Determining invariant measures for fractals in R^2 using magnitude functions

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Abstract

Few invariant measures are defined for fractals. Development of such measures will increase fractal geometry's utility. This investigation explored the possibility of developing a systematic approach for determining invariants. Our strategy was to work asymptotically towards developing scaled magnitude functions for fractals in R^2 by approximating a fractal with a finite sequence in a metric space, and attempting to take a limit of those measures.

Using the Euclidean metric, it was found that the magnitude functions resulting from even low order fractals are very large and do not immediately lend themselves to asymptotic analysis. General formulae for calculating weights for two and three orbit fractals were developed. In addition, a general form of Leinster and Willerton's (2009) Lemma 7 was proved. Solutions to some of the difficulties that were met when attempting to solve matrices and determine weights of higher order fractals will be presented. Suggestions for future research will be made.

Background

What are fractals? A fractal is a geometric curve defined by a distinct shape that recursively repeats itself on a decreasing scale. Fractal geometry can describe the shape of real-world objects like clouds, mountains, or systems of blood vessels as precisely as an architect might describe a building (Barnsley, 1988). This is beyond the descriptive capacity of classical geometry.

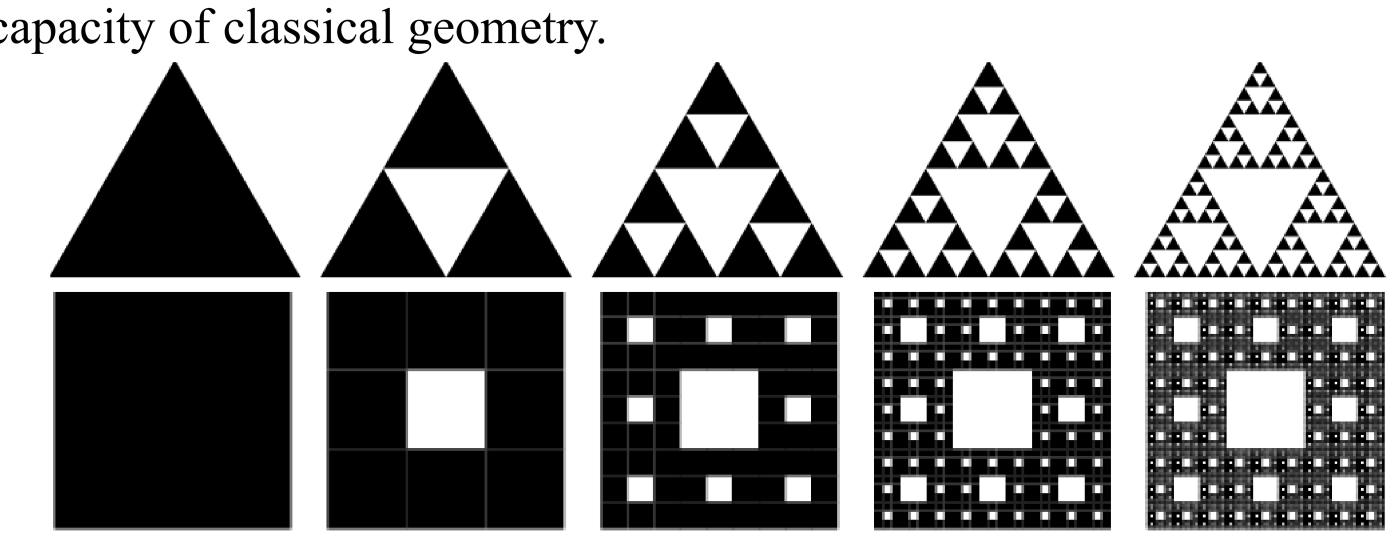

Figure 1. Depictions of Sierpinski triangle (top) and carpet (bottom) in Euclidian space (R^2), showing incremental application of self-similarity traits. The white spaces represent 'holes.'

Applications of fractals: Fractals are used in various fields. Examples include analyzing ultrasound scans of livers to detect diseases (Wen-Li et al., 2005), modeling soil water retention (Tyler and Wheatcraft, 1989) and analyzing non-uniform, naturally fractured reservoirs (Acuna et al., 1995).

What are invariant measures? Invariant measures are used as a means of distinguishing objects. Having well defined invariant measures could prove invaluable given fractal geometry's capacity to describe natural phenomena.

Invariant measures in fractals: Traditional measures – such as surface area, volume, and Euler characteristic – are not applicable to fractals. Thus, invariant measures must be defined specifically for fractals.

