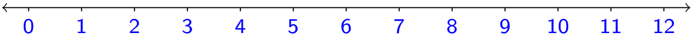
1. $\{x \in \mathbb{R} : x^2 > 1\}$.
2. $\{x \in (-2\pi, 2\pi] : \sin(x) \leq 0\}$.
3. $\{x \in [-2, 2] : x \notin \mathbb{Z}\}$.

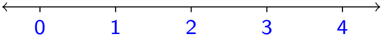




Example 1.10

1. Express $(2,8) \cup [3,10]$ as an interval.
2. Is the set $(0,\sqrt{2}] \cup [\frac{\pi}{2},3)$ an interval?





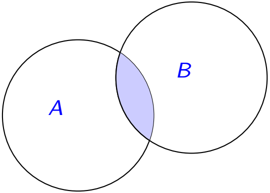
Homework 5

Express the set $A = \{x \in \mathbb{R} \mid x^2 + 1 > 37\}$ *as a union of intervals.*

Notice that we need to solve an inequality to answer this problem. This leads into our next sup-topic.

A busy intersection

- For sets A and B the intersection of A and B is the set of elements that are in both A and B.
- The intersection of A and B is written $A \cap B$.

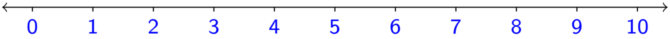


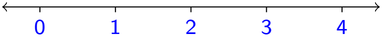
- In descriptive notation, the definition of intersection is

$$A \cap B = \{x \mid x \in A \text{ and } x \in B\}.$$

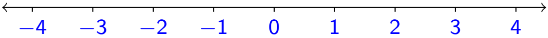
Example 1.11

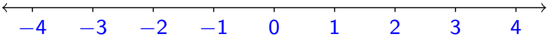
1. Express $(2,8) \cap [3,10]$ as an interval.
2. Express $(0,\sqrt{2}] \cap [\frac{\pi}{2},3)$ in the simplest possible way.
3. Express $\mathbb{Z} \cap [-\pi,\pi]$ in "list of elements" form.
4. Express $\mathbb{Z} \cap \{x \in \mathbb{R} \mid x^2 - 5 < 0\}$ in "list of elements" form.





Example continued





Example 1.12

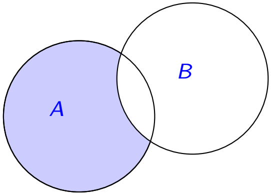
Express the following sets as intersections and as single set descriptions.

1. The set of reals with positive sine and negative cosine.
2. The set of integers with positive sine.

Example continued

Thanks for the complement!

- For sets A and B the relative complement (or difference) of A and B is the set of elements that are in A but not in B.
- The complement of A and B is written $A \setminus B$.



- In descriptive notation, the definition of set complement is

$$A \setminus B = \{x \in A \mid x \notin B\}.$$

But I thought everything was commutative!

Example 1.13

1. Find $(0, 2) \setminus (1, 3)$ and $(1, 3) \setminus (0, 2)$.
2. Is it generally true that $A \setminus B = B \setminus A$?

Example 1.14

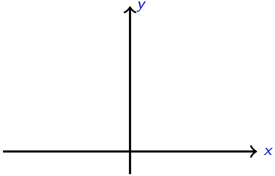
Write the set of real numbers for which each of the following expressions make sense as both a set complement and a union of intervals.

1. $\dfrac{x^5 - 2x + 4}{x^2 - 1}$.
2. $\dfrac{\log(x)}{x^2 - 1}$.

Example 1.15

Express each of the following sets as a set complement.

1. $\{x \in \mathbb{R} \mid x^2 > 1\}$.
2. $\{x \in [-2, 2] \mid x \notin \mathbb{Z}\}$.
3. $(-\infty, 0) \cup (0, \infty)$.



Homework 6

Express each of the following sets as a set complement.

1. $\{x \in [0, 2\pi] \mid \cos(x) > 0\}$.
2. $\{x \in \mathbb{R} \mid 1 - x^2 \leq x^2 - 1\}$.
3. *The set of irrational numbers.*

An excellent product . . .

- For sets A and B the Cartesian product of A and B is the set of ordered pairs with first component in A and second in B.
- The Cartesian product of A and B is written $A \times B$.
- Using descriptive notation, we can write:

$$A \times B = \{(x, y) \mid x \in A \text{ and } y \in B\}$$

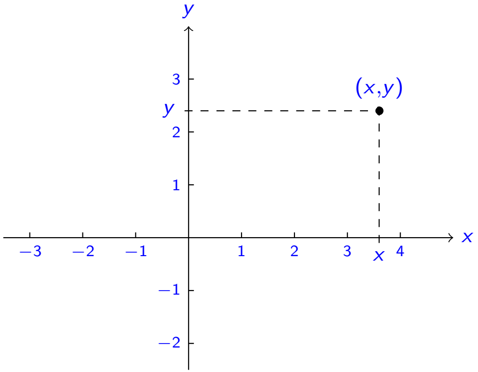
- When $A = B$ we usually use the abbreviated notation A^2 instead of $A \times A$.

Example 1.16

List the elements of $A \times B$ where $A = \{0, 1, 5\}$ and $B = \{e, \pi\}$.

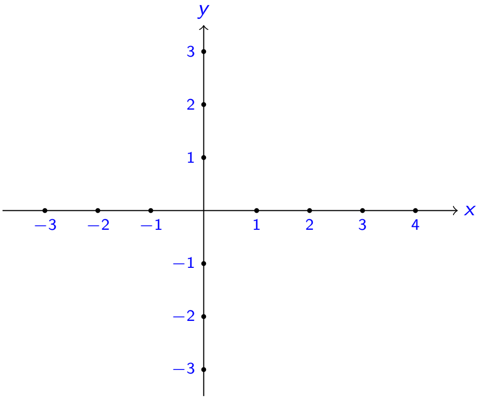
. . . and most excellent of all

- Probably the best known example of a Cartesian product is the plane, usually written as $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$.
- A typical point $(x, y) \in \mathbb{R}^2$ has horizontal coordinate x and vertical coordinate y.



Example 1.17

Sketch the Cartesian product $\mathbb{Z} \times \mathbb{Z} = \mathbb{Z}^2$ as a subset of $\mathbb{R}^2$.



Homework 7

Sketch the Cartesian products $\mathbb{Z} \times \mathbb{R}$ *and* $\mathbb{R} \times \mathbb{Z}$ *as subsets of* $\mathbb{R}^2$*.*
Is it always true that $A \times B = B \times A$*?*

An urgent product announcement!

- Two elements (a, b) and (c, d) of a Cartesian product $A \times B$ are *equal* precisely if

$$a = c \qquad \text{and} \qquad b = d.$$

- This is a key definition for Cartesian products.

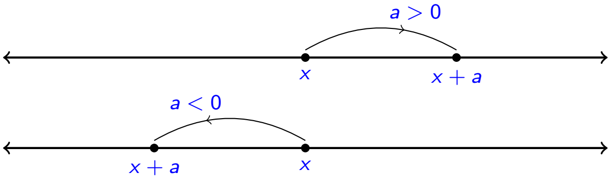
Working with inequalities

In calculus (and indeed in all areas of mathematics), we need to be able to work confidently with inequalities. This need arises, for example, when we use differentiation to find the intervals on which a function is increasing or decreasing or when we calculate the domain or range of a function.

The key to working with inequalities is understanding the algebraic rules applying to them. We start by considering the effects of adding and multiplying each side of an inequality. This leads naturally to the concepts of order preserving and order reversing functions and their use in solving inequalities.

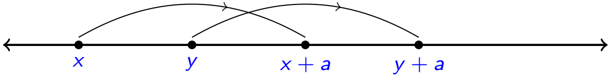
Adding a constant

- The geometric effect of adding a constant a to x is to shift x right if $a > 0$ and left if $a < 0$.



- Inequalities are therefore preserved when we add a constant:

$$x < y \Rightarrow x + a < y + a.$$

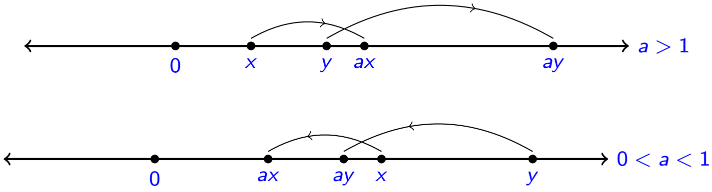


- The operation of adding a constant to each side of an inequality is order preserving - the direction of the inequality stays the same.

Multiplying by a positive constant

- Inequalities are preserved when we multiply by a positive constant a:

$$x < y \Rightarrow ax < ay.$$

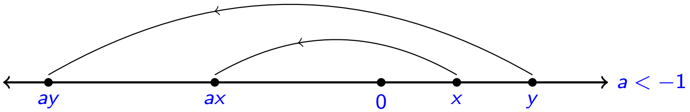


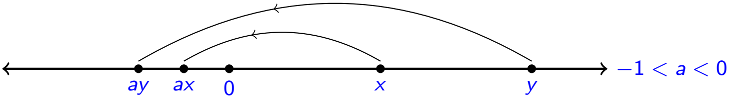
- Multiplying by a positive constant on each side of an inequality is order preserving.

Multiplying by a negative constant

- Inequalities are reversed when we multiply by a negative constant a:

$$x < y \Rightarrow ax > ay.$$





- Multiplying by a negative constant on each side of an inequality is order reversing.

Example 1.18

Express the set $\{x \in \mathbb{R} \mid -2 - \frac{1}{2}x > -4\}$ as an interval.

- This technique is often called solving an inequality (in this case, the inequality $-2 - \frac{1}{2}x > -4$).

Subtraction and division

- Since subtracting a constant a from each side of an inequality is the same as adding $-a$, this operation is order preserving.
- Since dividing by a constant a on each side of an inequality is the same as multiplying by $\frac{1}{a}$, this operation is
  - order preserving if $a > 0$.
  - order reversing if $a < 0$.

Example 1.19

Express the set $\{x \in \mathbb{R} \mid 1 - x < 3x + 2\}$ as an interval.

Summing up

Theorem 1.20 (Algebra of inequalities)

For any $x, y \in \mathbb{R}$,

1. *For any* a:
   - $x < y \Rightarrow x + a < y + a$.
   - $x < y \Rightarrow x - a < y - a$.
2. *When* $a > 0$:
   - $x < y \Rightarrow ax < ay$.
   - $x < y \Rightarrow \dfrac{x}{a} < \dfrac{y}{a}$.
3. *When* $a < 0$:
   - $x < y \Rightarrow ax > ay$.
   - $x < y \Rightarrow \dfrac{x}{a} > \dfrac{y}{a}$.

Homework 8

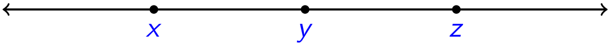
Write down corresponding Theorems for $\leq, \geq$ *and* $>$.

Putting inequalities together

- We frequently use the transitivity property of inequalities:

$$x < y \text{ and } y < z \Rightarrow x < z$$

which is completely obvious if you think geometrically:



Homework 9

Formulate transitivity properties for the cases

$$x < y \textit{ and } y \leq z$$

and

$$x \leq y \textit{ and } y < z.$$

Now we can do more

- We now know enough basic properties of inequalities to prove other useful properties.

Example 1.22 (Adding inequalities)

Prove that

$$x < y \text{ and } a < b \Rightarrow x + a < y + b$$

for any $a, b, x, y \in \mathbb{R}$.

Example continued

Order preserving and reversing functions . . .

- When solving inequalities, we often need to apply a function to each side of an inequality.
- We can apply an order preserving or reversing function to each side of an inequality.
- Geometrically speaking:
  - Order preserving functions have non-negative gradient at every point.
  - Order reversing functions have non-positive gradient at every point.
- In our study of differential calculus, we will use the derivative to establish that a function is order preserving or reversing.

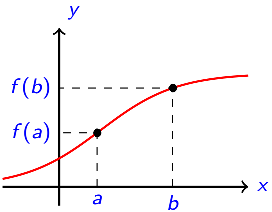
⚠️ When applying an order reversing function to each side of an inequality, don't forget to reverse the inequality.

. . . AKA increasing and decreasing functions

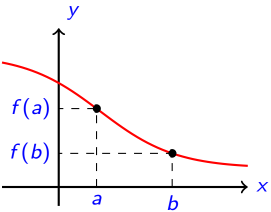
Definition 1.23 (Order preserving and reversing)

A function f defined on an interval I is

- increasing or order preserving on I if $a < b \Rightarrow f(a) < f(b)$ for all $a, b \in I$.
- decreasing or order reversing on I if $a < b \Rightarrow f(a) > f(b)$ for all $a, b \in I$.



Increasing function



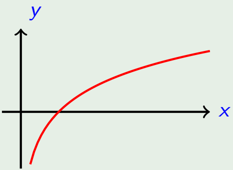
Decreasing function

Example 1.24

Use the fact that log is order preserving on its domain to express

$$\{x \in \mathbb{R} \mid 2e^{5x} - 1 < 5\}$$

as an interval.

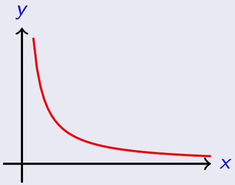


Homework 10

Use the fact that $f(x) = \dfrac{1}{x}$ *is order reversing for* $x > 0$ *to express the set*

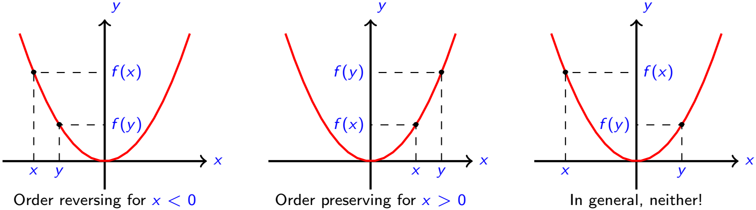
$$\left\{ x \in (0, \infty) \;\middle|\; \frac{1}{x^2 + 9} < \frac{1}{25} \right\}$$

as an interval.



⚠ Caution!

- Some functions are neither increasing nor decreasing over all of $\mathbb{R}$ (although they may be on a smaller interval).



- It may be possible to apply such a function to each side of an inequality on a restricted domain, but we need to be careful!
- We will explore this in a Practice Class.

Complex Numbers

Mathematicians created the complex numbers to provide a number system in which previously unsolvable polynomial equations have solutions. They did this by enlarging the real number system. Complex numbers are used extensively in physics and engineering, in areas such as electromagnetic waves and electric circuits and together with calculus, they form the mathematical study of complex analysis.

The key idea is to introduce a new number i (sometimes called an imaginary number) that has the property that

$$i^2 = -1,$$

so i is definitely not in $\mathbb{R}$ and then use this to extend the real numbers to a bigger number system in which the usual rules of algebra apply. This enables us to define the square roots of negative real numbers, but beyond this, the complex numbers allow us to do many new and wonderful things, many of them with remarkable applications in calculus.

I'd like you to meet i

- Many polynomial equations have no solutions in $\mathbb{R}$. EG:

$$\begin{aligned} x^2 + 1 &= 0 \qquad\qquad (*) \\ \Rightarrow x^2 &= -1. \end{aligned}$$

  but no $x \in \mathbb{R}$ has square -1, so $(*)$ has no real solutions.

- However, introducing a brand new number i obeying the usual rules of arithmetic, except that

$$i^2 = -1$$

  allows us to solve $(*)$. We get:

$$x = \pm\, i.$$

But i's not enough ...

- This new number i allows us to solve at least one polynomial equation we could previously not solve. But there's more!

Example 1.25

Assuming the usual rules of arithmetic apply, find two solutions of

$$x^2 + 4 = 0.$$

- To solve some polynomial equations, it seems we need numbers of the form yi where y is real.

. . . and yi's still not enough

Example 1.26

Assuming the usual rules of arithmetic apply, check that $1 + i$ is a solution of

$$x^2 - 2i = 0.$$

- So in order to solve some equations, it seems we need numbers of the form $x + iy$ where x and y are real.
- This turns out to be enough to solve all polynomial equations (!!!) and leads to the definition of the complex numbers.

At last, we have enough numbers!

Definition 1.27

A complex number (typically denoted z) is a quantity consisting of a real number added to a real multiple of i. That is:

$$z = x + iy$$

where $x, y \in \mathbb{R}$. The set of all complex numbers is denoted $\mathbb{C}$, so in set notation

$$\mathbb{C} = \{x + iy \mid x, y \in \mathbb{R}\}.$$

- A complex number written in the form $z = x + iy$ is said to be in Cartesian form.
- The definition tells us that that $\mathbb{R} \subseteq \mathbb{C}$. (Do you see why?)

Example 1.28

1. Using the number i, write down two square roots of -25.
2. Simplify i^7.

Just imagine

Definition 1.29

For any $z = x + iy \in \mathbb{C}$

- x is called the real part of z and is denoted $\mathrm{Re}(z)$;
- y is called the imaginary part of z and is denoted $\mathrm{Im}(z)$.

Note that both $\mathrm{Re}(z)$ and $\mathrm{Im}(z)$ are real numbers.

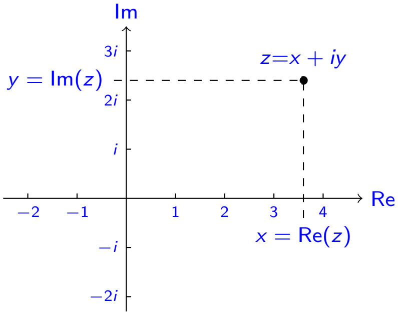
Example 1.30

For the complex number $z = 2 - 3i$, write down:

1. $\mathrm{Re}(z)$.
2. $\mathrm{Im}(z)$.
3. $\mathrm{Re}(z) - \mathrm{Im}(z)$.

The plane facts

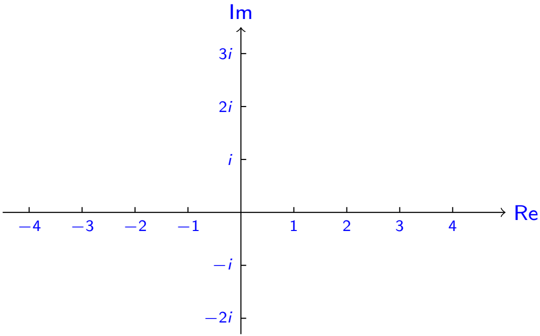
- Complex numbers can be represented graphically in the complex plane.
- A complex number $z = x+iy$ corresponds to a point in the plane with:
  - its real part measured on the horizontal axis (now called the real axis)
  - its imaginary part measured on the vertical axis (now called the imaginary axis).



- The complex plane is sometimes called the Argand plane, and a diagram of it is called an Argand diagram.

Example 1.31

Sketch the complex numbers $3 - i$, $-2 + 2i$, -4, and $3i$ in the complex plane.

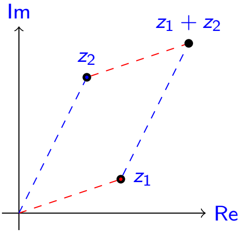


Complex arithmetic Part 1 - Addition and friends

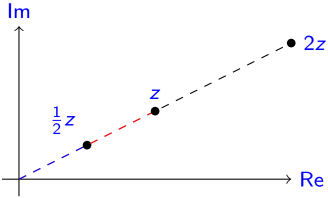
- Let $z_1 = a + ib$ and $z_2 = c + id$ be complex numbers.
  - Equality: z_1 and z_2 are equal precisely if
  $$a = c \qquad \text{and} \qquad b = d.$$
  - Addition: We add z_1 and z_2 as follows:
  $$z_1 + z_2 = (a + ib) + (c + id) = (a + c) + i(b + d).$$
  - Subtraction: We subtract z_2 from z_1 as follows:
  $$z_1 - z_2 = (a + ib) - (c + id) = (a - c) + i(b - d).$$
  - Multiplication by a real: We multiply z_1 by $k \in \mathbb{R}$ as follows:
  $$kz_1 = k(a + ib) = (ka) + i(kb).$$

Geometric interpretation

- $z_1 + z_2$ is the other vertex of the parallelogram with sides 0 to z_1 and 0 to z_2.

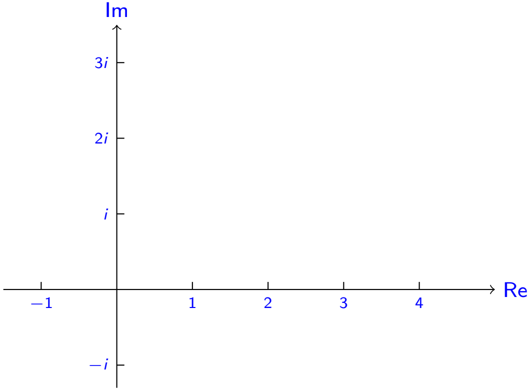


- For k positive, kz is found by stretching (or shrinking) z.



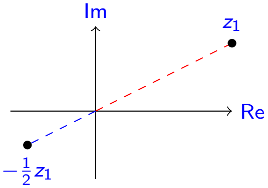
Example 1.32

Illustrate the geometric effect of adding complex numbers for $z_1 = 1 + 2i$ and $z_2 = 2 + i$.

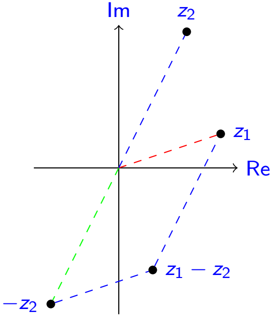


Geometric interpretation continued

- For k negative, kz_1 is found by stretching or shrinking z_1 and reflecting through 0.



- $z_1 - z_2$ is similarly found by expressing it as $z_1 + (-z_2)$.



- We will explore these ideas further when we study vectors.

Complex Arithmetic Part 2 - Multiplication

- We want the Complex numbers to obey the usual laws of arithmetic.
- Accordingly, we define multiplication using the familiar distributive law.
- This means we multiply complex numbers by simply "expanding the brackets" and remembering that $i^2 = -1$.

$(a + ib)(c + id) =$

The Complex Conjugate

To help us divide complex numbers (and perform other wonderful feats), we define the complex conjugate:

Definition 1.33

If $z = a + ib$ is a complex number, then the complex conjugate of z is denoted $\bar{z}$ ("z bar"), and is defined to be

$$\bar{z} = a - ib.$$

The real part stays the same and the imaginary part changes sign.

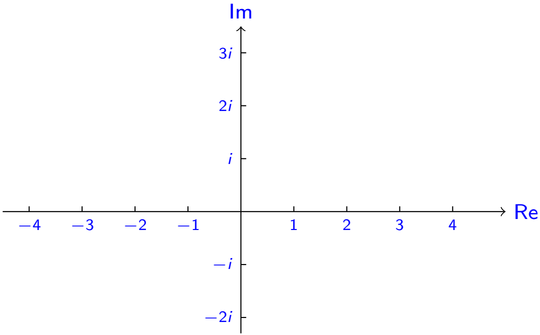
Example 1.34

Write down the complex conjugates of:

(a) $-3+7i$ (b) $2-5i$ (c) $3i$ (d) 4

Example 1.35

- If $z_1 = 1 + i$ and $z_2 = -3 - 2i$, plot z_1, z_2, $\bar{z}_1$ and $\bar{z}_2$ in the complex plane.
- What is the geometric relationship between z and $\bar{z}$?

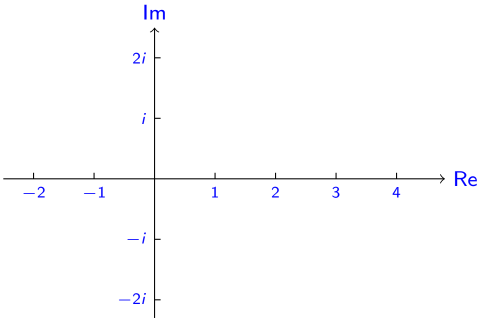


Homework 11

Plot $z_1 = 2i$, $z_2 = 3$, $\bar{z}_1$ *and* $\bar{z}_2$ *in the complex plane.*

Example 1.36

By completing the square, find the solutions of $z^2 - 6z + 10 = 0$ and plot them in the complex plane. What is the relationship between the solutions?



Properties of the conjugate

Theorem 1.37

Let $z = x + iy$ *and* $w = a + ib$ *be complex numbers.*
Prove the following properties of the complex conjugate.

1. $z + \bar{z} = 2x = 2\,\mathrm{Re}(z)$ *(real!);*
2. $z - \bar{z} = 2yi = 2\,\mathrm{Im}(z)i$ *(purely imaginary!);*
3. $z\bar{z} = x^2 + y^2$ *(real!);*
4. $\overline{z + w} = \bar{z} + \bar{w}$;

Proofs of properties of the conjugate continued

Homework 12

Prove part 4 of Theorem 1.37 for a pair of complex numbers $z = x + iy$ and $w = a + ib$.

Complex arithmetic Part 3 - Division

- Suppose we want to divide two complex numbers:

$$\frac{a + ib}{c + id}. \tag{1}$$

  We can get an answer in the form $x + iy$ by making the denominator real.

- We use of Part (3) of Theorem 1.37, which says that a complex number multiplied by its conjugate is real:

  $$\frac{a + ib}{c + id} = \frac{a + ib}{c + id} \times \frac{c - id}{c - id}$$

Example 1.38

Express the following in Cartesian form $x + iy$:

$$\frac{1+2i}{-1+3i}$$

Example 1.39

Find $\text{Re}\left(\dfrac{1+5i}{2-2i}\right)$ and $\text{Im}\left(\dfrac{1+5i}{2-2i}\right)$.

Homework 13

Let $z = 1 - 5i$ *and* $w = -2 + i$. *Express the following complex numbers in cartesian form* $a + ib$ *where* a *and* b *are real.*

(a) $w^2 z$ *(b)* $\dfrac{w}{w + 2z}$

Complex numbers in polar form

We have learned how to work with complex numbers written in Cartesian form

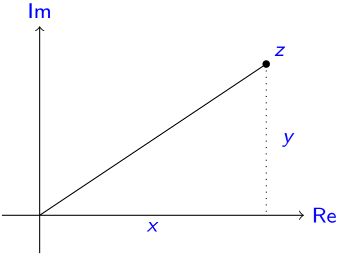
$$x + iy.$$

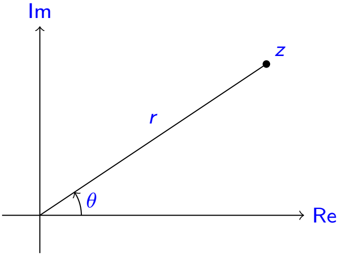
In this form, the task of adding and subtracting complex numbers is easy. Multiplication and division are reasonably easy in Cartesian form, but taking powers and roots is generally hard work. Our practice of representing complex numbers as points in the plane leads to another representation.

Instead of giving the x and y coordinates, we can represent a complex number z by saying how far it is from the origin and what angle we get when we draw a line from z to the origin. In this polar form finding powers and roots becomes much easier.

Modulus and Argument

- The position of a complex number z in the complex plane can be specified in two different ways:
  - Cartesian form: give the real and imaginary parts x and y, so that $z = x + iy$;
  - Polar form: give the distance r from the origin, and the angle θ from the positive real axis.





⚠ Distance r is always given as a non-negative real.

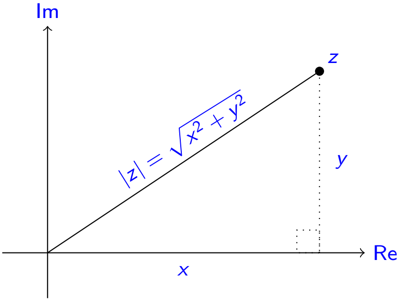
Modulus

Definition 1.40

The modulus of z, denoted $|z|$, is the distance r from z to the origin in the complex plane.

- Writing $z = x + iy$, we can find $|z|$ by Pythagoras' Theorem:

$$|z| = \sqrt{x^2 + y^2}.$$



Arguments can be beneficial

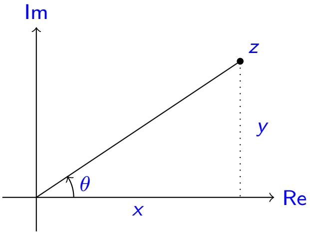
Definition 1.41

An argument of z is any angle θ that z makes with the positive real axis in the complex plane.

- To find an argument θ of

  $$z = x + iy$$

  draw z in the complex plane and use standard triangles (if possible) to find θ.



- If θ is not a standard angle, we note that $\tan(\theta) = \dfrac{y}{x}$ - but be careful - always draw a diagram!

⚠ Caution!

- Arguments of z are not unique, since adding multiples of 2π does not change the position of z in the complex plane.
- However, there is one unique value of the argument that satisfies

$$-\pi < \theta \leq \pi.$$

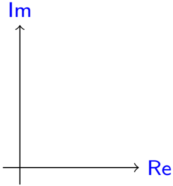
- This is called the principal argument of z and is sometimes denoted $\operatorname{Arg}(z)$ – with a capital A.
- In this subject, if we refer to the argument of z, we will always mean the principal argument.

Example 1.42

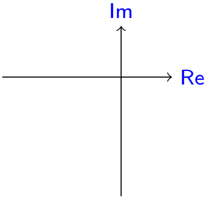
Find the modulus and an argument of

(a) $1+\sqrt{3}i$ (b) $-3-3i$ (c) $3+4i$ (d) -7

(a)

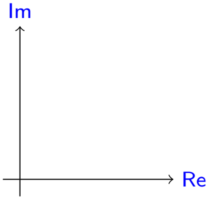


(b)

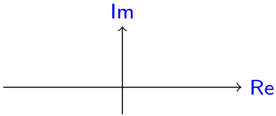


Example continued

(c)



(d)



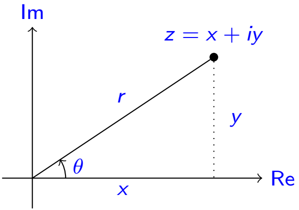
- Always draw a sketch when converting to polar form.
- Later, the inverse tangent function $\arctan(x)$, we give us another way of expressing the answer to (c).

Homework 14

Find the modulus and an argument of $z = 2 - 2i$.

Polar Form and Complex Exponential

- Recall that any complex number $z = x + iy$ can also be specified by its modulus r and an argument θ.



- From the diagram:

$$\cos(\theta) = \frac{x}{r} \text{ and } \sin(\theta) = \frac{y}{r} \Rightarrow x = r\cos(\theta) \text{ and } y = r\sin(\theta).$$

$$\Rightarrow z = x + iy = r\cos(\theta) + i\, r\sin(\theta).$$

Definition 1.43

The trigonometric polar form of the complex number z is:

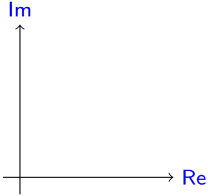
$$z = r(\cos(\theta) + i\sin(\theta)).$$

Example 1.44

Express the following complex numbers in trigonometric polar form.

