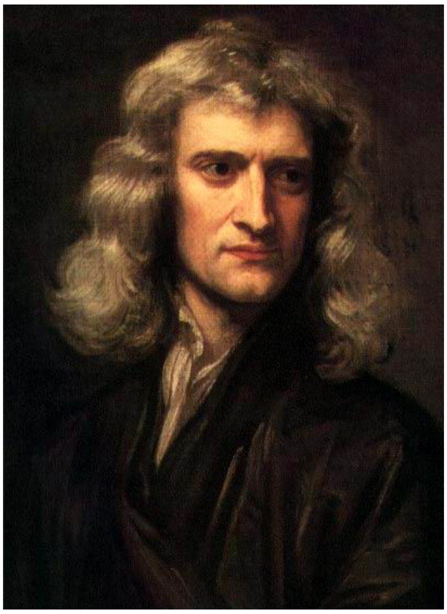
The University of Melbourne
School of Mathematics and Statistics

MAST10005
Calculus 1
Semester 2, 2018



Student name: ______________________________

Email address: ______________________________

Cover pictures: Gottfried Leibniz (left) and Sir Isaac Newton (right) independently discovered calculus and share the credit for its development. Leibniz (1646–1716) was a German mathematician, philosopher, physicist, natural scientist and engineer, who also wrote about politics, law, ethics, history and theology. Newton (1642–1726) was an English mathematician, physicist, astronomer, natural philosopher and alchemist, and is regarded by many as the greatest scientist in the history of mankind.

Contents

MAST10005 Calculus 1

Semester 2, 2018

Subject Organisation

MAST10005 Calculus 1 is a mathematics subject designed for students who have not completed a subject equivalent to VCE Specialist Mathematics 3/4. After successfully completing MAST10005 Calculus 1, you may continue their study of mathematics by proceeding to MAST10006 Calculus 2 and/or MAST10007 Linear Algebra. You may also choose to study statistics by choosing MAST10010 Data Analysis 1.

Syllabus

Calculus 1 builds on knowledge you will have acquired in previous mathematics subjects, such as VCE Mathematical Methods 3/4. It also introduces new topics.

One of the important assumed areas of knowledge is trigonometric functions and trigonometric identities including the compound and double angle formulae. It is therefore essential that you brush up on your trigonometry skills! This subject will also extend your knowledge of trigonometric functions to reciprocal trigonometric functions and inverse trigonometric functions.

You will make a careful study of set notation and proofs involving sets. You will learn about a new number system: the complex numbers. As you will see, this set is so large that it contains the real numbers as a proper subset. You will learn arithmetic of complex numbers and how to represent them in cartesian and polar form. We can then use them to find roots of polynomials that are not solvable over the reals (e.g., $x^2 + 1$).

You will make a careful study of *functions*, their domains, ranges and inverses. We will introduce you to the ideas of vectors and the important concept of vector valued functions. You will learn vector arithmetic and algebra, study linear independence, scalar products and explore some geometric applications.

You will learn how to extend differentiation techniques to implicit differentiation, derivatives of inverse trigonometric functions, and second and higher order derivatives. We will apply these techniques to curve sketching and some simple modelling and optimisation problems. You will be introduced to the calculus of vector valued functions.

You will also learn various new techniques of integration and apply integration techniques to calculate areas between curves and to solve simple ordinary differential equations.

Classes and Consultations

The subject MAST10005 Calculus 1 has

- three one-hour lectures per week;
- a one-hour practice class each week.
- a one-hour workshop each week.

All of these classes start in the first week of semester. Details of your classes are given on your personal timetable.

The lecturer and tutors will be available for individual consultation, starting in week 1. Consultation hours will be announced in lectures and be accessible from the LMS.

Lectures

There is only 1 lecture stream:

Stream	Lecturer	Times
1	Dr. John Banks	**Monday 9:00**, Carillo Gantner Theatre (Asia Centre). **Thursday 3:15**, Carillo Gantner Theatre (Asia Centre). **Friday 4:15**, Carillo Gantner Theatre (Asia Centre).

The subject coordinator is Dr. John Banks:

- **Office:** Room G41, Peter Hall Building
- **Phone:** 8344 3687
- **Email:** john.banks@unimelb.edu.au

Reading and Resources

Lecture Notes

Lecture notes will be available on the LMS. These contain an outline of the theory of each topic, with space left for examples to be completed in lectures. These worked examples are only available in the lectures.

Recommended Textbook

- Hass, Weir and Thomas - *University Calculus Early Transcendentals*, 3rd edition, Pearson 2016.

How to Use This Booklet

There are four problem sheets in this booklet, corresponding to the four major topics covered in lectures. Deliberately brief answers appear at the end. Working through these problems in your own time is the best way to ensure you have understood the lecture material. You should seek help in a consultation session or from the tutor-on-duty service (see below) if there are answers you still don't understand after an earnest attempt.

LMS

Learning materials such as the assignments, the consultation roster and other announcements will be available on the subject LMS site:

www.lms.unimelb.edu.au

If you have trouble viewing or printing any of the PDF documents in your browser, try 'right clicking' on the links and selecting 'save link as' or 'download link as file'. Then open the downloaded PDF in your favourite viewer.

Assessment

The assessment is composed of two parts:

- a three-hour exam at the end of the semester;
- ten written assignments.

Final mark = 80% Exam mark + 20% Assignment mark

There are no hurdle requirements.

Assignments

- The remaining assignments are due weekly on Fridays at 4 pm sharp, starting from Friday 10 March.
- The weekly assignment questions will be distributed during lectures, a week prior to their due date, and posted on the LMS.
- Make sure that you write your name, student number, tutor's name and tutorial time on each assignment.
- Your assignment should be placed into the appropriate assignment box, ground floor, Peter Hall Building (just around the corner from the vending machine). Please ensure that you place the assignment in your tutor's box for MAST10005 Calculus 1.
- Late assignments will be accepted only with a medical certificate (or similar documentation) and will not count towards assessment unless they are received by 10:00AM on the Monday following the due date. Under these circumstances, assignments must be taken to the coordinator, Dr. John Banks in room G41, Peter Hall Building.
- Assignment 1 must be submitted with a completed plagiarism cover sheet, which will cover all assignments in MAST10005 during semester 2. The assignments will not be marked until a completed plagiarism cover sheet has been submitted.

Calculators and Formula Sheets

Students are not permitted to use calculators and formula sheets (other than the one supplied) in the end of semester exam. Assessment in this subject concentrates on the testing of concepts and the ability to conduct procedures in simple cases. There is no formal requirement to possess a calculator for this subject. If you have a calculator, then you may find it useful occasionally for questions on the example sheets.

Getting help!

The first source of help is the person beside you in lectures and practice classes, who is doing the same problems as you. Remember though, that fellow students have no obligation to help you, nor you to help them. Forming a small study group of two to four people is a preferred mode of "trading secrets" for many students.

The next source is your tutor: students are expected to attend all practice classes, where you can gain valuable practice and help with problems.

The lecturer and some tutors have rostered consultation hours to provide help on an individual basis. Attendance is on a voluntary basis. Details will be announced in lectures and provided on the LMS.

You can also get help, particularly with basic mathematics skills from the tutor-on-duty in the *mathSpace* study area located near the Evan Williams Theatre in the Peter Hall Building. The hours for this service are posted in *mathSpace*.

If something major goes wrong, and you need to be absent for more than a couple of days, inform your lecturer or the subject coordinator.

Assumed Knowledge

The prerequisite for MAST10005 is VCE Mathematical Methods 3/4 or equivalent.

The expectation in this subject is that each student spend at least 4 hours per week outside class working on Calculus 1 material. This may consist of reading over lecture notes, doing problems from the booklet or textbook, doing the weekly assignment or reviewing practice class questions.

Useful Formulae

Pythagorean identity

$$\cos^2(x) + \sin^2(x) = 1$$

Compound angle formulae

$$\begin{aligned}
\sin(x+y) &= \sin(x)\cos(y) + \cos(x)\sin(y) \\
\cos(x+y) &= \cos(x)\cos y - \sin(x)\sin(y) \\
\tan(x+y) &= \frac{\tan(x)+\tan(y)}{1-\tan(x)\tan(y)}
\end{aligned}$$

Antiderivatives from inverse trigonometric functions

$$\begin{aligned}
\int \frac{1}{\sqrt{a^2-x^2}}\,dx &= \arcsin\left(\frac{x}{a}\right) + C \\
\int \frac{-1}{\sqrt{a^2-x^2}}\,dx &= \arccos\left(\frac{x}{a}\right) + C \\
\int \frac{1}{a^2+x^2}\,dx &= \frac{1}{a}\arctan\left(\frac{x}{a}\right) + C
\end{aligned}$$

where $a > 0$ is constant, and C is an arbitrary constant of integration.

Complex exponential formulae

$$\begin{aligned}
e^{i\theta} &= \cos(\theta) + i\sin(\theta) \\
\cos(\theta) &= \frac{1}{2}\left(e^{i\theta} + e^{-i\theta}\right) \\
\sin(\theta) &= \frac{1}{2i}\left(e^{i\theta} - e^{-i\theta}\right)
\end{aligned}$$

Vector projections

- The *vector projection* of $\mathbf{v}$ onto $\mathbf{u}$ is $\mathbf{v}_\parallel = (\hat{\mathbf{u}}\cdot\mathbf{v})\hat{\mathbf{u}} = k\mathbf{u}$, where $k \in \mathbb{R}$ is the unique solution of $\mathbf{u}\cdot(\mathbf{v} - k\mathbf{u}) = 0$.
- The *vector component of* $\mathbf{v}$ *perpendicular to* $\mathbf{u}$ is $\mathbf{v}_\perp = \mathbf{v} - \mathbf{v}_\parallel$.

Complex roots

The n-th roots of $w = se^{i\phi}$ are $s^{\frac{1}{n}}\, e^{i(\frac{1}{n}(\phi+2k\pi))}$ for $k = 0, 1, \ldots, n-1$.

Changes in speed

Provided $\mathbf{r}'(t) \neq \mathbf{0}$, the speed function $\|\mathbf{r}'(t)\|$ is decreasing when $\mathbf{r}'(t)\cdot\mathbf{r}''(t) < 0$ and increasing when $\mathbf{r}'(t)\cdot\mathbf{r}''(t) > 0$.

Topic 1: Numbers and Sets

Sets

1. Subsets. Decide whether each statement is true or false and prove that your decision is correct.

(a) $\{x \in \mathbb{R} \mid \tan(x) = 1\} \subseteq \{x \in \mathbb{R} \mid \sin(x) \leq \cos(x)\}$

(b) $\{x \in \mathbb{R} \mid e^x < 1\} \subseteq \{x \in \mathbb{R} \mid x^3 < 0\}$

(c) $\left\{\cos\left(\frac{k\pi}{3}\right) \,\middle|\, k \in \mathbb{Z}\right\} \subseteq \mathbb{Z}$

(d) $\left\{\cos\left(\frac{k\pi}{3}\right) \,\middle|\, k \in \mathbb{Z}\right\} \subseteq \mathbb{Q}$

2. Both ways. For the sets $A = \{z \in \mathbb{C} \mid \bar{z} = z\}$ and $B = \{z \in \mathbb{C} \mid z^2 \in \mathbb{R}\}$, which of the following is true?

(a) $A \subseteq B$.

(b) $B \subseteq A$.

In each case *prove* your answer.

3. Irrationals. (a) Prove that $\log_3(2)$ is irrational.

(b) Use (a) to prove that $\log_3(\frac{1}{2})$ is irrational.

4. More Irrationals. (a) What is the final digit of a positive integer power of 5?

(b) What final digits are possible for a positive integer power of 9.

(c) Use (a) and (b) to prove that $\log_9(5)$ is irrational.

5. Unions and complements. Express each of the following sets as a union of intervals and as a set complement. Drawing graphs may help.

(a) $\{x \in \mathbb{R} : 1 - x^2 < 3x^2\}$.

(b) $\{x \in \mathbb{R} : \sin(x) \geq 0\} \cap [0, 6\pi)$.

(c) $\{x \in \mathbb{R} : e^{-x} \geq e^x\}$.

(d) $\{x \in [0, 2\pi] \mid \sin(x) < \cos(x)\}$.

6. More Set Calculations. Express the following sets in the simplest possible form.

(a) $\mathbb{Q} \cap \{x \in \mathbb{R} : 3^x = 2\}$

(b) $\mathbb{Q} \cap \{x \in \mathbb{R} : 2x^2 - 3x + 1 = 0\}$

(c) $\mathbb{Z} \cap \{x \in \mathbb{R} : 2x^2 - 3x + 1 = 0\}$.

(d) $\{x \in \mathbb{R} : \sin(x) \in \mathbb{Z}\}$.

Inequalities

7. Solving inequalities. Solve the following inequalities, expressing your answers in valid set notation.

(a) $3 - 2x \leq -1 - x$

(b) $3 - 2x \leq -1 - 2x$

(c) $1 \leq \frac{1}{x^2} < 4$

8. More interesting inequalities Solve the following inequalities, expressing your answers in valid set notation. You may use the fact that log is order preserving on its entire domain.

(a) $e^{2x} - 2 \geq e^x$ [Hint: The quadratic formula will be useful.]

(b) $1 - \cos^2(x) \leq \sin^2(x)$

(c) $1 - \cos^2(x) < \sin^2(x)$

Complex Numbers

Note: In these problems, the principal argument should be used whenever the complex number is written in polar or exponential form, that is $-\pi < \theta \leq \pi$.

9. Square root of negative numbers. Represent the following square roots as imaginary numbers.

(a) $\sqrt{-4}$ (b) $\sqrt{-6}$ (c) $2\sqrt{-12}$

10. Real and Imaginary. In each case, find $\text{Re}(z)$, $\text{Im}(z)$, and $\text{Re}(z)$–$\text{Im}(z)$.

(a) $z = 3 - 4i$ (b) $z = 6i$ (c) $z = 2 + 7i$

(d) $z = \frac{5}{7}$ (e) $z = \frac{1}{2}i + 2$

Complex Arithmetic

11. Complex Arithmetic. Let $z_1 = 2 + 6i,\ z_2 = \frac{1}{2} + 2i,\ z_3 = -6i,\ z_4 = -3 + 2i$ and $z_5 = 5$. Simplify:

(a) $z_1 + z_4$ (b) $z_1 + z_3$ (c) $z_1 - 2z_2$

(d) $z_4 + z_5$ (e) $z_2 + z_4$ (f) $3z_2 - 2z_4$

(g) $z_5 + z_3$ (h) $z_3 + z_2$ (i) $2z_1 + z_2 + 3z_3$

12. The Argand Plane. Let $z_1 = 2 + 4i,\ z_2 = -2 + 4i,\ z_3 = 2 - 4i$ and $z_4 = -2 - 4i$.

(a) Sketch z_1, z_2, z_3 and z_4 on the same set of axes. (b) Sketch $z_1, 2z_1$ and $-3z_1$ on the same set of axes. What do you observe?

(c) Sketch $z_1 + z_2,\ z_1 + z_4$ and $z_2 + z_3$ on the same set of axes.

(d) Is it true that

(i) $z_1 = -z_2$? (ii) $z_1 = -z_3$?

(iii) $z_1 = -z_4$? (iv) $z_2 = -z_3$?

13. Powers of i. Express i^2, i^3 and i^4 in cartesian form. Use your answers to express the following in cartesian form:

(a) $i^{23} = (i^4)^5 \times i^3$ (b) i^{14} (c) i^{259}

14. Cartesian form. Simplify each of the following products. You should express your answer in the form $a+ib,\ a,b\in\mathbb{R}$.

(a) $i(2+3i)$ (b) $(3-2i)^2$ (c) $(-2-i)(2+i)$

(d) $3(2+3i)$ (e) $(2+i)(2-i)$ (f) $(1+i)^3$

(g) $(3-2i)(1+i)$ (h) $(2+i)(-2+i)$ (i) $i^2(3+2i)$

15. Rationalising the Denominator. Write each of the following expressions $\frac{z_1}{z_2}$ in the form $a+ib,\ a,b\in\mathbb{R}$. In each case you should first write down $\overline{z_2}$.

(a) $\dfrac{1}{3-2i}$ (b) $\dfrac{2}{2+i}$ (c) $\dfrac{i}{2+3i}$ (d) $\dfrac{5-2i}{1-i}$

(e) $\dfrac{4+3i}{-2+i}$ (f) $\dfrac{1-i}{-3-2i}$

16. Real and Imaginary. Find the real and imaginary parts:

(a) $\mathrm{Im}\left(\dfrac{1-5i}{4+i}\right)$ (b) $\mathrm{Re}\left(\dfrac{2+7i}{4-6i}\right)$

17. Harder Examples. Express each of the following complex numbers in the form

$a+ib,\ a,b\in\mathbb{R}$.