Two different approaches for determining invariant measures for fractals have been developed by Llorente and Winter (2006) and Leinster and Willerton (2009). Both methods appear sound, yet produce different results. This inconsistency highlights the present challenges in defining invariant measures for fractals.

Methods

1. Define some finite metric space $(A,d) \in R^2$ with n points, $a_1, \ldots, a_n$, where the metric d [extrinsic, Euclidean distance] between any two points is given by $d_{ij} = d(a_i,a_j)$.
2. Let $\mathbf{Z}'$ be an $n \times n$ matrix of exponential distances, scaled by t where $t \in R$: $t > 0$, between points in A:
$$\mathbf{Z}' = [Z'_{ij}] = e^{-t \cdot d_{ij}}$$
3. Define a weighting $w_a \in R$ for each point $a \in tA$, such that:
$$\sum_{a' \in A} e^{-t \cdot d(a,a')} \cdot w_{a'} = 1$$
4. If tA has a unique vector of weights w, and since generic square matrices are invertible, w_a is the sum of entries in the ith row of $(\mathbf{Z}')^{-1}$:
$$w_a := \sum_j (\mathbf{Z}')^{-1}_{i,j}$$
5. The magnitude $|tA|$ is defined to be the sum of the weights:
$$|tA| := \sum_{a' \in A} w_a$$
6. The sets $A \in R^2$ were constructed using the vertices from Sierpinksi triangle and carpet fractals (Fig. 1).
7. Define functions P and q such that $|tA| = P(tA) + q(tA)$, where $P(tA)$ is a function that satisfies the inclusion-exclusion principle, $P_{A \cup B}(t) = P_A(t) + P_B(t) - P_{A \cap B}(t)$, and $\lim_{t \to \infty} q(tA) = 0$. Our conjecture was that invariant measures for a fractal could be found in the lowest terms of $P(tA)$.

Results

Weight Formulae:

We define an orbit as a minimal set of points that remain unchanged under the action of the fractal's symmetry group. Figure 2 shows the results obtained for two orbit sets tA.

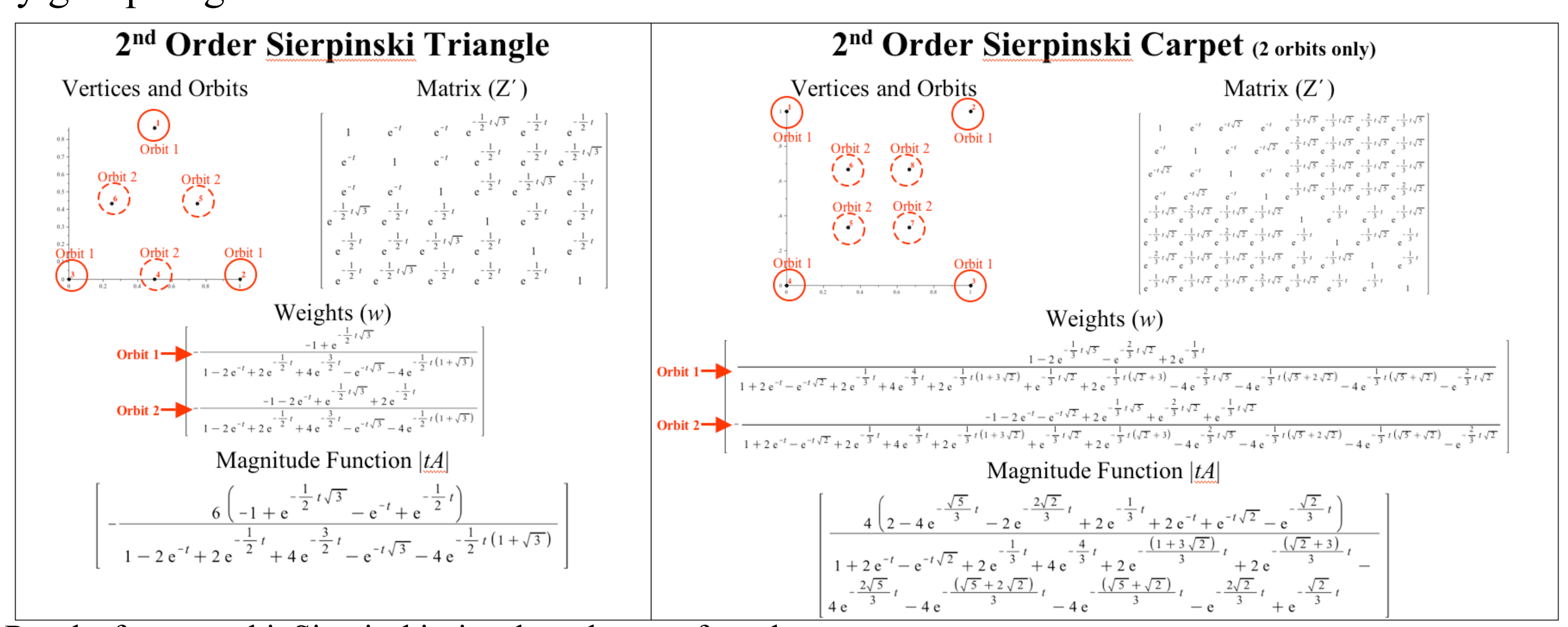

Figure 2. Results for two-orbit Sierpinski triangle and carpet fractals.