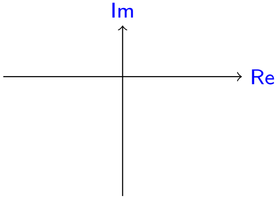
(a) $z = \sqrt{3} + i$ (b) $z = -1 - i$

(a)



Example continued

(b)

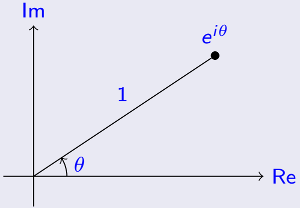


The Complex Exponential

Definition 1.45

For any θ, the complex exponential $e^{i\theta}$ is defined to be

$$e^{i\theta} = \cos\theta + i\sin\theta.$$



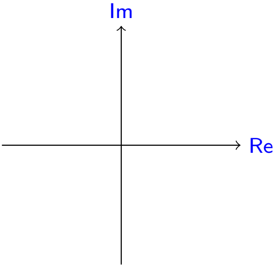
- We can now write a complex number z in exponential polar form

$$z = re^{i\theta}$$

  where $r = |z|$ and θ is (any) argument of z.
- As we shall see, this is a very useful of way of expressing a complex number.

Example 1.46

Express $z = -2 + 2\sqrt{3}i$ in exponential polar form $re^{i\theta}$.



Homework 15

Express $z = 1 - i$ *in exponential polar form.*

Example 1.47

Express $z = 5e^{i\frac{3\pi}{4}}$ in Cartesian form $x + iy$.

Homework 16

Express $z = 4e^{-i\frac{\pi}{3}}$ *in cartesian form.*

Exponentials and conjugates

Example 1.48

Use the basic trig identities $\cos(-\theta) = \cos(\theta)$ (even function) and $\sin(-\theta) = -\sin(\theta)$ (odd function) to prove that

$$e^{-i\theta} = \overline{e^{i\theta}}$$

for any θ.

- This unassuming little result yields a very powerful representation of $\cos$ and $\sin$.

Properties of the complex exponential

- From the definition of the complex exponential, we can prove:

Theorem 1.49

1. $e^{i0} = 1$
2. $e^{i\theta_1} e^{i\theta_2} = e^{i\theta_1 + i\theta_2} = e^{i(\theta_1 + \theta_2)}$
3. $\dfrac{e^{i\theta_1}}{e^{i\theta_2}} = e^{i\theta_1 - i\theta_2} = e^{i(\theta_1 - \theta_2)}$
4. $e^{i\theta_1} = e^{i\theta_2}$ *precisely if* $\theta_2 = \theta_1 + 2k\pi$ *for some* $k \in \mathbb{Z}$.

- Note that these properties are consistent with the usual index laws for exponentials in $\mathbb{R}$ (so they are easy to remember!).
- This is why we generally work with the exponential polar form $re^{i\theta}$ instead of the trigonometric polar form $r(\cos\theta + i\sin\theta)$.

Proving Property 2: Multiplication in polar form

- Recall the compound angle formulas for cos and sin:

$$\cos(\theta_1 + \theta_2) = \cos\theta_1 \cos\theta_2 - \sin\theta_1 \sin\theta_2$$
$$\sin(\theta_1 + \theta_2) = \sin\theta_1 \cos\theta_2 + \cos\theta_1 \sin\theta_2.$$

- You proved these formulae in a Practice Class.
- From this and the definition of the complex exponential:

$$\begin{aligned} e^{i\theta_1} e^{i\theta_2} &= (\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2) \\ &= (\cos\theta_1 \cos\theta_2 - \sin\theta_1 \sin\theta_2) \\ &\qquad + i\,(\cos\theta_1 \sin\theta_2 + \cos\theta_2 \sin\theta_1) \\ &= \cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2) \\ &= e^{i(\theta_1 + \theta_2)}. \end{aligned}$$

Theorem 1.50

For two complex numbers $z_1 = r_1 e^{i\theta_1}$ *and* $z_2 = r_2 e^{i\theta_2}$*, the product is given by*

$$z_1 z_2 = r_1 r_2 e^{i(\theta_1+\theta_2)}.$$

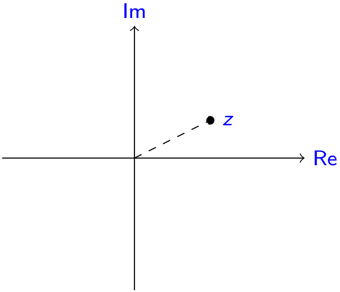
- **Geometrically:** the product of two complex numbers in polar form is obtained by
  - multiplying their moduli ($r_1 r_2$) adding their arguments ($\theta_1 + \theta_2$):
- For modulus, this gives the simple formula $|z_1 z_2| = |z_1||z_2|$.
- We can't write quite such a simple formula for the principal argument of a product.

Homework 17

Find $z_1, z_2 \in \mathbb{C}$ *such that* $\mathrm{Arg}(z_1 z_2) \neq \mathrm{Arg}(z_1) + \mathrm{Arg}(z_2)$.

Example 1.51

Describe geometrically what happens when a complex number z is multiplied by $w = i$.



Homework 18

Describe geometrically what happens when a complex number is multiplied by $w = -i$.

Property 3 – Division in polar form

Homework 19

Consider two complex numbers written in polar form:

$$z_1 = r_1 e^{i\theta_1} \qquad \textit{and} \qquad z_2 = r_2 e^{i\theta_2} \neq 0.$$

Show that

$$\frac{z_1}{z_2} = \frac{r_1}{r_2}\, e^{i(\theta_1 - \theta_2)}.$$

In Summary

- **Geometric interpretation:** The quotient of two complex numbers in polar form is obtained by
  - dividing their moduli ($\frac{r_1}{r_2}$) and
  - subtracting their arguments ($\theta_1 - \theta_2$).
- For modulus, this gives the simple formula $\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$.
- We can't write quite such a simple formula for the principal argument of a quotient.

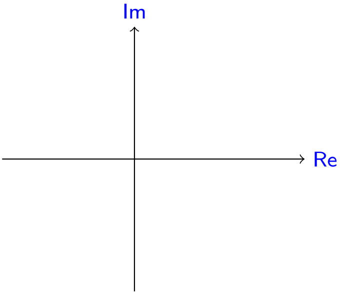
Homework 20

Find a pair of complex numbers of z_1 and z_2 such that

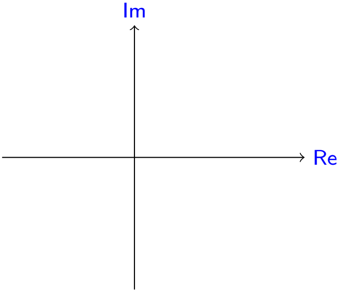
$$\operatorname{Arg}\left(\frac{z_1}{z_2}\right) \neq \operatorname{Arg}(z_1) - \operatorname{Arg}(z_2).$$

Example 1.52

If $z = \sqrt{3} + i$ and $w = 1 - \sqrt{3}i$, use exponential polar form to find $\frac{1}{z}$ and zw .



Example continued



Homework 21

Describe geometrically what happens when a complex number z *is divided by* $w = \frac{1}{2} + \frac{\sqrt{3}}{2}i$.

Properties of the modulus and argument

- We can prove the following properties of the modulus and argument of complex numbers z and w.

Theorem 1.53

1. $|zw| = |z||w|$
2. $\left|\dfrac{z}{w}\right| = \dfrac{|z|}{|w|}$
3. *An argument of* zw *is* $\mathrm{Arg}(z) + \mathrm{Arg}(w)$
4. *An argument of* $\dfrac{z}{w}$ *is* $\mathrm{Arg}(z) - \mathrm{Arg}(w)$.

- You will prove Property 1 in a Practice Class.
- These results are useful for calculating moduli and arguments.

Homework 22

Use Property 1 to prove Property 2.

Example 1.54

Using the above properties, evaluate

$$\left|\frac{-2(3-i)(5+2i)}{(1+3i)(7-i)}\right|$$

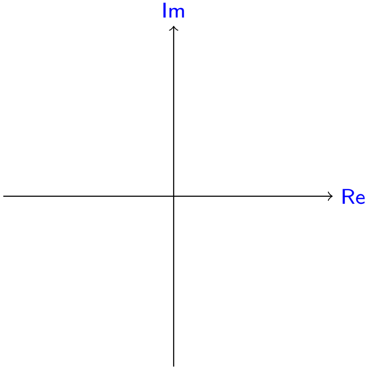
Example 1.55

Using the above properties, find an argument of

1. $z_1 = (1+i)(-1+\sqrt{3}i)$
2. $z_2 = \dfrac{-i}{-2+2i}$

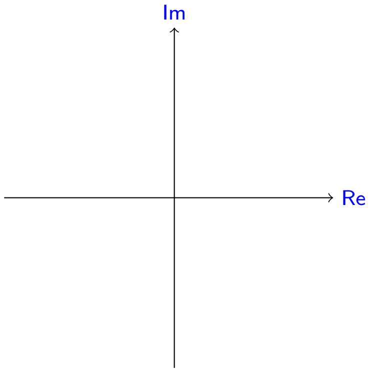
and hence find $\mathrm{Arg}(z_1)$ and $\mathrm{Arg}(z_2)$.

1.



Example continued

2.



Complex representation of trig functions

- The complex exponential can be used to express cos and sin in a way that makes a lot of calculations easier.

Example 1.56

Use Theorem 1.37(Part 1) and Example 1.48 to prove that

$$\cos(\theta) = \frac{1}{2}\left(e^{i\theta} + e^{-i\theta}\right)$$

for every $\theta \in \mathbb{R}$.

- Amazing! The expression $\frac{1}{2}\left(e^{i\theta} + e^{-i\theta}\right)$ is always real!

And now for the other one . . .

Homework 23

Prove that $\sin(\theta) = \dfrac{1}{2i}\left(e^{i\theta} - e^{-i\theta}\right)$.

Theorem 1.57

$\cos(\theta) = \dfrac{1}{2}\left(e^{i\theta} + e^{-i\theta}\right)$ *and* $\sin(\theta) = \dfrac{1}{2i}\left(e^{i\theta} - e^{-i\theta}\right)$.

- We can use these identities to express powers of $\cos(\theta)$ or $\sin(\theta)$ in terms of $\cos(n\theta)$ or $\sin(n\theta)$.
- This will be very useful in our study of integration.

Binomial memories

- To put these representations to good use, we need the expansion of the binomial

$$(a+b)^n.$$

- We use Pascal's triangle:

$$\begin{array}{ccccccccccc}
 & & & & & 1 & & & & & \\
 & & & & 1 & & 1 & & & & \\
 & & & 1 & & 2 & & 1 & & & \\
 & & 1 & & 3 & & 3 & & 1 & & \\
 & 1 & & 4 & & 6 & & 4 & & 1 & \\
1 & & 5 & & 10 & & 10 & & 5 & & 1 \\
 & & & & & \vdots & & & & &
\end{array}$$

where each entry is obtained by summing the two entries above it in the previous row.

Thanks Pascal!

Theorem 1.58 (Binomial formula)

$$(a+b)^n = \binom{n}{0}a^n + \binom{n}{1}a^{n-1}b + \binom{n}{2}a^{n-2}b^2 + \dots + \binom{n}{n-2}a^2b^{n-2} + \binom{n}{n-1}ab^{n-1} + \binom{n}{n}b^n,$$

where $\binom{n}{j}$ is the number in the corresponding row of Pascal's triangle.

- Using Pascal's triangle to find binomial coefficients is much easier than using the definition in terms of factorials:

$$\binom{n}{j} = \frac{n!}{j!(n-j)!}.$$

- To expand $(a-b)^n$, regard it as $(a+(-b))^n$. This means that the signs of the terms will alternate $+, -, +, -, \dots$

Example 1.59

Expand $(a+b)^5$ and $(a-b)^5$.

- Applying the binomial formula to our exponential formulas for $\sin$ and $\cos$ converts expressions $\sin^m(\theta)$ and $\cos^n(\theta)$ into a form where we can integrate them.

Example 1.60

Express $\sin^4(\theta)$ as a sum of sines or cosines of multiples of θ.

Example continued

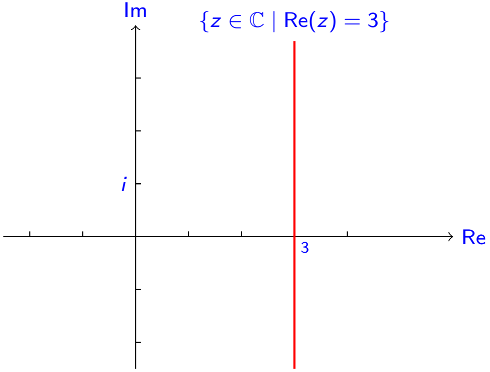
This approach always converts expressions of the form $\sin^m(\theta)\cos^n(\theta)$ to an equivalent form that is easy to integrate.

Homework 24

Express $\cos^4(\theta)\sin^2(\theta)$ *as a sum of sines and/or cosines of multiples of* θ.

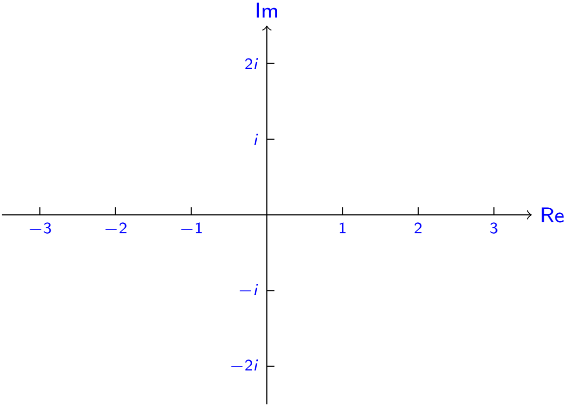
Sketching subsets of the Complex Plane

- The set of complex numbers that satisfy certain conditions corresponds to some region in the complex plane.
- **Example:** The set $\{z \in \mathbb{C} \mid \mathrm{Re}(z) = 3\}$ of all complex numbers with real part 3, is represented in the complex plane by the vertical line intersecting the real axis at 3.



Example 1.61

Sketch the set $\{z \in \mathbb{C} \mid \mathrm{Im}(z) < 2\}$ in the complex plane.

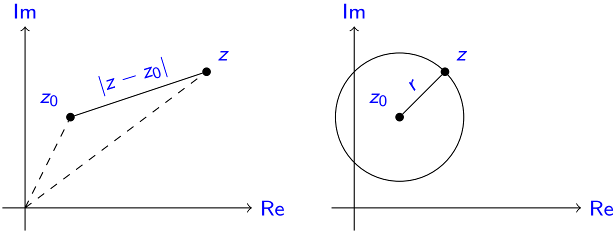


Keeping our distance

- If z and z_0 are complex numbers, then $|z - z_0|$ is simply the distance between z and z_0 in the complex plane.
- For fixed $z_0 \in \mathbb{C}$ and $r > 0$ this tells us that the set

$$\{z \in \mathbb{C} : |z - z_0| = r\}$$

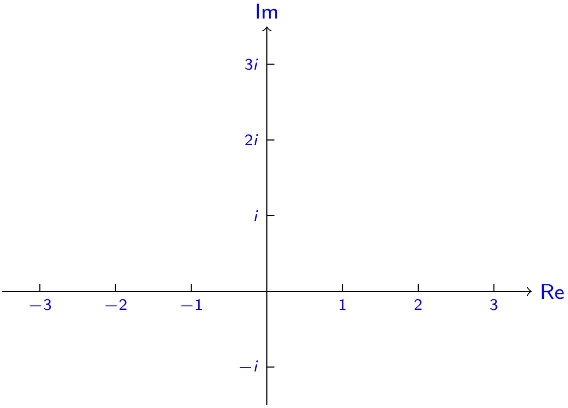
is a circle of radius r and centre z_0.



- We will pursue this idea further when we study vectors.

Example 1.62

Sketch the set $C = \{z \in \mathbb{C} : |z - i| = 2\}$.



Example continued

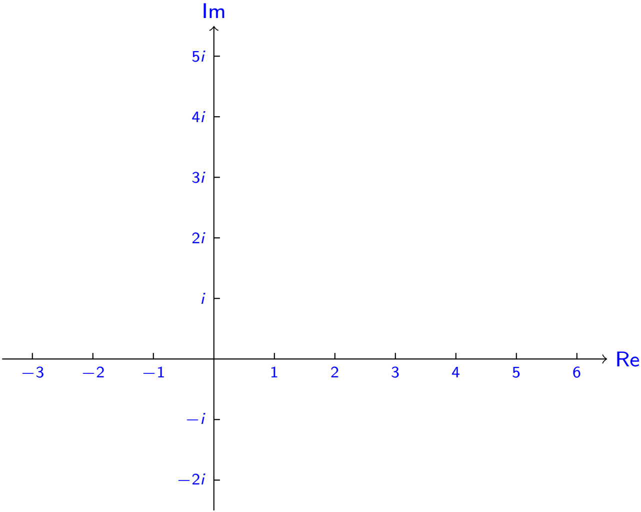
Find the Cartesian equation (i.e. an equation in terms of x and y) of this curve, by substituting $z = x + iy$ into

$$|z - i| = 2$$

and simplifying.

Example 1.63

Sketch the set $\{z \in \mathbb{C} : |z - 3 - 2i| \leq 3\}$.



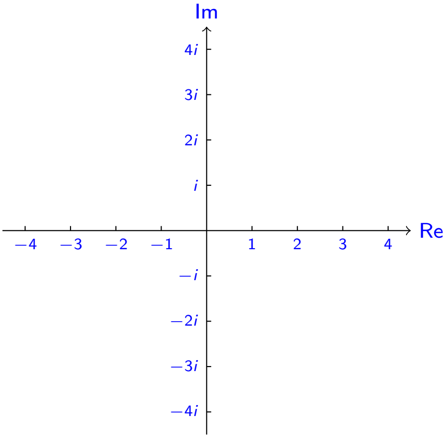
Homework 25

Find the cartesian equation of this region by substituting $z = x + iy$*. Hence find the points where the boundary of this set intersects the real axis.*

Example 1.64

Sketch the region

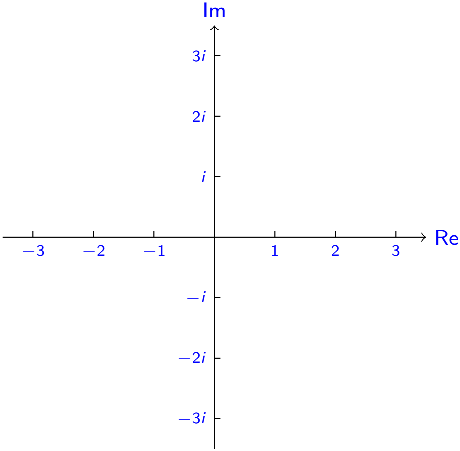
$$\left\{ z \in \mathbb{C} \;\middle|\; 2 \leq |z| \leq 4 \quad \text{and} \quad \frac{\pi}{4} < \operatorname{Arg}(z) < \frac{\pi}{2} \right\}.$$



Example 1.65

Sketch the set $\{z \in \mathbb{C} : |z+2| = |z+i|\}$.

Example continued

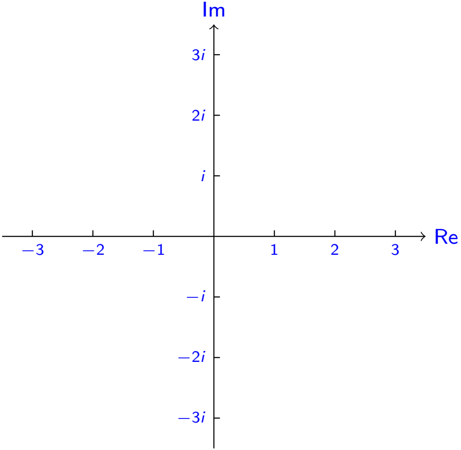


Homework 26

Find the cartesian equation of the set in Example 1.65.

Example 1.66

Sketch the set $\{z \in \mathbb{C} \mid z = i\bar{z}\}$.



Homework 27

Sketch the region $\{z \in \mathbb{C} \mid z\bar{z} = 16\}$.

Powers and roots of complex numbers

We have learned how to multiply complex numbers in Cartesian form. In principle, this means we know how to calculate integer powers of complex numbers. However, calculating large integer powers directly from the definition of multiplication would be a horrifying task! Just think how much work it would take to find

$$(\cos(1) + i\sin(1))^{100}$$

directly! Luckily for us, there is a much easier way to calculate powers, by first expressing complex numbers in their exponential polar form. In the other direction, complex numbers in exponential form also yields a relatively easy way to find their n^{th} roots.

We have the power

Theorem 1.67 (De Moivre's theorem)

Let $z = re^{i\theta}$. For any integer n:

$$z^n = r^n e^{in\theta}.$$

"Proof":

This formula is consistent with the usual exponent laws

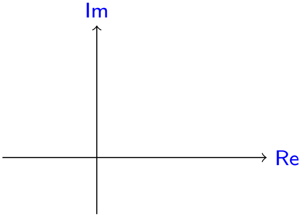
$$z^n = (re^{i\theta})^n = r^n (e^{i\theta})^n = r^n e^{in\theta}.$$

Powers are easy

De Moivre's theorem can be used to avoid expanding the brackets when finding large powers of complex numbers.

Example 1.68

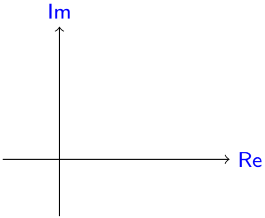
Use de Moivre's theorem to find $(1+i\sqrt{3})^8$ in both exponential and Cartesian form.



Example continued

Example 1.69

Find $\left(\dfrac{2}{1+i}\right)^{14}$ in exponential and Cartesian form.



Example continued

Roots of Complex Numbers

- Suppose we wish to find the n-th roots of a given complex number w, so we seek z such that

$$z = w^{\frac{1}{n}} \Rightarrow z^n = w.$$

- Let $z = re^{i\theta}$ and $w = se^{i\phi}$ be the respective exponential polar forms, so the equation becomes

$$(re^{i\theta})^n = se^{i\phi} \Rightarrow r^n e^{in\theta} = se^{i\phi}$$

- Equating modulus and argument gives:

$$\begin{aligned} r^n = s \quad \text{and } e^{in\theta} = e^{i\phi} \\ \Rightarrow r = s^{\frac{1}{n}} \quad \text{and } n\theta = \phi + 2k\pi, \quad k \in \mathbb{Z} \\ \Rightarrow r = s^{\frac{1}{n}} \text{ and } \theta = \frac{1}{n}(\phi + 2k\pi), \quad k \in \mathbb{Z}. \end{aligned}$$

Knowing when to stop

- The values $k = 0, 1, \ldots, n-1$ give n distinct n-th roots of w.
- We stop at $n-1$ since for $k = n$ we would be adding 2π to the argument, giving the same answer as $k = 0$:

Theorem 1.70 (Root finding formula)

The n-th roots of $w = se^{i\phi}$ are:

$$w^{\frac{1}{n}} = s^{\frac{1}{n}}\, e^{i(\frac{1}{n}(\phi+2k\pi))} \qquad \text{for } k = 0, 1, \ldots, n-1.$$

or in set notation

$$\left\{z \in \mathbb{C} \mid z^n = w\right\} = \left\{s^{\frac{1}{n}}\, e^{i(\frac{1}{n}(\phi+2k\pi))} \;\middle|\; k = 0, 1, \ldots, n-1\right\}$$

- You do not need to memorise the theorem.
- It is easy to derive the n-th roots of w by expressing $z^n = w$ in exponential form and solving for z.

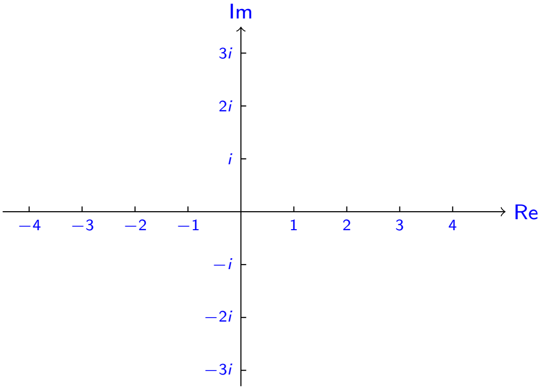
Example 1.71

Find the cube roots of 8, sketch them in the complex plane, and express them in cartesian form.

- Recall that over the reals, 2 is the only cube root of 8.
- However, over the complex numbers there are three!

Example continued

Sketch the three cube roots of 8:



Beautiful symmetry

- We see that the cube roots of 8 all have modulus 2, and are evenly spaced around a circle of radius 2 in the complex plane.
- This is a general property—the n-th roots of any complex number are evenly spaced around the origin in the complex plane. Can you see why?

Example continued

Express the three cube roots of 8 in Cartesian form:

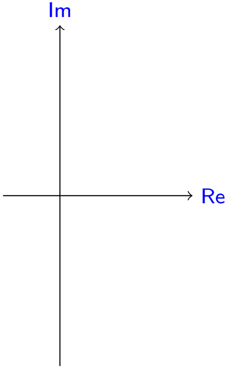
Homework 28

Find the 6-th roots of 1, and sketch these in the complex plane. You may keep your answers in exponential polar form.

Note: *The n-th roots of 1 are often called 'roots of unity'.*

Example 1.72

Find the 4-th roots of $1 - i\sqrt{3}$.



Example continued

Example continued

Homework 29

Find the cube roots of $1 + i$.

Factorising polynomials

We can apply our knowledge of complex numbers to the well known problem of factorising polynomials. Our method for finding roots of complex numbers supports new techniques for finding roots of polynomials and hence factorising them. An ability to factorise over the complex numbers often enables us to factors in the more familiar setting of real valued polynomials.

This has many important applications in calculus. For example, we later learn how to integrate functions of the form

$$\frac{p(x)}{q(x)}$$

where $p(x)$ and $q(x)$ are polynomials using the technique of partial fractions. A key first step in this technique is factorisation of $q(x)$.

Polynomials over $\mathbb{C}$

- A real polynomial $p : \mathbb{R} \longrightarrow \mathbb{R}$ defined by

$$p(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$$

extends to a complex polynomial $q : \mathbb{C} \longrightarrow \mathbb{C}$ defined by

$$q(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0$$

- Since these two polynomials agree exactly on $\mathbb{R}$, solving $q(z) = 0$, also yields the solutions of $p(x) = 0$.
- Surprisingly, solving polynomials over $\mathbb{C}$ is sometimes easier that solving polynomials over $\mathbb{R}$.
- It is often convenient to allow the coefficients

$$a_n, a_{n-1}, \ldots, a_1, a_0$$

of a complex polynomial to be general complex numbers.

Roots of polynomials

- Finding the roots or zeros of a complex polynomial $P(z)$ means (as usual) solving $P(z) = 0$.
- Recall that over the reals $\mathbb{R}$, some quadratics have roots because they can be factorised into linear factors, like

$$x^2 + 3x + 2 = (x + 2)(x + 1),$$

- Others have no real roots and cannot be factorised over $\mathbb{R}$:

$$x^2 - 2x + 2.$$

Here the discriminant $b^2 - 4ac = -4$ is negative.

The quadratic formula rocks!

- It is easy to solve and hence factorise a quadratic over $\mathbb{C}$ using the quadratic formula. This always gives linear factors!

Example 1.73

Use the quadratic formula to factorise $p(z) = z^2 - 2z + 2$.

- We interpret $\pm\sqrt{b^2 - 4ac}$ last the two square roots of $b^2 - 4ac$ guaranteed to exist by the root finding formula.
- The formula also works when the coefficients are complex.

Mission accomplished!

- We have now achieved our aim of constructing system in which all polynomial equations have solutions!

Theorem 1.74 (The Fundamental Theorem of Algebra)

Every complex polynomial $P(z)$ of degree n can be factorised into n linear factors over $\mathbb{C}$, that is

$$P(z) = a(z - \alpha_1)(z - \alpha_2)\dots(z - \alpha_n),$$

where $a, \alpha_1, \alpha_2, \dots, \alpha_n \in \mathbb{C}$.

- This also tells us that a polynomial of degree n has $\leq n$ roots over $\mathbb{C}$.
- There might be some repeated roots and hence repeated factors, so the number of roots could be $< n$.

They travel in pairs

- We will prove the following theorem in a Practice Class.

Theorem 1.75 (Conjugate Pairs)

If the coefficients of a complex polynomial $P(z)$ are all in $\mathbb{R}$, the non-real roots of $P(z)$ occur in complex conjugate pairs z and $\bar{z}$.

- The requirement that all coefficients are real is indispensable.

Example 1.76

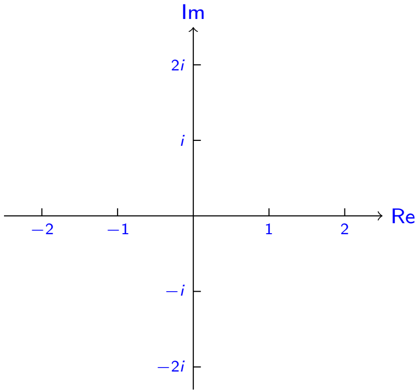
Use the quadratic formula to solve $z^2 - 2iz + 1 = 0$.

Example 1.77

Consider the polynomial $P(z) = z^3 - 3iz^2 - 2z$.

1. Factorise $P(z)$.
2. Hence find the roots of $P(z)$ and sketch them.
3. Verify your answers by checking that $P(z) = 0$ in each case.

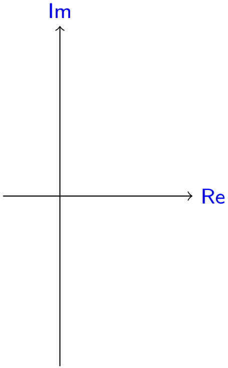
Example continued



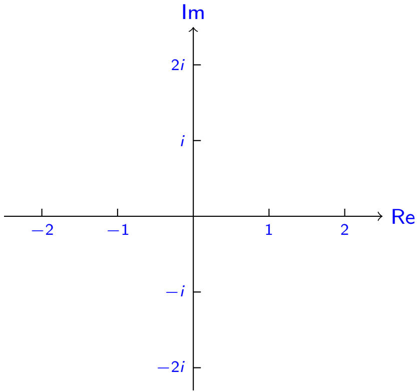
All's fair in love and factorisation

Example 1.78

Sketch the set $\{z \in \mathbb{C} \mid \; z^4 - 2z^2 + 4 = 0\}$.



Example continued



Homework 30

Sketch the set $\{z \in \mathbb{C} \mid \; z^4 + z^2 - 12 = 0\}$.

Making progress

Theorem 1.79 (Geometric Progression)

For any $z \in \mathbb{C} \setminus \{1\}$

$$z^n + z^{n-1} + \cdots + z + 1 = \frac{z^{n+1} - 1}{z - 1}$$

Proof:

- This lets us find the roots of $z^n + z^{n-1} + \cdots + z + 1$.

Example 1.80

Solve $z^3 + z^2 + z + 1 = 0$.

Example continued

The key factors

- The following consequence of the Conjugate Pairs Theorem has important consequences for our study of integration.
- We will use complex numbers to prove this in a Practice Class.

Theorem 1.81 (Real Polynomials)

If the coefficients of a polynomial $p(x)$ are all real, then $p(x)$ can be written as a product of linear and quadratic factors with real coefficients.

Example 1.82

Express the polynomial $p(z) = z^3 + 8$ as a product of linear and quadratic factors with all coefficients real .