(a) $\dfrac{i(1+2i)}{3(2-i)}$ (b) $\dfrac{3(2-i)}{i(1+2i)}$

(c) $\dfrac{(2+i)^2}{(1+3i)(2+3i)}$ (d) $\dfrac{(3+2i)(5-i)}{i^2(2-i)}$

18. Conjugates to Cartesian. Express the following numbers in the cartesian form $a+ib$ with a, b real:

(a) $\overline{12+7i}$ (b) $(1+i)^2+(\overline{1+i})$ (c) $(\overline{3+2i})(4+5i)$

(d) $(\overline{3-i})(2-i)(-1-3i)$ (e) $\overline{i^2(1-i)^3}$ (f) $\overline{\left(\dfrac{2-3i}{1+5i}\right)}$

19. Complex conjugates. Let $z=x+iy$ with x and y real, and let $\bar{z}$ be the complex conjugate. Show the following:

(a) $z\bar{z}=x^2+y^2$ (b) $z+\bar{z}=2\mathrm{Re}(z)$ (c) $z-\bar{z}=2i\mathrm{Im}(z)$

Sketch the following points in the complex plane:

(d) z and $\bar{z}$ (e) z and $-z$ (f) z and $\overline{-z}$

20. Complex Trigonometric. Express the following in cartesian form $a + ib$ with a, b real.

(a) $(\cos\theta + i\sin\theta)(\cos\theta - i\sin\theta)$

(b) $\dfrac{\cos\theta + i\sin\theta}{\cos\phi + i\sin\phi}$

(c) $\dfrac{1}{1 + \cos\alpha + i\sin\alpha}$

Modulus and Argument

21. Modulus and Argument. For each of the following complex numbers write down $|z|$ (the modulus of z) and Arg(z) (remember this is an angle in the range $(-\pi, \pi]$). You may find it useful to sketch the complex numbers on an argand diagram first.

(a) $z = 6$

(b) $z = -3$

(c) $z = -\sqrt{3} + i$

(d) $z = 4i$

(e) $z = -2i$

(f) $z = -1 - \sqrt{3}i$

(g) $z = 7i$

(h) $z = 1 + i$

(i) $z = 4 - 4i$

22. Something more complicated.

(a) Find the modulus of each of the following complex numbers without multiplying them into cartesian form:

(i) $\dfrac{5 + 2i}{2 + 5i}$

(ii) $\dfrac{-27i(8 + 2i)(2 + i)}{(4 + i)(4 - 3i)(4 - 8i)}$

(iii) $\dfrac{(2 - 3i)(-2 - 2i)(-5)}{(1 + i)(3 + 2i)}$

(b) Find an argument θ, $-\pi < \theta \leq \pi$, for the following complex numbers:

(i) $\dfrac{-2}{1 + i\sqrt{3}}$

(ii) $\dfrac{i}{-2 - 2i}$

(iii) $(1 + i)^4$

Regions in the Complex Plane

23. Regions in the Complex Plane. Sketch the following regions in the complex plane:

(a) $\mathrm{Re}(z) = 1$

(b) $\mathrm{Im}(z) > 1$

(c) $|z - 2i| \leq 1$

(d) $|z - 3| = |z + 2i|$

(e) $|z| \leq 4$, $0 \leq \arg(z) \leq \pi/4$

(f) $\mathrm{Re}(z^2) = 4$

Complex Exponential

24. Exponential Polar Form. Express each of the following complex numbers in polar form $re^{i\theta}$. In each case, chose an angle θ, $-\pi < \theta \leq \pi$:

(a) -1

(b) $5 + 5i$

(c) $(1 - i)^2$

(d) $2 + 5i$

25. Powers. Simplify the following powers of complex numbers:

(a) $(1 + i)^{20}$

(b) $(2\sqrt{3} + 2i)^5$

(c) $\left(\dfrac{1+i}{\sqrt{3}+i}\right)^{12}$

(d) $\left(\dfrac{1-i}{\sqrt{3}-i}\right)^{7}$

Finding Complex Solutions

26. Complete the square. Find all solutions for z in each of the following equations, completing the square where necessary.

(a) $z^2 + 1 = 0$

(b) $z^2 + 16 = 0$

(c) $z^2 - 4 = 0$

(d) $z^2 + 4z + 5 = 0$

(e) $z^2 + 3z + 3 = 0$

27. Quadratic formula. Find all solutions for z in each of the following equations by using the quadratic formula.

(a) $z^2 + 3z + 2 = 0$

(b) $z^2 + 3z + 4 = 0$

(c) $2z^2 - z + 3 = 0$

(d) $3z^3 + 2z^2 + z = 0$

28. Any way you like.

Find all solutions for z in each of the following equations. These can be done by factorising, or by rearranging and finding the n^{th} root.

(a) $z^4 - 81 = 0$

(b) $z^8 - 256 = 0.$

29. Roots and Factors. For each of the following find the roots of the given equations and sketch the roots in the complex plane:

(a) cube roots $z^3 = 1$

(b) square roots $z^2 = i$

(c) sixth roots $z^6 = -64$

(d) fifth roots $z^5 = 32e^{5\pi i/3}$

30. Roots of Polynomials. Find the roots of the following polynomials, using the complex exponential and roots of unity where necessary:

(a) $z^4 + 4z^2 + 4 = 0$

(b) $z^4 + 4z^2 + 16 = 0$

(c)$(z+i)^5-(z-i)^5=0$ (d) $z^4+z^3+z^2+z+1=0$

Powers of Sin and Cos

31. Polynomials in Cos and Sin. Using de Moivre's Theorem and equate real and imaginary parts to express the following as the sum of powers of $\cos\theta$ and $\sin\theta$ as indicated:

(a) $\sin(3\theta)$ in terms of $\sin\theta$

(b) $\cos(4\theta)$ in terms of $\cos\theta$ and $\sin\theta$

(c) $\sin(6\theta)$ in terms of $\sin\theta$ and $\cos\theta$

32. Sin and Cos of Multiple Angles. Using the complex exponential, express the following as sums of sines and cosines of multiple angles ($\cos n\theta$ and $\sin n\theta$):

(a) $\sin^3\theta$ (b) $\cos^4\theta$ (c) $\sin^7\theta$

Topic 2: Functions and Vectors

Basics of Functions

33. Functions or not? For each of the following, decide whether it is a function or not and give a short explanation:

(a) $f(x) = x^2 + x - 3$

(b) $f\colon \mathbb{R} \longrightarrow \mathbb{R}$ given by $f(x) = \dfrac{\sin(x)}{x-1}$

(c) $f\colon (-\pi, \pi) \longrightarrow \mathbb{R}$ given by $f(x) = \tan(x/2)$

34. Image of a Set under a Function. Find the image of the interval $(0, \pi]$ under each of the following functions:

(a) $f\colon \mathbb{R} \longrightarrow \mathbb{R}$ given by $f(x) = 3 - 2x$

(b) $g\colon \mathbb{R} \longrightarrow \mathbb{R}$ given by $g(x) = e^x$

(c) $h\colon \mathbb{R} \longrightarrow \mathbb{R}$ given by $h(x) = \sin(x)$

35. Injective Functions. For each of the following functions, find the largest intervals on which it is injective:

(a) $f\colon \mathbb{R} \longrightarrow \mathbb{R}$ given by $f(x) = 3 - 2x$

(b) $g\colon \mathbb{R} \longrightarrow \mathbb{R}$ given by $g(x) = e^{|x|}$

(c) $h\colon \mathbb{R}\backslash\{k\pi/2 \mid k \in \mathbb{Z}\} \longrightarrow \mathbb{R}$ given by $h(x) = \operatorname{cosec}(2x)$

36. Counting Functions. Suppose $n \in \mathbb{N}$. How many functions $f\colon \{a, b, c\} \longrightarrow \{1, 2, \ldots, n\}$ are there?

37. Counting Injective Functions. Suppose $n \in \mathbb{N}$. How many injective functions $f\colon \{a, b, c\} \longrightarrow \{1, 2, \ldots, n\}$ are there:

(a) if $n < 3$?

(b) if $n \geq 3$?

38. Counting Bijective Functions. Suppose $n \in \mathbb{N}$. How many bijective functions $f\colon \{1, 2, \ldots, n\} \longrightarrow \{a, b, c\}$ are there:

(a) if $n < 3$?

(b) if $n = 3$?

(c) if $n > 3$?

39. Inverse functions. For each of the following functions, either find the inverse or explain why the inverse does not exist:

(a) $f\colon (-\infty, 1] \longrightarrow [0, \infty)$ given by $f(x) = x^2 - 3x + 2$

(b) $f\colon (0, \infty) \longrightarrow [0, \infty)$ given by $f(x) = |\log(x)|$

(c) $f\colon \mathbb{R} \longrightarrow (0, \infty)$ given by $f(x) = 2e^x + 1$

Inverse Trigonometric Functions

40. Inverse Trigonometric. Evaluate each of the following in radians:

(a) $\arcsin\left(\frac{1}{2}\right)$ (b) $\arccos\left(\frac{\sqrt{3}}{2}\right)$ (c) $\arctan\left(-\sqrt{3}\right)$

41. Simplification. Simplify the following:

(a) $\sin\left(\arcsin\left(\frac{1}{2}\right)\right)$

(b) $\tan\left(\arcsin\left(\frac{1}{\sqrt{2}}\right)\right)$

(c) $\arccos\left(\sin\left(\frac{5\pi}{4}\right)\right)$

(d) $\cos\left(\arcsin\left(\frac{1}{2}\right)\right)$

(e) $\arctan\left(2\cos\left(\frac{5\pi}{6}\right)\right)$

(f) $\arccos\left(\cos\left(\frac{11\pi}{6}\right)\right)$

42. Graphing Inverse Trigonometric Functions. On the same graph sketch

(a) $\arccos(x)$

(b) $\arccos(x/3)$

(c) $\arccos(x/3 - 1/3)$

(d) $2 - \arccos(x/3 - 1/3)$

Domain and Range

43. Step by Step. For each of part (a) and (b)

(i) Find the implied domain and range of f.
(ii) Find the implied domain and range of g.
(iii) Find $\mathrm{dom}\,(f) \cap \mathrm{range}\,(g)$.
(iv) Use your answers to (i), (ii) and (iii) to find the implied domain and range of $f \circ g$.

(a) $f(x) = \sqrt{x}$; $g(x) = 4 - x^2$; $(f \circ g)(x) = \sqrt{4 - x^2}$.

(b) $f(x) = \cot(x)$; $g(x) = \arcsin x$; $(f \circ g)(x) = \cot(\arcsin(x))$.

44. Standard. State the (i) implied domain and (ii) range, of each of the following:

(a) $y = \dfrac{1}{1-x}$

(b) $y = \log(1+x)$

(c) $y = \sqrt{4-x^2}$

(d) $y = \dfrac{1}{\sqrt{4-x^2}}$

(e) $y = \sqrt{2\sin(x)}$

(f) $y = 2\log\left(\cos(x)\right)$

45. Inverse Trigonometric. State the (i) implied domain and (ii) range, of each of the following:

(a) $y = \arcsin(1 - x)$ (b) $y = \arccos(2x + 3)$

(c) $y = \arctan(4 - x)$ (d) $y = \arccos(\sin(2x))$

(e) $y = \tan(2\arcsin(x))$ (f) $y = \arccos(\sqrt{3}\tan(x))$

46. An incremental example. State the (i) implied domain and (ii) range, of each of the following:

(a) $y = |x|$ (b) $y = 2|x|$

(c) $y = 2|x| - 1$ (d) $y = \arcsin(2|x| - 1)$

47. Trigonometric. Express the following as algebraic functions of x:

(a) $f(x) = \cos(\arccos(x))$ (b) $g(x) = \sin(\arccos(x))$

State the implied domain and range of each function.

Vector Algebra

48. Introduction to vectors. Let $\mathbf{a} = 2\mathbf{i} + \mathbf{j}$ and $\mathbf{b} = -\mathbf{i} + \mathbf{j}$.

On the same set of x-y axes draw the vectors:

(i) $\mathbf{a}$ (ii) $\mathbf{b}$ (iii) $-\mathbf{a}$ (iv) $2\mathbf{b}$ (v) $\mathbf{a} + \mathbf{b}$ (vi) $\mathbf{b} - 3\mathbf{a}$

49. Vector Arithmetic. Let $\mathbf{u} = 3\mathbf{i} + \mathbf{j}$, $\mathbf{v} = 2\mathbf{i} + \mathbf{k}$, $\mathbf{w} = -\mathbf{i} - 2\mathbf{j} + 3\mathbf{k}$ be vectors in $\mathbb{R}^3$. Find

(a) $2\mathbf{u}$ (b) $-3\mathbf{u}$ (c) $\frac{1}{2}\mathbf{w}$

(d) $\mathbf{u} + \mathbf{v}$ (e) $\mathbf{w} - \mathbf{v}$ (f) $\mathbf{u} - 5\mathbf{v} + 2\mathbf{w}$

50. The Hexagon. $ABCDEF$ is a regular hexagon with centre O. If $\overrightarrow{OA} = \mathbf{a}$ and $\overrightarrow{OC} = \mathbf{c}$, express the vectors $\overrightarrow{OB}$, $\overrightarrow{FB}$, $\overrightarrow{DA}$, $\overrightarrow{AC}$ and $\overrightarrow{EA}$ in terms of $\mathbf{a}$ and $\mathbf{c}$.

Position and Length

51. Position and Length. For each of the following pairs of points find:

(i) the position vector in the direction from the first point to the second point

(ii) the length of this vector.

(a) $(0, 2)$ $(4, -5)$ (b) $(2, 3)$ $(5, 4)$ (c) $(4, -5)$ $(0, 2)$

(d) $(5, 4)$ $(2, 3)$ (e) $(3, 7)$ $(5, 7)$ (f) $(7, -3)$ $(3, -3)$

52. Unit Length, Unit Vector. Calculate unit vectors in the direction of each of the position vectors you have found in the previous question.

53. Let's Go Backwards. For each pair of vectors above, find the position vector in the direction from the second point to the first point.

54. More Vector Lengths. Let $\mathbf{u} = 3\mathbf{i} - 4\mathbf{j} + \mathbf{k}$ and let $\mathbf{v}$, the vector from the point $(2, 1, -1)$ to $(3, 2, -1)$, be vectors in $\mathbb{R}^3$. Find

(a) $\|\mathbf{u}\|$ (b) $\|\mathbf{v}\|$ (c) $\hat{\mathbf{u}}$

(d) $\hat{\mathbf{v}}$ (e) $2\mathbf{u} - \mathbf{v}$ (f) $\|2\mathbf{u} - \mathbf{v}\|$

(g) A unit vector parallel and in the same direction as $2\mathbf{u} - \mathbf{v}$

Dots and Angles

55. Dot product. Find $\mathbf{u} \cdot \mathbf{v}$ for each of the following pairs of vectors:

(a) $\mathbf{u} = 2\mathbf{i} + 3\mathbf{j} + \mathbf{k}$ and $\mathbf{v} = \mathbf{i} + 2\mathbf{j} - \mathbf{k}$ (b) $\mathbf{u} = 3\mathbf{i} - 2\mathbf{j} - \mathbf{k}$ and $\mathbf{v} = 2\mathbf{i} + \mathbf{j} + \mathbf{k}$

(c) $\mathbf{u} = \mathbf{i} + \mathbf{j} + \mathbf{k}$ and $\mathbf{v} = 3\mathbf{i} - \mathbf{j} - \mathbf{k}$ (d) $\mathbf{u} = \mathbf{i} + \mathbf{j}$ and $\mathbf{v} = 2\mathbf{i} - \mathbf{j} - \mathbf{k}$

(e) $\mathbf{u} = \mathbf{j} + \mathbf{k}$ and $\mathbf{v} = \mathbf{i} + \mathbf{j}$ (f) $\mathbf{u} = \mathbf{i} + \mathbf{j}$ and $\mathbf{v} = \mathbf{k}$

56. Angles. Find the angle between each pair of vectors $\mathbf{u}$ and $\mathbf{v}$ above.

57. Dot and Angle. If $\mathbf{a} = 3\mathbf{i} + \mathbf{j} - 2\mathbf{k}$ and $\mathbf{b} = \mathbf{i} + 3\mathbf{j} + \mathbf{k}$, find:

(a) $\mathbf{a} \cdot \mathbf{b}$ (b) the angle between $\mathbf{a}$ and $\mathbf{b}$

58. Perpendicular. Find a constant a in each case below, such that the pairs of vectors $\mathbf{u}$ and $\mathbf{v}$ are perpendicular.

(a) $\mathbf{u} = 3\mathbf{i} + a\mathbf{j}$ and $\mathbf{v} = 2\mathbf{i} + 6\mathbf{j}$ (b) $\mathbf{u} = a\mathbf{i} - 3a\mathbf{j} + 2\mathbf{k}$ and $\mathbf{v} = 4\mathbf{i} + 3\mathbf{j} - 2\mathbf{k}$

59. Parallel and Perpendicular. If $\mathbf{u} = 3\mathbf{i} + 4\mathbf{j} + 5\mathbf{k}$ which of the following are either parallel or perpendicular to $\mathbf{u}$?