As a result of our efforts, a general formula for determining the weights of points in sets with two or three orbits was developed (Fig. 3).

Definitions and Notation:

With $\mathcal{O}_n$ as the set of points belonging to the nth orbit in A, and where $a_i \in \mathcal{O}_i$ and $a_j \in \mathcal{O}_j$ and $a_k \in \mathcal{O}_k$, the distance $\mathcal{D}$ between orbits are defined and represented as follows:

$$\mathcal{D}(\mathcal{O}_i \to \mathcal{O}_j) = \sum_j e^{-d(a_i,a_j)^t}$$

$$\mathcal{D}(\mathcal{O}_i \leftrightarrows \mathcal{O}_j) = \left(\sum_j e^{-d(a_i,a_j)^t}\right) \cdot \left(\sum_i e^{-d(a_j,a_i)^t}\right)$$

$$\mathcal{D}\left(\mathcal{O}_i \overset{\nearrow \mathcal{O}_j \searrow}{\longleftarrow} \mathcal{O}_k\right) = \left(\sum_j e^{-d(a_i,a_j)^t}\right) \cdot \left(\sum_k e^{-d(a_j,a_k)^t}\right) \cdot \left(\sum_i e^{-d(a_k,a_i)^t}\right)$$

Using the Speyer formula (Leinster and Willerton, 2009), which defines the weight of points in a set with one orbit as:

$$w_a = \frac{1}{\sum_{a' \in A} e^{-d(a,a')}}$$

We define the weight of points in a single orbit, within a multi – orbit set as:

$$W(n) = \frac{1}{\mathcal{D}(\mathcal{O}_i \to \mathcal{O}_i)}$$

Formulae:

Our formula for the weight w_n of the points in the n^{th} orbit of a **two orbit** set tA (where n' represents the other orbit in tA):

$$\boldsymbol{w_n} = W(n) \cdot \frac{1 - W(n') \cdot \mathcal{D}(\mathcal{O}_{n'} \to \mathcal{O}_n)}{1 - W(n) \cdot W(n') \cdot \mathcal{D}(\mathcal{O}_n \leftrightarrows \mathcal{O}_{n'})}$$

Our formula for the weight w_n of the points in the n^{th} orbit of a **three orbit** set tA (where n' and n" represent the other orbits in tA):

$$\boldsymbol{w_n} = W(n) \cdot \frac{1 - W(n') \cdot \mathcal{D}(\mathcal{O}_{n'} \to \mathcal{O}_n) - W(n'') \cdot \mathcal{D}(\mathcal{O}_{n''} \to \mathcal{O}_n) + W(n') \cdot W(n'') \cdot \mathcal{D}(\mathcal{O}_{n''} \to \mathcal{O}_{n'} \to \mathcal{O}_n) + W(n') \cdot W(n'') \cdot \mathcal{D}(\mathcal{O}_{n'} \to \mathcal{O}_{n''} \to \mathcal{O}_n)}{1 - W(n'') \cdot W(n) \cdot \mathcal{D}(\mathcal{O}_{n''} \leftrightarrows \mathcal{O}_n) - W(n') \cdot W(n) \cdot \mathcal{D}(\mathcal{O}_{n'} \leftrightarrows \mathcal{O}_n) - W(n') \cdot W(n'') \cdot \mathcal{D}(\mathcal{O}_{n'} \leftrightarrows \mathcal{O}_{n''}) + W(n') \cdot W(n'') \cdot W(n) \cdot \mathcal{D}\left(\mathcal{O}_{n'} \overset{\nearrow \mathcal{O}_{n''} \searrow}{\longleftarrow} \mathcal{O}_n\right) + W(n') \cdot W(n'') \cdot W(n) \cdot \mathcal{D}\left(\mathcal{O}_{n'} \overset{\nearrow \mathcal{O}_{n''} \searrow}{\longleftarrow} \mathcal{O}_n\right)}$$

Figure 3. Weight formulae for two and three orbit fractals, along with definitions and notation.