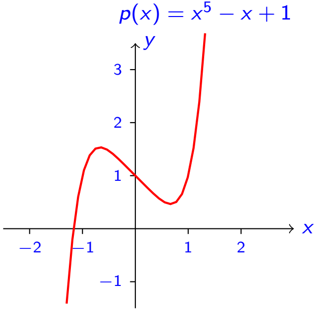
Example continued

Homework 31

Express the polynomial $p(z) = (z-2)^3 + 8$ *as a product of linear and quadratic factors with real coefficients.*

Bad news on factorisation

- Even though the theorem guarantees linear/quadratic factors exist, they can be very difficult to find in general.
- For polynomials of degree 5 or more, it is often impossible!
- For example, we can't express the factors of $p(x) = x^5 - x + 1$ in simple terms, even though $p(x)$ clearly has a real root.



- These results belong a field of mathematics called Galois Theory, studied in *MAST30005 Algebra*.

But does any of this make sense?

- **Q:** How can we be sure that the complex numbers really exist?
- **A:** We can construct them from the reals!

Definition 1.83 (Constructing the complex numbers)

Let $\mathcal{C}$ denote the Cartesian product $\mathbb{R}^2$ and for each pair $(a,b),(c,d) \in \mathcal{C}$ define:

- $(a,b)+(c,d) = (a+c, b+d)$.
- $(a,b)*(c,d) = (ac-bd, ad+bc)$.

and define $\mathcal{R} = \{(a,0) \mid a \in \mathbb{R}\}$ so $\mathcal{R} \subseteq \mathcal{C}$

- From these definitions, we can prove
  1. $\mathcal{R}$ is algebraically identical to $\mathbb{R}$, with each $x \in \mathbb{R}$ corresponding to $(x,0) \in \mathcal{R}$.
  2. $\mathcal{C}$ is algebraically identical to $\mathbb{C}$, with each $x+iy \in \mathbb{C}$ corresponding to $(x,y) \in \mathcal{C}$.
- In this construction $i \in \mathbb{C}$ corresponds to $(0,1) \in \mathcal{C}$.

Topic 2 - Functions

Functions are a fundamental concept of mathematics and a cornerstone of most of its applications. While sets allow us to organise objects, functions are where all the action is. They allow us to relate elements of different sets, and to model phenomena where certain quantities are changing.

In this topic, we explore the concept of function in its modern mathematical incarnation and study properties and general operations on functions. This will serve as preparation for the later topics, where we will consider infinitesimal operations on functions (differentiation and integration).

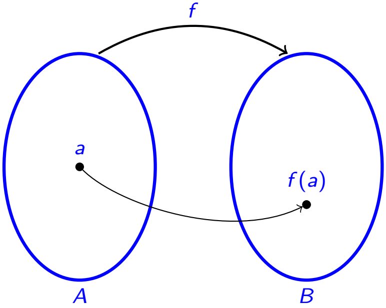
We also introduce the notion of vectors, explore their basic properties, learn about some vector operations and discuss briefly functions whose values are vectors.

Topic 2 Contents

- What is a function? 199
- Key properties of functions 212
- Composition 229
- Inverse of a function 233
- Introduction to vectors 261
- The scalar product 285
- Vector projections 301
- Parametric curves 313

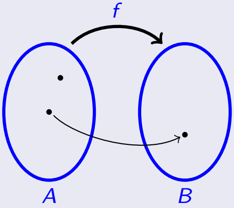
What is a function?

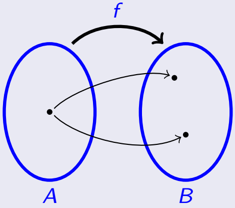
- A function f consists of:
  - a nonempty set A called the domain of f;
  - a nonempty set B called the codomain of f;
  - a rule that associates to each element a of the set A a unique element $f(a)$ of the set B.
- The notation $f : A \longrightarrow B$ means f is a function with domain A and codomain B.
- We can also say "f maps A into B".



Homework 32

For each of the following pictures, explain why the rule f is not a function:





Example 2.1

Is $g\colon \mathbb{R} \longrightarrow \mathbb{R}$ given by

$$g(x) = \frac{x+1}{x^2+4}.$$

a function?

Example 2.2

Is $h \colon \mathbb{C} \longrightarrow \mathbb{C}$ given by

$$h(z) = \frac{z+1}{z^2+4}$$

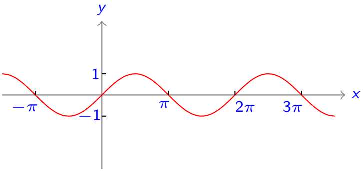
a function? Why not? What modifications can turn it into a function?

Graph of a function $\mathbb{R} \longrightarrow \mathbb{R}$

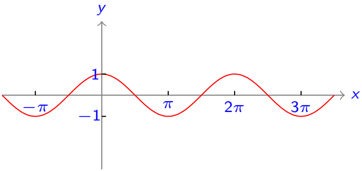
- When the domain and codomain are subsets of $\mathbb{R}$, we can also visualise a function $f\colon A \longrightarrow B$ via its graph.
  Formally, the graph is the set

$$\{(a, f(a)) \mid a \in A\} \subseteq A \times B.$$

- For instance, recall the graphs of the basic trigonometric functions:



$\sin\colon \mathbb{R} \longrightarrow \mathbb{R}$



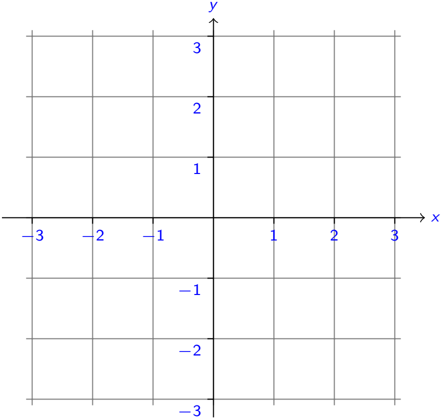
$\cos\colon \mathbb{R} \longrightarrow \mathbb{R}$

Example 2.3

Consider the floor function $f : \mathbb{R} \longrightarrow \mathbb{Z}$ given by

$$f(x) = \text{the largest integer } n \text{ such that } n \leq x.$$

Is it a function? What is its graph?



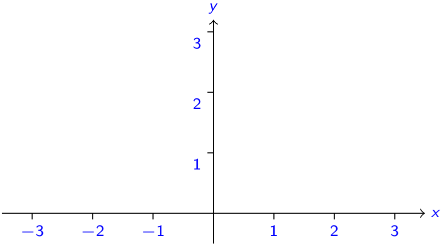
Constant functions

- A function that takes the same value everywhere is called a constant function.

Example 2.4

Sketch the graphs of the following functions on the same set of axes

1. $f : \mathbb{R} \longrightarrow \mathbb{R}$ defined by $f(x) = 1$ for all $x \in \mathbb{R}$.
2. $f : \mathbb{R} \longrightarrow \mathbb{Z}$ defined by $f(x) = 2$ for all $x \in \mathbb{R}$.
3. $f : \mathbb{Z} \longrightarrow \mathbb{Z}$ defined by $f(x) = 3$ for all $x \in \mathbb{R}$.



Homework 33

Consider $f : \mathbb{R} \longrightarrow \mathbb{R}$ *given by*

$$f(x) = \textit{the solution } t \textit{ of the equation } t^2 = x.$$

Is this a function?
What if we change it to $f : \mathbb{R} \longrightarrow \mathbb{C}$*, with the same rule?*

A non-numerical function

- There is no reason to restrict the domain and codomain of a function to be sets of numbers.

Example 2.5

Let W denote the set of all English words and A the Roman alphabet. Consider $\alpha : W \longrightarrow A$ given by

$$\alpha(w) = \text{the first letter in } w.$$

Homework 34

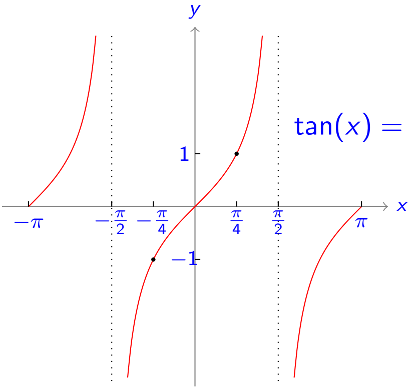
Find three more examples of functions where the domain and/or the codomain are not sets of numbers.

Six little trig functions

- We already reviewed the graphs of the basic trig functions sin and cos. Their quotient:

$$\tan\colon \mathbb{R} \setminus \{(2k+1)\,\tfrac{\pi}{2} \mid k \in \mathbb{Z}\} \longrightarrow \mathbb{R}$$

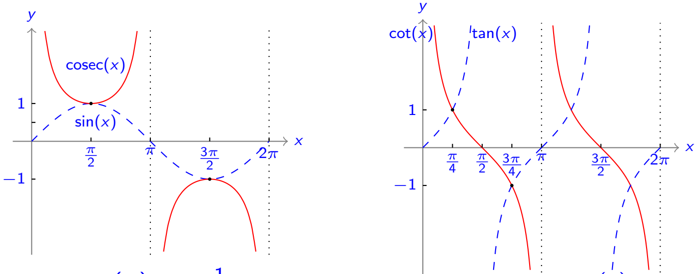
called the tangent function, is given by



$$\tan(x) = \frac{\sin(x)}{\cos(x)}$$

Reciprocal trigonometric functions

- It is also useful to define and study the reciprocals of sine, cosine and tangent (well almost):



$\mathrm{cosec}(x) = \frac{1}{\sin(x)}$ $\qquad$ $\cot(x) = \frac{\cos(x)}{\sin(x)}$

- If we were to define $\cot(x) = \dfrac{1}{\tan(x)}$, the domain and range would change slightly. Do you see why?

Homework 35

Find the domain and sketch the graph of the function
$\sec(x) = \dfrac{1}{\cos(x)}$.

The image of a set under a function

- Let $f\colon A \longrightarrow B$ be a function and let S be a subset of the domain A.
- The image of S under f is the set

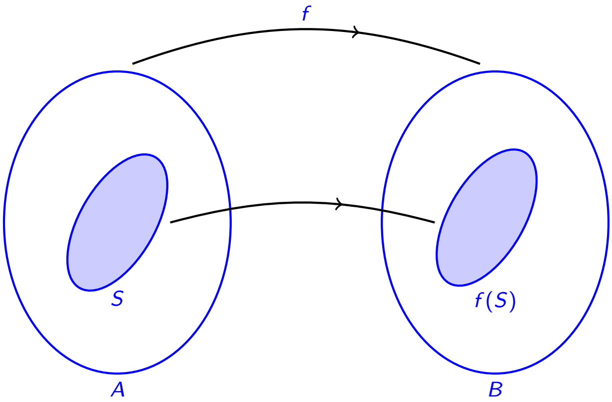
$$\begin{aligned} f(S) &= \{b \in B \mid b = f(s) \text{ for some } s \in S\} \\ &= \{f(s) \mid s \in S\}. \end{aligned}$$

- As a special case, we can consider the image of the entire domain A under f:

$$\begin{aligned} f(A) &= \{b \in B \mid b = f(a) \text{ for some } a \in A\} \\ &= \{f(a) \mid a \in A\}. \end{aligned}$$

- This is known as the range of f, and it is the smallest valid codomain we could use when defining f.

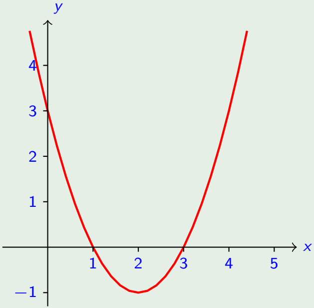
In pictures



Example 2.6

Consider the function $f : \mathbb{R} \longrightarrow \mathbb{R}$ given by

$$f(x) = x^2 - 4x + 3.$$



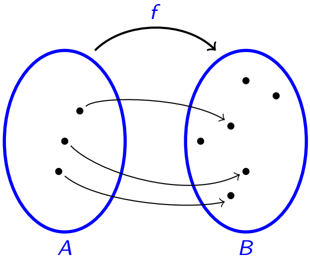
1. What is $f([3,4])$?
2. What is $f([1,4])$?
3. What is the range of f?

Homework 36

Find the image of the interval $(-\frac{\pi}{6}, \pi]$ *under the function* $f : \mathbb{R} \longrightarrow \mathbb{R}$ *given by* $f(x) = \sin(x) + 1$.

Injective (one-one) functions

- A function $f: A \longrightarrow B$ is said to be injective or one-to-one if the equality $f(a_1) = f(a_2)$ implies that $a_1 = a_2$.
- Equivalently, this means that $a_1 \neq a_2$ implies that $f(a_1) \neq f(a_2)$
- In other words, f keeps the elements of A separated: no two distinct elements of A are given the same value.
- Pictorially:



Monotone functions are injective

- A function is called monotone if it is either increasing (order preserving) or decreasing (order reversing) on the whole of its domain.
- If A and B are subsets of $\mathbb{R}$, then every monotone function $f\colon A \longrightarrow B$ is injective.
- Indeed, suppose f is increasing and suppose $a_1 \neq a_2$. There are two possibilities:
  - If $a_1 > a_2$, then $f(a_1) > f(a_2)$ because f is increasing. In particular, $f(a_1) \neq f(a_2)$.
  - If $a_2 > a_1$, then $f(a_2) > f(a_1)$ because f is increasing. In particular, $f(a_1) \neq f(a_2)$.

Example 2.7

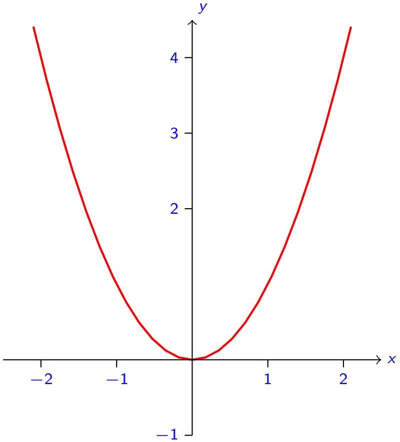
The function $\log\colon (0, \infty) \longrightarrow \mathbb{R}$ is increasing, therefore injective.

Homework 37

Adapt the argument from the previous slide to show that if $f : A \longrightarrow B$ *is decreasing then* f *is injective.*
Using this, or directly, prove that $f : (0, \infty) \longrightarrow \mathbb{R}$ *given by* $f(x) = \frac{1}{x}$ *is injective.*

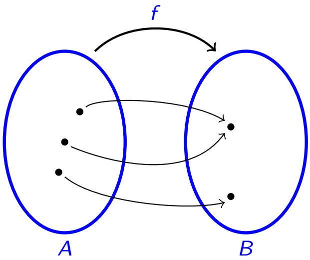
Example 2.8

Consider $f\colon \mathbb{R} \longrightarrow \mathbb{R}$ given by $f(x) = x^2$.
Is f injective? Find the largest intervals on which f is injective.



Surjective (onto) functions

- A function $f : A \longrightarrow B$ is said to be surjective or onto if $f(A) = B$.
- In other words, B is the range of f.
- In other words, for every $b \in B$ there exists at least one $a \in A$ such that $f(a) = b$.
- Pictorially:



Functions with finite domains

- When the domain of a function is finite, we can use a table (instead of a formula) to define it.

Example 2.9

Use a table to define the function $f : \{0, 1, 2\} \longrightarrow \{0, 1, 2\}$ where

$$f(x) = \begin{cases} x + 1 & x \neq 2 \\ 0, & x = 2 \end{cases}.$$

Example 2.10

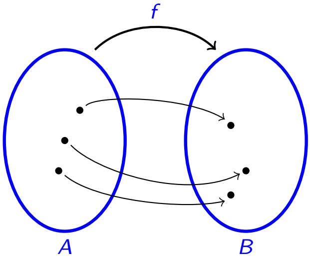
Give definitions of all of the surjective functions
$\{1, 2, 3\} \longrightarrow \{a, b\}$ in a single table.

Homework 38

Find all the surjective functions $\{a, b\} \longrightarrow \{1, 2, 3\}$.

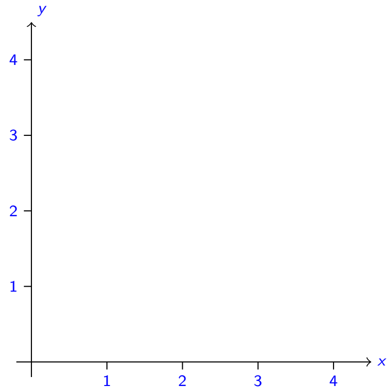
Bijective functions

- A function $f: A \longrightarrow B$ is said to be bijective if it is both injective and surjective.
- In other words, f gives a precise correspondence between the elements of A and the elements of B.
- Pictorially:



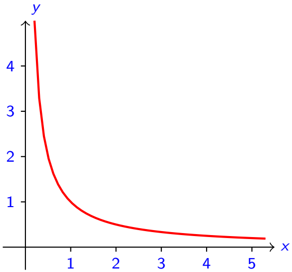
Example 2.11

What's the simplest bijective function $f\colon (0,\infty) \longrightarrow (0,\infty)$ you can think of?



Homework 39

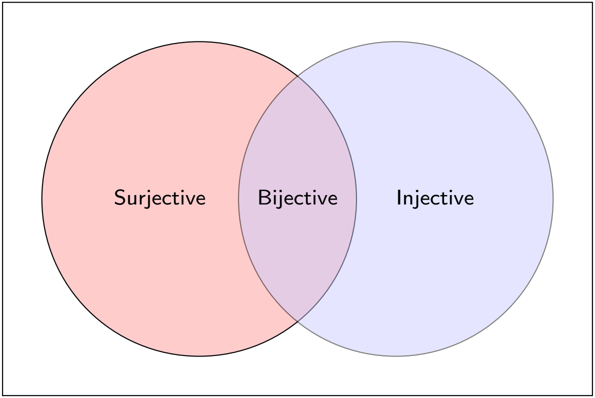
Give an example of a bijective function $g : (0, \infty) \longrightarrow (0, \infty)$ *that is order reversing.*



Everything that could happen does happen

Example 2.12

Give examples of functions $\mathbb{R} \longrightarrow \mathbb{R}$ in every part of the following Venn diagram.



When are functions equal?

- For two functions to be equal, all of the data attached to one of the functions must be identical to the data attached to the other function.
- In other words, saying that $f\colon A \longrightarrow B$ and $g\colon C \longrightarrow D$ are equal functions amounts to guaranteeing that
  - $A = C$;
  - $B = D$;
  - for any $a \in A = C$, we have $f(a) = g(a)$.

Example 2.13

Are the functions $f, g\colon \mathbb{R} \longrightarrow \mathbb{R}$ given by

$$f(x) = \sin(2x)$$
$$g(x) = 2\sin(x)\cos(x)$$

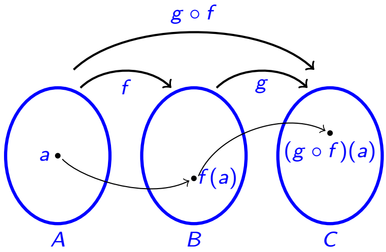
equal?

Composition of functions

- We have seen how to produce new functions from old by applying arithmetic operations.
- But things become really fascinating when we take the resulting values of one function and use them as input for a second function.
- This operation is called composition, and is defined formally as follows: starting with functions $f\colon A \longrightarrow B$ and $g\colon B \longrightarrow C$, the function $g \circ f\colon A \longrightarrow C$ is given by

$$(g \circ f)(x) = g(f(x)).$$

- Pictorially:



Does not commute!

- Based on our experience with multiplication of numbers, it is natural to wonder whether $f \circ g = g \circ f$.
- Even when domains and codomains match up nicely, composition of functions is rarely commutative.

Example 2.14

Consider $f, g \colon \mathbb{R} \longrightarrow \mathbb{R}$ given by $f(x) = x^2$ and $g(x) = e^x$.
Compare $f \circ g$ to $g \circ f$.

Homework 40

Consider $f, g\colon \mathbb{C} \longrightarrow \mathbb{C}$ *given by*

$$\begin{aligned} f(z) &= i \qquad [\textit{constant function}] \\ g(z) &= z. \end{aligned}$$

Compare $f \circ g$ *to* $g \circ f$.

Some good news

Theorem 2.15 (Composition is associative)

If $f\colon A \longrightarrow B$, $g\colon B \longrightarrow C$ *and* $h\colon C \longrightarrow D$, *then*

$$h \circ (g \circ f) = (h \circ g) \circ f$$

so the expression $h \circ g \circ f$ *is unambiguous.*

Example 2.16

Check that associativity holds by evaluating both expressions at an arbitrary point $x \in \mathrm{dom}(f)$.

Inverse of a function

- Let $f \colon A \longrightarrow B$ be a function.
- An inverse of f is a function $g \colon B \longrightarrow A$ such that

$$g \circ f(x) = x$$

and

$$f \circ g(y) = y$$

for every $x \in A$ and every $y \in B$.

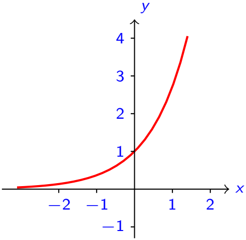
Theorem 2.17

A function f has an inverse if and only if f is bijective and the inverse is unique.

Some familiar examples

1. The function $f\colon \mathbb{R} \longrightarrow \mathbb{R}$ given by $f(x) = 3x - 4$ has inverse $g\colon \mathbb{R} \longrightarrow \mathbb{R}$ given by
$$g(x) = \frac{x+4}{3}$$
2. The function $f\colon \mathbb{R} \longrightarrow (0, \infty)$ given by $f(x) = e^x$ has inverse $g\colon (0, \infty) \longrightarrow \mathbb{R}$ given by $g(x) = \log(x)$.

- Note that in the second example, $f\colon \mathbb{R} \longrightarrow \mathbb{R}$ given by $f(x) = e^x$ does not have an inverse, because it is not surjective.



The importance of being invertible

- The usual way to solve equations is to use inverses.

Example 2.18

Solve the equation $e^{x^3+1} = e^3$.

Example 2.19

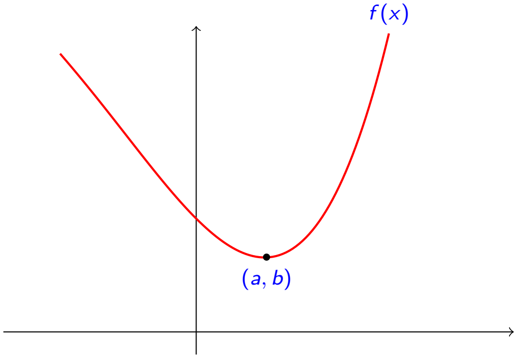
Is the function $f : \mathbb{C} \longrightarrow \mathbb{C}$ defined by $f(z) = \bar{z}$ bijective? If so, what is its inverse?

Becoming surjective

- There are a lot of interesting functions that are neither injective nor surjective.
- By the previous theorem, such functions do not have inverses.
- The failure of surjectivity is easily dealt with:
  - replace the function $f\colon A \longrightarrow B$ with the closely related function $f_1\colon A \longrightarrow \mathrm{range}(f)$ given by the formula $f_1(a) = f(a)$.
- So in the rest of this section we may assume that our functions are surjective.
- This still makes sense theoretically, even when we don't know the range exactly.

Finding ranges precisely can be difficult!

- For $f : \mathbb{R} \longrightarrow \mathbb{R}$ defined by $f(x) = \frac{1}{2}x^2 + e^{-x}$



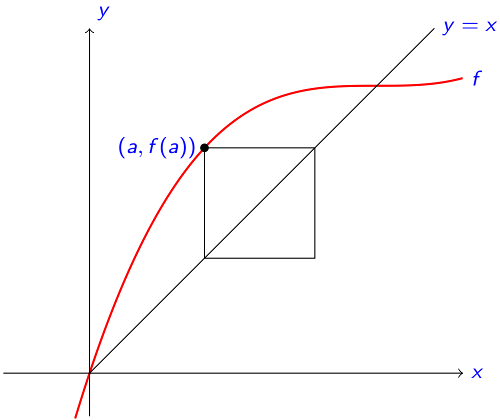
  the range is clearly $[b, \infty)$, but what is b?
- This is why do not want to be forced to define $f : \mathbb{R} \longrightarrow T$ where $T = \text{range}(f)$.

Becoming injective

- To resolve non-injectivity, we have to restrict the function $f\colon A \longrightarrow B$ to some subset C of A on which f is injective.
- In other words, we define a function $f_C\colon C \longrightarrow B$ given by $f_C(c) = f(c)$ for all $c \in C$.
- This function is called the restriction of f to C.
- Then f_C is bijective and therefore has an inverse $g_C\colon B \longrightarrow C$.
- Notes:
  - As indicated by the notation, the inverse g_C depends heavily on the choice of subset $C \subseteq A$.
  - Since we want to make our inverses as powerful as possible when solving equations, we choose C as large as possible.

Finding the inverse graphically

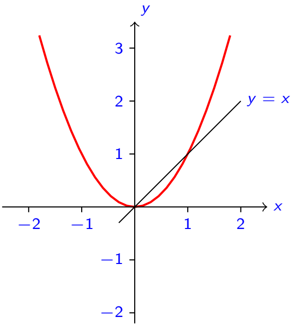
- For bijective $f : I \longrightarrow J$ where $I, J \subseteq \mathbb{R}$, we obtain the graph of the inverse g by reflecting the graph of f across the line $y = x$ (the diagonal).
- Lets see why this works:

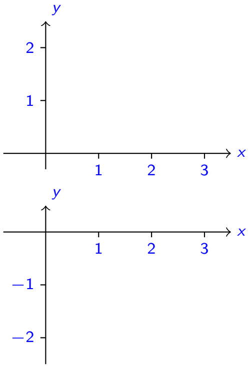


Example 2.20

Consider the function $f : \mathbb{R} \longrightarrow [0, \infty)$ given by $f(x) = x^2$.

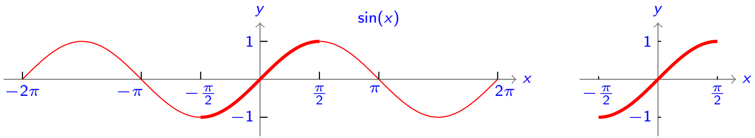
- It is not injective on $\mathbb{R}$, but its restriction $f_{[0,\infty)}$ is injective. What is the inverse of $f_{[0,\infty)}$?
- Is there another possible choice of subset of $\mathbb{R}$? What is the corresponding inverse?





Inverse trigonometric functions

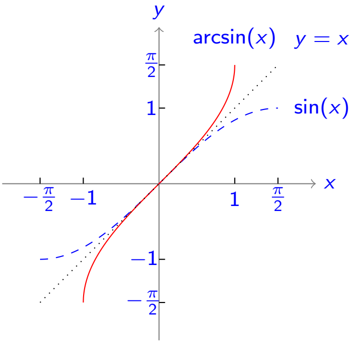
- Clearly, sine is not injective because it is periodic.

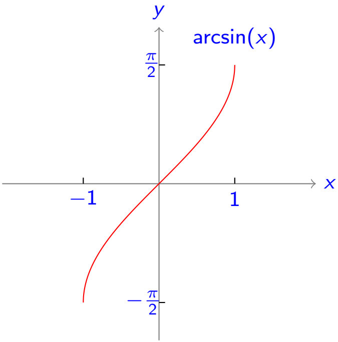


- We need to restrict the domain of sin to a subset (as big as possible) on which it is injective.
- There are infinitely many possible choices, but the convention is to restrict to $[-\frac{\pi}{2}, \frac{\pi}{2}]$.
- The range of this restricted function is still $[-1, 1]$, so $f : [-\frac{\pi}{2}, \frac{\pi}{2}] \longrightarrow [-1, 1]$.

Graph of inverse sine

- Now we can obtain the graph of the inverse function by reflecting through the diagonal $y = x$.
- We call this function arcsine. It is denoted arcsin.
- Its domain is $[-1, 1]$ and its range is $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.





⚠ Our notation

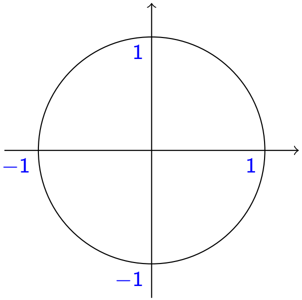
- This function is also referred to in some texts as Sin^{-1} where the capital S denotes the restricted domain and the superscript -1 means inverse function rather than reciprocal.
- In this subject we will only use the arcsin notation.
- This avoids potential confusion between $\mathrm{Sin}^{-1}(x)$ and $\frac{1}{\sin(x)}$.
- Similarly, we never use the notation $\mathrm{Cos}^{-1}(x)$ or $\mathrm{Tan}^{-1}(x)$.

Properties of arcsine

- For any $x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$,
$$\arcsin(\sin(x)) = x$$
- For any $y \in [-1, 1]$,
$$\sin(\arcsin(y)) = y.$$
- When we are solving equations, we can always use
$$\theta = \arcsin(x) \Rightarrow \sin(\theta) = x$$
but we can only use
$$\sin(\theta) = x \Rightarrow \theta = \arcsin(x)$$
if we know for sure that $\theta \in \text{range}(\arcsin) = \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

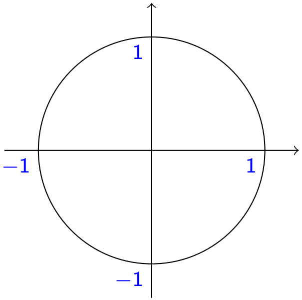
Example 2.21

Evaluate $\arcsin\left(\frac{1}{2}\right)$.



Example 2.22

Simplify $\arcsin\left(\sin\left(\frac{3\pi}{4}\right)\right)$.

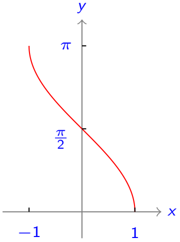


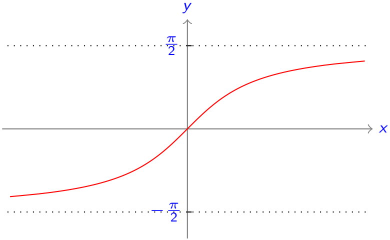
Homework 41

Evaluate $\arcsin\left(\sin\left(\frac{2\pi}{3}\right)\right)$.

Arccosine, arctangent

- It should not come as a surprise that we can perform the same analysis with the trigonometric functions cos and tan.
- Appropriately restricted, they have the following inverses:





$\text{arccos}\colon [-1,1] \longrightarrow [0,\pi]$

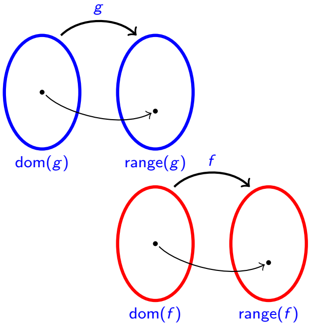
$\text{arctan}\colon \mathbb{R} \longrightarrow \left(-\frac{\pi}{2},\frac{\pi}{2}\right)$

Domain and range of compositions

- For the expression $f(x) = \log(x)$, we know that the largest possible domain is $(0, \infty)$ and the corresponding range is $\mathbb{R}$.
- Similarly, given $g(x) = 1 - x^2$, we know that the largest possible domain is $\mathbb{R}$ and the corresponding range is $(-\infty, 1]$.
- But what happens when we compose these two expressions and consider
$$(f \circ g)(x) = \log(1 - x^2)?$$
- Can we deduce the largest possible domain for $f \circ g$, and the corresponding range?