(a) $-3\mathbf{i} - 4\mathbf{j} - 5\mathbf{k}$ (b) $5\mathbf{i} + 4\mathbf{j} + 3\mathbf{k}$ (c) $-3\mathbf{i}$

(d) $-5\mathbf{i} + 3\mathbf{k}$ (e) $-3\mathbf{j} + 5\mathbf{k}$

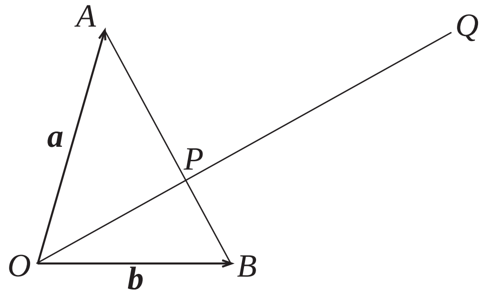
60. Vector Angles. Let $\mathbf{a} = (2, -1, 6)$ and $\mathbf{b} = (1, -1, -1)$.

(a) Show that $\mathbf{a} + \mathbf{b}$ is perpendicular to $\mathbf{b}$.

(b) Find the angle between the vectors $\mathbf{a} + \mathbf{b}$ and $\mathbf{a} - \mathbf{b}$.

Vector Geometry

61. Triangle. In the triangle OAB, $\overrightarrow{OA} = \mathbf{a}$ and $\overrightarrow{OB} = \mathbf{b}$. P is a point on AB such that the length of AP is twice the length of BP. Q is a point such that $\overrightarrow{OQ} = 3\overrightarrow{OP}$.



(a) Find each of the following in terms of $\mathbf{a}$ and $\mathbf{b}$:

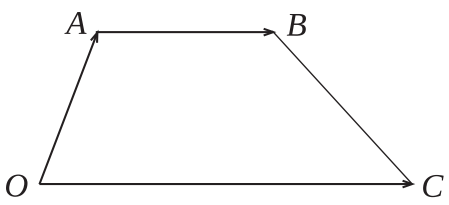
(i) $\overrightarrow{AB}$ (ii) $\overrightarrow{AP}$ (iii) $\overrightarrow{OP}$

(iv) $\overrightarrow{OQ}$ (v) $\overrightarrow{AQ}$

(b) Hence explain why $\overrightarrow{AQ}$ is parallel to $\overrightarrow{OB}$.

62. Trapezium. A trapezium is a quadrilateral with two parallel sides. Show that the points $A(4, 3, 0)$, $B(5, 2, 3)$, $C(4, -1, 3)$ and $D(2, 1, -3)$ form a trapezium and state the ratio of the parallel sides.

63. Another Trapezium. $OABC$ is a trapezium with $\overrightarrow{OC} = 2\overrightarrow{AB}$. Suppose $\overrightarrow{OA} = (2, -1, -3)$ and $\overrightarrow{OC} = (6, -3, 2)$.



(a) Find $\overrightarrow{AB}$.

(b) Express $\overrightarrow{BC}$ in terms of $\overrightarrow{OC}$, $\overrightarrow{OA}$ and $\overrightarrow{AB}$.

(c) Hence find $\overrightarrow{BC}$.

64. Parallelogram. $ABCD$ is a parallelogram. A, B and C are represented by the position vectors $\mathbf{i} + 2\mathbf{j} - \mathbf{k}$, $2\mathbf{i} + \mathbf{j} - 2\mathbf{k}$ and $4\mathbf{i} - \mathbf{k}$ respectively.

(a) Draw a sketch of the parallelogram with the vertices labelled with their coordinates.

(b) Find $\overrightarrow{AD}$.

(c) Find the angle at vertex A of the parallelogram.

65. Diagonals. $OABC$ is a parallelogram, where O is at the origin and A and C are defined by $\mathbf{a} = (2, 2, -1)$ and $\mathbf{c} = (2, -6, -3)$.

(a) Draw a diagram, and explain why the diagonals of the parallelogram are given by the vectors $\mathbf{a} + \mathbf{c}$ and $\mathbf{a} - \mathbf{c}$.

(b) Find:

(i) $\|\mathbf{a}+\mathbf{c}\|$ (ii) $\|\mathbf{a}-\mathbf{c}\|$ (iii) $(\mathbf{a}+\mathbf{c})\cdot(\mathbf{a}-\mathbf{c})$

(c) Hence find an angle between the diagonals of the parallelogram.

66. Right-Angled Triangle. ABC is a right-angled triangle with the right angle at B. If $\overrightarrow{AC} = 2\mathbf{i}+4\mathbf{j}$ and $\overrightarrow{AB}$ is parallel to $\mathbf{i}+\mathbf{j}$, find $\overrightarrow{AB}$.

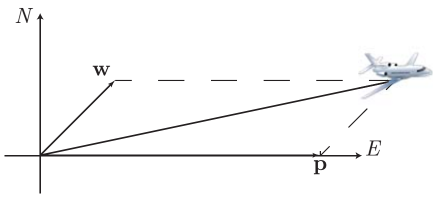
HINT: What is a simple expression for $\overrightarrow{AB}$? Use this to first find an expression for $\overrightarrow{BC}$.

67. Triangle and Parallelogram. Let OAB be a triangle such that the midpoint of $\overline{OA}$ is P, the midpoint of $\overline{OB}$ is Q, and the midpoint of $\overline{AB}$ is R. Let $\mathbf{a} = \overrightarrow{OA}$ and $\mathbf{b} = \overrightarrow{OB}$.

Use vector methods to show that $APQR$ is a parallelogram.

68. Flying Direction. If a plane encounters a wind, the velocity of the plane and the velocity of the wind can be described by vectors $\mathbf{p}$ and $\mathbf{w}$ respectively as shown below. The magnitude of the vector $\mathbf{p}+\mathbf{w}$ gives the ground speed of the plane and the direction of $\mathbf{p}+\mathbf{w}$ gives the true direction of the plane.

(a) Assume that a plane is flying due east with a velocity of 700 km/hour and is hit by a gust of wind blowing exactly north east with a velocity of 100 km/hour. Find the ground speed and the true course of this plane.



(b) Assume that the wind is blowing exactly north east with a steady velocity of $30\sqrt{2}$ km/hour. What direction and velocity should the pilot choose if he wishes to fly with a true course of due east and a ground speed of 700 km/hour?

Scalar and Vector Projections

69. Simple Projections. Let $\mathbf{a} = (-3, 4)$, $\mathbf{b} = (0, 2)$ and $\mathbf{c} = (3, -1)$.

(a) Find the scalar projection of $\mathbf{b}$ on $\mathbf{a}$.

(b) Find the scalar projection of $\mathbf{c}$ on $\mathbf{a}$.

(c) Find $\text{proj}_{\mathbf{a}}\mathbf{b}$, the vector projection of $\mathbf{b}$ in the direction parallel to $\mathbf{a}$.

(d) Find $\text{proj}_{\mathbf{a}}\mathbf{c}$, the vector projection of $\mathbf{c}$ in the direction parallel to $\mathbf{a}$.

(e) On the same set of x-y axes draw the vectors $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, $\text{proj}_{\mathbf{a}}\mathbf{b}$ and $\text{proj}_{\mathbf{a}}\mathbf{c}$.

70. Scalar Projection. For each of the following pairs of vectors, find (i) the scalar projection of $\mathbf{u}$ onto $\mathbf{v}$, (ii) the scalar projection of $\mathbf{v}$ onto $\mathbf{u}$

(a) $\mathbf{u} = 3\mathbf{i} + 2\mathbf{j} + \mathbf{k}$ and $\mathbf{v} = 4\mathbf{i} + \mathbf{j}$

(b) $\mathbf{u} = \mathbf{i} - 2\mathbf{j} + 3\mathbf{k}$ and $\mathbf{v} = 3\mathbf{i} - 2\mathbf{j} + \mathbf{k}$

(c) $\mathbf{u} = 2\mathbf{i} - 3\mathbf{k}$ and $\mathbf{v} = 2\mathbf{i} + \mathbf{j} - \mathbf{k}$

(d) $\mathbf{u} = 5\mathbf{i} + \mathbf{j} - 3\mathbf{k}$ and $\mathbf{v} = 2\mathbf{i} + 2\mathbf{k}$

(e) $\mathbf{u} = \mathbf{j} - 2\mathbf{k}$ and $\mathbf{v} = \mathbf{i} - \mathbf{k}$

(f) $\mathbf{u} = \mathbf{i} + \mathbf{j} + \mathbf{k}$ and $\mathbf{v} = \mathbf{j}$

71. Vector Projection. For each of the pairs of vectors above, find (i) the vector projection of $\mathbf{v}$ in the direction parallel to $\mathbf{u}$, (ii) the vector projection of $\mathbf{v}$ in the direction perpendicular to $\mathbf{u}$.

72. Scalar and Vector Projections. Let $\mathbf{u} = 3\mathbf{i}+4\mathbf{j}+2\mathbf{k}$ and $\mathbf{v} = \mathbf{i}+2\mathbf{j}-\mathbf{k}$. Calculate

(a) the scalar projection of $\mathbf{u}$ on $\mathbf{v}$

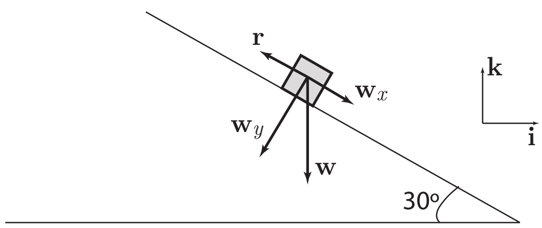
(b) the scalar projection of $\mathbf{v}$ on $\mathbf{u}$

(c) the vector projection of $\mathbf{v}$ in the direction parallel to $\mathbf{u}$

(d) the vector projection of $\mathbf{u}$ in the direction perpendicular to $\mathbf{v}$.

You should give a sketch describing your answers.

73. Gravity. [Optional question.] Assume that a box with mass $m = 50$ kgs is on a ramp as shown below. The magnitude of the weight force, $\mathbf{w}$, on the box is mg Newtons/sec in a direction vertically downwards, where $g = 9.8$ m/sec^2. In order to stop the box sliding a force of $\mathbf{r}$ is applied in the direction shown where $\|\mathbf{r}\| = \|\mathbf{w}_x\|$ and $\mathbf{w}_x =$ the vector projection of $\mathbf{w}$ in the direction parallel to the ramp. Find $\mathbf{r}$ and $\|\mathbf{r}\|$.



Parametric Curves

74. Equation of a Path. Let a particle's position as a function of time be as given below. Find the equation of the path for time $t \geq 0$. Sketch the graph of the path.

(a) $\mathbf{u} = t\mathbf{i} - 2t\mathbf{j}$

(b) $\mathbf{u} = t^2\mathbf{i} + t\mathbf{j}$

(c) $\mathbf{u} = (t-4)\mathbf{i} + 3t^2\mathbf{j}$

(d) $\mathbf{u} = t^3\mathbf{i} + t\mathbf{j}$

(e) $\mathbf{u} = (t+1)\mathbf{i} + (t^2 - 2t)\mathbf{j}$

75. Periodic Motion. For each of the following functions describing a particle's position at time t, (i) find the equation of the path, (ii) sketch its graph, (iii) state the period of motion of the path.

(a) $\mathbf{u} = \cos(2t)\mathbf{i} + \sin(2t)\mathbf{j}$

(b) $\mathbf{u} = 2\cos(2t)\mathbf{i} + 2\sin(2t)\mathbf{j}$

(c) $\mathbf{u} = (2 + \cos(t))\mathbf{i} + (-3 + \sin(t))\mathbf{j}$

76. Another Path. Let $\mathbf{r}(t) = (e^t - e^{-t})\mathbf{i} + (e^t + e^{-t})\mathbf{j} = x\mathbf{i} + y\mathbf{j}$ where $t \in \mathbb{R}$.

(i) find $y^2 - x^2$.

(ii) Hence find the equation of the path and sketch this.

Topic 3: Differential Calculus

Second and Higher Order Derivatives

1. First and Second Derivatives. Find the first and second derivatives of the functions:

(a) $f(x) = x^4 - 3x^3 + 16x$ (b) $h(x) = \sqrt{x^2+1}$ (c) $f(x) = \dfrac{x}{1-x}$

(d) $y = (1-x^2)^{3/4}$ (e) $f(x) = e^{2x-\sin(5x)}$ (f) $g(x) = \operatorname{cosec}(x)$

2. nth Derivatives. For the following functions evaluate the first few derivatives. Deduce the pattern and write down an expression for the n^{th} derivative $f^{(n)}(x)$:

(a) $f(x) = x^n$ (b) $f(x) = \dfrac{1}{3x^3}$

3. Related Derivatives. Assuming $g(x)$ is twice differentiable, that is, we can write down $g'(x)$ and $g''(x)$, write down expressions for $f''(x)$ if:

(a) $f(x) = xg(x^2)$ (b) $f(x) = g(\sqrt{x})$

Graph sketching

4. Graph sketching. For each of the following functions:

(a) $f(x) = \dfrac{1+x^2}{1-x^2}$ (b) $f(x) = \dfrac{x^3}{x^2+1}$ (c) $f(x) = x\sqrt{9-x^2}$

(d) $f(x) = e^{1/x}$ (e) $f(x) = \dfrac{1}{x^2-1}$

Determine where possible

- domain of f
- asymptotes of f
- x and y intercepts of the graph $y = f(x)$
- local extrema of f
- intervals over which f is increasing
- intervals over which f is decreasing
- intervals over which f is concave up
- intervals over which f is concave down
- points of inflection of f

Use these results to sketch the graph of $y = f(x)$.

Implicit Differentiation

5. Implicit Derivatives. Find $\dfrac{dy}{dx}$ by implicit differentiation:

(a) $x^2 - xy + y^3 = 8$

(b) $4\cos x \sin y = 1$

(c) $x \log_e y + \sqrt{y} = \log_e x$

(d) $x^2 y = e^{xy}$

(e) $\cos(x-y) = xe^x$

(f) $\log_e \frac{x}{y} = \cot(xy)$

6. Second Derivative. If $x^4 + y^4 = 16$, use the following steps to find y'': (a) Use implicit differentiation to find y'. (b) Use the quotient rule to differentiate the expression for y' from part (a). Express your answer in terms of x and y only. (c) Use the fact that x and y must satisfy the original equation $x^4 + y^4 = 16$ to simplify your answer to part (b) to $y'' = -48\dfrac{x^2}{y^7}$.

7. More Implicit Derivatives Suppose y is given implicitly as a function of x by the equation

$$x^2 + axy^2 + by^3 = 1.$$

Given that this curve passes through the point $(3, 2)$ and that the tangent line that touches the curve at this point has a gradient of -1, find a and b.

8. Implicit Tangent. Show, by implicit differentiation, that the equation of the tangent line to the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \ ,$$

at the point (x_0, y_0) is

$$\frac{x_0 x}{a^2} + \frac{y_0 y}{b^2} = 1 \ .$$

Inverse Trigonometric Functions

9. Derivatives. Find the derivative with respect to x of each of the following expressions:

(a) $\arcsin\left(\frac{x}{5}\right)$

(b) $\arccos\left(\frac{x}{4}\right)$

(c) $\arctan\left(\frac{x}{\sqrt{7}}\right)$

(d) $\arccos(4x)$

(e) $\arctan(7x)$

(f) $\arcsin(2x)$

10. More derivatives. Find $f'(x)$ if $f(x)$ is equal to:

(a) $\arccos\left(\frac{7x}{4}\right)$

(b) $\arcsin\left(\frac{8x}{5}\right)$

(c) $\arctan\left(\frac{7x}{2}\right)$

(d) $\arccos(4x-3)$

(e) $\arcsin\left(\frac{3-4x}{5}\right)$

(f) $\arctan\left(\frac{5x-3}{4}\right)$

11. Even more derivatives. Find $f'(x)$ if $f(x)$ is equal to:

(a) $\arccos(x^3)$ (b) $\arctan(e^x + 5)$ (c) $\arcsin(\sqrt[3]{x})$

(d) $\arccos(2\log(x))$ (e) $\arctan(\cos(3x) + 1)$ (f) $\arcsin\left(\frac{2x^2+1}{3}\right)$

Applications of Differentiation

12. Flying High. The height h, in metres, of a kite above the ground t minutes after it is let off is given by

$$h(t) = 12t - 2t^2 \ .$$

(a) Find when the kite reaches its maximum height, and the maximum height reached.

(b) Find the time when the kite falls to the ground.

13. Flying Higher. In a second attempt, the kite now flies at height

$$h(t) = \frac{1}{6}t^3 - \frac{9}{4}t^2 + 9t \qquad \text{for } 0 \le t \le 10.$$

(a) What is the maximum height reached by the kite in the first 5 minutes?