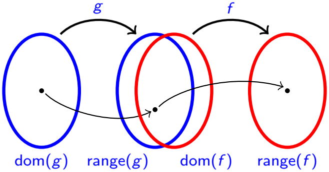
Finding the domain . . .

- Let's rephrase our question in more general terms.
- We are given two functions $f\colon \operatorname{dom}(f) \longrightarrow \operatorname{range}(f)$ and $g\colon \operatorname{dom}(g) \longrightarrow \operatorname{range}(g)$:



- For the composition to have any chance, we must have that $\operatorname{range}(g)$ and $\operatorname{dom}(f)$ intersect.
- Otherwise, the composition $f \circ g$ does not exist (as its domain would be empty).

. . . and the range



- The picture indicates that the implied (or maximal) domain of $f \circ g$ is given by

$$\mathrm{dom}(f \circ g) = \{x \in \mathrm{dom}(g) \mid g(x) \in \mathrm{dom}(f)\},$$

while the range of $f \circ g$ is given by the image

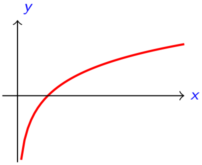
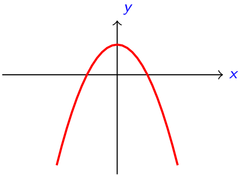
$$\mathrm{range}(f \circ g) = f\big(\mathrm{range}(g) \cap \mathrm{dom}(f)\big).$$

- Note: it is preferable to understand the picture and how to translate it into the two equations, rather than memorise the equations themselves.

Example 2.23

Let's see how this unfolds in the example we started with:

$$(f \circ g)(x) = \log(1 - x^2)$$

Graphs:  

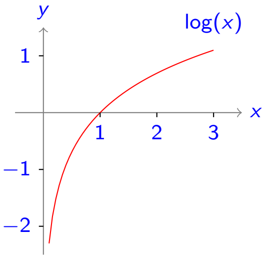
We have

$$\operatorname{dom}(f \circ g) = \{x \in \operatorname{dom}(g) \mid g(x) \in \operatorname{dom}(f)\}$$
$$=$$

Example continued

Finally

$$\begin{aligned}\text{range}(f \circ g) &= f\,(\text{range}(g) \cap \text{dom}(f)) \\ &= \end{aligned}$$



Example 2.24

1. Find the domain and range of the expression

$$\sqrt{\cos(x) - 1}$$

2. How about $\sqrt{\cos(x) - 2}$?

Example continued

Homework 42

Find range of the expression

$$\arcsin\left(\sqrt{2}\,\sin(x)\right)$$

The domain of this function is the union of infinitely many intervals. Describe a typical interval.

Introduction to vectors

There are many different types of quantifiable objects, for example, time, length, speed and height. We can quantify these things with a single (real) number which we call a scalar.

Other quantities, however, require more than just a single magnitude to be completely specified, for example: displacement, velocity and force. These quantities also have a direction. We call these quantities vectors.

Arithmetic and properties of vectors

- Recall, from our work on numbers and sets, the set

$$\mathbb{R}^2 = \mathbb{R} \times \mathbb{R} = \{(a_1, a_2) \mid a_1 \in \mathbb{R} \text{ and } a_2 \in \mathbb{R}\}.$$

- More generally, for any natural number n we consider the set

$$\mathbb{R}^n = \{(a_1, a_2, \ldots, a_n) \mid a_1, a_2, \ldots, a_n \in \mathbb{R}\},$$

  usually called n-dimensional space.
- In this subject, a vector is simply an element of $\mathbb{R}^n$.
- For example:
  - $(3, 1, 0)$ and $(-1, -1, -1)$ are vectors in $\mathbb{R}^3$;
  - $(0, 0)$ and $(1, 5)$ are vectors in $\mathbb{R}^2$;
  - $(0, 1, 2, 3, 4, 5, 6, 7, 8, 9)$ is a vector in $\mathbb{R}^{10}$.

Vector arithmetic

- The following arithmetical operations on vectors are defined component-wise:
  1. Addition:

     $$(a_1, a_2, \ldots, a_n)+(b_1, b_2, \ldots, b_n) = (a_1+b_1, a_2+b_2, \ldots, a_n+b_n)$$

  2. Multiplication by a scalar (a real number):

     $$\lambda(a_1, a_2, \ldots, a_n) = (\lambda a_1, \lambda a_2, \ldots, \lambda a_n) \qquad [\lambda \in \mathbb{R}]$$

  3. Subtraction:

     $$(a_1, a_2, \ldots, a_n)-(b_1, b_2, \ldots, b_n) = (a_1-b_1, a_2-b_2, \ldots, a_n-b_n).$$

Example 2.25

Let $\mathbf{v} = (3, -2)$ and $\mathbf{w} = (1, 5)$. Calculate

(a) $\mathbf{v} + \mathbf{w}$

(b) $\mathbf{v} - \mathbf{w}$

(c) $\mathbf{w} - \mathbf{v}$

(d) $3\mathbf{v}$

Homework 43

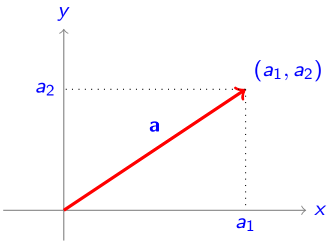
Let $\mathbf{v} = (1, -3, 2)$ and $\mathbf{w} = (-1, 0, -4)$. Repeat (a)–(d) from the previous page.

⚠ One may only perform arithmetic with pairs of vectors of the same dimension! For instance, the following does not have any mathematical meaning:

$$(1, 2, 3, 4) + (5, 6) = ???$$

Representation as arrows

- In applications, it often helps to visualise vectors as directed line segments or arrows.
- This is particularly useful when developing physical models.
- For example, $(a_1, a_2) \in \mathbb{R}^2$ is represented by the arrow from the origin $(0, 0)$ to (a_1, a_2).



- The zero vector $(0, 0, \ldots, 0)$ is simply a point. It has length 0, but has no specific direction.

Position, position, position

- It is sometimes convenient to interpret arrows that do not start at the origin as vectors.
- Given points $A = (x_1, x_2, \ldots, x_n)$ and $B = (y_1, y_2, \ldots, y_n)$ in $\mathbb{R}^n$, the vector $\overrightarrow{AB}$ is defined by

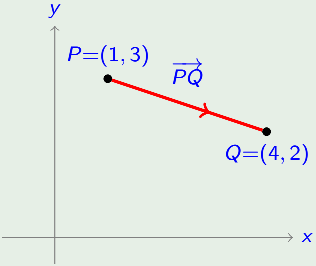
$$\begin{aligned} \overrightarrow{AB} &= (y_1, y_2, \ldots, y_n) - (x_1, x_2, \ldots, x_n) \\ &= (y_1 - x_1, y_2 - x_2, \ldots, y_n - x_n) \end{aligned}$$

- In this notation, a point $P \in \mathbb{R}^n$ may be written as $\overrightarrow{OP}$, where $O = (0, 0, \ldots, 0)$ denotes the origin.
- $\overrightarrow{OP}$ is sometimes called the position vector of P and we may write

$$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}.$$

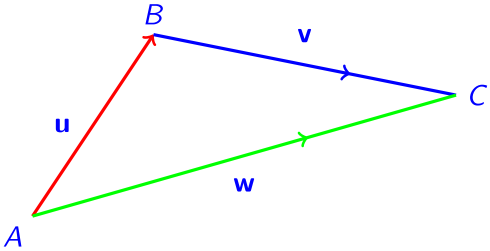
Example 2.26

Express the directed line segment $\overrightarrow{PQ}$ shown in the picture as a vector by subtracting position vectors.



Geometric addition and subtraction

- Moving the base point away from the origin allows a powerful geometric interpretation of vector operations.
- Suppose we have two vectors $\mathbf{u} = \overrightarrow{AB}$ and $\mathbf{v} = \overrightarrow{BC}$.
- If we move the base of $\mathbf{v}$ to the end of $\mathbf{u}$ ...

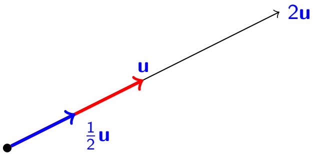


...the vector $\mathbf{w} = \mathbf{u} + \mathbf{v}$ is represented by the arrow starting at the base of $\mathbf{u}$ and ending at the tip of $\mathbf{c}$.

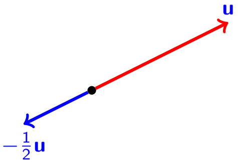
- Once we know how to add vectors geometrically, we can subtract $\mathbf{v}$ from $\mathbf{u}$ simply by adding $-\mathbf{v}$:

Geometric scalar multiplication

- Multiplying by $\lambda > 0$ stretches or shrinks $\mathbf{u}$.



- Multiplying by $\lambda < 0$ stretches or shrinks $\mathbf{u}$ and reflects through 0.

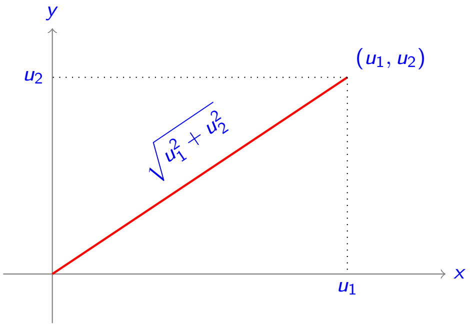


Definition 2.27 (Parallel vectors)

We say that non-zero vectors $\mathbf{u}$ and $\mathbf{v}$ are parallel if $\mathbf{u} = \lambda\mathbf{v}$ for some $\lambda \neq 0$.

Length of a vector

- Intuitively, the length of a vector is the length of the line segment that represents the vector graphically.
- In $\mathbb{R}^2$, we can find this length by using Pythagoras's Theorem.

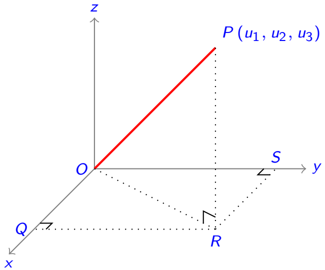


- The length of $\mathbf{u}$ (also called the norm of $\mathbf{u}$), denoted $\|\mathbf{u}\|$, is given by:

$$\|\mathbf{u}\| = \sqrt{u_1^2 + u_2^2}$$

Higher dimensions

- We can extend this definition to a vector $\mathbf{u}$ in $\mathbb{R}^n$ (for any $n \in \mathbb{N}$):
$$\|\mathbf{u}\| = \sqrt{u_1^2 + u_2^2 + \cdots + u_n^2}$$
- Let's check that this makes sense in $\mathbb{R}^3$:



- The diagram gives $\left\|\overrightarrow{OR}\right\| = \sqrt{u_1^2 + u_2^2}$.

Example 2.28

Use the formula for $\left\|\overrightarrow{OR}\right\|$ to find $\left\|\overrightarrow{OP}\right\|$.

Example 2.29

Find the length of the vectors (a) $(1,2)$ (b) $(-1,3,-2)$.

Homework 44

Find the length of the vectors (a) $(-2,-1)$ *(b)* $(-3,6,-2)$.

Properties of length

1. for any $\mathbf{u} \in \mathbb{R}^n$, $\|\mathbf{u}\| \geq 0$ and $\|\mathbf{u}\| = 0$ precisely if $\mathbf{u} = \mathbf{0}$.
2. $\|\lambda \mathbf{u}\| = |\lambda| \|\mathbf{u}\|$ for any $\lambda \in \mathbb{R}$ and any vector $\mathbf{u}$.
3. $\|\mathbf{u} + \mathbf{v}\| \leq \|\mathbf{u}\| + \|\mathbf{v}\|$ for any vectors $\mathbf{u}$ and $\mathbf{v}$.

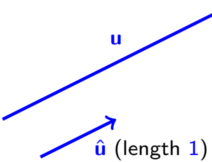
Example 2.30

Prove Property 2 for a general vector $\mathbf{u} = (u_1, u_2, u_3) \in \mathbb{R}^3$.

Unit vectors

- Any vector of length 1 is called a unit vector.
- To construct a unit vector in the same direction as a nonzero vector $\mathbf{u}$, we simply multiply $\mathbf{u}$ by the scalar $\frac{1}{\|\mathbf{u}\|}$.
- We denote this unit vector in the direction of $\mathbf{u}$ by $\hat{\mathbf{u}}$:

$$\hat{\mathbf{u}} = \frac{1}{\|\mathbf{u}\|}\mathbf{u} = \frac{\mathbf{u}}{\|\mathbf{u}\|}$$



- We can easily check that $\hat{\mathbf{u}}$ has length 1:

Example 2.31

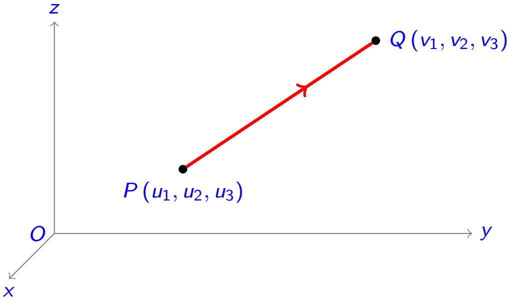
Is the vector $\mathbf{v} = (1, -2, 1)$ a unit vector? If it is not, find a unit vector in the same direction as $\mathbf{v}$.

Example 2.32

Let $\mathbf{v}$ be the vector from $A(2, 0, -1)$ to $B(1, 2, -3)$. Find two unit vectors parallel to $\mathbf{v}$.

Distance between two points

- If $P\,(u_1, u_2, u_3)$ and $Q\,(v_1, v_2, v_3)$ are two points in $\mathbb{R}^3$, then the distance between them is the length of the vector $\overrightarrow{PQ}$:



- We know that $\overrightarrow{PQ} = (v_1 - u_1, v_2 - u_2, v_3 - u_3)$, so the distance between P and Q is

$$\left\|\overrightarrow{PQ}\right\| = \sqrt{(v_1 - u_1)^2 + (v_2 - u_2)^2 + (v_3 - u_3)^2}.$$

- Clearly we can extend this definition to $\mathbb{R}^n$ in the obvious way.

Example 2.33

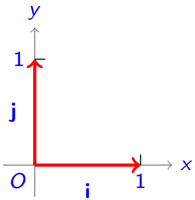
Find the distance between the points $P(-2, 1, 0)$ and $Q(3, -1, 1)$.

Homework 45

Find the distance between the points $A(-3, 1, -1)$ *and* $B(2, 0, -1)$.

Standard unit vectors in $\mathbb{R}^2$

- The unit vectors $\mathbf{i} = (1, 0)$ and $\mathbf{j} = (0, 1)$ are called the standard unit vectors in $\mathbb{R}^2$.



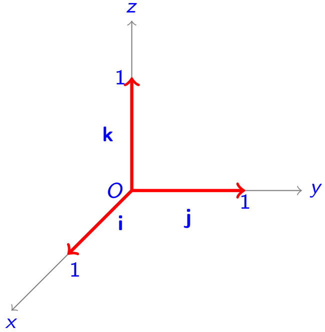
- Every vector in $\mathbb{R}^2$ can be written in terms of $\mathbf{i}$ and $\mathbf{j}$.

Example 2.34

Express the vectors $\mathbf{v} = (1, 2)$ and $\mathbf{u} = (-1, 3)$ in terms of $\mathbf{i}$ and $\mathbf{j}$.

Standard unit vectors in $\mathbb{R}^3$

- In $\mathbb{R}^3$, the standard unit vectors are $\mathbf{i} = (1, 0, 0)$, $\mathbf{j} = (0, 1, 0)$ and $\mathbf{k} = (0, 0, 1)$.



Homework 46

Write $(-3, 1, 5) \in \mathbb{R}^3$ in terms of $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$.

The scalar product

We know how to multiply a vector by a scalar but have not discussed the possibility of multiplying two vectors. In vector algebra there are several types of vector product. One is called the scalar product (or dot product), which we discuss now. A key application of the scalar product is to the calculation of angles between vectors.
Another product of vectors, the cross product, is specific to vectors in $\mathbb{R}^3$ and will be introduced in *MAST10007 Linear Algebra*.

Scalar product definition

- Let $\mathbf{u}$ and $\mathbf{v}$ be vectors in $\mathbb{R}^n$ (for some $n \in \mathbb{N}$), with components
  $\mathbf{u} = (u_1, u_2, \ldots, u_n)$ and $\mathbf{v} = (v_1, v_2, \ldots, v_n)$.
- The scalar product of $\mathbf{u}$ and $\mathbf{v}$ is defined as:

$$\mathbf{u} \cdot \mathbf{v} = u_1 v_1 + u_2 v_2 + \ldots + u_n v_n.$$

- Note that the scalar product is a sum of real numbers, therefore it is a scalar quantity (not a vector)!
- For instance, in $\mathbb{R}^3$ we have $\mathbf{u} = (u_1, u_2, u_3)$, $\mathbf{v} = (v_1, v_2, v_3)$ and

$$\mathbf{u} \cdot \mathbf{v} = u_1 v_1 + u_2 v_2 + u_3 v_3.$$

Example 2.35

Let $\mathbf{u} = (2, 3, -1)$ and $\mathbf{v} = (4, 5, 0)$. Calculate $\mathbf{u} \cdot \mathbf{v}$.

Homework 47

Calculate $\mathbf{u} \cdot \mathbf{v}$ *if* $\mathbf{u} = (3, 2, 1)$ *and* $\mathbf{v} = (1, 1, 3)$.

Properties of the scalar product

Theorem 2.36

Let $\mathbf{u}$, $\mathbf{v}$ *and* $\mathbf{w}$ *be vectors in* $\mathbb{R}^n$ *and let* $\lambda \in \mathbb{R}$.

1. *The scalar product is commutative; that is*
$$\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}.$$
2. *The scalar product is distributive; that is*
$$\mathbf{v} \cdot (\mathbf{u} + \mathbf{w}) = \mathbf{v} \cdot \mathbf{u} + \mathbf{v} \cdot \mathbf{w}.$$
3. $\lambda(\mathbf{u} \cdot \mathbf{v}) = (\lambda \mathbf{u}) \cdot \mathbf{v} = \mathbf{u} \cdot (\lambda \mathbf{v})$.
4. *The scalar product of a vector with itself yields* $\mathbf{u} \cdot \mathbf{u} = \|\mathbf{u}\|^2$.

A couple of proofs

Example 2.37

Prove Parts 3 and 4 of Theorem 2.36 for vectors

$$\mathbf{u} = (u_1, u_2, \ldots, u_n), \mathbf{v} = (v_1, v_2, \ldots, v_n) \in \mathbb{R}^n$$

and $\lambda \in \mathbb{R}$.

Example continued

Homework 48

Prove Part 2 of Theorem 2.36 for general vectors

$$\begin{aligned}\mathbf{u} &= (u_1, u_2, \ldots, u_n),\\ \mathbf{v} &= (v_1, v_2, \ldots, v_n),\\ \mathbf{w} &= (w_1, w_2, \ldots, w_n) \in \mathbb{R}^n\end{aligned}$$

and $\lambda \in \mathbb{R}$.

Example 2.38

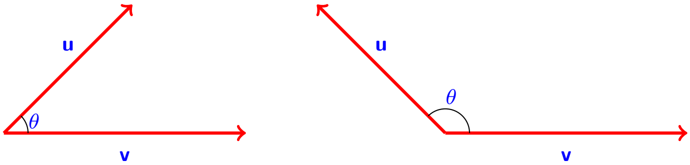
Use relevant parts of Theorem 2.36 to simplify

$$\|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 - \|\mathbf{v} - \mathbf{u}\|^2$$

for any vectors $\mathbf{v}, \mathbf{u} \in \mathbb{R}^n$.

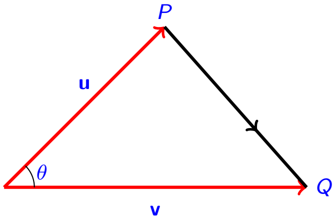
Scalar product and the angle between vectors

- Let $\mathbf{u}$ and $\mathbf{v}$ be non-zero vectors in $\mathbb{R}^n$ and assume that they are positioned so that their tails meet.
- We define the angle between $\mathbf{u}$ and $\mathbf{v}$ to be the angle θ such that $0 \leq \theta \leq \pi$ as pictured below.



- The notion of angle between $\mathbf{u}$ and $\mathbf{v}$ is closely related to the scalar product $\mathbf{u} \cdot \mathbf{v}$.
- To see this, we need the Law of Cosines, which we will prove in a Practice Class.

Law of Cosines



- The Law of Cosines says that

$$\|\overrightarrow{PQ}\|^2 = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 - 2\|\mathbf{u}\|\|\mathbf{v}\|\cos(\theta).$$

- Since $\overrightarrow{PQ} = -\mathbf{u} + \mathbf{v} = \mathbf{v} - \mathbf{u}$, this can be rearranged to give:

$$\cos(\theta) = \frac{\|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 - \|\mathbf{v} - \mathbf{u}\|^2}{2\|\mathbf{u}\|\|\mathbf{v}\|} = \frac{2\mathbf{u} \cdot \mathbf{v}}{2\|\mathbf{u}\|\|\mathbf{v}\|} = \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\|\|\mathbf{v}\|}$$

using Example 2.38.

Theorem 2.39

If θ is the angle in $[0, \pi]$ between non-zero vectors $\mathbf{u}$ *and* $\mathbf{v}$*, then:*

$$\cos(\theta) = \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\|\|\mathbf{v}\|}.$$

Example 2.40

Find the angle between $\mathbf{u} = (1, -2, 0)$ and $\mathbf{v} = (3, 1, -2)$.

Homework 49

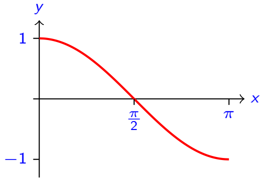
Find the angle between $\mathbf{u} = (-3, 0, 1)$ *and* $\mathbf{v} = (2, -2, 1)$.

Acute, obtuse or somewhere in between?

- We have seen that if θ is the angle in $[0, \pi]$ between non-zero vectors $\mathbf{u}$ and $\mathbf{v}$, then:

$$\cos(\theta) = \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\|\|\mathbf{v}\|}$$

  and $\|\mathbf{u}\|\|\mathbf{v}\| > 0$.
- The graph of $\cos(\theta)$ restricted to $[0, \pi]$ therefore tells us that:
  - θ is acute precisely if $\mathbf{u} \cdot \mathbf{v} > 0$.
  - θ is obtuse precisely if $\mathbf{u} \cdot \mathbf{v} < 0$.
  - θ is $\dfrac{\pi}{2}$ precisely if $\mathbf{u} \cdot \mathbf{v} = 0$.



- In the case where $\mathbf{u} \cdot \mathbf{v} = 0$ (so $\mathbf{u}$ and $\mathbf{v}$ are perpendicular), the Law of Cosines reduces to Pythagoras' Theorem.

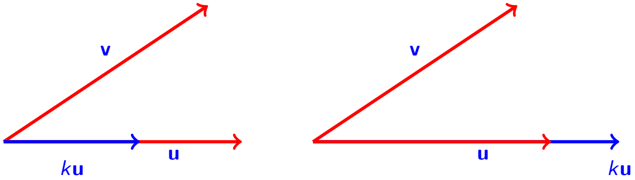
Vector projections

In many applications involving vectors, it is useful to choose a specific vector $\mathbf{u}$ as a reference direction and to then express other vectors in terms of $\mathbf{u}$ as the sum of a component in the direction of $\mathbf{u}$ and another component perpendicular to $\mathbf{u}$.

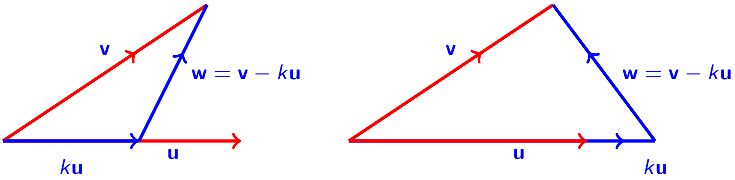
We now study the general technique for achieving this decomposition, known as vector projection. A key application of this idea is finding the closest point on a line L to a given point P.

Breaking up is never easy . . .

- For non-zero vectors $\mathbf{u}$ and $\mathbf{v}$ an any vector $k\mathbf{v}$ parallel to $\mathbf{u}$ . . .



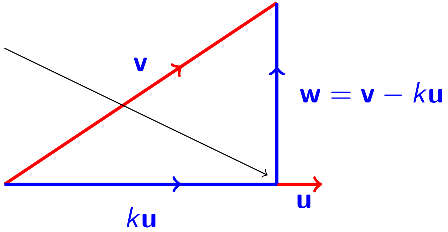
- . . . we can express $\mathbf{v}$ as $k\mathbf{u} + \mathbf{w}$ by letting $\mathbf{w} = \mathbf{v} - k\mathbf{u}$.



...but there should be a sweet spot ...

- We want to write $\mathbf{v}$ as the sum of a vector $\mathbf{v}_{\parallel}$ parallel to $\mathbf{u}$ and another vector $\mathbf{v}_{\perp}$ perpendicular to $\mathbf{u}$.

Choose k to make angle $\frac{\pi}{2}$.



- Provided we can find the unique k for which $k\mathbf{u}$ is perpendicular to $\mathbf{w}$, we can define:

$$\mathbf{v}_{\parallel} = k\mathbf{v},$$
$$\mathbf{v}_{\perp} = \mathbf{w} = \mathbf{v} - k\mathbf{u}$$

...if only we can find it

- We use the fact that that vectors are perpendicular precisely when their dot product is 0 to find k.
- We simply solve:

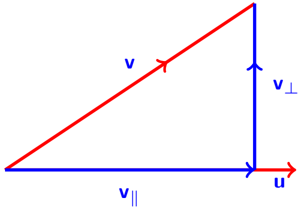
$$\begin{aligned}
&(k\mathbf{u}) \cdot (\mathbf{v} - k\mathbf{u}) = 0 \\
&\Rightarrow k(\mathbf{u} \cdot (\mathbf{v} - k\mathbf{u})) = 0 \\
&\Rightarrow k = 0 \quad \text{or} \quad \mathbf{u} \cdot (\mathbf{v} - k\mathbf{u}) = 0
\end{aligned}$$

- We can ignore the solution $k = 0$. Do you see why?
- So we seek the solution of

$$\mathbf{u} \cdot (\mathbf{v} - k\mathbf{u}) = 0.$$

The vector projections

- Once we solve for k, an easy calculation gives $\mathbf{v}_{\parallel}$ and $\mathbf{v}_{\perp}$.



- **In Summary:** The parallel and perpendicular projections of $\mathbf{v}$ onto $\mathbf{u}$ are given by

$$\mathbf{v}_{\parallel} = k\mathbf{u},$$
$$\mathbf{v}_{\perp} = \mathbf{v} - k\mathbf{u}$$

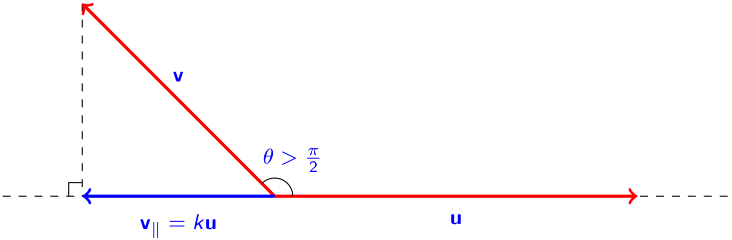
  where k is the unique solution of $\mathbf{u} \cdot (\mathbf{v} - k\mathbf{u}) = 0$.
- In a tutorial, we will use properties of the scalar product to find a general formula for k.

Example 2.41

Let $\mathbf{u} = (3, 1, -2)$ and $\mathbf{v} = (1, 0, 5)$. Find the parallel and perpendicular projections of $\mathbf{v}$ onto $\mathbf{u}$.

Projecting backwards

- In the previous example, we obtained a negative k value.
- This means that the vector $k\mathbf{u}$ points in the opposite direction to $\mathbf{u}$, so the angle between $\mathbf{u}$ and $\mathbf{v}$ is obtuse.



- When k is positive, the angle θ is acute.

Homework 50

Let $\mathbf{u} = (1, 1, 0)$ *and* $\mathbf{v} = (1, -2, 3)$. *Find the vector projections of* $\mathbf{v}$ *onto* $\mathbf{u}$.

Points and lines

- We can use vector projections to find the closest point on a line L to a given point P.

Example 2.42

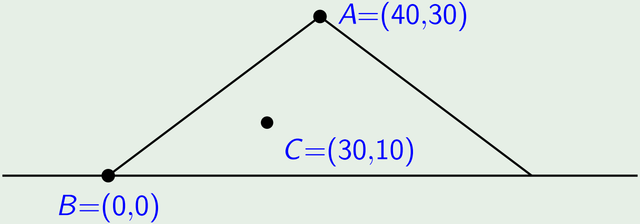
Let $P = (1, 3)$ and let L be the line passing through $(0, 1)$ and $(4, 2)$. Find the closest point on L to P and hence find the distance from P to L.

Example continued

A typical application

Example 2.43

Indiana Jones believes there is a secret chamber in an ancient pyramid at the position C shown in the diagram (measurements in metres).



Indiana plans to dig a shaft from the surface of the pyramid to the chamber. To minimise the length of the shaft

1. Where should he start digging?
2. How long will the shaft be?

Example continued

Parametric curves

Up to this point we have seen vectors of the form $\mathbf{u} = x\mathbf{i} + y\mathbf{j}$, where the components of the vectors have been constant. Suppose now that these components are functions of time; that is,

$$\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j}, \qquad \text{where } t \in \mathbb{R}.$$

Here $\mathbf{r}$ is a function $\mathbf{r}\colon \mathbb{R} \longrightarrow \mathbb{R}^2$, since for each $t \in \mathbb{R}$, $\mathbf{r}(t)$ is a vector in $\mathbb{R}^2$. This is called a vector-valued function of a real variable (which we often think of as time).

The functions $x(t)$ and $y(t)$ are called parametric equations, since they depend on the parameter t. The resulting curve that $\mathbf{r}(t)$ traces out in $\mathbb{R}^2$ is called a parametric curve. Such vector-valued functions are particularly useful in applications, for example, in describing the motion of a particle at any time t.

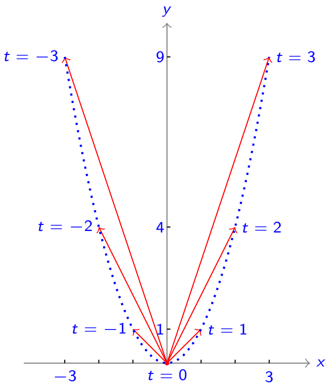
A simple example

- Consider the formula $\mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j}$, for $t \in \mathbb{R}$. In this example $x(t) = t$ and $y(t) = t^2$.
- To sketch the graph of $\mathbf{r}(t)$, we need to know the values of x and y for different values of t.
- A simple way to do this is to construct a table to give us a rough picture of what the curve looks like.

t	$x(t) = t$	$y(t) = t^2$	$\mathbf{r}(t)$
-3	-3	9	$-3\mathbf{i} + 9\mathbf{j}$
-2	-2	4	$-2\mathbf{i} + 4\mathbf{j}$
-1	-1	1	$-\mathbf{i} + \mathbf{j}$
0	0	0	$\mathbf{0}$
1	1	1	$\mathbf{i} + \mathbf{j}$
2	2	4	$2\mathbf{i} + 4\mathbf{j}$
3	3	9	$3\mathbf{i} + 9\mathbf{j}$

- So for each point t we get a vector $\mathbf{r}(t)$.

Joining the dots



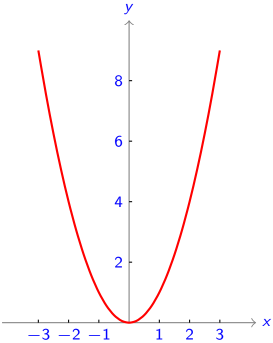
- The path (actually range($\mathbf{r}$)) of $\mathbf{r}$ is the curve traced out by the heads of the vectors as t varies through dom($\mathbf{r}$)
- In the diagram the path is indicated by the dotted line.

Finding the equation of a path

- Sometimes, we can find the Cartesian equation of the path of a parametric curve by solving the equations simultaneously.
- The aim is to eliminate the parameter t and hence obtain a relationship between x and y.
- For our example $\mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j}$, the x component is t and the y component is t^2, so:

  $$t = x \quad \text{and} \quad y = t^2 \Rightarrow y = x^2.$$

  so the Cartesian equation of the curve is $y = x^2$.



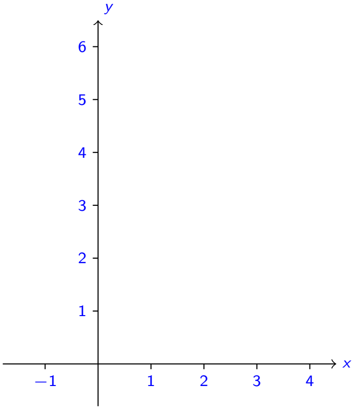
- In many cases, it is difficult or impossible to find a Cartesian equation for the path of a parametric curve.

Example 2.44

Find the equation of the path of a particle whose position is given by

$$\mathbf{r}(t) = (t^2 - t)\mathbf{i} + 3t\,\mathbf{j}, \qquad \text{for } t \geq 0.$$

Sketch the path, indicating the direction of increasing t.



Example continued

Homework 51

Find the equation of the path of a particle whose position is given by $\mathbf{r}(t) = t\mathbf{i} + (t^2+1)\mathbf{j}$, *for* $t \geq 0$.

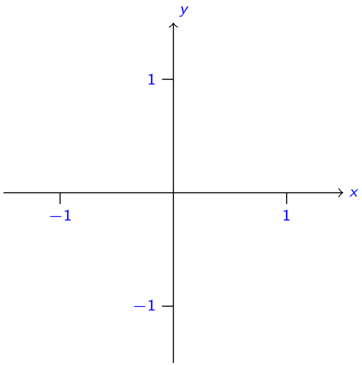
Example 2.45

Find the equation of the path of a particle whose position is given by

$$\mathbf{r}(t) = \cos(t)\mathbf{i} + \sin(t)\mathbf{j}, \qquad \text{for } t \in \mathbb{R}.$$

Sketch the graph of the path.

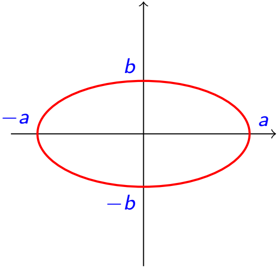
Example continued



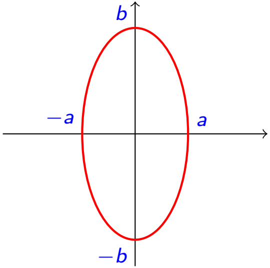
Elliptical comments

- In many applications (E.G. planetary motion), the path of the parametric curve is an ellipse.
- For $a, b > 0$, an ellipse centred at the origin with x-intercepts $-a$ and a and y-intercepts $-b$ and b has Cartesian equation:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \qquad (a, b > 0).$$



$a > b$



$a < b$

- a and b are sometimes called the axes of the ellipse.

Example 2.46

Show that for $a, b \neq 0$ the path of $\mathbf{r} : \mathbb{R} \longrightarrow \mathbb{R}^2$ defined by $\mathbf{r}(t) = a\cos(t)\mathbf{i} + b\sin(t)\mathbf{j}$ is an ellipse with axes $|a|$ and $|b|$.

Example 2.47

Find the equation of the path of a particle whose position is given by $\mathbf{r}(t) = \cos(t)\mathbf{i} + 2\sin(t)\mathbf{j}$, for $t \in \mathbb{R}$.

Paths vs parameterisations

- In Example 2.45, the path was a circle, which we could have simply sketched from the Cartesian equation $x^2 + y^2 = 1$.
- However, the parametric curve tells us much more about the motion of the particle. E.G.:
  - At time $t = 0$ the particle is at the point $(1, 0)$.
  - As t increases from 0, $x = \cos(t)$ decreases, while $y = \sin(t)$ increases (consider the graphs of $\cos$ and $\sin$).
  - The point $\mathbf{r}(t) = \cos(t)\mathbf{i} + \sin(t)\mathbf{j}$ is at angle t from the positive x-axis, so as t increases, $\mathbf{r}(t)$ moves anticlockwise
  - Hence the particle takes 2π to travel around the circle, so the period of motion is 2π.
  - etc ...
- In summary, the parametric curve tells us how the particle is moving.
- In the next section we will learn how to calculate the speed and velocity of the particle as it moves along the path.

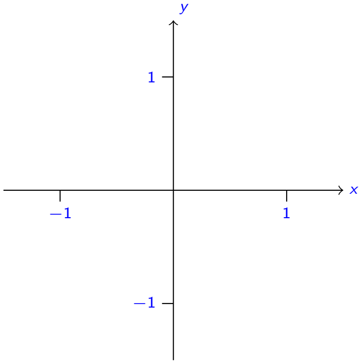
Example 2.48

Find the equation of the path of a particle whose position is given by

$$\mathbf{r}(t) = \sin(2t)\mathbf{i} + \cos(2t)\mathbf{j}, \qquad \text{for } t \in \mathbb{R}.$$

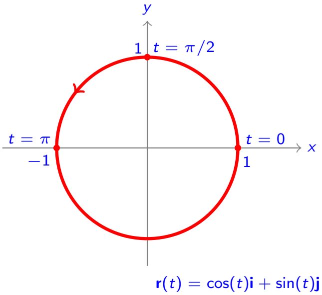
and hence sketch the path.

Example continued

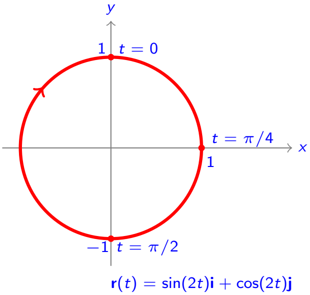


Same path, different motion

- We described the same path as in Examples 2.45 and 2.48, but for any given t we may not be at the same point on the path.
- For instance, when $t = 0$, in Example 2.45 we are at the point $(1, 0)$, whereas in Example 2.48 we are at $(0, 1)$.



Example 2.45:
Anticlockwise motion with period 2π



Example 2.48:
Clockwise motion with period π.

Example 2.49

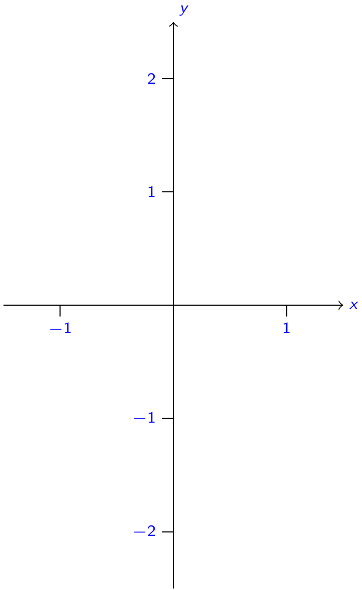
The motion of a particle is described by $\mathbf{r} : \mathbb{R} \longrightarrow \mathbb{R}^2$.

$$\mathbf{r}(t) = \cos(2t)\mathbf{i} - 2\sin(2t)\mathbf{j}, \qquad \text{for } t \geq 0.$$

Sketch:

1. the Cartesian equation of the path;
2. the position of the particle at times
   (i) $t = 0$ (ii) $t = \frac{\pi}{4}$ (iii) $t = \frac{\pi}{2}$;
3. the time taken by the particle to return to its original position;
4. the direction of motion.

Example continued



Crash!

- In many applications, we are interested in detecting collisions between particles following different parametric curves

$$\mathbf{r}_1(t) = x_1(t)\mathbf{i} + y_1(t)\mathbf{j}$$
$$\mathbf{r}_2(t) = x_2(t)\mathbf{i} + y_2(t)\mathbf{j}$$

- Collisions occur at t-values for which $\mathbf{r}_1(t) = \mathbf{r}_2(t)$, so if to find collisions we must determine any t-values for which both

$$x_1(t) = x_2(t) \quad \text{and} \quad y_1(t) = y_2(t).$$

- There may be other points where the paths cross, but where the particles do not collide since they are not there at the same time.
- Imagine two people walking around a room. Their paths may cross, but they will only bump into each other if they are at the same point at the same time.

Example 2.50

The motion of two particles is given by the equations

$$\mathbf{r}_1(t) = (t+1)\mathbf{i} + (t^2 - 4t)\mathbf{j} \quad \text{and} \quad \mathbf{r}_2(t) = (2t)\mathbf{i} + (6t-9)\mathbf{j}.$$

Determine:

(a) the point(s) at which the particles collide;

(b) the distance between the particles when $t = 2$.

Example continued

Example continued

- Note: Since the particles collide at the point $(2, -3)$ (when $t = 1$), the two paths must cross at this point.

Homework 52

Find the Cartesian equations of the two paths in this example and hence find all points where the paths cross.

Topic 3 - Differential Calculus

In this section we advance our knowledge of differential calculus. This is a fundamental topic in calculus, and is the foundation for our next topics of integral calculus and differential equations.

All these topics have major real world applications in science, engineering and economics. As an example, we conclude our discussion of differentiation of parametric functions with the application to projectile motion.

Differential Calculus

- Differentiation and stationary points 339
- Second and higher order derivatives 369
- Asymptotes and graph sketching 389
- Implicit differentiation 405
- Derivatives of inverse functions 421
- Differentiating parametric curves 437
- Application: Projectile motion 463

Differentiation from first principles

- We start by reviewing the definition of a derivative as a limit.

Definition 3.1

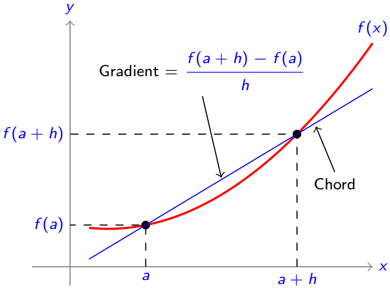
Let $f : I \longrightarrow \mathbb{R}$, where I is an open interval. The derivative of f at $a \in I$ is the gradient of the tangent line to the graph of f at $(a, f(a))$, given by the limit:

$$f'(a) = \lim_{h \longrightarrow 0} \frac{f(a+h) - f(a)}{h}.$$

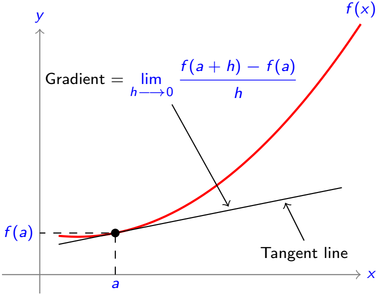
If this limit exists, f is said to be differentiable at a.

In pictures

- $$\frac{f(a+h)-f(a)}{h}$$
  gives the gradient of the chord connecting $(a, f(a))$ and $(a+h, f(a+h))$.



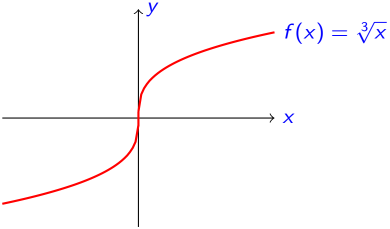
- Taking the limit as
  $$h \longrightarrow 0$$
  gives the gradient of the tangent line to graph at $(a, f(a))$.



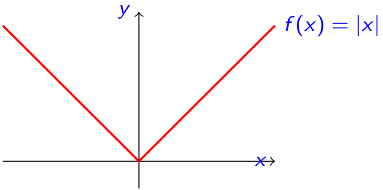
Continuous but non-differentiable

- There are two typical reasons why a continuous function fails be differentiable at a point in its domain:

1. The tangent line is vertical.



2. The graph has a "sharp" point.



Well known derivatives

- We will frequently use the standard derivatives you learned at school, and the key rules of differentiation.
- Another notation for $f'(x)$ is $\dfrac{d}{dx}[f(x)]$. This is particularly useful when $f(x)$ does not have a well known "name".

$$
\begin{aligned}
&\frac{d}{dx}[x^a] = ax^{a-1} && \text{for every } a \in \mathbb{R} \setminus \{0\} \\
&\frac{d}{dx}[\sin(x)] = \cos(x) && (\text{or } \sin'(x) = \cos(x)) \\
&\frac{d}{dx}[\cos(x)] = -\sin(x) && (\text{or } \cos'(x) = -\sin(x)) \\
&\frac{d}{dx}[e^x] = e^x && (\text{or } \exp'(x) = \exp(x)) \\
&\frac{d}{dx}[\log(x)] = \frac{1}{x} && (\text{or } \log'(x) = \frac{1}{x})
\end{aligned}
$$

- We will soon add derivatives of other trigonometric functions to our list.

Linearity of Derivatives

Theorem 3.2 (Differentiation is a linear operator)

If f and g are both differentiable at x and $c \in \mathbb{R}$:

$$\frac{d}{dx}\left[cf(x)\right] = c\frac{d}{dx}\left[f(x)\right],$$
$$\frac{d}{dx}\left[f(x)+g(x)\right] = \frac{d}{dx}\left[f(x)\right] + \frac{d}{dx}\left[g(x)\right].$$

Example 3.3

If $f : \mathbb{R} \setminus \{0\} \longrightarrow \mathbb{R}$ with $f(x) = 2e^x + 3x^{-7}$, find $f'(x)$.

Homework 53

If $f : [0, \infty) \longrightarrow \mathbb{R}$ *with* $f(x) = -9\cos(x) + 2\sqrt{x}$*, find* $f'(x)$.

Product Rule

Theorem 3.4 (Product rule)

If f and g are differentiable at x, then so is the product fg, and:

$$\begin{aligned}\frac{d}{dx}\left[f(x)g(x)\right] &= \frac{d}{dx}\left[f(x)\right]g(x) + f(x)\frac{d}{dx}\left[g(x)\right] \\ &= f'(x)g(x) + f(x)g'(x).\end{aligned}$$

Example 3.5

For $f\colon (0,\infty) \longrightarrow \mathbb{R}$ with $f(x) = (x^4 - 3x)\log(x)$, find $f'(x)$.

Example continued

Homework 54

For $f : \mathbb{R} \longrightarrow \mathbb{R}$ *with* $f(x) = e^x(2x^3 - x^2)$, *find* $f'(x)$.

Quotient Rule

Theorem 3.6 (Quotient rule)

If f and g are differentiable at x and if $g(x) \neq 0$, then $\frac{f}{g}$ is differentiable at x and:

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{f'(x)g(x) - f(x)g'(x)}{(g(x))^2}.$$

Example 3.7

If $y = \dfrac{x^3}{x^2+1}$ for all $x \in \mathbb{R}$, find $\dfrac{dy}{dx}$.

Example continued

Homework 55

If $f : \mathbb{R} \setminus \{0\} \longrightarrow \mathbb{R}$ *with* $f(x) = \dfrac{e^x}{x^3}$*, find* $f'(x)$.

More derivatives of trigonometric functions

- Unless otherwise stated, our convention is that all functions are defined on their implied domain.
- For example, we assume $\operatorname{dom}(\tan) = \mathbb{R} \setminus \left\{ k\pi + \frac{\pi}{2} \,\middle|\, k \in \mathbb{Z} \right\}$.

Example 3.8

Find $\tan'(x)$.

Example continued

Homework 56

Find $\sec'(x)$.

Homework 57

Find $\operatorname{cosec}'(x)$.

Chain Rule

Theorem 3.9

If g is differentiable at x and f is differentiable at $g(x)$, then the composition $f \circ g$ is differentiable at x and:

$$\frac{d}{dx}\left[f(g(x))\right] = f'(g(x))\, g'(x).$$

This is called the chain rule.
If we let $y = f(u)$ where $u = g(x)$, the chain rule can be written:

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}.$$

Example 3.10

Find $\dfrac{dy}{dx}$ if $y = \sin(3x^2 + 8)$.

Homework 58

Find $\frac{dy}{dx}$ *if* $y = \sqrt{e^x + 4}$.

Homework 59

Use the chain rule and the product rule to prove the quotient rule.

Derivatives and graphs

Definition 3.11 (Revisited from topic 1)

A function f defined on an interval I is:

- increasing or order preserving on I if $a < b \Rightarrow f(a) < f(b)$.
- decreasing or order reversing on I if $a < b \Rightarrow f(a) > f(b)$.

- There are many derivative tests for deciding whether a function is increasing or decreasing on an interval.
- The following theorem gives a test that is particularly useful for graph sketching.

Theorem 3.12

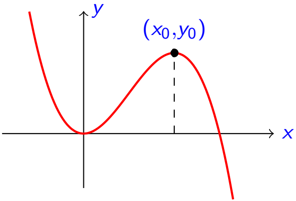
Let f be differentiable on the interval $I = [a, b]$ (so $I \subseteq \mathrm{dom}(f)$).

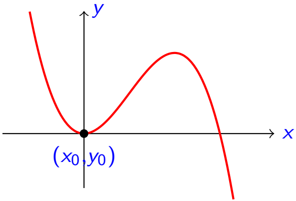
1. *If $f'(x) > 0$ for all $x \in (a, b)$ then f is increasing on I.*
2. *If $f'(x) < 0$ for all $x \in (a, b)$ then f is decreasing on I.*

Local extrema and stationary points

Definition 3.13

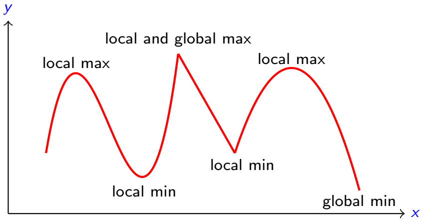
- A function f is said to have:
  - a local maximum at (x_0, y_0) if there is an open interval J containing x_0 on which $y_0 = f(x_0)$ is the largest value.
  - a local minimum at (x_0, y_0) if there is an open interval J containing x_0 on which $y_0 = f(x_0)$ is the smallest value.
- Local maxima and minima are also called local extrema.
- The points where a function attains its overall largest and smallest values are called its global extrema.





Stationary points

- The diagram illustrates two types of local extrema:
  1. Points where the graph has a horizontal tangent.
  2. Points where the function is not differentiable (has a sharp point).



- We focus on the first type:

Definition 3.14

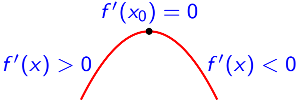
A stationary point of f is any point $(x, f(x))$ such that

$$f'(x) = 0.$$

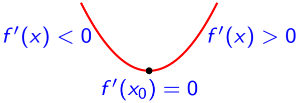
Types of stationary points

- A stationary point can be:

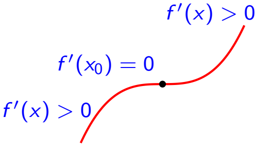
1. a local maximum:

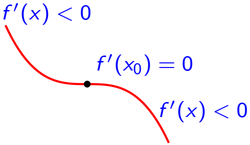


2. a local minimum:



3. or neither:



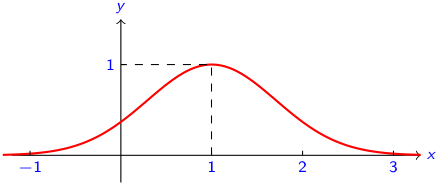


Example 3.15

Find the stationary points of the function $f : \mathbb{R} \longrightarrow \mathbb{R}$ with $f(x) = e^{-(x-1)^2}$. On what intervals is f increasing? Decreasing?

Example continued:

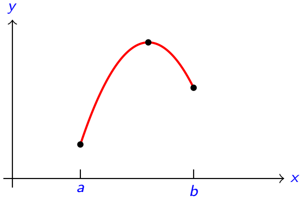
- Our findings are verified by looking at the graph of f:



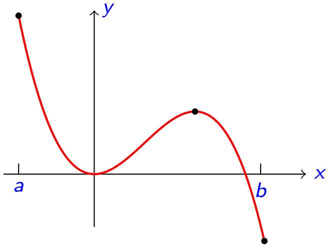
Don't forget the endpoints!

- To find the global maximum or minimum of a function f defined on an interval $[a, b]$, it is important to consider both:
  - the value of f at any stationary points of f
  - the value of f at the endpoints of the domain $[a, b]$

  because the global max/min could occur at either type of point:



Global max at a stationary point inside $[a, b]$.

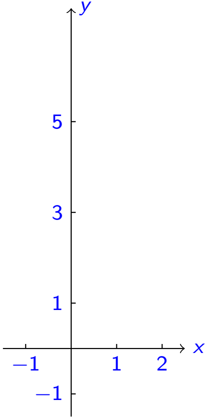


Global max is at endpoint a, not a stationary point.

Example 3.16

Find the global maximum and minimum of $f : [-1, \sqrt{3}] \longrightarrow \mathbb{R}$ defined by

$$f(x) = \sqrt{3}(x^3 - x).$$



Example continued

Second and higher derivatives

Differentiation of a function f produces a formula for a new function f' on some set $I \subseteq \text{dom}(f)$. This new function may be differentiated again to produce another function f'', which we call the second derivative. We can't then differentiate again to obtain the third derivative f''' and so on ...

This is more than just differentiation practice! The second derivative is very useful. It gives a lot of information about the graph of f and gives a way of expressing the acceleration of an object in physical applications.

Second derivatives

- If f' is differentiable at x, differentiating again gives the second derivative of $f(x)$ at x, which is denoted $f''(x)$.
- We say that f is twice differentiable at x.
- Several different notations are commonly used for the second derivative:

Definition 3.17

$$f''(x) = \frac{d}{dx}\left[f'(x)\right] = \frac{d}{dx}\left(\frac{d}{dx}f(x)\right) = \frac{d^2}{dx^2}\left[f(x)\right].$$

or putting $y = f(x)$:

$$\frac{d}{dx}\left[\frac{dy}{dx}\right] = \frac{d^2y}{dx^2}.$$

Example 3.18

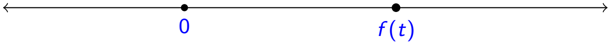
Let $f, g : \mathbb{R} \longrightarrow \mathbb{R}$ be given by $f(x) = \frac{1}{2}(e^x + e^{-x})$ and $g(x) = \frac{1}{2}(e^x - e^{-x})$.

Find $f'(x)$, $f''(x)$, $g'(x)$ and $g''(x)$. What do you notice?

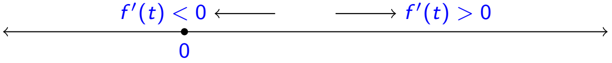
Example continued

Rectilinear motion

- We represent the position (or displacement) of an object moving along a line by a function f. The position at time t is $f(t)$:



- $f'(t)$ gives velocity (rate of change in displacement) at time t.
- $f'(t) > 0$ when the object is moving right and $f'(t) < 0$ when the object is moving left.
- If the object changes direction at time t, we have $f'(t) = 0$.



- $f''(t)$ gives the acceleration (rate of change in the velocity) of the object at time t.

Example 3.19

Assuming negligible air resistance, the position of an object dropped from a 40m cliff is given by

$$f(t) = 40 - 4.9t^2$$

after t seconds. Find the velocity and acceleration at time t. What is the velocity when the object hits the ground?

Example continued

Concavity

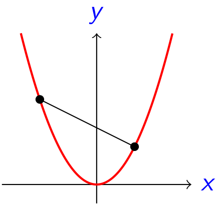
Definition 3.20

If f is differentiable on an interval I, then:

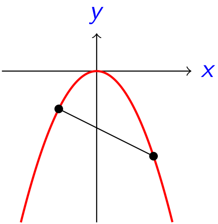
1. f is concave up on I if f' is increasing on I;
2. f is concave down on I if f' is decreasing on I.

- Concave up and concave down are geometric terms that describe the direction of curvature of the graph of a function.
- Recall that line segments joining two distinct points on the graph of a function are called chords to the graph.
- We can visualise concavity in terms of chords to the graph:
  - If f is concave up on I, every chord will lie above the graph.
  - If f is concave down on I, every chord will lie below the graph.

Concavity in pictures



Graphs of x^2 and e^x are concave up. Chords lie above graph.



Graphs of $-x^2$ and $\log(x)$ are concave down. Chords lie below graph.

Calculus to the rescue!

- Trying to prove which way the concavity goes from this definition is typically quite hard.
- For twice differentiable functions, we can determine concavity using the second derivative:

Theorem 3.21

Suppose f is twice differentiable on an interval I.

1. *If $f''(x) > 0$ for all x in I, then f is concave up on I.*
2. *If $f''(x) < 0$ for all x in I, then f is concave down on I.*

Example 3.22

Use the second derivative to determine where the following functions $f, g, h : \mathbb{R} \longrightarrow \mathbb{R}$ are concave up and where they are concave down.

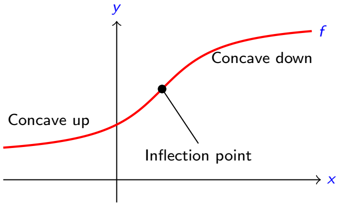
(a) $f(x) = x^4$. (b) $g(x) = x^3$. (c) $h(x) = x^{\frac{1}{3}}$.

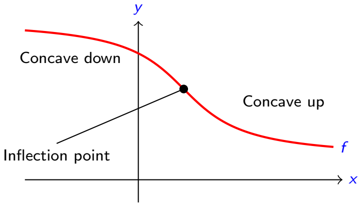
Example continued

Inflection points

Definition 3.23

A point of inflection is a point in $\mathrm{dom}(f)$ where f changes between being concave up and concave down.





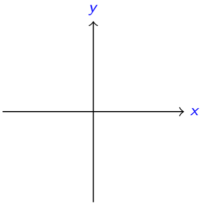
⚠ Changes in concavity - not so simple!

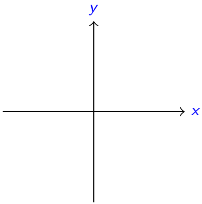
- It is NOT enough to simply solve $f''(x) = 0$ to find points of inflection, because:
  - There are cases where $f''(x) = 0$ but x is not a point of inflection (see Example 3.24(a)).
  - There are cases where concavity changes at a point missing from the domain of f'' (see Example 3.24(c)).
- There are also cases where concavity changes at a point missing from the domain of f (see Example 3.31).

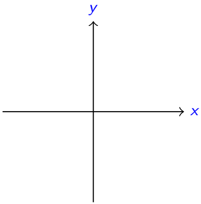
Example 3.24

Find any points of inflection of $f, g, h : \mathbb{R} \longrightarrow \mathbb{R}$ given by:

(a) $f(x) = x^4$. (b) $g(x) = x^3$. (c) $h(x) = x^{\frac{1}{3}}$.







Example continued

Higher order derivatives

- Just as we defined the first derivative of f at x to be $f'(x)$, and the second derivative to be $f''(x)$, it is often possible to continue taking derivatives:

$$f'''(x) = f^{(3)}(x), f''''(x) = f^{(4)}(x), f'''''(x) = f^{(5)}(x), \dots$$

Example 3.25

Write down the first five derivatives of $f(x) = \sin x$.

Example 3.26

If $f : \mathbb{R} \longrightarrow \mathbb{R}$ with $f(x) = ax^3 + bx^2 + cx + d$, find $f^{(n)}(x)$ for $n \leq 6$.

Homework 60

Find a third degree polynomial $P(x)$ such that $P(1) = 1$, $P'(1) = 3$, $P''(1) = 6$ and $P'''(1) = 12$.

Homework 61

If $f(x) = \frac{1}{x}$*, compute* f', f'', f''' *and* $f^{(4)}$ *and use these to guess a formula for the* n*-th derivative* $f^{(n)}(x)$*.*

Asymptotes and graphs

An asymptote of a function f is a straight line the graph of f approaches either as $x \longrightarrow x_0$ or as $x \longrightarrow \pm\infty$. Three types of straight lines in the plane give rise to:

- vertical asymptotes,
- horizontal asymptotes,
- oblique (slant) asymptotes.

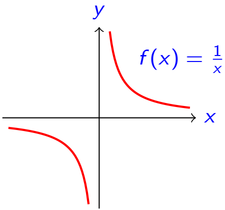
Combining our knowledge of asymptotic behaviour together with our knowledge of extrema and concavity puts us in a position to understand the graphs of functions.

Vertical asymptotes

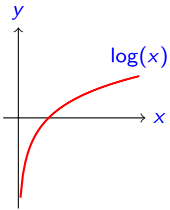
- We often encounter vertical asymptotes when a ‘zero denominator’ occurs. That is, when

$$f(x) = \frac{g(x)}{h(x)}$$

and $h(x) = 0$ for some values of x.



- As we will see in Example 3.27, a ‘zero denominator’ does not guarantee a vertical asymptote.
- Vertical asymptotes may also arise in situations that do not involve quotients.
- **Example:** The function $f(x) = \log(x)$ does not involve quotients, but it has a vertical asymptote at $x = 0$..

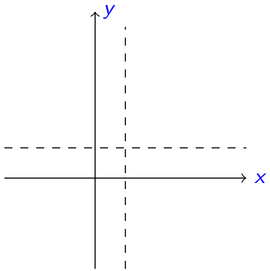


Beware of common factors!

Example 3.27

Find the asymptotes of

$$f(x) = \frac{x^2 + 2x + 1}{x^2 - 1}$$



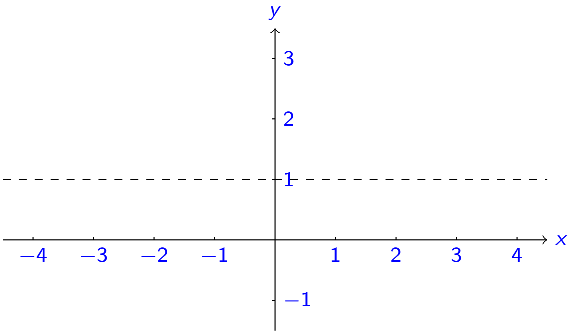
Example continued

Horizontal asymptotes

- Horizontal asymptotes are associated with limiting behaviour as $x \longrightarrow \pm\infty$.
- We see this kind of behaviour with exponential functions.

Example 3.28

On the same figure sketch the graphs of $f, g : \mathbb{R} \longrightarrow \mathbb{R}$ given by $f(x) = e^x + 1$ and $g(x) = 1 - e^{-x}$.