(b) What is the maximum height reached by the kite in the whole 10 minutes?

14. Cylinder. The radius, r cm, and height, h cm, of a solid circular cylinder vary in such a way that the volume of the cylinder is always 250π cm^3. (a) Draw the cylinder and write down expressions for its total surface area, A cm^2, and its volume, V cm^3, in terms of r and h.

(b) Show that the total surface area can be rewritten as:

$$A = 2\pi r^2 + \frac{500\pi}{r} \ .$$

(c) Find the minimum total surface area of the cylinder.

15. Rectangle. Find the maximum possible area of a rectangle which has its base on the x-axis and its upper vertices on the semi-circle $y = \sqrt{9 - x^2}$, $-3 \le x \le 3$.

16. Manufacturing. The sales revenue (in dollars) that a manufacturer receives for selling x units of a certain product can be approximated by the function

$$R(x) = 900\log\left(1 + \frac{x}{300}\right) \ .$$

However, each unit costs the manufacturer one dollar to produce, and the initial cost of adjusting the machinery for the production is \$200, so the total cost of production (in dollars) of x units is

$$C(x) = 200 + x \ .$$

(a) Write down the profit, $P(x)$ dollars, obtained by the production and sales of x units.

(b) Hence find the number of units which should be produced and sold for maximum profit, and calculate this maximum profit.

17. Rock n Roll. A travelling rock band is having a trailer made to carry their equipment. The roadies say the trailer must have a square base, an open top, and a volume of 32m^3. Find the dimensions (length of base side and the height) for the trailer to be made with the least amount of sheet metal.

Differentiating Parametric Functions

18. Velocity, speed and acceleration. For each of the following parametric curves $\mathbf{r} : \mathbb{R} \longrightarrow \mathbb{R}^2$, find formulas for the velocity, speed and acceleration at time t.

(a) $\mathbf{r}(t) = \cos(t)\mathbf{i} + \sin(3t)\mathbf{j}$.

(b) $\mathbf{r}(t) = t^5\mathbf{i} + t^7\mathbf{j}$.

(c) $\mathbf{r}(t) = e^{-t^2}\mathbf{i} + t^3\mathbf{j}$.

(d) $\mathbf{r}(t) = e^{-t^2}\mathbf{i} + e^{-t^3}\mathbf{j}$.

19. Cusps and smoothness. Decide whether each of the following parametric curves $\mathbf{r} : \mathbb{R} \longrightarrow \mathbb{R}^2$ is smooth. For those that are not smooth, find any cusps.

(a) $\mathbf{r}(t) = e^{-t^2}\mathbf{i} + t^3\mathbf{j}$.

(b) $\mathbf{r}(t) = t^5\mathbf{i} + t^7\mathbf{j}$

(c) $\mathbf{r}(t) = \cos(t)\mathbf{i} + \sin(3t)\mathbf{j}$.

(d) $\mathbf{r}(t) = \cos(2t)\mathbf{i} + \sin(t)\mathbf{j}$.

(e) $\mathbf{r}(t) = (2\cos(t) + \cos(2t))\mathbf{i} + (2\sin(t) - \sin(2t))\mathbf{j}$.

20. Variation in speed. For each of the following parametric curves, find the set of t values for which the speed is (i) increasing, (ii) decreasing and (iii) stationary.

(a) $\mathbf{r}(t) = \frac{1}{2}t^2\mathbf{i} - (\frac{1}{3}t^3 - t)\mathbf{j}$.

(b) $\mathbf{r}(t) = a\cos(t)\mathbf{i} + b\sin(t)\mathbf{j}$ where $a > b > 0$.

21. Putting it all together. Suppose that the motion of two particles are given by the parametric curves $\mathbf{r}_1 : \mathbb{R} \longrightarrow \mathbb{R}$ and $\mathbf{r}_2 : \mathbb{R} \longrightarrow \mathbb{R}$, where:

$$\begin{aligned}\mathbf{r}_1(t) &= \left(t^3 - t^2\right)\mathbf{i} + \left(t^3 + t^2\right)\mathbf{j}\\ \mathbf{r}_2(t) &= \left(t^2 + 3t\right)\mathbf{i} + (t + 1)\,\mathbf{j}\end{aligned}$$

(a) Find the *time* and *position* where the particles collide.

(b) Find the intervals on which the speed of the particle given by $\mathbf{r}_1$ is:

(i) increasing. (ii) decreasing.

(c) Decide whether the speed of each of the particle is increasing or decreasing the time of collision.

Topic 4: Integration and Differential Equations

Integration from Standard Integrals

1. Simple Integrals. Find an antiderivative of each of the following:

(a) $7x^3 + 6x^2 - 4x + 3$ (b) $\sin(\pi x)$ (c) e^{3x}

(d) $\dfrac{3}{2x^2}$ (e) $\cos\left(\dfrac{2\pi x}{3}\right)$ (f) $\dfrac{4}{x}$

(g) $\sec^2\left(\dfrac{x}{2}\right)$ (h) $2x - \dfrac{1}{2x}$ (i) $e^{-\frac{x}{5}}$

2. Equation of Curve. The slope of a certain curve at any point (x, y) on the curve is given by $(x-1)^3 - 2$, and the curve passes through the point $(3, 0)$. Find the equation of the curve.

3. Inverse Trigonometric. Integrate the following:

(a) $\dfrac{-2}{\sqrt{16 - x^2}}$ (b) $\dfrac{4}{25 + x^2}$ (c) $\dfrac{7}{\sqrt{3 - x^2}}$

Integration by Substitution

4. Derivative Present. Find each of the following:

(a) $\displaystyle\int 2x(x^2+1)^5\,\mathrm{d}x$ (b) $\displaystyle\int 3x^2\cos(x^3+5)\,\mathrm{d}x$ (c) $\displaystyle\int \frac{x}{(x^2+1)^3}\,\mathrm{d}x$

(e) $\displaystyle\int (8x+6)e^{2x^2+3x}\,\mathrm{d}x$ (f) $\displaystyle\int 5x\sqrt{9+x^2}\,\mathrm{d}x$ (g) $\displaystyle\int \frac{6\sin 3x}{\cos^2 3x}\,\mathrm{d}x$

(h) $\displaystyle\int \frac{5}{x}\sin(\log_e(x))\,\mathrm{d}x$

5. Complete the Square. Find antiderivatives of the following:

(a) $\dfrac{1}{x^2+2x+2}$ (b) $\dfrac{1}{\sqrt{-x^2-4x+21}}$ (c) $\dfrac{-1}{\sqrt{7-x^2+6x}}$

6. Linear Substitutions. Evaluate the following:

(a) $\displaystyle\int x\sqrt{1-x}\,\mathrm{d}x$

(b) $\displaystyle\int \frac{2x-1}{(x-1)^2}\,\mathrm{d}x$

(c) $\displaystyle\int (x+3)(x+4)^{\frac{1}{3}}\,\mathrm{d}x$

(d) $\displaystyle\int (2x+1)(x+3)^{20}\,\mathrm{d}x$

Integration using Trigonometric Identities

7. Trigonometric Powers. Find the following:

(a) $\displaystyle\int 2\sin^4 2x\cos 2x\,\mathrm{d}x$

(b) $\displaystyle\int \sin^2 x\,\mathrm{d}x$

(c) $\displaystyle\int \cos^5 2x\,\mathrm{d}x$

(d) $\displaystyle\int \cos^2 7x\,\mathrm{d}x$

(e) $\displaystyle\int \cos^2 5x\sin^2 5x\,\mathrm{d}x$

(f) $\displaystyle\int \sin^4 x\cos^5 x\,\mathrm{d}x$

Integration by Partial Fractions

8. Partial Fractions. Antidifferentiate the following:

(a) $\displaystyle\frac{3x}{(x-2)(x+4)}$

(b) $\displaystyle\frac{3}{x^2-5x+4}$

(c) $\displaystyle\frac{2x+1}{x^2+4x+4}$

(d) $\displaystyle\frac{x+3}{x^2-3x+2}$

(e) $\displaystyle\frac{2x+1}{x^2-1}$

(f) $\displaystyle\frac{2x}{x^2-2x+10}$

9. Long Division. Find antiderivatives of the following, after first applying polynomial long division.

(a) $\displaystyle\frac{x^3+x^2-3x+3}{x+2}$

(b) $\displaystyle\frac{4x^3+8x^2+5x+13}{4x^2+5}$

(c) $\displaystyle\frac{x^3+3}{x^2-x}$

(d) $\displaystyle\frac{x^3+4x^2+11x+2}{x^2+4x+10}$

(e) $\displaystyle\frac{4x^3+42x+1}{x^2+10}$

10. Mixed Integrals. Find the following:

(a) $\displaystyle\int \cos^2 x \sin^3 x \, \mathrm{d}x$ (b) $\displaystyle\int \frac{x^3+3x-2}{x^2-x}\, dx$ (c) $\displaystyle\int \cos 2x \, e^{\sin 2x}\, dx$

(d) $\int \cos^4 x \, dx$ (e) $\displaystyle\int \tan^2 5x \sec^4 5x \, \mathrm{d}x$ (f) $\displaystyle\int \frac{1}{\sqrt{5+x}}\, \mathrm{d}x$

(g) $\displaystyle\int \frac{-2x-3}{x^2-x}\, dx$ (h) $\displaystyle\int \frac{x-2}{\sqrt{x+1}}\, dx$ (i) $\displaystyle\int \frac{1}{8+2x^2}\, dx$

(j) $\displaystyle\int \frac{1}{x^2+6x+10}\, \mathrm{d}x$ (k) $\displaystyle\int \frac{x^4+6x^3+10x^2+11x+13}{x^2+6x+10}\, \mathrm{d}x$

Definite Integrals

11. Simple Type. Evaluate the following definite integrals:

(a) $\displaystyle\int_0^2 (3x^2+2x+4)\, \mathrm{d}x$ (b) $\displaystyle\int_0^{\pi/4} (\cos\theta + 2\theta)\ d\theta$

(c) $\displaystyle\int_{-1}^1 (e^t - e^{-t})\ dt$ (d) $\displaystyle\int_1^2 \frac{1}{3t}\ dt$

12. Substitution. Evaluate the following:

(a) $\displaystyle\int_0^1 x\sqrt{1-x}\, \mathrm{d}x$ (b) $\displaystyle\int_0^{\pi/4} \sin^3 t \cos t \ dt$

(c) $\displaystyle\int_0^3 x\sqrt{x^2+16}\, \mathrm{d}x$ (d) $\displaystyle\int_e^{e^2} \frac{1}{\log x}\cdot\frac{1}{x}\, \mathrm{d}x$

13. Mixed Type. Evaluate the following:

(a) $\displaystyle\int_2^3 \frac{3}{x^3}\, \mathrm{d}x$ (b) $\displaystyle\int_0^{\pi/2} \sin^2 2\theta \ d\theta$ (c) $\displaystyle\int_0^{1/2} \frac{3}{\sqrt{1-x^2}}\, \mathrm{d}x$

(d) $\displaystyle\int_1^2 \frac{1}{x(x+1)}\, \mathrm{d}x$ (e) $\displaystyle\int_2^3 \frac{2x+6}{(x-1)^2}\, \mathrm{d}x$ (f) $\displaystyle\int_{-1}^1 \frac{e^t}{e^t+1}\ dt$

(g) $\displaystyle\int_0^{\pi/4} \sec^2 x \, \mathrm{d}x$ (h) $\displaystyle\int_0^1 \frac{x^3+x^2+4x+1}{x^2+1}\, dx$

(i) $\displaystyle\int_{-\pi/6}^{\pi/3} \frac{\sin x}{10+6\cos x+\cos^2 x}\, dx$

Areas

14. Area under the graph. Find the areas indicated below. (a) Area under $y = x + \dfrac{1}{x^2}$ from $x = 2$ to $x = 3$.

(b) Area under $y = \dfrac{6}{4 + x^2}$ from $x = -2$ to $x = 2$.

(c) Area under $y = \dfrac{2}{x} \log x$ from $x = 1$ to $x = e$.

(d) Area enclosed by $y = -1 + \dfrac{2}{x^2 + 1}$ and the x-axis.

15. With respect to x. For each pair of curves below, sketch them and find their intersection points. Hence find the area enclosed between the curves. (a) $y = x^2$ and $y = 2x$.

(b) $y = 4x^2$ and $y = x^2 + 3$.

(c) $y^2 = x$ and $y = \dfrac{1}{3}x$.

16. With respect to y. Sketch the pairs of curves below and find the areas enclosed by them.

(a) $x = \sqrt{y}$ and $x = \dfrac{y}{2}$.

(b) $x = y^2$ and $x = 3y$.

(c) $x = -y^2 + 2$ and $y = x$.

17. You decide! Consider the two curves

$$y = x - 2 \qquad \text{and} \qquad x = y^2 \ .$$

(a) Draw the region enclosed by these curves, including all relevant intersection points.

(b) Write down (but do not solve) the integrals that give the area of this region:

(i) With respect to y,

(ii) With respect to x.

HINT: One of these should be a sum of two integrals.

(c) Find the area of the region by evaluating **one** of the integrals in (b).

Verification of Solutions

18. Verification. For each of the following differential equations, verify that the given function is a solution (i.e. substitute the function into both sides of the differential equation). The given functions are the *general solutions* of the differential equations.

(a) $\dfrac{dy}{dx} = 1\ ,\ y = x + C$

(b) $\dfrac{dy}{dx} = y\ ,\ y = Ce^x$

(c) $\dfrac{dy}{dx} = -\dfrac{x}{y}\ ,\ x^2 + y^2 = C$ (implicit)

(d) $\dfrac{dy}{dx} = 2y - 4x\ ,\ y = Ce^{2x} + 2x + 1$

(e) $\dfrac{dy}{dx} = -\dfrac{2y}{x}\ ,\ y = \dfrac{C}{x^2}$

19. Constant Solutions. Find the *constant solutions* for the following separable differentiable equations.

(a) $\dfrac{dy}{dx} = e^x y.$

(b) $\dfrac{dy}{dx} = xe^y.$

(c) $\dfrac{dy}{dx} = \sin(x)(y^2 + 1).$

(d) $\dfrac{dy}{dx} = (x^2 - 1)\sin(2y).$

20. Initial Value Problems. Solve the following initial value problems, by starting with the general solutions verified in the Question 18 (so you do not need to solve them from scratch).

(a) $\dfrac{dy}{dx} = 1\ ,$ with $y(0) = 3.$

(b) $\dfrac{dy}{dx} = y\ ,$ with $y(0) = 2.$

(c) $\dfrac{dy}{dx} = -\dfrac{x}{y}\ ,$ with $y(3) = 4.$

(d) $\dfrac{dy}{dx} = 2y - 4x\ ,$ with $y(-1) = 0.$

(e) $\dfrac{dy}{dx} = -\dfrac{2y}{x}\ ,$ with $y(2) = 64.$

21. Second Order. Verify that the given function is a solution of the given second order differential equation. (a) $\dfrac{d^2y}{dx^2} - \dfrac{dy}{dx} - 6y = 0\ ,\quad y = e^{-2x} + e^{3x}$ (b) $\dfrac{d^2y}{dt^2} = -n^2x\ ,\quad x = C\sin(nt)$

(c) $\dfrac{d^2y}{dx^2} = \dfrac{2}{y}\left(\dfrac{dy}{dx}\right)^2 ,\quad y = \dfrac{4}{x+1}$

22. Cubic Solution. Find the constants a, b, c, d if $y = ax^3 + bx^2 + cx + d$ is a solution of the differential equation

$$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + y = x^3\ .$$

23. Exponential Solution. Find the possible values of k if $f(x) = e^{kx}$ satisfies the differential equation

$$f''(x) + 3f'(x) + 2f(x) = 0\ .$$

Solving by Direct Antidifferentiation

24. First Order. Find the general solution of the following differential equations using direct antidifferentiation. For those with an initial condition, find the corresponding particular solution.

(a) $\dfrac{dy}{dx} = \dfrac{1}{\sqrt{x}}$

(b) $\dfrac{dy}{dt} = \dfrac{t^2 + 3t - 1}{t}$

(c) $\dfrac{dy}{dt} = \sin(3t + \pi)$, $\quad y(0) = 1$

(d) $\dfrac{dy}{dx} = \dfrac{1}{2x - 1}$, $\quad y(1) = 3$

25. Second Order. Find the general solution of the following by direct antidifferentiation.

(a) $\dfrac{d^2y}{dx^2} = e^{\frac{x}{2}}$

(b) $\dfrac{d^2y}{dt^2} = \sqrt{1 - t}$

(c) $\dfrac{d^2y}{dx^2} = \dfrac{1}{(x + 1)^2}$

Separable Differential Equations

26. *y* only. Solve the following separable differential equations whose right hand sides depend only on y. For those with initial conditions, find the particular solution.