Oblique (slant) asymptotes

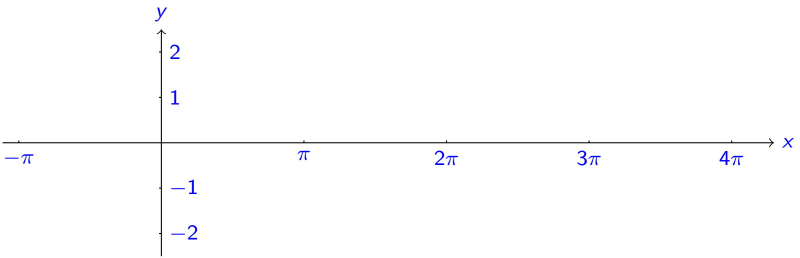
- Sometimes the graph of a function approaches a slant line $y = \alpha x + \beta$ with $\alpha \in \mathbb{R} \setminus \{0\}$ as $x \longrightarrow \pm\infty$.
- We call this line an oblique asymptote.
- These asymptotes often occur for rational functions – functions of the form $f(x) = \dfrac{p(x)}{q(x)}$ where p and q are polynomials.
- To find an oblique or horizontal asymptote of a rational function, we apply polynomial long division.
- This makes it easier to analyse the behaviour of the function as $x \longrightarrow \pm\infty$ and also helps us find stationary and inflection points.

Crossing a line

- It is incorrect to think of an asymptote as "a line that may not be crossed".
- This is only true of vertical asymptotes.
- A function may cross a horizontal or oblique asymptote many times.

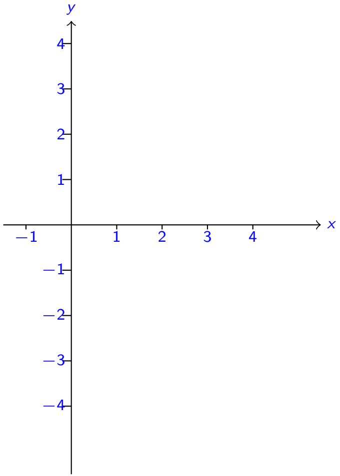
Example 3.29

Sketch the graph of $f(x) = e^{-x}\sin(x)$.



Example 3.30

Find and sketch any asymptotes of $f(x) = \dfrac{2x^3 - 3x^2 + 2x - 2}{x^2 + 1}$.



Example continued

Homework 62

Find and sketch any asymptotes of $f(x) = \dfrac{2x^2 + 3x - 2}{x^2 - 1}$.

Putting it all together

Example 3.31

Consider the function $f(x) = \dfrac{x^2}{x+1}$. Find:

1. the domain of f;
2. the asymptotes of f;
3. the x and y intercepts of the graph $y = f(x)$;
4. the stationary points of f;
5. the intervals where f is increasing;
6. the intervals where f is decreasing;
7. the local maxima/minima of f;
8. the intervals where f is concave up;
9. the intervals where f is concave down;
10. the points of inflection of f.

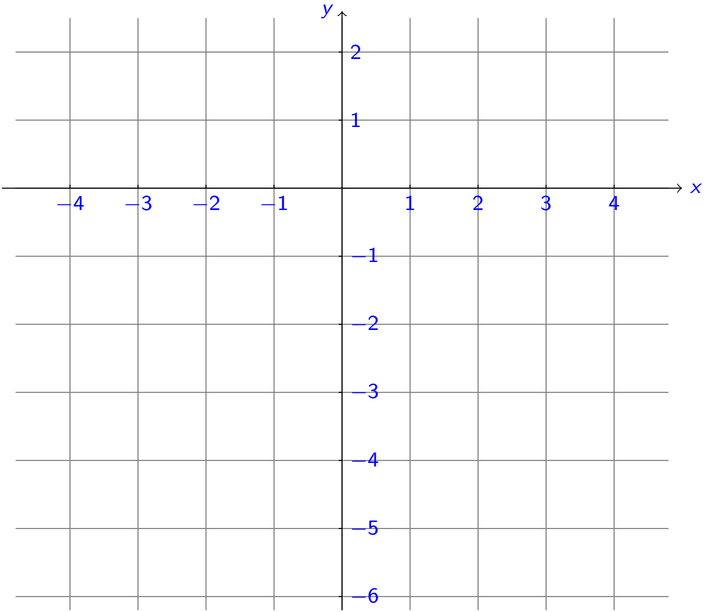
Now use all of this information to sketch the graph $y = f(x)$.

Example continued

Example continued

Example continued

Example continued



Implicit Differentiation

- Usually, to find the derivative $\dfrac{dy}{dx}$ we are given (or can express) y in terms of x and apply differentiation rules.
- However, sometimes we cannot express y as a function of x.

Example 3.32

For the curve $x^2 - xy + y^4 = 16$, can you express y in terms of x?

- This equation describes an implicit dependence of y on x.
- Despite this, we can still find the derivative $\dfrac{dy}{dx}$ using a technique called implicit differentiation.

A gentle start

- We begin with an easy example where we can express y as a function of x.

Example 3.33

Find a formula for $\frac{dy}{dx}$ where

$$x^2y = 1.$$

Example continued

- This answer is given in terms of both x and y and this is the best we can hope for in general ...
- ...but sometimes we an express $\frac{dy}{dx}$ entirely in terms of x:

$$x^2 y = 1 \Rightarrow y = \frac{1}{x^2},$$

so we can plug this into our answer to get:

Implicit differentiation in general

For an equation involving both x and y where y depends implicitly on x, the steps are as follows:

1. Take the derivative of each side with respect to x.
2. Use the usual differentiation rules to simplify each side.
3. Rearrange to solve for $\frac{dy}{dx}$. This will usually give a formula for $\frac{dy}{dx}$ in terms of both x and y.

⚠ When differentiating y, we must treat it as a function of x, so:

- we must apply the chain rule to terms of the form $g(y)$.
- we must apply the product and chain rules rule to terms of the form $f(x)g(y)$.
- etc ...

Example 3.34

Assuming y is implicitly dependent on x, find the following derivatives:

1. $\dfrac{d}{dx}\left[\sin(y)\right]$
2. $\dfrac{d}{dx}\left[y^3\right]$
3. $\dfrac{d}{dx}\left[xe^y\right]$
4. $\dfrac{d}{dx}\left[x^2\log(y)\right]$

Example continued

Back to where we started from

- For our original problem, we cannot solve for y in terms of x, so we use implicit differentiation to find $\dfrac{dy}{dx}$.
- We can then use $\dfrac{dy}{dx}$ to obtain information about the curve.

Example 3.35

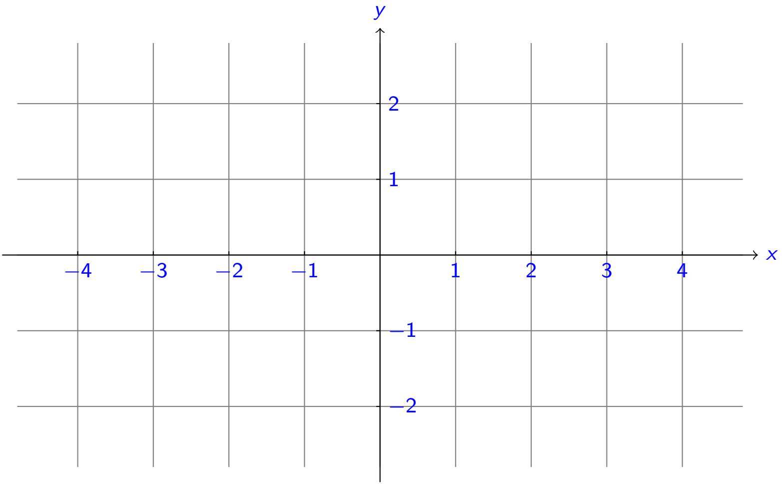
For the curve $x^2 - xy + y^4 = 16$:

1. Find $\dfrac{dy}{dx}$.
2. Find the x and y intercepts and hence find the gradient of the curve at each intercept.
3. Find the points on the curve where $x = y$ and hence find the gradient of the curve at each of these points.
4. Use this information to draw a rough sketch of the curve.

Example continued

Example continued

Example continued



Homework 63

For the curve in Example 3.35, find the x-values for which the gradient is 0 (so the curve is horizontal).

Vertical tangent

- Implicit differentiation typically expresses $\frac{dy}{dx}$ as a quotient.
- For values of x and y giving denominator 0 and numerator $\neq 0$, the derivative is not defined because the tangent line is vertical.

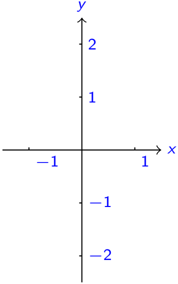
Example 3.36

Find the points where the tangent line to the curve

$$\frac{y^4}{16} + x^4 = 1$$

is vertical.

Example continued



⚠ If the numerator and denominator are both 0, further analysis is required. The tangent could be vertical, horizontal, sloping or even undefined!

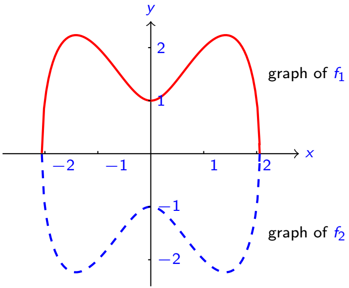
Homework 64

For the curve $\cos(y) = x^2$*:*

1. *Find* $\dfrac{dy}{dx}$.
2. *Find the range of possible* x*-values of points on this curve.*
3. *Find the points where the tangent line to the curve is horizontal.*
4. *Find the points where the tangent line to the curve is vertical.*

Why does it work?

- But does it even make sense to think of y as a function of x in the Cartesian equation of a curve?
- Yes, if we break the curve into suitable pieces!



$$y^2 = 1 + 4x^2 - x^4$$

- The points where the functions change are usually the points where the tangent is vertical.

Derivatives of inverse functions

We can use implicit differentiation to find the derivatives of inverses of well known functions. For mutual inverses like exp and log we already know the derivative of both functions. In other cases, like the trigonometric functions, for example, we need to use the inverse relationship to discover the derivative of the inverse.

In this section, we use implicit differentiation to find the derivatives of the inverses of the standard trigonometric functions.

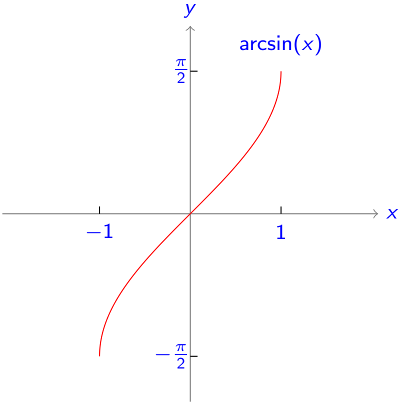
An easy example

- To find the derivative of $y = \log(x)$, we could rewrite this relation as
$$e^y = x$$
and apply implicit differentiation: (use the chain rule on the left):

- This gives the well known derivative of $\log$.
- Note that we needed to use the relation $y = \log(x)$ to express our answer as a function of x.

Derivatives of inverse trigonometric functions

- This method enables us to find the derivatives of the inverse trigonometric functions arcsin, arccos and arctan.
- Recall the function $\arcsin\colon [-1,1] \longrightarrow \mathbb{R}$ with graph



Derivative of arcsin

- To find the derivative of $y = \arcsin(x)$, we rewrite this relation as
$$\sin(y) = x.$$
- Implicit differentiation with respect to x gives:
- And we can write $\cos(y)$ in terms of x:

Derivative of arcsin continued

- Substituting this into the implicit derivative gives:

Theorem 3.37

$$\arcsin'(x) = \frac{1}{\sqrt{1-x^2}}\,, \qquad \textit{for } -1 < x < 1.$$

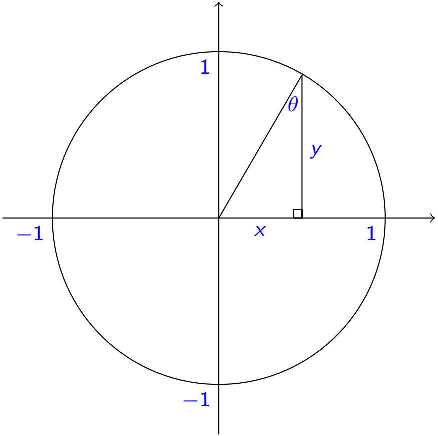
Homework 65

How can we be sure that the identity $\cos(y) = \sqrt{1-\sin^2(y)}$ *holds in the previous calculation?*

In pictures

Example 3.38

Show geometrically why $\cos(\arcsin(x)) = \sqrt{1 - x^2}$.



Homework 66

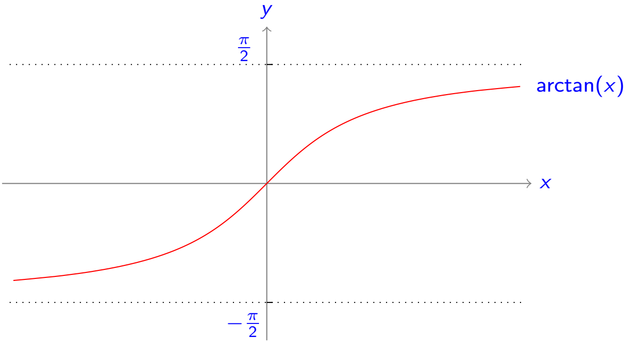
What can we say about the tangent line and hence derivative of $\arcsin$ *when* $x = -1$ *and* $x = 1$*?*

Homework 67

Find $\dfrac{d}{dx}\big(\arcsin(2x+1)\big)$.

Derivative of arctan

Recall the function $\arctan\colon \mathbb{R} \longrightarrow \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ with graph



We now use the identity $\sec^2(x) = \tan^2(x) + 1$ find its derivative.

Example 3.39

Use implicit differentiation to find $\arctan'(x)$.

Example continued

Theorem 3.40

$\arctan'(x) = \dfrac{1}{1+x^2}$.

Homework 68

Find $\dfrac{d}{dx}\big(\arctan(x^5)\big)$.

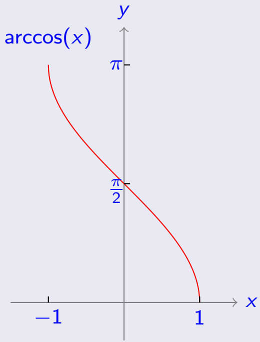
Derivative of arccos

Homework 69

Consider the function

$$\arccos\colon [-1,1] \longrightarrow [0,\pi]$$

with graph as shown. Use implicit differentiation to show that



$$\frac{d}{dx}\big(\arccos(x)\big) = \frac{-1}{\sqrt{1-x^2}} = -\arcsin'(x)$$

for $-1 < x < 1$. *What happens when* $x = -1$ *and* $x = 1$*?*

Homework 70

Find $\dfrac{d}{dx}\big(\arccos(e^x)\big)$.

Differentiating parametric curves

We can apply the definition of the derivative as a limit to vector valued functions like parametric curves. This gives us a vector valued derivative which yields important information about motion along the curve. In a similar way to derivatives of real valued functions, the derivative computes the instantaneous rate of change with respect to the parameter t.

In physical applications, parametric curves are used to specify the position of a moving particle or object as a function of time t. In this context, differentiation allows us to determine the velocity, speed and acceleration of the particle at each point in time. We illustrate this with the example of projectile motion.

Differentiation, pretty much as we know it!

- The familiar definition of the derivative of real-valued function extends naturally to a parametric functions.

Definition 3.41

Let $\mathbf{r} : I \longrightarrow \mathbb{R}^2$ where I is an interval. The derivative of $\mathbf{r}$ at t is defined by the limit:

$$\mathbf{r}'(t) = \lim_{h \longrightarrow 0} \frac{1}{h}\big(\mathbf{r}(t+h) - \mathbf{r}(t)\big).$$

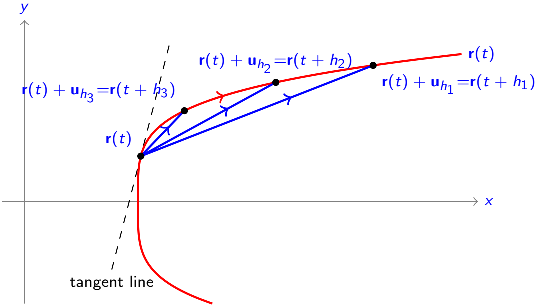
If this limit exists, f is said to be differentiable at t.

- Since $\mathbf{v}_h = \frac{1}{h}(\mathbf{r}(t+h) - \mathbf{r}(t))$ is a vector and $\mathbf{r}'(t)$ is a limit of the $\mathbf{v}_h$'s, it is a vector:

$$\mathbf{r}'(t) = a\mathbf{i} + b\mathbf{j}.$$

Tangency

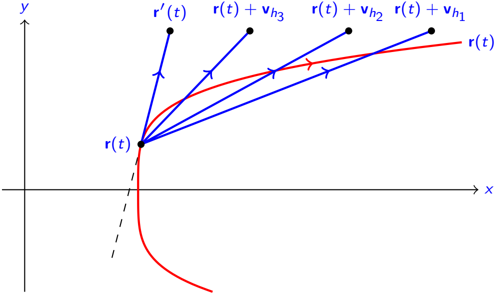
- Just like for the graph of a function, the direction of a parametric curve at a point is given by the tangent line.
- Moving the vector $\mathbf{u}_h = \mathbf{r}(t+h) - \mathbf{r}(t)$ to $\mathbf{r}(t)$ by adding $\mathbf{r}(t)$ gives a chord to the curve:



- As $h \longrightarrow 0$, the direction of $\mathbf{u}_h$ clearly approaches the direction of the tangent line at $\mathbf{r}(t)$.

Varying the length

- Moving $\mathbf{v}_h = \frac{1}{h}(\mathbf{r}(t+h) - \mathbf{r}(t))$ to $\mathbf{r}(t)$ does not generally give a chord because scaling by $\frac{1}{h}$ changes the length ...

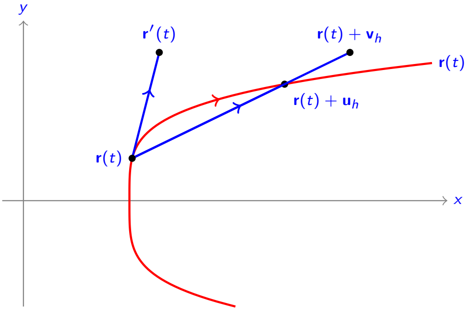


... but the direction of $\mathbf{v}_h$ still approaches the tangent.

- The vector $\mathbf{r}'(t)$ is still tangent to the curve at $\mathbf{r}(t)$.

One Direction ...

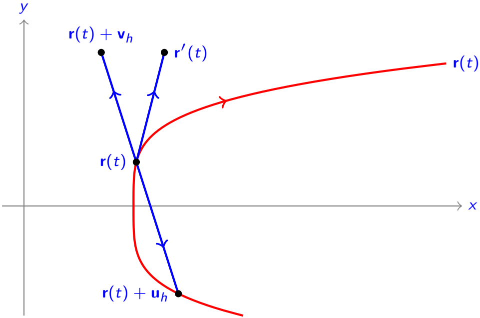
- The vectors $\mathbf{u}_h = \mathbf{r}(t+h) - \mathbf{r}(t)$ and $\mathbf{v}_h = \frac{1}{h}(\mathbf{r}(t+h) - \mathbf{r}(t))$ are parallel.
- For $h > 0$, both $\mathbf{u}_h$ and $\mathbf{v}_h$ follow the curve's direction of motion.



- So taking the limit just for $h > 0$, we would expect $\mathbf{r}'(t)$ to follow the direction of motion.
- But what happens when $h < 0$?

. . . same direction

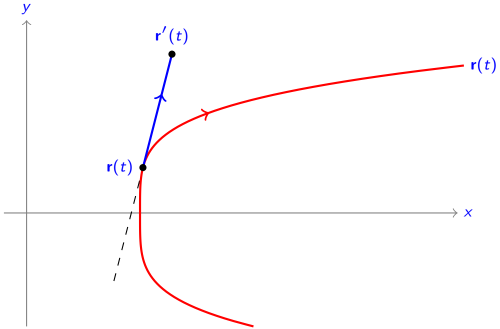
- When $h < 0$, the vector $\mathbf{u}_h$ points opposite to the curve's direction of motion . . .
- . . . but $\mathbf{v}_h = \frac{1}{h}\mathbf{u}_h$ and $\frac{1}{h} < 0$, so $\mathbf{v}_h$ follows the direction of motion again!



- So when we take the limit for $h < 0$, we expect $\mathbf{r}'(t)$ to follow the direction of motion.
- Hence $\mathbf{r}'(t)$ always follows the direction of motion. Yay!

In summary

- The vector $\mathbf{r}'(t)$ is tangent to the curve and follows the direction of motion along the curve.



- We later consider the geometric and physical meaning of the length of $\mathbf{r}'(t)$.

Differentiation is too easy!

Theorem 3.42

Let $\mathbf{r} : I \longrightarrow \mathbb{R}^2$ *with* $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j}$. *Provided* x *and* y *are differentiable*

$$\mathbf{r}'(t) = x'(t)\mathbf{i} + y'(t)\mathbf{j}.$$

- This makes differentiation very easy because we already know how to differentiate the functions $x : I \longrightarrow \mathbb{R}$ and $y : I \longrightarrow \mathbb{R}$.

Example 3.43

For each curve, give a formula for $\mathbf{r}'(t)$:

1. $\mathbf{r}(t) = (t^2 + 1)\mathbf{i} + (\frac{1}{3}t^3 + 1)\mathbf{j}$.
2. $\mathbf{r}(t) = \mathbf{i} + 3t^3\mathbf{j}$.
3. $\mathbf{r}(t) = 2t\mathbf{i} - (3t^3 + 1)\mathbf{j}$.
4. $\mathbf{r}(t) = 2\cos(t)\mathbf{i} + \sin(t)\mathbf{j}$.

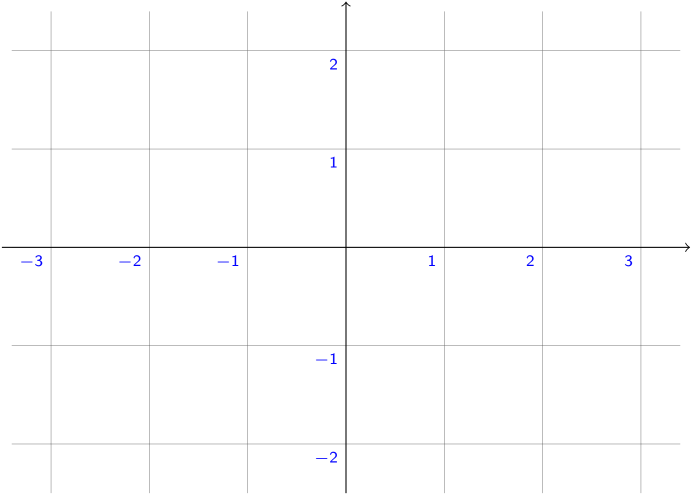
Example 3.44

For the curve given by $\mathbf{r}(t) = 2\cos(t)\mathbf{i} + \sin(t)\mathbf{j}$, calculate the vectors

(a) $\mathbf{r}'(0)$ (b) $\mathbf{r}'\left(\frac{\pi}{2}\right)$ (c) $\mathbf{r}'(\pi)$ (d) $\mathbf{r}'\left(\frac{3\pi}{2}\right)$

and sketch them based at the appropriate point on the curve defined by $\mathbf{r}(t)$.

Example continued



Homework 71

For the parametric curve $\mathbf{r}(t) = (t^4 + 1)\mathbf{i} + t\mathbf{j}$, *calculate the vectors*

(a) $\mathbf{r}'(-1)$ *(b)* $\mathbf{r}'(0)$ *(c)* $\mathbf{r}'\left(\frac{1}{2}\right)$ *(d)* $\mathbf{r}'(1)$

and sketch them based at the appropriate point on the curve.

Velocity versus speed

- For a particle whose motion is described by a curve $\mathbf{r}$, the vector $\mathbf{r}'(t)$ gives the instantaneous velocity at time t.
- This vector quantity expresses the velocity in the $\mathbf{i}$ and $\mathbf{j}$ directions at time t.
- The length $\|\mathbf{r}'(t)\|$ of the velocity vector gives the speed of motion along the curve:

$$\begin{aligned}\text{Speed} = \|\mathbf{r}'(t)\| &= \|x'(t)\mathbf{i} + y'(t)\mathbf{j}\| \\ &= \sqrt{(x'(t))^2 + (y'(t))^2}\end{aligned}$$

- Speed measures distance covered per unit time along the curve.
- This corresponds to our intuitive idea of the speed at which an object is moving at a given instant in time.

Example 3.45

For each curve, give a formula for the speed at time t:

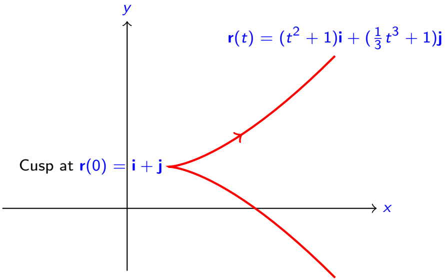
1. $\mathbf{r}(t) = (t^2 + 1)\mathbf{i} + (\frac{1}{3}t^3 + 1)\mathbf{j}$.
2. $\mathbf{r}(t) = \mathbf{i} + 3t^3\mathbf{j}$.
3. $\mathbf{r}(t) = 2t\mathbf{i} - (3t^3 + 1)\mathbf{j}$.
4. $\mathbf{r}(t) = 2\cos(t)\mathbf{i} + \sin(t)\mathbf{j}$.

and find any times at which the speed is 0.

Example continued

Right on the cusp

- One of the curves in Example 3.45 had a point where the velocity was $\mathbf{0}$ (and hence the speed was 0).
- **Q:** What is the tangent line in this situation?
- **A:** There isn't one!



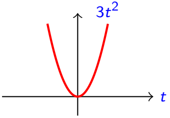
- If $\mathbf{r}'(t) = 0$ for some t, the curve usually has a sharp point called a cusp at $\mathbf{r}(t)$.
- In spite of the cusp $\mathbf{r}$ is still differentiable at 0. Does this surprise you?

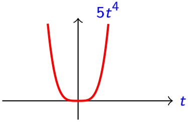
Definition 3.46

The curve $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j}$ has a cusp at t precisely if $x'(t) = y'(t) = 0$ and at least one of x' and y' change sign at t.
A curve with no cusps is called smooth.

Example 3.47

Is the curve given by $\mathbf{r}(t) = t^3\mathbf{i} + t^5\mathbf{j}$ smooth?





Homework 72

Find the cusps of the curve given by

$$\mathbf{r}(t) = (t - \sin(t))\mathbf{i} + (1 - \cos(t))\mathbf{j}.$$

(There are infinitely many).

Curves like this, called cycloids, play an important role in the solution of many physical problems.

How can this be?

- Of course, the graph of a differentiable function $f : \mathbb{R} \longrightarrow \mathbb{R}$ cannot have any sharp points.
- The path of a parametric curve $\mathbf{r} : \mathbb{R} \longrightarrow \mathbb{R}^2$ is not its graph.
- The graph

$$\{(t, \mathbf{r}(t)) \mid t \in \mathbb{R}\} = \{(t, (x(t), y(t))) \mid t \in \mathbb{R}\}$$

  of $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j}$ is really three dimensional because we can equate it with

$$\{(t, x(t), y(t)) \mid t \in \mathbb{R}\}.$$

- If $\mathbf{r}$ is differentiable, its graph has no sharp points, but its path may have cusps.

Acceleration

- Since $\mathbf{r}' : I \longrightarrow \mathbb{R}^2$, we can differentiate again to obtain $\mathbf{r}'' : I \longrightarrow \mathbb{R}^2$.

Definition 3.48

At each t the vector $\mathbf{r}''(t)$ is called the acceleration vector.

- The vector $\mathbf{r}''(t)$ expresses the rate of change in velocity in the $\mathbf{i}$ and $\mathbf{j}$ directions at time t.

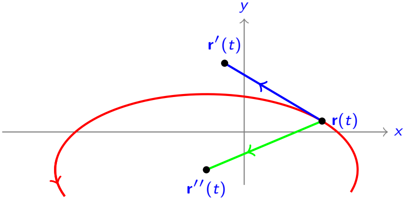
Example 3.49

For each curve, give a formula for the acceleration at time t:

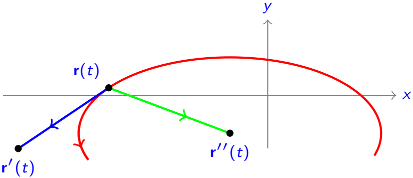
1. $\mathbf{r}(t) = (t^2 + 1)\mathbf{i} + (\frac{1}{3}t^3 + 1)\mathbf{j}$,
2. $\mathbf{r}(t) = 2t\mathbf{i} - (3t^3 + 1)\mathbf{j}$,
3. $\mathbf{r}(t) = (t - \sin(t))\mathbf{i} + (1 - \cos(t))\mathbf{j}$.

Acceleration and Velocity

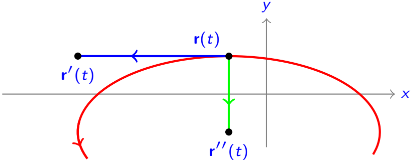
- When the angle between $\mathbf{r}''(t)$ and $\mathbf{r}'(t)$ is acute, the speed is increasing.



- When this angle is obtuse, the speed is decreasing.



- When $\mathbf{r}'(t)$ and $\mathbf{r}''(t)$ are perpendicular, the speed is at a turning point.



Speeding up our calculations

- These results about speed are consequences of the following Theorem, which we will prove in a Practice Class.

Theorem 3.50

Suppose $\mathbf{r}'(t) \neq \mathbf{0}$*. The speed function* $\|\mathbf{r}'(t)\|$ *is:*

- *Decreasing if* $\mathbf{r}'(t) \cdot \mathbf{r}''(t) < 0$*.*
- *At a turning point if* $\mathbf{r}'(t) \cdot \mathbf{r}''(t) = 0$*.*
- *Increasing if* $\mathbf{r}'(t) \cdot \mathbf{r}''(t) > 0$*.*

- This Theorem is proved by determining the sign of

$$0 = \frac{d}{dt}\left[\|\mathbf{r}'(t)\|\right] = \frac{d}{dt}\left[\sqrt{(x'(t))^2 + (y'(t))^2}\right]$$

Example 3.51

For each curve, find the t values for which the speed is increasing:

1. $\mathbf{r}(t) = (t^2 + 1)\mathbf{i} + (\frac{1}{3}t^3 + 1)\mathbf{j}$
2. $\mathbf{r}(t) = 2t\mathbf{i} - (3t^3 + 1)\mathbf{j}$
3. $\mathbf{r}(t) = (t - \sin(t))\mathbf{i} + (1 - \cos(t))\mathbf{j}$.

Example continued

Homework 73

For a particle with motion given by $\mathbf{r}(t) = \cos(t)\mathbf{i} + 3\sin(t)\mathbf{j}$ *find the times at which:*

1. *The speed is at a maximum.*
2. *The speed is at a minimum.*

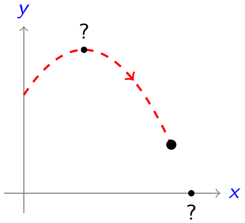
Express your answers as sets.

Application: Projectile Motion

A simple example illustrating speed and velocity along a parametric curve is the motion under gravity of a projectile, launched from a given starting position and starting velocity. We can express the trajectory of such a projectile (like a ball, missile or *Angry Bird*) in a simple way as a parametric curve.

This will enable us to answer many obvious questions, eg:

- When and where will it land?
- How high will it go?
- At what speed will it hit the ground?

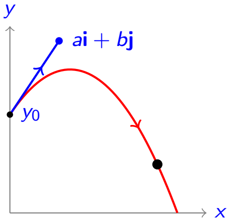


Actually, it is rocket science!

- Ignoring air resistance, projectile motion is given by the parametric curve:

$$\mathbf{r}(t) = at\mathbf{i} + (y_0 + bt - \frac{1}{2}gt^2)\mathbf{j}$$

  where:
  - $a\mathbf{i} + b\mathbf{j}$ is the velocity vector at time $t = 0$.
  - y_0 is the height above the ground at time $t = 0$.
  - g is a gravitational constant expressing acceleration due to gravity.



- For simplicity, we let $x = 0$ when $t = 0$.
- Near the surface of the Earth $g \doteqdot 9.8$ metres per second per second (m/s^2).
- For other planetary bodies, gravitational constants differ.

Example 3.52

Suppose Captain Risky is fired from a cannon at ground level in a flat plain with initial velocities $20\ m/s$ in both the horizontal and vertical directions.

1. Find formulas for the position, velocity and acceleration of Risky at time t.
2. At what angle (from the ground) is Risky launched?
3. Find the maximum altitude and the time at which this altitude is attained.
4. Find the time at which Risky hits the ground and his crash speed.

Example continued

Example continued

Example continued

Homework 74

Suppose Captain Risky drives his car off a $40\ m$ *high cliff at initial velocity* $40\ km/h$.

1. *Express the initial velocity in metres per second.*
2. *Hence find formulas for the position, velocity and acceleration of Risky's car at time* t.
3. *Find the time at which Risky's car hits the ground and the speed at which he is travelling.*

Our future trajectory

- It is customary to express the initial velocities in projectile motion problems by stating the launch angle and launch speed.
- In a Practice Class, we will explore the conversion between this approach and the method presented here.
- We will also find general formulas for
  - velocity, speed and acceleration,
  - landing time and distance,
  - maximum altitude and time at which it is attained,
  - minimum speed and time at which it is attained,

  in Practice Classes and assignments.

Topic 4 - Integration and Differential Equations

You are are already familiar with the ideas. After reviewing the basic ideas of integration and antidifferentiation, with a focus on the definition of the integral as an area, we move on to learn the particularly useful technique of integration by substitution and explore the range of integrals that it can be used to calculate. We also explore the integration of rational functions using partial fractions.

With these new integration techniques at our disposal, we are ready to study the topic of differential equations. A grasp of differential equations puts us in a position to understand the derivation of mathematical models that arise in many branches of science and economics.

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- Partial fractions 533
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- Separable differential equations 589
- Population growth 613
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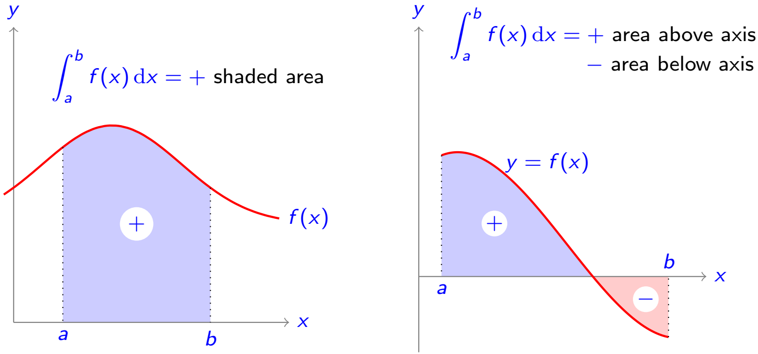
The definite integral

- The definite integral of a function f from a to b, denoted

$$\int_a^b f(x)\,\mathrm{d}x,$$

  is defined to be the sum of the signed areas bounded by:
  - the curve $y = f(x)$,
  - the x-axis,
  - the vertical lines $x = a$ and $x = b$, where $a \leq b$.
- Signed area means that when calculating the integral:
  - the areas under the graph above the x-axis are added,
  - the areas above the graph below the x-axis are subtracted.
- The values a and b are called the terminals of the integral and $[a, b]$ is called the interval of integration.

In pictures



- Definite integrals share the linearity properties of derivatives:

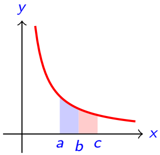
  1. $$\int_a^b f(x) + g(x)\,\mathrm{d}x = \int_a^b f(x)\,\mathrm{d}x + \int_a^b g(x)\,\mathrm{d}x;$$
  2. $$\int_a^b k\,f(x)\,\mathrm{d}x = k\int_a^b f(x)\,\mathrm{d}x;$$

Special properties of integrals

- Integrals also have the following properties (motivated by considering areas):

3. $$\int_a^a f(x)\,\mathrm{d}x = 0;$$

4. $$\int_a^c f(x)\,\mathrm{d}x = \int_a^b f(x)\,\mathrm{d}x + \int_b^c f(x)\,\mathrm{d}x;$$



5. $$\int_b^a f(x)\,\mathrm{d}x = -\int_a^b f(x)\,\mathrm{d}x.$$

- Property 4 is known as additivity.
- Property 5 is really a definition. Without it Property 4 would only work for the case $a \leq b \leq c$.

Example 4.1

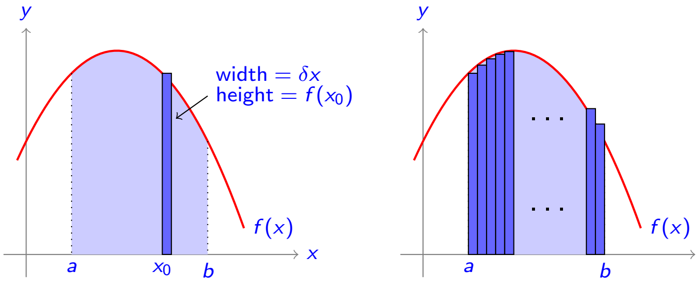
Evaluate the definite integral

$$\int_0^4 \sqrt{16 - x^2}\,\mathrm{d}x$$

by interpreting it as the area of a known geometric figure.

Back to first principles

- Dividing the area under the graph into thin vertical rectangles of width δx, we can approximate the area by vertical strips.
- One way to find the area under the graph as the limit of this approximation as $\delta x \longrightarrow 0$ (or number of strips $\longrightarrow \infty$).



- Calculating these limits is difficult if not impossible in all but the simplest cases. Somebody help!

We are saved!

- The the link between definite integrals and derivatives saves the day!
- For a function f, we call another function F an antiderivative of f on an interval I if F is differentiable on I and

$$F'(x) = f(x) \qquad \text{for all } x \in I.$$

Theorem 4.2 (Fundamental Theorem of Calculus)

If f is continuous on $[a, b]$ and F is an antiderivative of f on $[a, b]$, then

$$\int_a^b f(x)\,\mathrm{d}x = F(b) - F(a) = [F(x)]_a^b\,.$$

- It is therefore essential to be able to find antiderivatives.

So many choices!

- Notice that antiderivatives are not unique. For any F that satisfies

$$F'(x) = f(x),$$

the function $F(x) + C$ also has this property:

$$\frac{d}{dx}[F(x) + C] = F'(x) = f(x).$$

- We often write the operation of anti-differentiation using an integral sign $\int$ without terminals.
- This is also called the indefinite integral. We write:

$$\int f(x)\,\mathrm{d}x = F(x) + C,$$

where F is some antiderivative of f.

Some easy examples:

1. $\int x^2 \, \mathrm{d}x = \frac{1}{3}x^3 + C$, since

$$\frac{d}{dx}\left[\frac{1}{3}x^3 + C\right] = \frac{1}{3} \cdot 3x^2 = x^2.$$

2. $\int \cos x \, \mathrm{d}x = \sin x + C$, since

$$\frac{d}{dx}[\sin x + C] = \cos x.$$

Many antiderivatives can be guessed directly in this way.

Some standard (easy to guess) antiderivatives ...

$$\int x^n \, \mathrm{d}x = \frac{x^{n+1}}{n+1} + C \qquad \text{for } n \neq -1$$

$$\int \frac{1}{x} \, \mathrm{d}x = \log|x| + C$$

$$\int e^{kx} \, \mathrm{d}x = \frac{1}{k} e^{kx} + C \qquad \text{for } k \in \mathbb{R}$$

$$\int \sin kx \, \mathrm{d}x = -\frac{1}{k} \cos kx + C \qquad \text{for } k \in \mathbb{R}$$

$$\int \cos kx \, \mathrm{d}x = \frac{1}{k} \sin kx + C \qquad \text{for } k \in \mathbb{R}$$

$$\int \sec^2 kx \, \mathrm{d}x = \frac{1}{k} \tan kx + C \qquad \text{for } k \in \mathbb{R}$$

. . . plus a few new ones

- Since we have learnt how to differentiate inverse trigonometric functions, we also have:

$$\int \frac{1}{\sqrt{a^2 - x^2}} \, \mathrm{d}x = \arcsin\left(\frac{x}{a}\right) + C$$

$$\int \frac{-1}{\sqrt{a^2 - x^2}} \, \mathrm{d}x = \arccos\left(\frac{x}{a}\right) + C$$

$$\int \frac{1}{a^2 + x^2} \, \mathrm{d}x = \frac{1}{a} \arctan\left(\frac{x}{a}\right) + C$$

where $a > 0$.

Homework 75

Check these antiderivatives by differentiating the right hand side (you will need the chain rule).

Properties of antiderivatives

- Antiderivatives, like derivatives and definite integrals, are linear. That is:

$$\int af(x)\,\mathrm{d}x = a\int f(x)\,\mathrm{d}x$$

$$\int (f(x)+g(x))\,\mathrm{d}x = \int f(x)\,\mathrm{d}x + \int g(x)\,\mathrm{d}x$$

Example 4.3

Find $\int 5e^{2x} - \frac{3}{x}\,\mathrm{d}x, \qquad x \neq 0.$

⚠ No product or quotient rule!

- BEWARE of the following common errors:

$$\int f(x)g(x)\,\mathrm{d}x \neq f(x)\int g(x)\,\mathrm{d}x$$

$$\int f(x)g(x)\,\mathrm{d}x \neq \int f(x)\,\mathrm{d}x \int g(x)\,\mathrm{d}x$$

$$\int \frac{f(x)}{g(x)}\,\mathrm{d}x \neq \frac{\int f(x)\,\mathrm{d}x}{\int g(x)\,\mathrm{d}x}$$

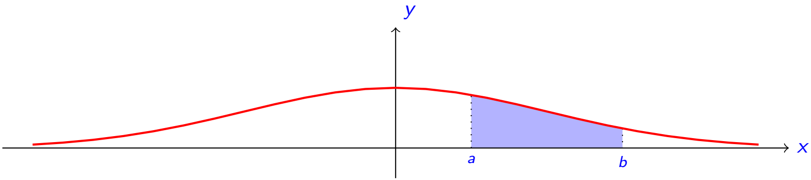
- There is no general rules for finding antiderivatives of products or quotients.
- Integration by substitution, our next topic, enables us to find antiderivatives of products in certain special cases.

Sometimes it is impossible!

- There are cases where we cannot express antiderivatives in terms of well-known functions.
- For example, the integral

$$\int_a^b \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}\ dx$$

is extremely important in statistics.



- This integral cannot be expressed in terms of elementary functions (trig, logs, powers, exponentials, etc.).

Integration by substitution

Integration by substitution works by using the chain rule "backwards". To illustrate this, we start with an example.

According to the chain rule:

$$\frac{d}{dx}\left(\sin^5(x)\right) = 5\sin^4(x)\cos(x).$$

Therefore, by the Fundamental Theorem of Calculus:

$$\int_a^b 5\sin^4(x)\cos(x)\,\mathrm{d}x = \int_a^b \frac{d}{dx}[\sin^5(x)]\,\mathrm{d}x = \sin^5(b) - \sin^5(a).$$

To turn this into an integration rule, we need to learn how to recognise integrands like $5\sin^4(x)\cos(x)$ as the derivative of a composite. The substitution rule provides a systematic way of turning the chain rule "backwards".

Our example analysed

- Notice that our integrand $5\sin^4(x)\cos(x)$ can be written

$$g(h(x))h'(x)$$

where $g(x) = 5x^4$ and $h(x) = \sin(x)$.

- We can integrate any function of this form provided we know an antiderivative G of g:

$$\begin{aligned}
\int_a^b g\big(h(x)\big)\,h'(x)\,\mathrm{d}x &= \int_a^b G'\big(h(x)\big)\,h'(x)\,\mathrm{d}x \\
&= \int_a^b \frac{d}{dx}\big[G(h(x))\big]\,\mathrm{d}x \qquad \text{[Chain rule]} \\
&= \big[G(h(x))\big]_a^b = G(h(b)) - G(h(a))
\end{aligned}$$

- It is more convenient to split this process into two easier steps.

Stepping back to admire our work

- We achieve this by working backwards from our previous answer in a slightly different way:

$$
\begin{aligned}
G(h(b)) - G(h(a)) &= \big[G(x)\big]_{h(a)}^{h(b)} \\
&= \int_{h(a)}^{h(b)} G'(x)\,\mathrm{d}x = \int_{h(a)}^{h(b)} g(x)\,\mathrm{d}x.
\end{aligned}
$$

- Combining these two expressions for $G(h(b)) - G(h(a))$ yields Step 1, the integration by substitution rule:

Substitution rule (definite integral form)

$$\int_a^b g\big(h(x)\big)\,h'(x)\,\mathrm{d}x = \int_{h(a)}^{h(b)} g(x)\,\mathrm{d}x.$$

So what's Step 2?

Example 4.4

Find $\displaystyle\int_1^{e^{\frac{\pi}{2}}} \frac{\sin(\log(x))}{x}\,\mathrm{d}x.$

- After applying the substitution rule, Step 2 is to integrate the function g. Hopefully, that will be easier.

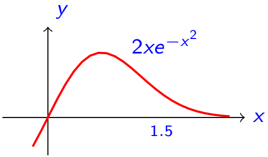
A grey area

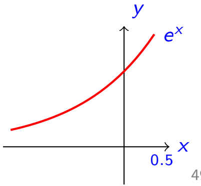
Example 4.5

Use integration by substitution to simplify

$$\int_0^1 2xe^{-x^2}\,\mathrm{d}x$$

and illustrate this in terms of areas on the graphs below.





Homework 76

Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ *be a function such that* $\displaystyle\int_0^1 f(x)dx = \frac{\pi}{2}$.
Use integration by substitution to evaluate

$$\int_1^e \frac{2f(\log(x))}{x}\ dx.$$

Homework 77

Suppose $h : \mathbb{R} \longrightarrow \mathbb{R}$ *is differentiable,* $h(4) = 1$ *and* $h(7) = 0$*. Find*

$$\int_4^7 (h^2(x) + 4h(x) + 7)h'(x)\,\mathrm{d}x.$$

Not so definite

- Setting $u = h(x)$ gives a version of the substitution rule that helps us to find antiderivatives:

Substitution rule (indefinite integral form)

$$\int g(u)\frac{du}{dx}\,\mathrm{d}x = \int g(u)\,\mathrm{d}u.$$

- This is the most commonly used form of the rule for integration by substitution.
- It will be particularly useful when we study differential equations.

But what does it really mean?

- The u in the indefinite integral form of rule is not really a variable. So what does $\int g(u)\,\mathrm{d}u$ actually mean?
- This becomes clearer when we use this form of the rule to find a definite integral. Let G be any antiderivative of g:

$$\begin{aligned}\int_a^b g(u)\frac{du}{dx}\,\mathrm{d}x &= \int_a^b g(u)\,\mathrm{d}u = \int_a^b G'(u)\,\mathrm{d}u \\ &= \big[G(u)\big]_a^b = \big[G(h(x))\big]_a^b \\ &= G(h(b)) - G(h(a))\end{aligned}$$

- See what we did there? So $\int g(u)\,\mathrm{d}u$ means:

  "treat u as a variable, calculate $\int g(u)\,\mathrm{d}u$ and then replace u by $h(x)$ in the answer."

Using the rule to find antiderivatives

- Find a composite function $g(h(x))$ as a factor in the integrand.
- Set $u = h(x)$ and try to express the remaining factor in the integrand as
$$\frac{du}{dx} = h'(x).$$
- If we can do this, the substitution rule gives a new integral in the "variable" u - hopefully easier to integrate.
- Find a formula for this new integral as a function of u.
- Use the relation $u = h(x)$ to express the result in terms of x.
- For definite integrals, we can use the definite integral form directly.

Example 4.6

Find $\int 2x(x^2 - 5)^4 \, \mathrm{d}x$.

Example continued

Example 4.7

Find $\displaystyle\int \cos(3x)\,\sqrt{\sin(3x)+4}\,\mathrm{d}x$.

Homework 78

Find $\int (x^4 + 1)\, e^{x^5+5x}\, \mathrm{d}x$.

Homework 79

Find $\int \frac{x^2+1}{\sqrt{x^3+3x}}\,\mathrm{d}x.$

Linear Substitutions

- Sometimes, after substituting $u = h(x)$ and using $\dfrac{du}{dx} = h'(x)$, the integrand still contains the variable x.
- Completing the substitution would then require inverting the relation $u = h(x)$, which is typically a bad sign.
- A special situation where this can be done successfully is when $h(x)$ is a linear function in x.

Example 4.8

Find $\displaystyle\int (2x+1)\sqrt{x-3}\,\mathrm{d}x$.

Example continued

Example 4.9

Find $\int \frac{2x}{(x+1)^{10}}\,\mathrm{d}x$.

Homework 80

Find $\int 2x(x-5)^7 \, \mathrm{d}x$.

The simplest substitution of all

- A particularly simple form of linear substitution is in play when integrating $f(kx)$, where k is a constant.
- Suppose F is an antiderivative of f, that is

$$\int f(x)\,\mathrm{d}x = F(x) + C.$$

- We want to find an antiderivative of $f(kx)$.
- Letting $u = kx$ gives $\dfrac{du}{dx} = k \Rightarrow \dfrac{1}{k}\dfrac{du}{dx} = 1$, so:

$$\begin{aligned}\int f(kx)\,\mathrm{d}x &= \int f(u)\left(\frac{1}{k}\frac{du}{dx}\right)\,\mathrm{d}x = \int f(u)\frac{1}{k}\,\mathrm{d}u \quad \text{[Substitution]}\\ &= \frac{1}{k}F(u) + C = \frac{1}{k}F(kx) + C.\end{aligned}$$

- **Examples:** $\sin(kx)$, e^{kx}, $\sec^2(kx)$.

A classic substitution ...

- An integral like
$$\int \sin^4(x)\cos(x)\,\mathrm{d}x$$
is tailor-made for substitution.
- We recognise the composite function $\sin^4(x) = (\sin(x))^4$ and set $u = \sin(x)$, so $\dfrac{du}{dx} = \cos(x)$:

$$\begin{aligned}\int \sin^4(x)\cos(x)\,\mathrm{d}x &= \int u^4 \frac{du}{dx}\,\mathrm{d}x \\ &= \int u^4\,\mathrm{d}u \qquad \text{[Substitution rule]} \\ &= \frac{u^5}{5} + C = \frac{1}{5}\sin^5(x) + C.\end{aligned}$$

… extends to a neat trick

- What if we modify the original integral slightly:

$$\int \sin^4(x)\cos^3(x)\,\mathrm{d}x?$$

- The trick is to separate one factor of $\cos(x)$ from the others and use it as our $\dfrac{du}{dx}$.
- The remaining factor $\cos^2(x)$ can be expressed as

$$\cos^2(x) = 1 - \sin^2(x) \Rightarrow \cos^3(x) = (1 - \sin^2(x))\cos(x).$$

- The integral then becomes

$$\int \sin^4(x)\left(1 - \sin^2(x)\right)\cos(x)\,\mathrm{d}x$$

- This is ripe for substitution with $u = \sin(x)$.

- We can now complete the calculation ...

$$\int \sin^4(x)\left(1-\sin^2(x)\right)\cos(x)\,\mathrm{d}x =$$

The goodness of oddness

- More generally, the trick makes it easier to find integrals of the form
$$\int \sin^m(x)\cos^n(x)\,\mathrm{d}x$$
where at least one of the integers m and n is odd.
- If both m and n are odd, we can choose either $u = \sin(x)$ or $u = \cos(x)$ (but a bad choice can make calculation harder).
- When m and n are both even, we can use the complex exponential forms of sin and cos to express
$$\sin^m(x)\cos^n(x)$$
in a form that is easy to integrate.

Example 4.10

Find $\int \sin^{25}(x)\cos^{3}(x)\,\mathrm{d}x$.

⚠ What would have happened if we had chosen $u = \cos(x)$?

Example 4.11

Use the result of Example 1.60 to find $\int \sin^4(x)\,\mathrm{d}x$.

Homework 81

Use the result of Homework 24 to find $\int \cos^4(x)\sin^2(x)\,\mathrm{d}x$.

Homework 82

Find $\int \sin^7(2x)\,\mathrm{d}x$.

The antiderivative of $1/x$

- A classic integration by substitution involves the log function.
- For $x > 0$ we know that
$$\int \frac{1}{x}\,\mathrm{d}x = \log(x) + C,$$
but what happens when $x < 0$? The answer cannot be $\log(x)$ since $\mathrm{dom}(\log) = (0, \infty)$.
- For $x < 0$, consider
$$\int \frac{1}{x}\,\mathrm{d}x = \int \frac{1}{-x}(-1)\,\mathrm{d}x.$$
and write $u = -x$ so $u > 0$ and $\dfrac{du}{dx} = -1$.
- Hence
$$\int \frac{1}{x}\,\mathrm{d}x = \int \frac{1}{u}\frac{du}{dx}\;\mathrm{d}x = \int \frac{1}{u}\,\mathrm{d}u = \log(u) + C = \log(-x) + C.$$

Combining the cases

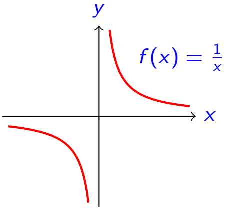
- We now have $\int \frac{1}{x}\,\mathrm{d}x = \begin{cases} \log(x) + C, & \text{when} \quad x > 0, \\ \log(-x) + C, & \text{when} \quad x < 0, \end{cases}$ and combining gives

$$\int \frac{1}{x}\,\mathrm{d}x = \log(|x|) + C \;, \qquad x \neq 0.$$

⚠ This formula holds on $(-\infty, 0)$ and $(0, \infty)$, but expressions like

$$\int_{-1}^{1} \frac{1}{x}\;\mathrm{d}x$$

don't make sense!



Rational functions with linear denominator

- The substitution rule allows us to find antiderivatives of most rational functions with denominator of degree one or two.
- We have just seen one example:

$$\int \frac{1}{x}\,\mathrm{d}x = \log(|x|) + C\ , \qquad x \neq 0.$$

- This lets us approach any integral of the type

$$\int \frac{1}{ax+b}\,\mathrm{d}x$$

by using the substitution $u = ax + b$. Easy!

Example 4.12

Find $\displaystyle\int \frac{1}{2x-3}\,\mathrm{d}x$

Rational functions with quadratic denominators

- For quadratic denominators, the integration technique depends on how the denominator $q(x)$ factorises.
- There are three possibilities:
  1. $q(x)$ does not factorise over the real numbers;
  2. $q(x)$ is the square of a linear factor;
  3. $q(x)$ is a product of two distinct linear factors.
- The first two cases can be integrated by substitution
  - In the first case, we rewrite the denominator by completing the square and using the antiderivative

$$\int \frac{1}{a^2 + u^2}\,\mathrm{d}u = \frac{1}{a}\arctan\left(\frac{u}{a}\right) + C.$$

  - The second case is most easily done by linear substitution.
- The last case requires a new technique (partial fractions), which we discuss soon.
- We first deal with the case where $q(x)$ does not factorise.

Completing the square

- Recall that a quadratic $p(x) = ax^2 + bx + c$ can be expressed as a perfect square plus a constant by completing the square:

$$ax^2 + bx + c = a\left(x + \frac{b}{2a}\right)^2 + r$$

where $r = c - \frac{b^2}{4a}$.

Example 4.13

Show that if $a > 0$ and $p(x)$ has no (real) roots, the remainder r is positive.

- **Question:** What can we do if $a < 0$ and $p(x)$ does not factorise?

Example 4.14

Find $\int \frac{1}{x^2 + 2x + 2} \, \mathrm{d}x$

⚠ This method will fail if the denominator has real linear factors. Do you see why?

Homework 83

Find $\int \frac{6}{x^2 + 2x + 5}\,\mathrm{d}x$

Linear numerator- the lucky case!

Example 4.15

Find $\displaystyle\int \frac{x+1}{x^2+2x+2}\,\mathrm{d}x$

Example continued

- This was easy because the numerator was a multiple of the derivative of the denominator. We are not always so lucky!

Linear numerator - the typical case

- If the numerator is a linear function of x, we first express the numerator in terms of the derivative of the denominator.

Example 4.16

Find $\displaystyle\int \frac{x-2}{x^2+2x+2}\,\mathrm{d}x$

Example continued

Homework 84

Find $\displaystyle\int \frac{3x+5}{x^2-4x+5}\,\mathrm{d}x$

Repeated linear factors - the easiest case!

Example 4.17

Find $\displaystyle\int \frac{2x+1}{x^2+2x+1}\,\mathrm{d}x$

Example continued

Homework 85

Find $\displaystyle\int \frac{2}{4x^2 - 12x + 9}\,\mathrm{d}x$

Partial fractions

The method of partial fractions is used to decompose a rational function into a sum of simpler fractions that we already know how to integrate. For instance

$$\frac{6x^4 - 6x^3 - 4x^2 + 12x - 14}{(x-1)^2(x+2)(x^2+1)} = \frac{3}{x-1} - \frac{1}{(x-1)^2} + \frac{2}{x+2} + \frac{x-4}{x^2+1}$$

You can check that an expression like this is correct by bringing the right hand side to a common denominator and then comparing the two sides. You should also check that we know how to integrate each of the terms on the RHS of the above example.

But how can we discover the right decomposition in the first place? There is a systematic way to do so, based on the way in which the denominator factorises.

Preliminaries

When integrating a rational function $\frac{p(x)}{q(x)}$ where p and q are polynomials:

- We need only know how to handle the case where the degree of p is strictly smaller than the degree of q.
- If not, polynomial long division gives a polynomial part plus a remainder that satisfies this requirement.
- The first step is to factorise q.
- As we have seen, q always factorises into linear and quadratic factors with real coefficients (but we may not be able to find them).
- The Partial Fractions Theorem tells us how to use this factorisation to break $\frac{p(x)}{q(x)}$ into pieces we can integrate.

The case of distinct factors

Theorem 4.18 (Partial Fractions with Distinct Factors)

Provided $\deg(p) < \deg(q)$

$$\frac{p(x)}{(x-a_1)\ldots(x-a_j)(x^2+b_1x+c_1)\ldots(x^2+b_kx+c_k)}$$

can always be written as

$$\frac{A_1}{x-a_1}+\cdots+\frac{A_j}{x-a_j}+\frac{B_1x+C_1}{x^2+b_1x+c_1}+\cdots+\frac{B_kx+C_k}{x^2+b_kx+c_k}$$

where $A_1,\ldots,A_j$, $B_1,\ldots,B_k$, $C_1,\ldots,C_k$ *are real constants.*

- Each linear factor $x-a$ gives a term $\dfrac{A}{x-a}$.
- Each quadratic factor x^2+bx+c gives a term $\dfrac{Bx+C}{x^2+bx+c}$.
- Our task is to find the A_i's, B_i's and C_i's.

Example 4.19

Decompose $\dfrac{9x+1}{(x-3)(x+1)}$ into partial fractions and hence find

$$\int \frac{9x+1}{(x-3)(x+1)}\,\mathrm{d}x.$$

Example continued

Example continued

⚠ What would happen if we wrote the numerator in terms of the derivative of the denominator $(x-3)(x+1) = x^2 - 2x - 3$?

Homework 86

Find $\displaystyle\int \frac{7}{x^2 + 3x - 10}\,\mathrm{d}x$

Example 4.20

Find $\displaystyle\int \frac{3x^2-2x+1}{(x+1)(x^2+2x+2)}\,\mathrm{d}x$

Example continued

Repeat offenders

- Recall that each distinct factor in the denominator gives a term in the partial fraction decomposition as follows:

$$\begin{aligned} (x-a) \quad &\text{gives} \quad \frac{A}{x-a} \\ (x^2+bx+c) \quad &\text{gives} \quad \frac{Bx+C}{x^2+bx+c} \end{aligned}$$

- On he other hand, where factors are repeated:

$$\begin{aligned} (x-a)^k \quad &\text{gives} \quad \frac{A_1}{x-a} + \frac{A_2}{(x-a)^2} + \cdots + \frac{A_k}{(x-a)^k} \\ (x^2+bx+c)^k \quad &\text{gives} \quad \frac{B_1x+C_1}{x^2+bx+c} + \frac{B_2x+C_2}{(x^2+bx+c)^2} + \\ &\qquad \cdots + \frac{B_kx+C_k}{(x^2+bx+c)^k} \end{aligned}$$

Uh oh!

- The integration techniques we have learned so far do not work for partial fraction terms of the following form:

$$\frac{C}{(x^2 + bx + c)^n}$$

  where $n > 1$ and $b^2 - 4c < 0$, so the denominator does not factorise over $\mathbb{R}$.
- Such terms can be integrated using techniques covered in MAST10006.

Example 4.21

Find $\displaystyle\int \frac{3x+1}{x^2+4x+4}\,\mathrm{d}x$

Example continued

A quicker way?

Example 4.22

Solve Example 4.21 using the linear substitution $u = x + 2$.

Homework 87

Find $\displaystyle\int \frac{2x-1}{x^2-6x+9}\,\mathrm{d}x$

- Polynomial long division may be required before attempting the partial fractions decomposition.

Example 4.23

Find $\int f(x)\,\mathrm{d}x$ where $f(x) = \dfrac{2x^3 - 3x^2 - 8x + 24}{x^2 - 4}$.

Example continued

Example continued

Homework 88

Practice your polynomial long division on these:

(a) $$\frac{2x^4 - 6x^3 + 14x^2 - 10x + 19}{x^2 - 3x + 5}$$

(b) $$\frac{5x^5 + 11x^4 - 3x^3 - 2x^2 - 2x + 1}{x^2 + 2x - 1}$$

Homework 89

Write down the partial fractions decomposition you would use for each of the following:

(a) $\dfrac{x+2}{x^2+4x-5}$

(b) $\dfrac{1-2x}{x^2+6x+9}$

(c) $\dfrac{3}{x^2-4}$

(d) $\dfrac{4x}{x^3-x^2}$

Homework 90

If you have studied linear algebra, use your knowledge of solving systems of equations to find the partial fraction decomposition of $\dfrac{-x+1}{(x+1)^2(x^2+2x+2)}$.