(a) $\dfrac{dy}{dx} = \dfrac{1}{y^2}$

(b) $\dfrac{dy}{dx} = 1 + y^2$

(c) $\dfrac{dy}{dx} = \sqrt{y}$, $\quad y(3) = 1$

(d) $\dfrac{dy}{dx} = y - 4$, $\quad y(0) = 5$

27. Separable. Find the general solutions of the following separable differential equations. In part (e) you do not need to express x as a function of t.

(a) $\dfrac{dy}{dx} = 5y^2 \cos x$

(b) $\dfrac{dy}{dx} = e^x e^{-2y}$

(c) $\dfrac{dy}{dx} = \dfrac{\sqrt{1 - y^2}}{x}$

(d) $\dfrac{dy}{dt} = 3\sqrt{9 - y^2} \sin^4 t \cos t$

(e) $\dfrac{dx}{dt} = \dfrac{3t + e^{2t}}{x^2 + e^{-x}}$

(f) $\dfrac{dy}{dt} = \dfrac{(y^2 + 1)\cos^2 3t}{2y}$

28. Separable Initial Value. Find the particular solutions of the following initial value problems.

(a) $\dfrac{dy}{dx} = 3xy$, $\quad y(0) = 3$

(b) $\dfrac{dx}{dt} = \dfrac{x}{t(t + 1)}$, $\quad x(1) = 1$

Applications of Differential Equations

29. Projectile Motion. The height of a projectile fired vertically up from the ground (on a planet slightly more massive than Earth) with an initial speed of 50m/s satisfies the differential equation

$$\frac{d^2h}{dt^2} = -10.$$

(a) Express the two initial conditions of this problem mathematically.

(b) Find the height at any time t.

(c) What is the maximum height reached?

30. Balloon. The volume $V\text{cm}^3$ of a balloon is increasing at a rate that is inversely proportional to its current volume. (a) Write down the differential equation governing the balloon's volume.

(b) If the initial volume of the balloon is 10cm^3 and after 5 seconds it is 40cm^3, find the volume V at any time t.

(c) What is the volume after 8 seconds?

31. Radioactive Decay. The rate of decay of a radioactive substance is proportional to the amount Q present at any time. (a) Write down the differential equation governing the amount of radioactive substance.

(b) Find the amount Q present at any time, given that initially $Q = 100$ and when $t = 10$, $Q = 50$.

Population Models

32. Something Fishy. The number of fish in a lake is growing according to

$$\frac{dF}{dt} = 0.1F$$

where F is the number of fish t weeks after observation is started. (a) Find an expression for $F(t)$ if there are initially 10 fish in the lake.

(b) How many weeks after observation begins will the number of fish reach 1000?

33. Something Moosey. The population P of moose in a certain forest increases at a rate proportional to the current population. (a) Write down the differential equation governing the moose population.

(b) Find the general solution of this differential equation.

(c) If the initial population was 100 and after 2 years the population had risen to 110, determine the population at any time.

(d) What is the population after 5 years?

Newton's Law of Cooling

34. Metal rod. A metal rod that has been heated to 110°C is placed into a large tank of water whose temperature is a constant 10°C. After 2 minutes the temperature of the rod is 70°C. Assume that Newton's Law of Cooling applies, so that the temperature T of the rod satisfies

$$\frac{dT}{dt} = -k(T - T_S)$$

where T_S is the temperature of the surroundings of the rod, and k is a positive constant. (a) Find an expression for the temperature of the rod at any time $t > 0$.

(b) Find the temperature of the rod after a further 2 minutes.

Answers

Topic 1: Numbers and Sets

1. Subsets.

(a) True. Requires proof.

(b) True. Requires proof.

(c) False. EG: For $k = 1$ we have $\cos\left(\dfrac{k\pi}{3}\right) = \frac{1}{2} \notin \mathbb{Z}$.

(d) True. Requires proof.

2. Both ways.

(a) True. Requires proof.

(b) False. Requires proof.

3. Irrationals.

(a) Requires proof by contradiction.

(b) Requires proof by contradiction.

4. More Irrationals.

(a) 5

(b) 1 and 9

(c) Requires proof by contradiction.

5. Unions and complements.

(a) $(-\infty, -\frac{1}{2}) \cup (\frac{1}{2}, \infty) = \mathbb{R} \setminus [-\frac{1}{2}, \frac{1}{2}]$

(b) $[0, \pi] \cup [2\pi, 3\pi] \cup [4\pi, 5\pi] = [0, 5\pi] \setminus ((\pi, 2\pi) \cup (3\pi, 4\pi))$

(d) $\left[0, \dfrac{\pi}{4}\right) \cup \left(\dfrac{\pi}{4}, 2\pi\right]$.

6. More Set Calculations.

(a) $\varnothing$

(b) $\{\frac{1}{2}, 1\}$.

(c) $\{1\}$.

(d) $\{k\pi \mid k \in \mathbb{Z}\} \cup \{k\pi + \frac{\pi}{2} \mid k \in \mathbb{Z}\} = \{k\pi \mid 2k \in \mathbb{Z}\}$.

7. Solving inequalities.

(a) $x \in [4, \infty)$

(b) $x \in \varnothing$

(c) $x \in [-1, -\frac{1}{2}) \cup (\frac{1}{2}, 1]$

8. More interesting inequalities

(a) $e^{2x} - 2 \geq e^x$ [Hint: The quadratic formula will be useful.]

(b) $1 - \cos^2(x) \leq \sin^2(x)$

(c) $1 - \cos^2(x) < \sin^2(x)$

(a) $x \in [\log(2), \infty)$

(b) $\mathbb{R}$

(c) $\varnothing$

9. Square root of negative numbers.

(a) $2i$ (b) $\sqrt{6}i$ (c) $4\sqrt{3}i$.

10. Real and Imaginary.

(a) $\mathrm{Re}(z) = 3$, $\mathrm{Im}(z) = -4$, $\mathrm{Re}(z) - \mathrm{Im}(z) = 7$

(b) $\mathrm{Re}(z) = 0$, $\mathrm{Im}(z) = 6$, $\mathrm{Re}(z) - \mathrm{Im}(z) = -6$

(c) $\text{Re}(z) = 2$, $\text{Im}(z) = 7$, $\text{Re}(z) - \text{Im}(z) = -5$ (d) $\text{Re}(z) = \frac{5}{7}$, $\text{Im}(z) = 0$, $\text{Re}(z) - \text{Im}(z) = \frac{5}{7}$

(e) $\text{Re}(z) = 2$, $\text{Im}(z) = \frac{1}{2}$, $\text{Re}(z) - \text{Im}(z) = \frac{3}{2}$.

11. Complex Arithmetic.

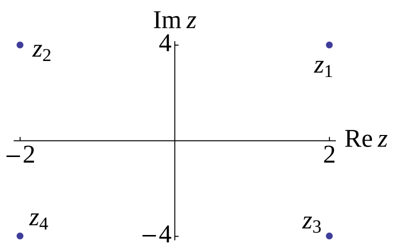
(a) $-1 + 8i$ (b) 2 (c) $1 + 2i$ (d) $2 + 2i$

(e) $-\frac{5}{2} + 4i$ (f) $\frac{15}{2} + 2i$ (g) $5 - 6i$ (h) $\frac{1}{2} - 4i$

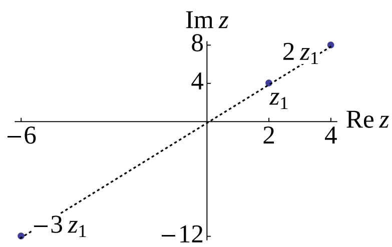
(i) $\frac{9}{2} - 4i$.

12. The Argand Plane.

(a)

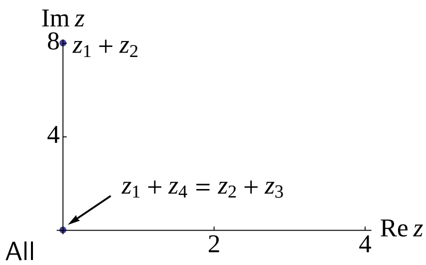


(b)



points are on the same line.

(c)



(d) (i) No (ii) No (iii) Yes (iv) Yes.

13. Powers of i.

$i^2 = -1$, $i^3 = -i$, $i^4 = 1$ (a) $-i$ (b) -1 (c) $-i$

14. Cartesian form.

(a) $-3 + 2i$ (b) $5 - 12i$ (c) $-3 - 4i$ (d) $6 + 9i$

(e) 5 (f) $-2 + 2i$ (g) $5 + i$ (h) -5

(i) $-3 - 2i$.

15. Rationalising the Denominator.

(a) $\frac{3}{13} + \frac{2}{13}i$ (b) $\frac{4}{5} - \frac{2}{5}i$ (c) $\frac{3}{13} + \frac{2}{13}i$ (d) $\frac{7}{2} + \frac{3}{2}i$

(e) $-1 - 2i$ (f) $-\frac{1}{13} + \frac{5}{13}i$.

16. Real and Imaginary. (a) -21/17 (b) -17/26

17. Harder Examples.

(a) $-\frac{1}{3}$ (b) -3 (c) $\frac{3}{26} - \frac{11}{26}i$ (d) $-\frac{27}{5} - \frac{31}{5}i$.

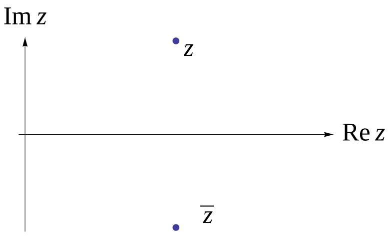
18. Conjugates to Cartesian.

(a) $12 - 7i$ (b) $1 + i$ (c) $22 + 7i$ (d) $-10 - 20i$

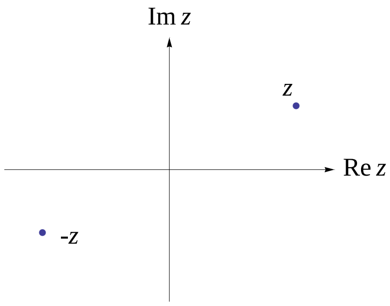
(e) $2 - 2i$ (f) $-\frac{1}{2} + \frac{1}{2}i$.

19. Complex conjugates.

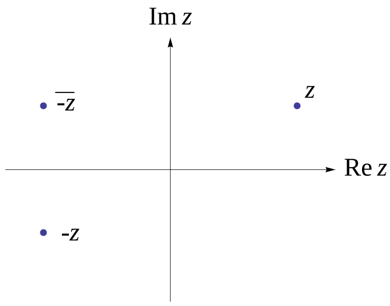
(d) reflection in x axis



(e) reflection in origin



(f) reflection in y axis.



20. Complex Trigonometric. (a) 1 (b) $\cos(\theta-\phi)+i\sin(\theta-\phi)$ (c) $\frac{1}{2}(1-i\frac{\sin\alpha}{1+\cos\alpha})$

21. Modulus and Argument.

(a) $|z|=6$, Arg$(z)=0$ (b) $|z|=3$, Arg$(z)=\pi$ (c) $|z|=2$, Arg$(z)=\frac{5\pi}{6}$

(d) $|z|=4$, Arg$(z)=\frac{\pi}{2}$ (e) $|z|=2$, Arg$(z)=-\frac{\pi}{2}$ (f) $|z|=2$, Arg$(z)=-\frac{2\pi}{3}$

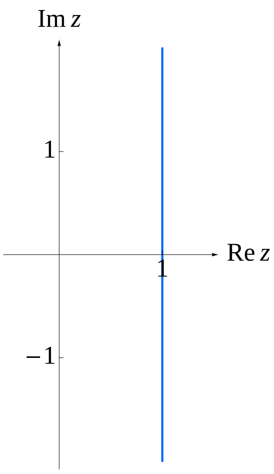
(g) $|z|=7$, Arg$(z)=\frac{\pi}{2}$ (h) $|z|=\sqrt{2}$,Arg$(z)=\frac{\pi}{4}$ (i) $|z|=4\sqrt{2}$, Arg$(z)=-\frac{\pi}{4}$.

22. Something more complicated.

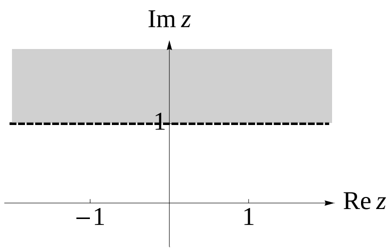
(a) (i) 1 (ii) 27/10 (iii) 10 (b) (i) $2\pi/3$ (ii) $-3\pi/4$ (iii) π

23. Regions in the Complex Plane.

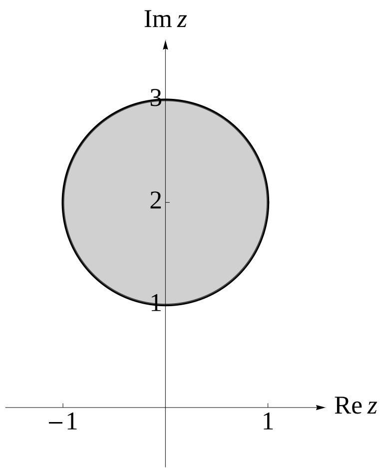
(a) Line $x=1$



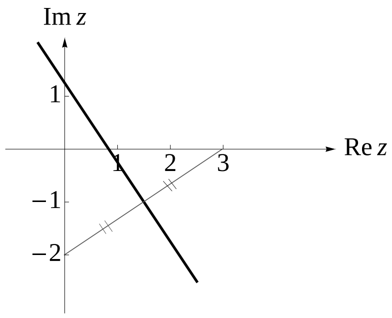
(b) half plane $y>1$



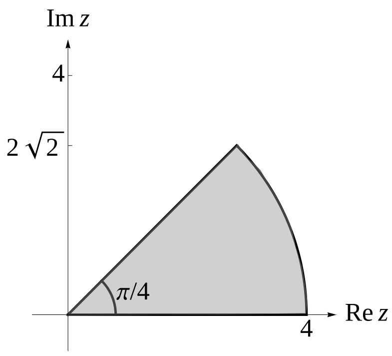
(c) boundary and interior of the circle $x^2+(y-2)^2=1$



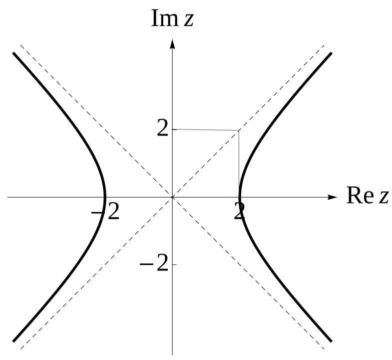
(d) line $6x + 4y = 5$



(e) Sector of circle, centred at the origin, radius 4, angle $\pi/4$ anticlockwise from the real axis.



(f) Hyperbola $x^2 - y^2 = 4$.



24. Exponential Polar Form.

(a) $e^{i\pi}$ (b) $5\sqrt{2}e^{i\pi/4}$ (c) $2e^{-i\pi/2}$ (d) $\sqrt{29}e^{i\alpha}$, where $\alpha = \arctan(5/2)$

25. Powers. (a) -1024 (b) $-512\sqrt{3} + 512i$ (c) $-1/64$ (d) $\frac{1}{2^{7/2}}e^{-7\pi i/12}$

26. Complete the square.

(a) $z_1 = i$, $z_2 = -i$ (b) $z_1 = 4i$, $z_2 = -4i$ (c) $z_1 = 2$, $z_2 = -2$ (d) $z_1 = -2 + i$, $z_2 = -2 - i$
(e) $z_1 = -\frac{3}{2} + \frac{\sqrt{3}}{2}i$, $z_2 = -\frac{3}{2} - \frac{\sqrt{3}}{2}i$.

27. Quadratic formula.

(a) $z_1 = -2$, $z_2 = -1$ (b) $z_1 = -\frac{3}{2} + \frac{\sqrt{7}}{2}i$, $z_2 = -\frac{3}{2} - \frac{\sqrt{7}}{2}i$ (c) $z_1 = \frac{1}{4} + \frac{\sqrt{23}}{4}i$, $z_2 = \frac{1}{4} - \frac{\sqrt{23}}{4}i$

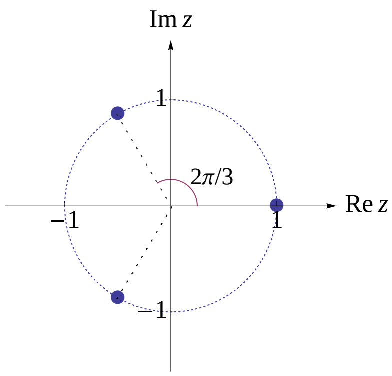
(d) $z_1 = -\frac{1}{3} + \frac{\sqrt{2}}{3}i$, $z_2 = -\frac{1}{3} - \frac{\sqrt{2}}{3}i$, $z_3 = 0$

28. Any way you like.

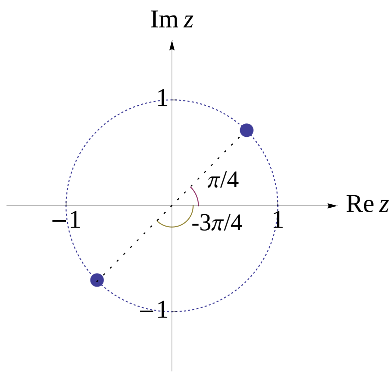
(a) $z_1 = 3$, $z_2 = -3$, $z_3 = 3i$, $z_4 = -3i$
(b) $z_1 = 2$, $z_2 = -2$, $z_3 = 2i$, $z_4 = -2i$, $z_5 = -\sqrt{2} - \sqrt{2}i$, $z_6 = \sqrt{2} + \sqrt{2}i$, $z_7 = \sqrt{2} - \sqrt{2}i$, $z_8 = -\sqrt{2} + \sqrt{2}i$.

29. Roots and Factors.

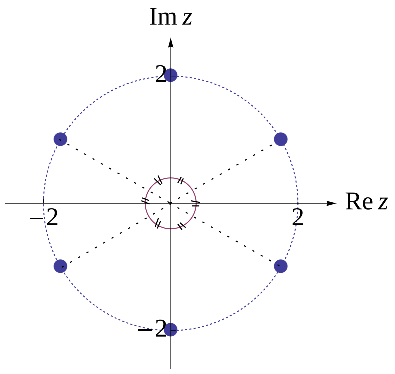
(a) 1, $e^{2\pi i/3}$, $e^{-2\pi i/3}$



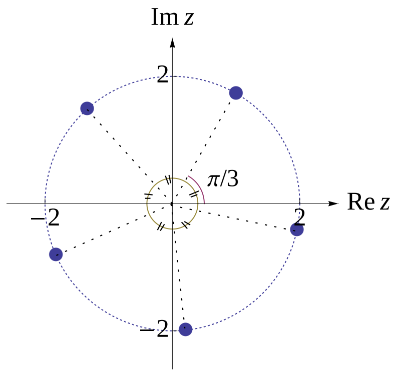
(b) $e^{\pi i/4}$, $e^{-3\pi i/4}$



(c) $2e^{\pi i/6}$, $2e^{\pi i/2}$, $2e^{5\pi i/6}$, $2e^{-5\pi i/6}$, $2e^{-\pi i/2}$, $2e^{-\pi i/6}$



(d) $2e^{\pi i/3}$, $2e^{11\pi i/15}$, $2e^{-13\pi i/15}$, $2e^{-7\pi i/15}$, $2e^{-\pi i/15}$



30. Roots of Polynomials.

(a) $\pm i\sqrt{2}$ repeated (b) $\pm 1 \pm i\sqrt{3}$ (c) $z = \pm\sqrt{(5+2\sqrt{5})/5},\ \pm\sqrt{(5-2\sqrt{5})/5}$

(d) $e^{2\pi i/5}$, $e^{4\pi i/5}$, $e^{-2\pi i/5}$, $e^{-4\pi i/5}$

31. Polynomials in Cos and Sin.

(a) $3\sin\theta - 4\sin^3\theta$ (b) $\cos^4\theta - 6\cos^2\theta\sin^2\theta + \sin^4\theta$ (c) $6\cos^5\theta\sin\theta - 20\cos^3\theta\sin^3\theta + 6\cos\theta\sin^5\theta$

32. Sin and Cos of Multiple Angles.

(a) $\frac{3}{4}\sin\theta - \frac{1}{4}\sin(3\theta)$ (b) $\frac{1}{8}(\cos(4\theta) + 4\cos(2\theta) + 3)$ (c) $\frac{1}{64}(-\sin(7\theta) + 7\sin(5\theta) - 21\sin(3\theta) + 35\sin\theta)$

Answers

Topic 2: Functions and Vectors

33. Functions or not?

(a) not a function: no domain and codomain given; (b) not a function: undefined at $x = 1$; (c) function.

34. Image of a Set under a Function. (a) $[3 - 2\pi, 3)$; (b) $(1, e^{\pi}]$; (c) $[0, 1]$.

35. Injective Functions.

(a) $\mathbb{R}$; (b) $(-\infty, 0]$ and $[0, \infty)$; (c) $\ldots(-\pi/2, \pi/4] \cup [-\pi/4, 0) \cup (0, \pi/4] \cup [\pi/4, \pi/2) \cup (\pi/2, 3\pi/4]\ldots$.

36. Counting Functions. n^3.

37. Counting Injective Functions. (a) None. (b) $n(n-1)(n-2)$.

38. Counting Bijective Functions. (a) None. (b) $6 = 1 \cdot 2 \cdot 3$. (c) None.

39. Inverse functions.

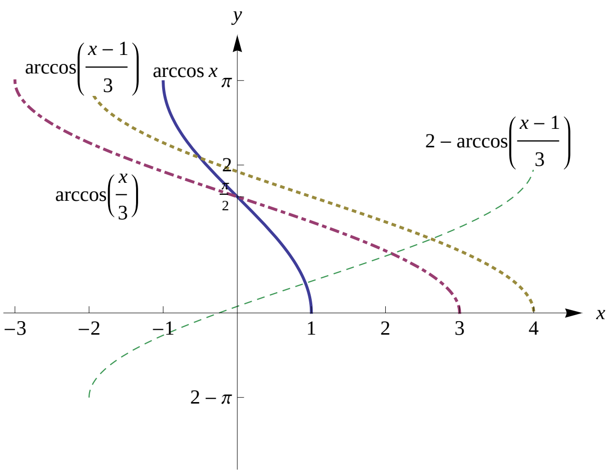
(a) $g(x) = \dfrac{3 - \sqrt{1 + 4x}}{2}$. (b) No inverse because f is not injective: $f(e) = f(e^{-1}) = 1$.

(c) No inverse because f is not surjective: the range of f is $(1, \infty) \subsetneq (0, \infty)$.

40. Inverse Trigonometric. (a) $\frac{\pi}{6}$ (b) $\frac{\pi}{6}$ (c) $-\frac{\pi}{3}$

41. Simplification. (a) $\frac{1}{2}$ (b) 1 (c) $\frac{3\pi}{4}$ (d) $\frac{\sqrt{3}}{2}$ (e) $-\frac{\pi}{3}$ (f) $\frac{\pi}{6}$

42. Graphing Inverse Trigonometric Functions.



43. Step by Step. Let $\mathrm{dom}(f)$ denote the domain of f and $\mathrm{range}(f)$ denote the range of f.

(a)

(i) $\mathrm{dom}(f) = [0, \infty)$; $\mathrm{range}(f) = [0, \infty)$ (ii) $\mathrm{dom}(g) = \mathbb{R}$; $\mathrm{range}(g) = (-\infty, 4]$

(iii) $[0, \infty) \cap (-\infty, 4] = [0, 4]$ (iv) $\mathrm{dom}(f \circ g) = [-2, 2]$; $\mathrm{range}(f \circ g) = [0, 2]$

(b) [Note that $\mathbb{Z}$ is the set of all integers.]

(i)$\mathrm{dom}(f) = \mathbb{R}\backslash\{k\pi : k \in \mathbb{Z}\} = \{(k\pi, \pi + k\pi) : k \in \mathbb{Z}\}$; $\mathrm{range}(f) = \mathbb{R}$
(ii) $\mathrm{dom}(g) = [-1, 1]$; $\mathrm{range}(g) = [-\pi/2, \pi/2]$
(iii) $\{(k\pi, \pi + k\pi) : k \in \mathbb{Z}\} \cap [-\pi/2, \pi/2] = [-\pi/2, 0) \cup (0, \pi/2]$
(iv) $\mathrm{dom}(f \circ g) = [-1, 0) \cup (0, 1]$; $\mathrm{range}(f \circ g) = \mathbb{R}$

44. Standard.

(a) (i) $\mathbb{R}\backslash\{1\}$ (ii) $\mathbb{R}\backslash\{0\}$ (b) (i) $(-1, \infty)$ (ii) $\mathbb{R}$ (c) (i) $[-2, 2]$ (ii) $[0, 2]$

(d) (i) $(-2, 2)$ (ii) $[\frac{1}{2}, \infty)$ (e) (i) $[2k\pi, 2k\pi + \pi]$, $k \in \mathbb{Z}$ (ii) $[0, \sqrt{2}]$

(f) (i) $(-\frac{\pi}{2} + 2k\pi, \frac{\pi}{2} + 2k\pi)$, $k \in \mathbb{Z}$ (ii) $(-\infty, 0]$

45. Inverse Trigonometric.

(a) (i) $[0, 2]$ (ii) $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ (b) (i) $[-2, -1]$ (ii) $[0, \pi]$ (c) (i) $\mathbb{R}$ (ii) $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

(d) (i) $\mathbb{R}$ (ii) $[0, \pi]$ (e) (i) $[-1, 1]\backslash\{\pm\frac{1}{\sqrt{2}}\}$ (ii) $\mathbb{R}$

(f) (i) $\left[-\frac{\pi}{6} + k\pi, \frac{\pi}{6} + k\pi\right]$, $k \in \mathbb{Z}$ (ii) $[0, \pi]$

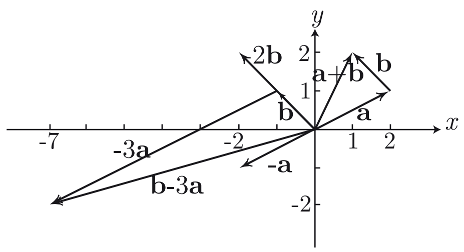
46. An incremental example.

(a) (i) $\mathbb{R}$ (ii) $[0, \infty)$ (b) (i) $\mathbb{R}$ (ii) $[0, \infty)$ (c) (i) $\mathbb{R}$ (ii) $[-1, \infty)$ (d) (i) $[-1, 1]$ (ii) $[-\pi/2, \pi/2]$

47. Trigonometric.

(a) $f(x) = x$; domain is $[-1, 1]$; range is $[-1, 1]$ (b) $g(x) = \sqrt{1 - x^2}$; domain is $[-1, 1]$; range is $[0, 1]$

48. Introduction to vectors.



49. Vector Arithmetic.

(a) $6\mathbf{i} + 2\mathbf{j}$ (b) $-9\mathbf{i} - 3\mathbf{j}$ (c) $-\frac{1}{2}\mathbf{i} - \mathbf{j} + \frac{3}{2}\mathbf{k}$

(d) $5\mathbf{i} + \mathbf{j} + \mathbf{k}$ (e) $-3\mathbf{i} - 2\mathbf{j} + 2\mathbf{k}$ (f) $-9\mathbf{i} - 3\mathbf{j} + \mathbf{k}$

50. The Hexagon. $\overrightarrow{OB} = \mathbf{a} + \mathbf{c}$, $\overrightarrow{FB} = \mathbf{a} + \mathbf{2c}$, $\overrightarrow{DA} = 2\mathbf{a}$, $\overrightarrow{AC} = \mathbf{c} - \mathbf{a}$, $\overrightarrow{EA} = 2\mathbf{a} + \mathbf{c}$

51. Position and Length.

(a) (i) $4\mathbf{i} - 7\mathbf{j}$ (ii) $\sqrt{65}$

(b) (i) $3\mathbf{i} + \mathbf{j}$ (ii) $\sqrt{10}$

(c) (i) $-4\mathbf{i} + 7\mathbf{j}$ (ii) $\sqrt{65}$

(d) (i) $-3\mathbf{i} - \mathbf{j}$ (ii) $\sqrt{10}$

(e) (i) $2\mathbf{i}$ (ii) 2

(f) (i) $-4\mathbf{i}$ (ii) 4

52. Unit Length, Unit Vector.

(a) $\frac{1}{\sqrt{65}}(4\mathbf{i} - 7\mathbf{j})$

(b) $\frac{1}{\sqrt{10}}(3\mathbf{i} + \mathbf{j})$

(c) $\frac{1}{\sqrt{65}}(-4\mathbf{i} + 7\mathbf{j})$

(d) $\frac{1}{\sqrt{10}}(-3\mathbf{i} - \mathbf{j})$

(e) $\mathbf{i}$

(f) $-\mathbf{i}$

53. Let's Go Backwards.

(a) $-4\mathbf{i} + 7\mathbf{j}$

(b) $-3\mathbf{i} - \mathbf{j}$

(c) $4\mathbf{i} - 7\mathbf{j}$

(d) $3\mathbf{i} + \mathbf{j}$

(e) $-2\mathbf{i}$

(f) $4\mathbf{i}$

54. More Vector Lengths.

(a) $\sqrt{26}$

(b) $\sqrt{2}$

(c) $\frac{3}{\sqrt{26}}\mathbf{i} - \frac{4}{\sqrt{26}}\mathbf{j} + \frac{1}{\sqrt{26}}\mathbf{k}$

(d) $\frac{1}{\sqrt{2}}\mathbf{i} + \frac{1}{\sqrt{2}}\mathbf{j}$

(e) $5\mathbf{i} - 9\mathbf{j} + 2\mathbf{k}$

(f) $\sqrt{110}$

(g) $\frac{1}{\sqrt{110}}(5\mathbf{i} - 9\mathbf{j} + 2\mathbf{k})$

55. Dot product. (a) 7 (b) 3 (c) 1 (d) 1 (e) 1 (f) 0

56. Angles. (a) $\arccos\left(\frac{7}{\sqrt{84}}\right)$ (b) $\arccos\left(\frac{3}{\sqrt{84}}\right)$ (c) $\arccos\left(\frac{1}{\sqrt{33}}\right)$
(d) $\arccos\left(\frac{1}{\sqrt{12}}\right)$ (e) $\frac{\pi}{3}$ (f) $\frac{\pi}{2}$

57. Dot and Angle. (a) 4 (b) $\arccos\left(\frac{4}{\sqrt{14}\sqrt{11}}\right)$

58. Perpendicular. (a) $a = -1$ (b) $a = -\frac{4}{5}$

59. Parallel and Perpendicular. (a) is parallel and (d) is perpendicular

60. Vector Angles. (a) $(\mathbf{a} + \mathbf{b}) \cdot \mathbf{b} = 0.$ (b) $\arccos\left(\frac{38}{\sqrt{38}\sqrt{50}}\right) = \arccos\left(\frac{\sqrt{19}}{5}\right)$

61. Triangle. (a) (i) $\mathbf{b} - \mathbf{a}$ (ii) $\frac{2}{3}(\mathbf{b} - \mathbf{a})$ (iii) $\frac{1}{3}(\mathbf{a} + 2\mathbf{b})$ (iv) $\mathbf{a} + 2\mathbf{b}$ (v) $2\mathbf{b}$

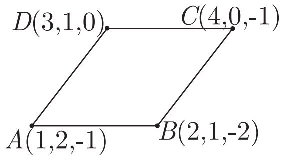
(b) Since $\overrightarrow{AQ}$ is a scalar multiple of $\overrightarrow{OB}$, they are parallel.

62. Trapezium. The side AB is parallel to DC, with the ratio being $AB : DC = 1 : 2$.

63. Another Trapezium. (a) $(3, -\frac{3}{2}, 1)$ (b) $-\overrightarrow{AB} - \overrightarrow{OA} + \overrightarrow{OC}$ (c) $(1, -\frac{1}{2}, 4)$

64. Parallelogram.

(a)

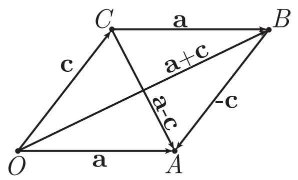


(b) $2\mathbf{i} - \mathbf{j} + \mathbf{k}$

(c) $\arccos\left(\frac{\sqrt{2}}{3}\right)$

65. Diagonals.

(a)



(b) (i) $4\sqrt{3}$ (ii) $2\sqrt{17}$ (iii) -40

(c) $\arccos\left(\frac{-5}{\sqrt{3}\sqrt{17}}\right)$

66. Right-Angled Triangle. $3\mathbf{i} + 3\mathbf{j}$

67. Triangle and Parallelogram. Proof required.

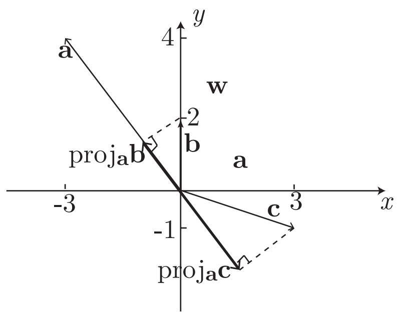
68. Flying Direction.

(a)The ground speed is $100\sqrt{50 + 7\sqrt{2}} \approx 773.95$ km/hour and the direction is $(700 + 50\sqrt{2})\mathbf{i} + 50\sqrt{2}\mathbf{j}$, approximately N84.76°E or 84.76 degrees east of north (.091 radians).