Hence find $\displaystyle\int \frac{-x+1}{(x+1)^2(x^2+2x+2)}\,\mathrm{d}x$.

Integration of rational functions - Summary

The technique of partial fractions gives a general procedure for integrating any rational function:

- If necessary, perform long division so that the numerator of the remainder has strictly smaller degree than the denominator.
- Factorise the denominator into linear and quadratic factors.
- Translate the factorisation into the appropriate form of the partial fraction decomposition.
- Solve to find the constants.
- Integrate each term of the partial fraction decomposition separately and put the results together.

⚠ Factorising the denominator can be difficult (or even impossible) if its degree is large.

⚠ Repeated quadratics in the denominator can give terms we don't yet know how to integrate.

Integrating the tems

- The trickiest terms we will need to integrate are of the form

$$\frac{p(x)}{q(x)} = \frac{\textit{linear}}{\textit{quadratic}}.$$

- **Case 1: Denominator factorises:** $q(x) = (x-a)(x-b)$
  - If $a \neq b$ use partial fractions.
  - If $a = b$ can use partial fractions but linear substitution is usually quicker.
- **Case 2: Denominator does not factorise:** First express as

$$\frac{mq'(x)+c}{q(x)} = \frac{mq'(x)}{q(x)} + \frac{c}{q(x)}$$

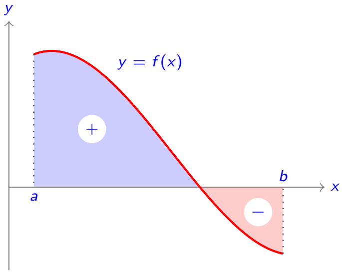
  - Integrate the $\frac{mq'(x)}{q(x)}$ term by substitution.
  - Integrate the $\frac{c}{q(x)}$ term by completing the square and using an arctan antiderivative.

Definite integrals and area between curves

Recall that the definite integral of a function f from a to b, denoted

$$\int_a^b f(x)\,\mathrm{d}x,$$

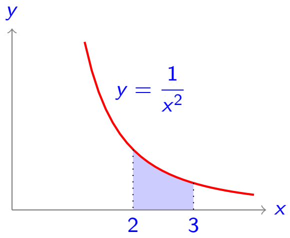
is defined to be the signed area between the graph of the function on $[a, b]$ and the x-axis.



We can use definite integrals to calculate the areas enclosed between curves. We compute the relevant definite integrals using the fundamental theorem of calculus.

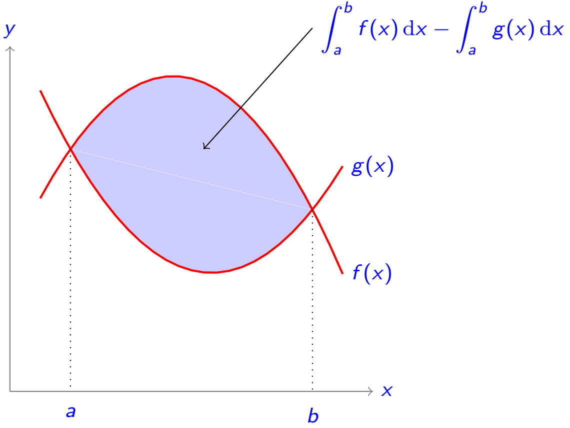
Example 4.24

Find the area enclosed by the curve $y = \frac{1}{x^2}$, the x-axis and the lines $x = 2$ and $x = 3$.



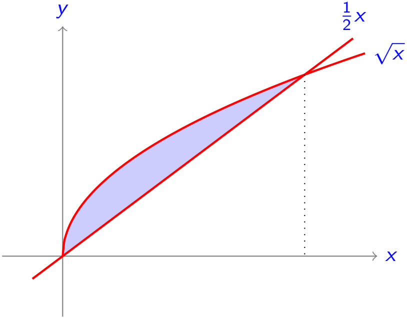
Area between two graphs

- The area enclosed by two graphs can be found by subtracting definite integrals.
- The terminals of these integrals occur where the curves cross.



Example 4.25

Find the area of the region enclosed by $y = \sqrt{x}$ and $y = \frac{1}{2}x$.



The curves intersect where:

Example continued

So area $=$

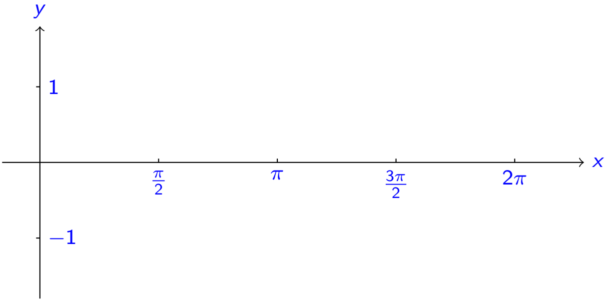
Always draw a sketch!

- Blindly evaluating a definite integral can lead to positive and negative areas (above/below the x-axis) cancelling out.
- Property 4 of definite integrals usually helps in this case.

Example 4.26

1. Evaluate the definite integral $\displaystyle\int_0^{2\pi} \sin(2x)\,\mathrm{d}x$.
2. Find the area enclosed by the curve $y = \sin(2x)$ and the x-axis for $0 \leq x \leq 2\pi$.

Example continued

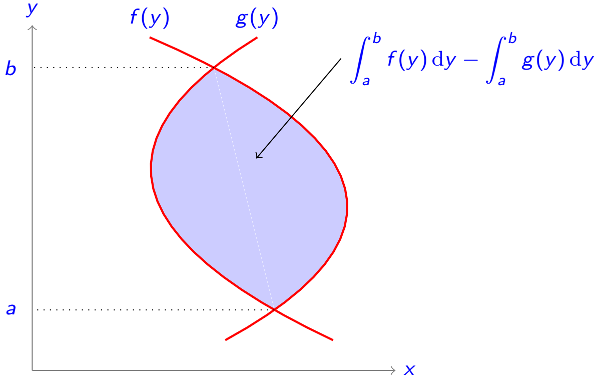


Homework 91

Find the area enclosed by the curves $y = x$ *and* $y = x^3 - x$.

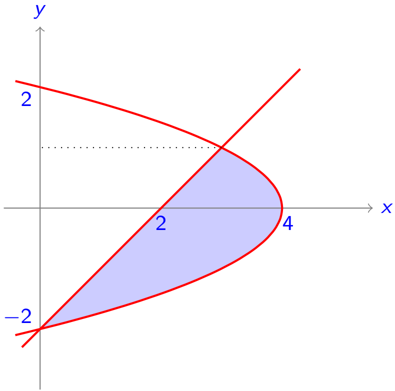
Going sideways

- Suppose we have two relations for which x is a function of y.
- Then we can find the area enclosed by the graphs of the two relations by integrating with respect to y.
- We still need to find where the curves intersect.



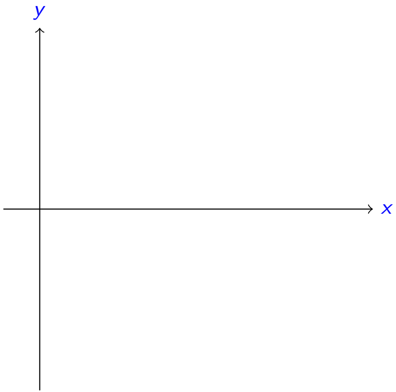
Example 4.27

Find the area enclosed by the curves $y = x - 2$ and $x = 4 - y^2$.



Example 4.28

Find the area enclosed by the curves $x^2(1 - y^2) = 1$ and $x = 2$.



Example continued

Introduction to differential equations

Now that we have mastered enough integration techniques we are ready to tackle what is arguably the most important part of calculus – the study of differential equations. These are a type of equation where the unknown is a function rather than a number or a set of numbers. Differential equations (DE's) arise in applications where our knowledge of a physical system is expressed in terms of the rate of change in some quantity.

Differential equations arise naturally in every branch of science and many areas of social science and economics as well. Indeed, a key reason why calculus is so indispensable to quantitative disciplines is the fact that differential equations are used to construct mathematical models that are essential to our understanding of quantitative relationships in so many fields.

What exactly is a DE?

- An equation involving a variable (say x), an unknown function (say y), and the derivatives of y with respect to x:

$$x, \; y, \; \frac{dy}{dx}, \; \frac{d^2y}{dx^2}, \; \ldots, \; \frac{d^ny}{dx^n}$$

  is called an (ordinary) differential equation or DE.

- The order of the highest derivative in the DE (n in the above list), is called the order of the DE.

Example 4.29

$\dfrac{dy}{dx} = -5y$ order =

$\dfrac{d^3y}{dt^3} + t\dfrac{dy}{dt} + (t^4 - 1)y = \sin(t)$ order =

$e^x f'(x) + f''(x) = f^3(x) + x$ order =

What exactly is a solution to an equation?

- To check a proposed solution to an algebraic equation is correct, we substitute it into both sides and see if it works.

Example 4.30

Consider the equation

$$x^4 + 3x^3 = 5 - x^5.$$

1. Is $x = 1$ a solution?
2. Is $x = 2$ a solution?

- Notice that we don't need to solve the equation here.

What exactly is a solution to a function equation?

- To check a proposed solution to a function equation we substitute it into both sides and see if it works for all x-values.

Example 4.31

Consider the following equation where y is a function of x:

$$y + \cos^2(x) = x^2 + 1.$$

1. Is $y = \sin^2(x) + x^2$ a solution?
2. Is $y = x^2$ a solution?

What exactly is a solution of a DE?

- A differential equation is really just a function equation involving some derivatives, so ...
- To check a proposed solution to a DE, we substitute it into both sides and see if it works for all x-values.

Example 4.32

Verify that $y = e^{3x}$ is a solution of the DE:

$$\frac{d^2y}{dx^2} = 15y - 2\frac{dy}{dx}.$$

Example continued

- ⚠ When we are asked to verify or prove that a given function is a solution of a DE, there is no need to solve the DE.
- In fact, it is logically incorrect to do so.

Homework 92

Use implicit differentiation to verify that any function y *for which*

$$\log(y) = xy^2 + C$$

is a solution of the differential equation

$$\frac{dy}{dx} = \frac{y^3}{1 - 2xy^2}.$$

Homework 93

Find the constants a, b *and* c *such that* $y = a + bx + cx^2$ *is a solution of the differential equation*

$$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 4y = 4x^2.$$

So many solutions

- **The bad news:** A typical DE has infinitely many solutions.
- **The good news:** The solutions typically have similar formulas, differing only by the presence of some constants.

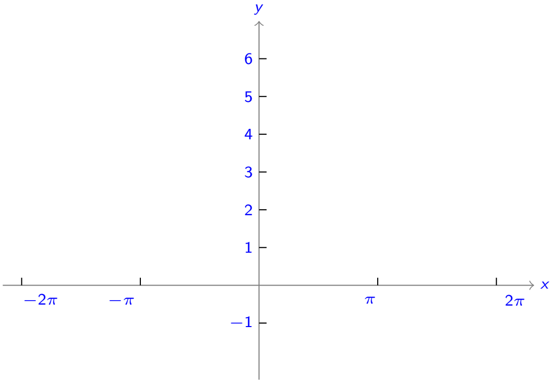
Example 4.33

Find a formula for all solutions of the following DE by integrating both sides:

$$\frac{dy}{dx} = \cos(x).$$

Example continued

- Sketch the graphs of some typical solutions of $\dfrac{dy}{dx} = \cos(x)$.



From the general . . .

- The general solution of a DE is a formula for all of the solutions of the DE.
- We can find the general solution a DE of the simple form

$$\frac{dy}{dx} = f(x).$$

where $f(x)$ is a known function by integrating both sides.
- This gives a constant of integration in the solution formula formula.
- Similarly, we can find the general solution a DE of the form

$$\frac{d^2y}{dx^2} = f(x).$$

by integrating both sides twice. This gives two constants of integration in the solution.
- . . . and so on.

...to the particular

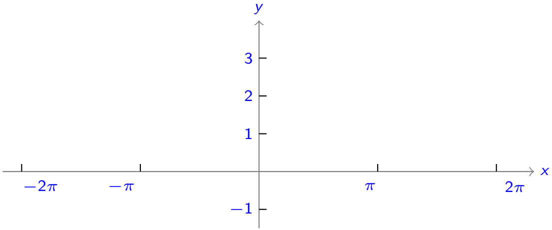
- Fixing the values of the constants of integration gives a particular solution.

Example 4.34

Sketch the particular solution of the DE

$$\frac{dy}{dx} = \cos(x).$$

with $C = 1$.



Initially yours

- In a first order DE arising in an application, we often know the value of the solution at some point.
- Such a constraint is called an initial condition or initial value.
- Knowing an initial value usually allows us solve for the constant of integration and hence find a specific solution.
- A DE together with one or more initial values is called an initial value problem.

Example 4.35

Find the *particular solution* of the DE

$$\frac{dy}{dx} = \cos(x),$$

subject to the initial condition $y = 3$ when $x = \frac{\pi}{2}$.

Example 4.36

Solve the initial value problem $f''(x) = x$ where $f(1) = 2$ and $f'(0) = 1$.

Example continued

⚠️ We must be careful about when to include our constants of integration. Just adding them at the end gives the wrong answer!

Homework 94

Suppose $y = Ae^{2x} + Bxe^{2x}$ *is the general solution of a second order DE. Find the particular solution satisfying the initial conditions* $y = 1$ *when* $x = 0$ *and* $\frac{dy}{dx} = 5$ *when* $x = 0$.

- We don't even need to know the DE to solve this problem!

Homework 95

Verify that the any function of the form

$$y = Ae^{3x} + Be^{-5x}$$

(where A and B are constants) is a solution of

$$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} - 15y = 0.$$

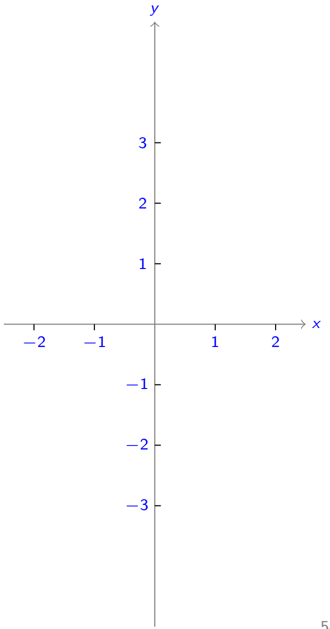
Example 4.37

Verify that the general solution of

$$f'(x) + \frac{1}{x} f(x) = 3x \quad \text{is} \quad f(x) = x^2 + \frac{C}{x}$$

where C is a constant. Sketch solutions with $C = -2, -1, 0, 1, 2$.

Example continued



Separable differential equations

As we shall see, DE's of the form

$$\frac{dy}{dx} = F(x)G(y)$$

where F and G are known functions, arise in many real world applications. They are called separable differential equations and they can usually be solved by a technique called separation of variables, which we now explore.

The first order DE's we solved in the preceding examples were all separable DE's in the special case $G(y) = 1$. In other words, they were of the (easy to solve) form

$$\frac{dy}{dx} = F(x)$$

We start with the (important) special case of separable DE's in which $F(x) = 1$, before moving on to the general case.

Autonomous differential equations

- First order DE's of the form:

$$\frac{dy}{dx} = G(y)$$

  (where G is a known function) are called autonomous.
- We can't just integrate both sides with respect to x because the RHS is now a function of y.
- Provided $G(y) \neq 0$, dividing both sides by $G(y)$:

$$\frac{1}{G(y)} \frac{dy}{dx} = 1. \tag{2}$$

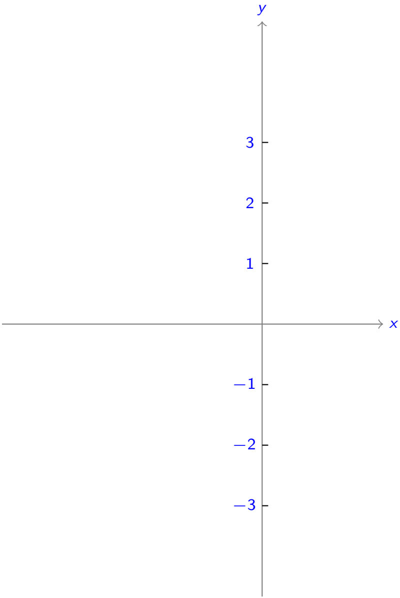
  allows us to integrate by substitution on the LHS and then (hopefully) integrate $\frac{1}{G(y)}$ with respect to y.
- We can then (hopefully) rearrange the resulting equation to find the formula for y as a function of x.
- We will consider what happens when $G(y) = 0$ later.

Example 4.38

Find the general solution of the following differential equation and sketch some typical solutions, assuming $y \neq 0$.

$$\frac{dy}{dx} = y.$$

Example continued

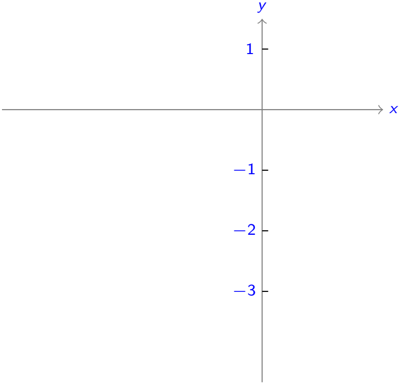


Example 4.39

Solve the initial value problem

$$\frac{dy}{dx} = y \qquad \text{where} \qquad y = -1 \text{ when } x = 0.$$

and sketch the solution.



The full story

- To solve a general separable DE
$$\frac{dy}{dx} = F(x)G(y).$$
our first step is to divide both sides by $G(y)$. Notice we are still assuming that $G(y) \neq 0$.
- This step, often called separating the variables, gives
$$\frac{1}{G(y)}\frac{dy}{dx} = F(x)$$
- We then integrate both sides with respect to x: and attempt to evaluate the integrals on each side.
- If possible, we solve the resulting function equation to obtain a formula for y (containing a constant of integration).

Example 4.40

Find the general solution of

$$\frac{dy}{dx} = -6xy^2.$$

Example 4.41

Find the general solution of

$$\frac{dy}{dx} = \frac{1+y}{x} \qquad \text{where} \quad y = 1 \text{ when } x = 1.$$

We can solve DE's using definite integrals

Example 4.42

Use definite integration to solve the initial value problem:

$$f'(x) = \frac{1}{2f(x)\sqrt{1-x^2}} \qquad \text{where} \quad f(0) = 3.$$

Example continued

What could possibly go wrong?

- Our method for solving separable DE's can fail at various steps:
  1. We may not be able to find a formula for $\int \frac{1}{G(y)}\, \mathrm{d}y$
  2. We may not be able to find a formula for $\int F(x)\, \mathrm{d}x$
  3. After integrating both sides, we may not be able to solve to obtain a formula for y as a function of x.

Example 4.43

Can we solve the following separable DE?

$$\frac{dy}{dx} = e^{x^2} e^{-y^2}$$

Example 4.44

Can we solve the following separable DE?

$$\frac{dy}{dx} = \frac{1}{6y^5 - 2y + 1}$$

In this example, we found an implicit equation for the solution.

⚠ Separable DE's in disguise

- Some DE's require a little rearrangement to put them into separable form.

Example 4.45

Show that the following DE is separable:

$$\frac{dy}{dx} = y^2 \sin(x) + y^2 - 4\sin(x) - 4$$

Homework 96

Solve the DE of Example 4.45. You will need to apply partial fractions to integrate the LHS. You may assume that $y > -2$.

Homework 97

Solve the initial value problem:

$$\frac{1}{3x^2y}\,\frac{dy}{dx} = \frac{1}{\sqrt{x^3-11}} \qquad \textit{where} \quad y = 1 \ \textit{when} \ x = 3.$$

Example continued

Constant solutions

- For any y value such that $G(y) = 0$, a general separable DE

$$\frac{dy}{dx} = F(x)G(y)$$

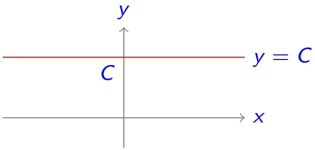
collapses to

$$\frac{dy}{dx} = 0.$$

which is very easy to integrate

$$\int \frac{dy}{dx}\,\mathrm{d}x = \int 0\,\mathrm{d}x \qquad \Rightarrow \qquad y = C.$$

- The solution in this case is a constant function - it takes the same value for every x.



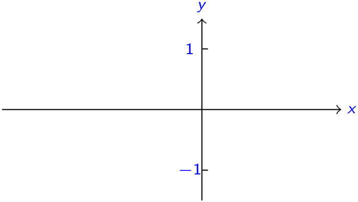
- This makes some initial value problems very easy to solve!

Example 4.46

Solve the initial value problem

$$\frac{dy}{dx} = y \qquad \text{where} \qquad y = 0 \text{ when } x = 0.$$

and sketch the solution.



- Any attempt to "separate the variables" here would mean dividing by zero.

⚠ When solving an initial value problem, first check for constant solutions.

Important, yet so very easy to find

- Constant solutions are important in the theory of differential equations.
- For separable DE's they are typically very, very, easy to find. We simply solve the equation $G(y) = 0$.

Example 4.47

Find any constant solutions of the following separable DE's and graph them.

1. $\dfrac{dy}{dx} = y^2 + y - 2$.
2. $\dfrac{dy}{dx} = e^y \sin(x)$.
3. $\dfrac{dy}{dx} = \cos^2(y) e^x$.
4. $\dfrac{dy}{dx} = y^{\frac{1}{3}}$.

Example continued

Solutions of Separable DE's

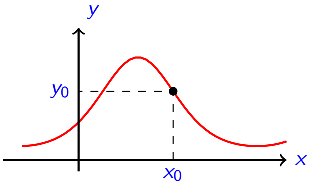
Theorem 4.48

Suppose F and G' are both defined and continuous on the interval I and let $x_0, y_0 \in I$. Then the initial value problem

$$\frac{dy}{dx} = F(x)G(y) \qquad y = y_0 \textit{ when } x = x_0$$

has a unique solution (on some interval $J \subseteq I$ containing x_0).

- A unique solution curve passes through (x_0, y_0).



Example 4.49

Use the constant solutions found in Example 4.47(1) and Theorem 4.48 to estimate the range of the solution of

$$\frac{dy}{dx} = y^2 + y - 2 \qquad \text{where} \quad y = 0 \text{ when } x = 0.$$

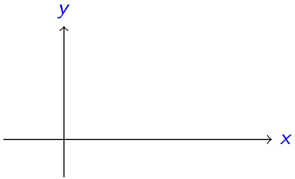
Example 4.50

Consider the initial value problem

$$\frac{dy}{dx} = y^{\frac{1}{3}} \qquad \text{where} \qquad y = 0 \text{ when } x = 0.$$

1. Find the constant solutions.
2. Find the general solution.
3. How many solution curves pass through the point $(0, 0)$?
4. Why does this not contradict Theorem 4.48?

Example continued



Population growth

The simplest model for population growth assumes that the population is not constrained by environmental limitations. In this case, the number of births and the number of deaths are both proportional to the current population, and so the overall growth rate is also proportional to the current population. This yields what is known as an exponential (or Malthusian) growth model, which forecasts exponential growth in population.

While exponential growth models have some validity when the constraints on growth are negligible, no population can grow exponentially forever. To obtain more realistic growth models, we must account for the effects of overcrowding and competition for resources. This leads to various models, the best known being the logistic growth model.

Exponential growth model

- Suppose the number of births and deaths in a population P are both in constant proportion to P at each point in time, so
  - Births $= bP$
  - Deaths $= dP$

  where b is the birth rate and d is the death rate per unit time.
- Then the rate of change at time t is given by

$$\frac{dP}{dt} = bP - dP = (b - d)P$$

- Letting $k = b - d$ gives the exponential (or Malthusian) growth model for a population:

$$\frac{dP}{dt} = kP$$

  an easily solved separable DE.

Example 4.51

The rate of growth of a certain mouse population is proportional to the current number of mice present, such that:

$$\frac{dM}{dt} = 0.2M,$$

where M is the number of mice t weeks after observation begins.

1. Given that there are initially 50 mice, determine the number of mice $M(t)$ at any time.
2. When will the mouse population reach 500?

Example continued

Homework 98

A certain culture of bacteria is growing at a rate proportional to the current number B of bacteria present. If initially there are 1000 bacteria and after 1 hour this number has doubled:

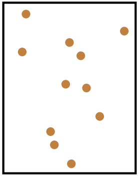
1. *Find the number of bacteria $B(t)$ present at any time t.*
2. *How many bacteria are present after $1\frac{1}{2}$ hours?*
3. *When will the number of bacteria reach 1 million?*

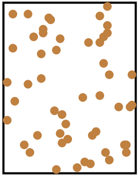
Logistic growth model

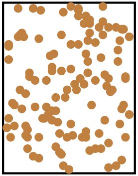
- In the long term, no population can grow exponentially, so the exponential growth model
$$\frac{dP}{dt} = kP$$
eventually breaks down.
- Adding an extra term gives a logistic growth model:
$$\frac{dP}{dt} = kP - rP^2.$$
- The term $-rP^2$ gives a reduced growth rate due to overcrowding and the DE is usually written as
$$\frac{dP}{dt} = kP\left(1 - \frac{r}{k}P\right).$$
- But why should the effects of overcrowding be proportional to the square of the population size?

Law of mass action

- The simplest model expresses overcrowding in terms of the probability of encounters between individuals.


Small population, few encounters.


Medium population, more encounters.


Large population, many encounters.

- Assuming individuals are placed randomly, this should be proportional to the number of pairs of distinct individuals.
- In a population of size P, the number of pairs of distinct individuals is approximately $\frac{1}{2}P^2$.
- Hence we model effects of overcrowding as proportional to P^2.
- This idea is similar to the law of mass action in chemistry.

Example 4.52

Use partial fractions to solve the logistic initial value problem

$$\frac{dP}{dt} = P\left(1 - \frac{P}{4}\right) \qquad \text{where} \qquad P = 1 \text{ when } t = 0.$$

You may assume that growth is positive.

Example continued

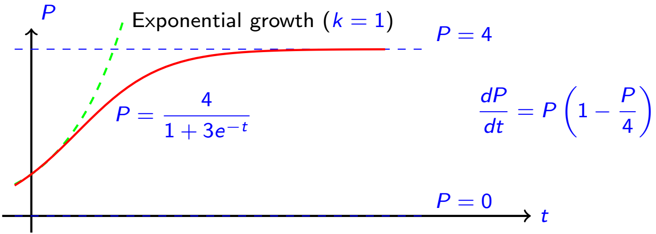
Example continued

Solutions of the logistic DE

- The general logistic growth DE

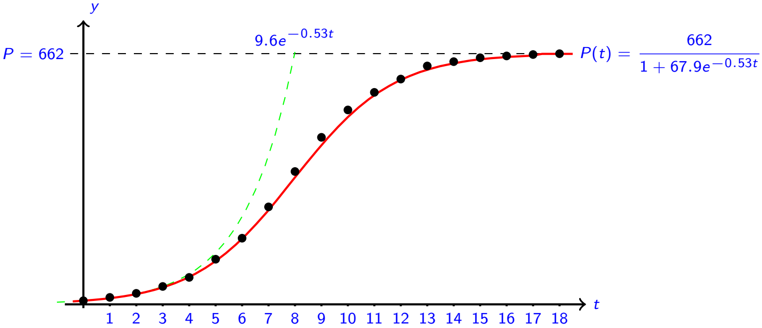
$$\frac{dP}{dt} = kP\left(1 - \frac{r}{k}P\right)$$

  has constant solutions $P = 0$ and $P = \frac{k}{r}$.
- The solution $P = \frac{k}{r}$ represents the theoretical maximum population called the carrying capacity.
- Solution curves are sigmoidal (elongated "S" shaped) and approximately exponential when the population is small.



Logistic Growth - a famous experiment

- In spite of its simplicity, the logistic model has been "unreasonably successful".
- It describes the growth of many experimental populations remarkably well.



- Yeast culture example from *Raymond Pearl, Biology of Population Growth, 1925*.

More separable equations

We conclude with two more applications of separable DE's:

- The heating or cooling of inert objects in an environment of a different temperature.
- The growth of a simple organism like a tree.

Newton's Law of Cooling

- The rate at which an object cools (or heats) is proportional to the difference between its temperature T and the temperature T_s of its immediate surroundings. That is,

$$\frac{dT}{dt} = -k(T - T_s),$$

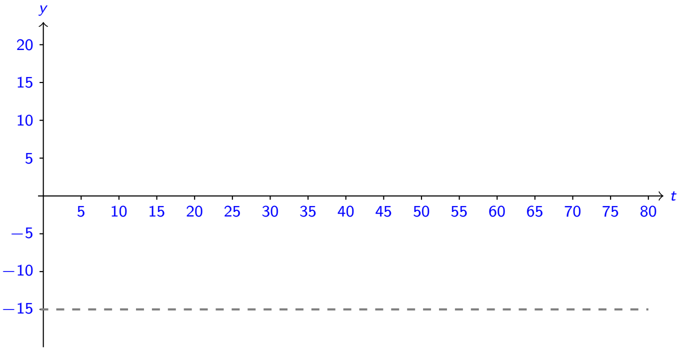
  where k is a positive constant.
- In applications, we can often determine k from experimental or observational data.

Example 4.53

A loaf of bread is placed in a freezer whose temperature is a constant -15°C. The bread obeys Newton's Law of Cooling. If the temperature T of the bread is initially 20°C and it takes 20 minutes for it to drop to 10°C, how long will it take for the bread to reach 0°C?

Example continued

Example continued



A Model for Tree Growth

- A simple initial value problem for growth of a tree

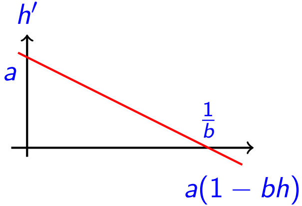
$$h'(t) = a(1 - bh(t)), \qquad h(0) = 0$$

is derived by assuming that per unit time:

1. Volume of photosynthetic nutrient production $\propto$ surface area.
2. Number of cell deaths $\propto$ mass.

- a and b are constants, typically depending on the species.

- The range of h values for which this model is relevant is $[0, \frac{1}{b})$.

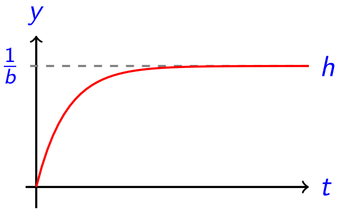


Tree growth model solution

- Solving the initial value problem for tree growth gives

$$h(t) = \frac{1}{b}(1 - e^{-abt})$$

- As $t \longrightarrow \infty$ the height asymptotically approaches the growth limit $\frac{1}{b}$.



Example 4.54

Verify this solution and find the growth rate when $t = 0$.

⚠ When verifying an initial value problem, don't forget to check that the initial value holds.

Example 4.55

Suppose that Ents grow to an average maximum height of 20m and that at 1 year of age their average height is 2m.

1. Find the average growth rate of an Ent at birth.
2. Hence find the average height of an Ent at age 2.

Example continued

End of lectures!

To finish the subject and prepare for the exam, you should ensure that you have:

- completed all lecture examples, and worked through all homework problems;
- finished all problems in the handbook;
- finished all practice class and workshop sheets.
- read the exam revision guide on the LMS.
- Use the pre-exam consultation sessions.