(b) $\mathbf{p} = 670\mathbf{i} - 30\mathbf{j}$. The required velocity $10\sqrt{4498} \approx 670.67$ km/hour and the required direction is $(67\mathbf{i} - 3\mathbf{j})/\sqrt{4498}$, approximately S87.43°E or 87.43 degrees east of south ($-.045$ radians).

69. Simple Projections. (a) $\frac{8}{5}$ (b) $-\frac{13}{5}$ (c) $\frac{8}{25}(-3, 4)$ (d) $-\frac{13}{25}(-3, 4)$

(e)



70. Scalar Projection.

(a) (i) $\frac{14}{\sqrt{17}}$ (ii) $\frac{14}{\sqrt{14}}$ (b) (i) $\frac{10}{\sqrt{14}}$ (ii) $\frac{10}{\sqrt{14}}$ (c) (i) $\frac{7}{\sqrt{6}}$ (ii) $\frac{7}{\sqrt{13}}$

(d) (i) $\frac{4}{\sqrt{8}}$ (ii) $\frac{4}{\sqrt{35}}$ (e) (i) $\frac{2}{\sqrt{2}}$ (ii) $\frac{2}{\sqrt{5}}$ (f) (i) 1 (ii) $\frac{1}{\sqrt{3}}$

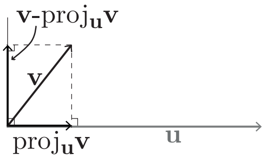
71. Vector Projection.

(a) (i) $\mathbf{u}$ (ii) $\mathbf{i} - \mathbf{j} - \mathbf{k}$ (b) (i) $\frac{5}{7}\mathbf{u}$ (ii) $\frac{1}{7}(16\mathbf{i} - 4\mathbf{j} - 8\mathbf{k})$ (c) (i) $\frac{7}{13}\mathbf{u}$ (ii) $\frac{1}{13}(12\mathbf{i} + 13\mathbf{j} + 8\mathbf{k})$

(d) (i) $\frac{4}{35}\mathbf{u}$ (ii) $\frac{1}{35}(50\mathbf{i} - 4\mathbf{j} + 82\mathbf{k})$ (e) (i) $\frac{2}{5}\mathbf{u}$ (ii) $\frac{1}{5}(5\mathbf{i} - 2\mathbf{j} - 1\mathbf{k})$ (f) (i) $\frac{1}{3}\mathbf{u}$ (ii) $\frac{1}{3}(-\mathbf{i} + 2\mathbf{j} - \mathbf{k})$

72. Scalar and Vector Projections.

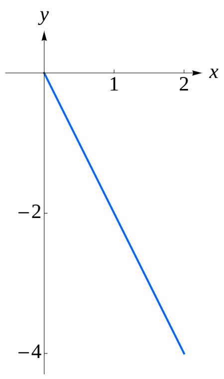
(a) $\frac{9}{\sqrt{6}} = \frac{3\sqrt{6}}{2}$ (b) $\frac{9}{\sqrt{29}}$ (c) $\frac{27}{29}\mathbf{i} + \frac{36}{29}\mathbf{j} + \frac{18}{29}\mathbf{k}$ (d) $\frac{3}{2}\mathbf{i} + \mathbf{j} + \frac{7}{2}\mathbf{k}$



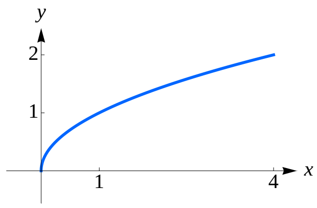
73. Gravity. [Optional question.] $\mathbf{w} = -490\mathbf{k}$; $\mathbf{r} = \frac{-245\sqrt{3}}{2}\mathbf{i} + \frac{245}{2}\mathbf{k}$; $\|\mathbf{r}\| = 245$

74. Equation of a Path.

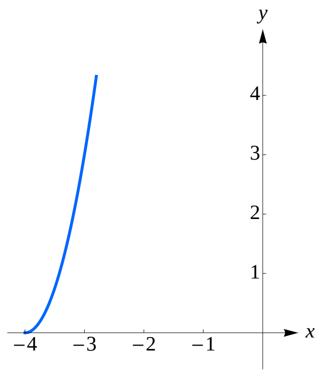
(a) $y = -2x$



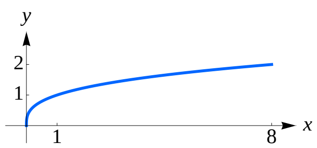
(b) $y = \sqrt{x}$



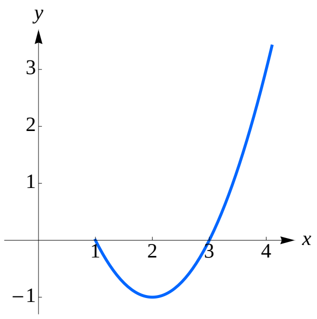
(c) $y = 3(x+4)^2$



(d) $y = x^{\frac{1}{3}}$

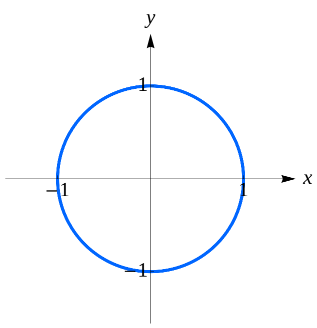


(e) $y = (x-3)(x-1)$

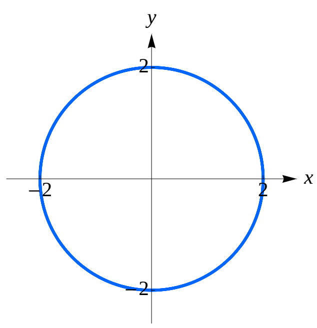


75. Periodic Motion.

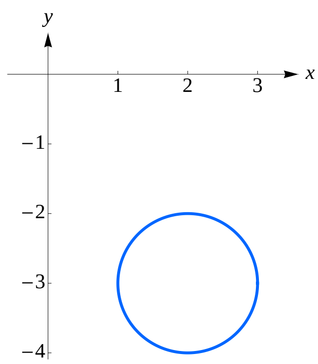
(a) $x^2 + y^2 = 1$, period π



(b) $x^2 + y^2 = 4$, period π

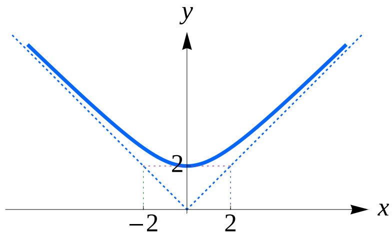


(c) $(x-2)^2 + (y+3)^2 = 1$, period 2π



76. Another Path.

(i) $y^2 - x^2 = 4$ (ii) This is a hyperbola and can be written $y^2/4 - x^2/4 = 1$. Note that $y \geq 0$ so it is only part of the hyperbola. The asymptotes are $y = \pm x$.



Answers

Topic 3: Differential Calculus

1. First and Second Derivatives.

(a) $f'(x) = 4x^3 - 9x^2 + 16$, $f''(x) = 12x^2 - 18x$ (b) $h'(x) = x/(\sqrt{x^2+1})$, $h''(x) = 1/(x^2+1)^{3/2}$

(c) $y' = 1/(1-x)^2$, $y'' = 2/(1-x)^3$ (d) $y' = -\frac{3}{2}x(1-x^2)^{-1/4}$, $y'' = \frac{3}{4}(1-x^2)^{-5/4}(x^2-2)$

(e) $f'(x) = (2 - 5\cos(5x))e^{2x-\sin(5x)}$, $f''(x) = (25\sin(5x) + (2-5\cos(5x))^2)e^{2x-\sin(5x)}$

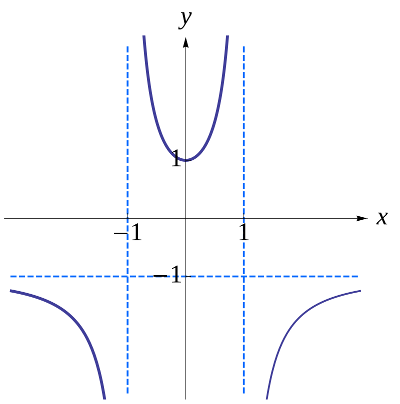
(f) $g'(x) = -\operatorname{cosec}(x)\cot(x)$, $g''(x) = \operatorname{cosec}^3(x) + \operatorname{cosec}(x)\cot^2(x)$

2. *n*th Derivatives. (a) $n! = n \times (n-1) \times (n-2) \times (n-3)...$ (b) $(-1)^n(n+2)!/(6x^{n+3})$

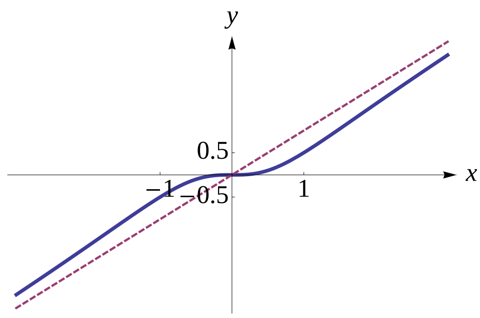
3. Related Derivatives. (a) $f''(x) = 6xg'(x^2) + 4x^3g''(x^2)$ (b) $f''(x) = \frac{\sqrt{x}g''(\sqrt{x}) - g'(\sqrt{x})}{4x\sqrt{x}}$

4. Graph sketching.

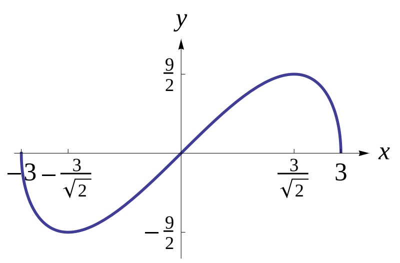
(a) Domain $\mathbb{R}$ except ± 1. y intercept at $y = 1$, no x intercept. Local minimum at $(0, 1)$. Increasing for $0 < x < 1$ and $x > 1$, decreasing for $x < -1$ and $-1 < x < 0$. Concave down for $x < -1$ and $x > 1$, concave up for $-1 < x < 1$. No points of inflection. Vertical asymptotes at $x = \pm 1$, horizontal asymptote at $y = -1$.



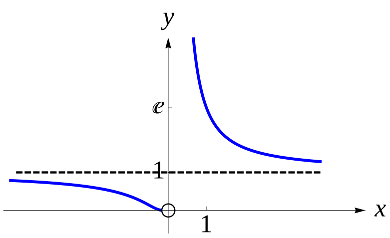
(b) Domain $\mathbb{R}$. Oblique asymptote $y = x$. x and y-intercept at the origin $(0, 0)$. Stationary point at $x = 0$. Increasing for all x. Concave up for $x < -\sqrt{3}$ and $0 < x < \sqrt{3}$. Concave down for $-\sqrt{3} < x < 0$ and $x > \sqrt{3}$. Points of inflection at $x = -\sqrt{3}, 0, \sqrt{3}$.



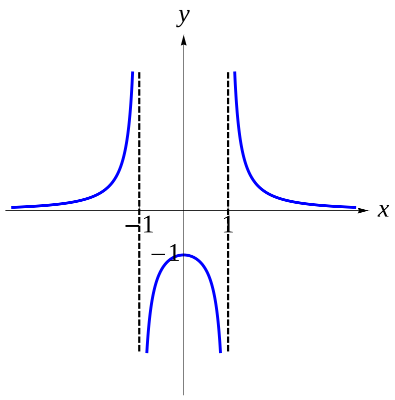
(c) Domain $-3 \le x \le 3$. Intercepts are (-3,0), (0,0), (3,0). Local minimum at $(-3\sqrt{2}/2, -9/2)$. Local maximum at $(3\sqrt{2}/2, 9/2)$. Decreasing for $x < -3\sqrt{2}/2$ and $x > 3\sqrt{2}/2$. Increasing on $-3\sqrt{2}/2 < x < 3\sqrt{2}/2$. Concave up for $x < 0$, concave down for $x > 0$. Point of inflection at $x = 0$.



(d) Domain $\mathbb{R}$ except 0. Vertical asymptote at $x = 0$, horizontal asymptote at $y = 1$. No x or y intercepts. No stationary points. Decreasing for all $x \in \mathbb{R}\backslash\{0\}$. Concave up for $-\frac{1}{2} < x < 0$ and $x > 0$, concave down for $x < -\frac{1}{2}$. Point of inflection at $x = -\frac{1}{2}$. As $x \longrightarrow 0$ from above, $f(x) \longrightarrow \infty$, while as $x \longrightarrow 0$ from below, $f(x) \longrightarrow 0$.



(e) Domain $\mathbb{R}$ except $\{-1,1\}$. Vertical asymptotes at $x=-1,1$, horizontal asymptote at $y=0$. No x intercept; y intercept $(0,-1)$. Local maximum at $(0,-1)$. Increasing on $(-\infty,-1)\cup(-1,0)$. Decreasing on $(0,1)\cup(1,\infty)$. No points of inflection. Concave up on $(-\infty,-1)\cup(1,\infty)$, concave down on $(-1,1)$.



5. Implicit Derivatives.

(a) $\dfrac{y-2x}{3y^2-x}$ (b) $\tan x\tan y$ (c) $\dfrac{2y-2xy\log_e y}{2x^2+x\sqrt{y}}$ (d) $\dfrac{ye^{xy}-2xy}{x^2-xe^{xy}}$

(e) $1+\dfrac{e^x(1+x)}{\sin(x-y)}$ (f) $\dfrac{y+y^2x\operatorname{cosec}^2 xy}{x-x^2y\operatorname{cosec}^2 xy}$

6. Second Derivative. (a) $y'=\dfrac{-x^3}{y^3}$ (b) $y''=\dfrac{-3x^2}{y^3}-\dfrac{3x^6}{y^7}$

7. More Implicit Derivatives $a=-9/5$ and $b=17/10$.

8. Implicit Tangent. Proof required.

9. Derivatives.

(a) $\dfrac{1}{\sqrt{25-x^2}}$ (b) $-\dfrac{1}{\sqrt{16-x^2}}$ (c) $\dfrac{\sqrt{7}}{7+x^2}$

(d) $-\dfrac{4}{\sqrt{1-16x^2}}$ (e) $\dfrac{7}{1+49x^2}$ (f) $\dfrac{2}{\sqrt{1-4x^2}}$

10. More derivatives.

(a) $-\dfrac{7}{\sqrt{16-49x^2}}$ (b) $\dfrac{8}{\sqrt{25-64x^2}}$ (c) $\dfrac{14}{4+49x^2}$

(d) $-\dfrac{4}{\sqrt{1-(4x-3)^2}}$ (e) $-\dfrac{4}{\sqrt{25-(3-4x)^2}}$ (f) $\dfrac{20}{16+(5x-3)^2}$

11. Even more derivatives.

(a) $-\dfrac{3x^2}{\sqrt{1-x^6}}$ (b) $\dfrac{e^x}{1+(e^x+5)^2}$ (c) $\dfrac{1}{3x^{2/3}\sqrt{1-x^{2/3}}}$

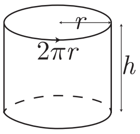
(d) $-\dfrac{2}{x\sqrt{1-4(\log(x))^2}}$ (e) $-\dfrac{3\sin(3x)}{1+(\cos(3x)+1)^2}$ (f) $\dfrac{4x}{\sqrt{9-(2x^2+1)^2}}$

12. Flying High. (a) $h=18$ at $t=3$ (b) $t=6$

13. Flying Higher. (a) $h = 11\frac{1}{4}$ at $t = 3$ (b) $h = 31\frac{2}{3}$ at $t = 10$

14. Cylinder.

(a) $A = 2\pi r^2 + 2\pi rh,\ \ V = \pi r^2 h$



(b) Proof required.

(c) 150π cm^2

15. Rectangle. Maximum area is 9, when one vertex is at $(\frac{3}{\sqrt{2}}, 0)$.

16. Manufacturing.

(a) $P(x) = 900\log(1 + \frac{x}{300}) - (200 + x)$ (b) $x = 600$ gives maximum profit of \$188.75.

17. Rock n Roll. Base: 4m by 4m, Height: 2m.

18. Velocity, speed and acceleration.

(a) Velocity: $\mathbf{r}'(t) = -\sin(t)\mathbf{i} + 3\cos(3t)\mathbf{j}$. Speed $\|\mathbf{r}'(t)\| = \sqrt{\sin^2(t) + 9\cos^2(3t)}$.

Acceleration: $\mathbf{r}''(t) = -\cos(t)\mathbf{i} - 9\sin(3t)\mathbf{j}$.

(b) Velocity: $\mathbf{r}'(t) = 5t^4\mathbf{i} + 7t^6\mathbf{j}$. Speed $\|\mathbf{r}'(t)\| = t^4\sqrt{25 + 49t^4}$.

Acceleration: $\mathbf{r}''(t) = 20t^3\mathbf{i} + 42t^5\mathbf{j}$.

(c) Velocity: $\mathbf{r}'(t) = -2te^{-t^2}\mathbf{i} + 3t^2\mathbf{j}$. Speed $\|\mathbf{r}'(t)\| = |t|\sqrt{4e^{-2t^2} + 9t^2}$.

Acceleration: $\mathbf{r}''(t) = 4t^2e^{-t^2}\mathbf{i} + 6t\mathbf{j}$.

(d) Velocity: $\mathbf{r}'(t) = -2te^{-t^2}\mathbf{i} - 3t^2e^{-t^3}\mathbf{j}$. Speed $\|\mathbf{r}'(t)\| = |t|e^{-t^2}\sqrt{4 + 9t^2}$.

Acceleration: $\mathbf{r}''(t) = 2e^{-t^2}(2t^2 - 1)\mathbf{i} + 3te^{-t^2}(3t^3 - 2e^{-t^2})\mathbf{j}$.

19. Cusps and smoothness.

(a) $\mathbf{r}'(t) = -2te^{-t^2}\mathbf{i} - 3t^2e^{-t^3}\mathbf{j} = \mathbf{0} \Rightarrow t = 0$ and $x'(t) = -2te^{-t^2}$ changes sign at $t = 0$, so there is a cusp at $\mathbf{r}(0) = \mathbf{i} + \mathbf{j}$.

(b) $\mathbf{r}'(t) = 5t^4\mathbf{i} + 7t^6\mathbf{j} = \mathbf{0} \Rightarrow t = 0$ and neither $x'(t) = 5t^4$ nor $x'(t) = 7t^6$ changes sign at $t = 0$, so there are no cusps. The curve is smooth.

(c) $\mathbf{r}'(t) = -\sin(t)\mathbf{i} + 3\cos(3t)\mathbf{j} = \mathbf{0} \Rightarrow \sin(t) = \cos(3t) = 0 \Rightarrow t \in \{k\pi \mid k \in \mathbb{Z}\} \cap \{\frac{\pi}{6} + \frac{k\pi}{3} \mid k \in \mathbb{Z}\}$. But $\{k\pi \mid k \in \mathbb{Z}\} \cap \{\frac{\pi}{6} + \frac{k\pi}{3} \mid k \in \mathbb{Z}\} = \varnothing$ (you should prove this), so there are no cusps. The curve is smooth.

(d) $\mathbf{r}'(t) = -4\sin(t)\cos(t)\mathbf{i} + \cos(t)\mathbf{j} = \mathbf{0} \Rightarrow \cos(t) = 0 \Rightarrow t \in \{\frac{\pi}{2} + k\pi \mid k \in \mathbb{Z}\}$ and $x'(t) = \cos(t)$ changes sign at each of these t values, so there are cusps at $\mathbf{r}(\frac{\pi}{2}) = -\mathbf{i} + \mathbf{j}$ and $\mathbf{r}(\frac{3\pi}{2}) = -\mathbf{i} - \mathbf{j}$.

(e) $\mathbf{r}'(t) = -2(\sin(t) + \sin(2t))\mathbf{i} + 2(\cos(t) - \cos(2t))\mathbf{j} = \mathbf{0} \Rightarrow \sin(t) = -\sin(2t)$ and $\cos(t) = \cos(2t)$.

Using the identities $\sin(2t) = 2\sin(t)\cos(t)$ and $\cos(2t) = 2\cos^2(t) - 1$ we can solve $\sin(t) = -\sin(2t)$ and $\cos(t) = \cos(2t)$ to obtain ($\sin(t) = 0$ or $\cos(t) = \frac{1}{2}$) and $\cos(t) \in \{-\frac{1}{2}, 1\}$ respectively.

These conditions hold for all $t \in \{2k\pi \mid k \in \mathbb{Z}\} \cap \{\frac{2\pi}{3} + 2k\pi \mid k \in \mathbb{Z}\} \cap \{\frac{-2\pi}{3} + 2k\pi \mid k \in \mathbb{Z}\}$, giving cusps at $2\mathbf{i}, -\frac{3}{2}\mathbf{i} + \frac{3\sqrt{3}}{2}\mathbf{j}$ and $-\frac{3}{2}\mathbf{i} - \frac{3\sqrt{3}}{2}\mathbf{j}$.

You should check that $2(\sin(t) + \sin(2t))$ changes sign at each of these points.

20. Variation in speed.

(a) $\mathbf{r}'(t)\cdot\mathbf{r}''(t) = 2t^3 - t = t(2t^2-1)$. From this we can show that speed is increasing on $(-\frac{1}{\sqrt{2}}, 0)\cup(\frac{1}{\sqrt{2}},\infty)$, decreasing on $(-\infty, -\frac{1}{\sqrt{2}})\cup(0,\frac{1}{\sqrt{2}})$ and stationary at $0, \pm\frac{1}{\sqrt{2}}$.

(b) $\mathbf{r}'(t)\cdot\mathbf{r}''(t) = a^2\sin(t)\cos(t) - b^2\sin(t)\cos(t) = \frac{1}{2}(a^2-b^2)\sin(2t))$ and we are given $a > b$ so the sign of $\mathbf{r}'(t)\cdot\mathbf{r}''(t)$ is given by the sign of $\sin(2t) = 0$. Thus, speed is increasing on each interval $(k\pi, k\pi+\frac{\pi}{2})$, decreasing on each interval $(k\pi+\frac{\pi}{2}, k\pi)$ and stationary at each point of $\{\frac{k\pi}{2} \mid k\in\mathbb{Z}\}$.

21. Putting it all together.

(a) Particles collide at $-2i$ when $t=-1$.

(b) (i) increasing on $(0,\infty)$ (ii) decreasing on $(-\infty,0)$

(c) Speed of particle given by $\mathbf{r}_1(t)$ decreasing when $t=-1$. Speed of particle given by $\mathbf{r}_2(t)$ is increasing when $t=-1$.

Answers

Topic 4: Integration and Differential Equations

1. Simple Integrals. (a) $\frac{7}{4}x^4 + 2x^3 - 2x^2 + 3x + C$ (b) $-\frac{1}{\pi}\cos(\pi x) + C$ (c) $\frac{1}{3}e^{3x} + C$

(d) $-\frac{3}{2x} + C$ (e) $\frac{3}{2\pi}\sin(\frac{2\pi x}{3}) + C$ (f) $4\log|x| + C$

(g) $2\tan(\frac{x}{2}) + C$ (h) $x^2 - \frac{1}{2}\log|x| + C$ (i) $-5e^{-\frac{x}{5}} + C$

2. Equation of Curve. $y = \frac{1}{4}(x-1)^4 - 2x + 2$

3. Inverse Trigonometric. (a) $2\arccos\frac{x}{4} + C$ (b) $\frac{4}{5}\arctan\frac{x}{5} + C$ (c) $7\arcsin\frac{x}{\sqrt{3}} + C$

4. Derivative Present. (a) $\frac{1}{6}(x^2+1)^6 + C$ (b) $\sin(x^3+5) + C$ (c) $\frac{-1}{4(x^2+1)^2} + C$ (d) $2e^{2x^2+3x} + C$

(g) $\frac{5}{3}(9+x^2)^{3/2} + C$ (h) $\frac{2}{\cos 3x} + C$ (i) $-5\cos(\log_e(x)) + C$

5. Complete the Square. (a) $\arctan(x+1) + C$ (b) $\arcsin\frac{x+2}{5} + C$ (c) $\arccos\frac{x-3}{4} + C$

6. Linear Substitutions. (a) $-\frac{2}{3}(1-x)^{3/2} + \frac{2}{5}(1-x)^{5/2} + C$ (b) $2\log|x-1| - \frac{1}{x-1} + C$

(c) $\frac{3}{7}(x+4)^{7/3} - \frac{3}{4}(x+4)^{4/3} + C$ (d) $\frac{1}{11}(x+3)^{22} - \frac{5}{21}(x+3)^{21} + C$

7. Trigonometric Powers. (a) $\frac{1}{5}\sin^5 2x + C$ (b) $\frac{1}{2}x - \frac{1}{4}\sin 2x + C$ (c) $\frac{1}{2}\sin 2x - \frac{1}{3}\sin^3 2x + \frac{1}{10}\sin^5 2x + C$

(d) $\frac{1}{2}x + \frac{1}{28}\sin 14x + C$ (e) $\frac{1}{8}x - \frac{1}{160}\sin 20x + C$ (f) $\frac{1}{5}\sin^5 x - \frac{2}{7}\sin^7 x + \frac{1}{9}\sin^9 x + C$

8. Partial Fractions. (a) $\log|x-2| + 2\log|x+4| + C$ (b) $\log|x-4| - \log|x-1| + C$ (c) $2\log|x+2| + \frac{3}{x+2} + C$

(d) $5\log|x-2| - 4\log|x-1| + C$ (e) $\frac{1}{2}\log|x+1| + \frac{3}{2}\log|x-1| + C$ (f) $\log(x^2-2x+10) + \frac{2}{3}\arctan\left(\frac{x-1}{3}\right) + C$

9. Long Division. (a) $\frac{1}{3}x^3 - \frac{1}{2}x^2 - x + 5\log|x+2| + C$ (b) $\frac{x^2}{2} + 2x + \frac{3\arctan(2x/\sqrt{5})}{2\sqrt{5}} + C$

(c) $\frac{1}{2}x^2 + x - 3\log|x| + 4\log|x-1| + C$ (d) $\frac{x^2}{2} + \frac{1}{2}\log(10+4x+x^2) + C$

(e) $2x^2 + \log(x^2+10) + \frac{\arctan(x/\sqrt{10})}{\sqrt{10}} + C$

10. Mixed Integrals. (a) $-\frac{1}{3}\cos^3 x + \frac{1}{5}\cos^5 x + C$ (b) $\frac{1}{2}x^2 + x + 2\log|x^2-x| + C$ (c) $\frac{1}{2}e^{\sin 2x} + C$

(d) $\frac{3x}{8} + \frac{1}{4}\sin(2x) + \frac{1}{32}\sin(4x) + C$ (e) $\frac{1}{25}\tan^5 5x + \frac{1}{15}\tan^3 5x + C$ (f) $2\sqrt{5+x} + C$

(g) $3\log|x| - 5\log|x-1| + C$ (h) $\frac{2}{3}(-8+x)\sqrt{1+x} + C$ (i) $\frac{1}{4}\arctan\frac{x}{2} + C$

(j) $\arctan(x+3) + C$ (k) $\frac{1}{3}x^3 + \frac{11}{2}\log(x^2+6x+10) - 20\arctan(x+3) + C$

11. Simple Type. (a) 20 (b) $\frac{1}{\sqrt{2}} + \frac{\pi^2}{16}$ (c) 0 (d) $\frac{1}{3}\log 2$

12. Substitution. (a) $\frac{4}{15}$ (b) $\frac{1}{16}$ (c) $\frac{61}{3}$ (d) $\log 2$

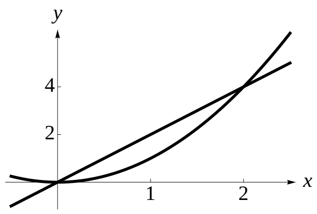
13. Mixed Type. (a) $\frac{5}{24}$ (b) $\frac{\pi}{4}$ (c) $\frac{\pi}{2}$ (d) $\log\frac{4}{3}$ (e) $\log 4 + 4$

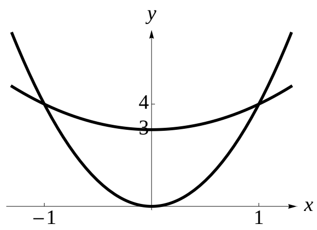
(f) 1 (g) 1 (h) $\frac{3}{2}(1 + \log 2)$ (i) $-\arctan(7/2) + \arctan(3 + \frac{\sqrt{3}}{2})$

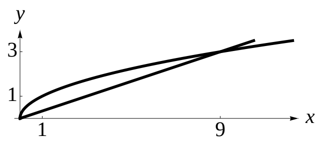
14. Area under the graph. (a) $\frac{8}{3}$ units^2 (b) $\frac{3\pi}{2}$ units^2 (c) 1 unit^2 (d) $\pi - 2$ units^2

15. With respect to x.

(a) $\frac{4}{3}$ (b) 4 (c) $\frac{9}{2}$

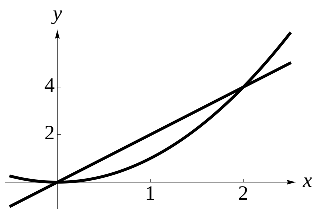


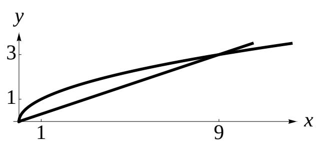


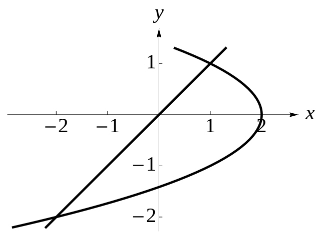


16. With respect to y.

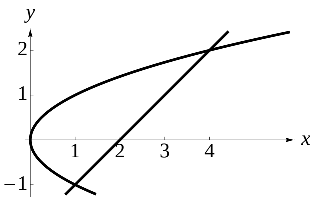
(a) $\frac{4}{3}$ (b) $\frac{9}{2}$ (c) $\frac{9}{2}$







17. You decide!

(a) 

(b) (i) Area $= \int_{-1}^{2} (y+2) - y^2 \, \mathrm{d}y$

(ii) Area $= \int_0^1 2\sqrt{x} \, \mathrm{d}x + \int_1^4 \sqrt{x} - x + 2 \, \mathrm{d}x$

(c) $\frac{9}{2}$

18. Verification. Verification required.

19. Constant Solutions.

(a) $y = 0.$

(b) No constant solutions since $e^y > 0.$

(c) No constant solutions since $y^2 + 1 > 0.$

(d) $y = \dfrac{k\pi}{2}.$

20. Initial Value Problems. (a) $y = x+3$ (b) $y = 2e^x$ (c) $y = \sqrt{25 - x^2}$ (d) $y = e^{2x+2}+2x+1$
(e) $y = \frac{256}{x^2}$

21. Second Order. Verification required.

22. Cubic Solution. $a = 1,\ b = -6,\ c = 18,\ d = -24$

23. Exponential Solution. $k = -1, -2$

24. First Order. (a) $y = 2\sqrt{x} + C$ (b) $x = \frac{1}{2}t^2 + 3t - \log|t| + C$ (c) $y = -\frac{1}{3}\cos(3t + \pi) + \frac{2}{3}$ (d) $y = \frac{1}{2}\log(2x - 1) + 3$

25. Second Order. (a) $y = 4e^{\frac{x}{2}} + Cx + D$ (b) $x = \frac{4}{15}(1 - t)^{\frac{5}{2}} + Ct + D$ (c) $y = -\log|x + 1| + Cx + D$

26. *y* only. (a) $y = \sqrt[3]{3(x - C)}$ (b) $y = \tan(x - C)$ (c) $y = \frac{1}{4}(x - 1)^2$ (d) $y = e^x + 4$

27. Separable. (a) $y = \frac{-1}{5\sin x + C}$ (b) $y = \frac{1}{2}\log(2e^x + D)$ (c) $y = \sin(\log|x| + C)$

(d) $y = 3\sin(\frac{3}{5}\sin^5 t + C)$ (e) $\frac{1}{3}x^3 - e^{-x} = \frac{3}{2}t^2 + \frac{1}{2}e^{2t} + C$

(f) $y = \pm\sqrt{\exp(t/2 + \sin 6t/12 + C) - 1} = \pm\sqrt{A\exp(t/2 + \sin 6t/12) - 1}$ where $A = e^C$.

28. Separable Initial Value. (a) $y = 3e^{\frac{3}{2}x^2}$ (b) $x = \frac{2t}{t+1}$

29. Projectile Motion. (a) $h(0) = 0$ (fired from ground level), $\dfrac{dh}{dt} = 50$ (initial speed 50m/s)

(b) $h = -5t^2 + 50t$ (c) 125m

30. Balloon. (a) $\dfrac{dV}{dt} = \frac{k}{V}$, where k is a constant. (b) $V = 10\sqrt{3t + 1}$ (c) 50cm^3

31. Radioactive Decay. (a) $\dfrac{dQ}{dt} = -kQ$, where k is a positive constant. (b) $Q = 100(\frac{1}{2})^{t/10}$

32. Something Fishy. (a) $F = 10e^{0.1t}$ (b) Approximately 46 weeks.

33. Something Moosey. (a) $\dfrac{dP}{dt} = kP$, where k is a constant. (b) $P = Ae^{kt}$ (A a constant)

(c) $P = 100e^{\frac{1}{2}\log(1.1)t} = 100(1.1)^{\frac{t}{2}}$ (d) $P(5) = 127$

34. Metal rod. (a) $T = 100e^{\frac{1}{2}\log(\frac{3}{5})t} + 10 = 100\left(\frac{3}{5}\right)^{\frac{t}{2}} + 10$ (b) 46°C