Magnitude Function Analysis:

A general form of Leinster and Willerton's (2009) Lemma 7 was proved, which should prove useful in determining invariant measures from magnitude functions:

For $P: \mathbb{R}^+ \to \mathbb{R}$, $\boldsymbol{P(ml) = nP(l) + C}$ for all $l > 0$, **if and only if** $\boldsymbol{P(l) = f(l) \cdot l^{\log_m(n)} + \frac{C}{(1-n)}}$

Where $f: \mathbb{R}^+ \to \mathbb{R}$ is a multiplicativity periodic function that satisfies $f(ml) = f(l)$, $l > 0$ and $n, m \in \mathbb{Z}^+$, $n > 1$.

Discussion

Difficulties were encountered when attempting to solve $\mathbf{Z}'$ to determine weightings of higher order fractals. However, this is not surprising. As shown by Fang and Havas (1997), algorithms used for solving large matrices are based on modified forms of Gaussian elimination which has the worst-case exponential space and time complexity for calculating Hermite normal forms.

By grouping the distances between orbits, $\mathbf{Z}'$ matrices were reduced to an $m \times m$ matrix S, where m is equal to the number of orbits in the set A, and each element $S_{i,j}$ represents the distance between orbits i and j:

$$S_{i,j} = \mathcal{D}(\mathcal{O}_i \to \mathcal{O}_j) = \sum_j e^{-d(a_i,a_j)^t}$$

Figure 4 provides an example of this matrix order reduction.

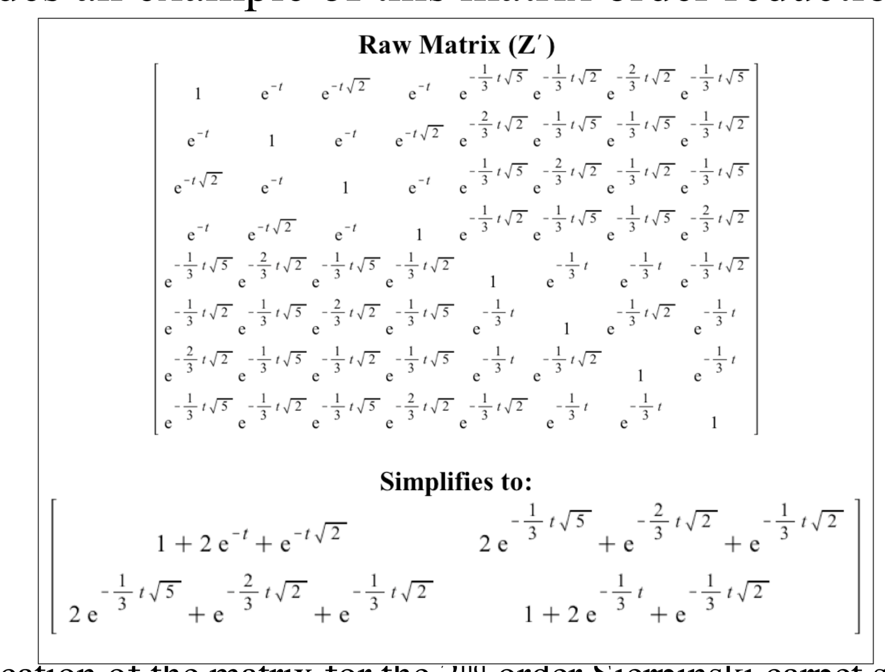

Figure 4. Simplification of the matrix for the 2nd order Sierpinski carpet shown in Figure 2 using orbit distances.

Magnitude functions resulting from even lower order fractals proved to be very large and cumbersome to work with (e.g. over 250 terms in $|tA|$ for 3rd order Sierpinski triangle). As such, the functions P and q outlined in Methods were not identified.

Conclusion

Further study is required to determine invariant measures for fractals. It was observed that if all points within A can be divided into orbits with common symmetry, points within the same an orbit will have the same weight. Thus, this observation may be useful for generalizing the formulation of weightings based on orbits. Additionally, using different metrics such as the Manhattan metric, or an intrinsic distance metric may allow for greater simplification of magnitude functions.

Literature Cited

Acuna, J., Ershaghi, I., and Yortsos, Y. 1995. Practical application of fractal pressure-transient analysis in naturally fractured reservoirs. *SPE Formation Evaluation*. 10(3):173-179.

Barnsley, M. 1988. *Fractals Everywhere*. San Diego, CA: Academic Press, Inc.

Fang, X. G., and Havas, G. 1997. On the worst-case complexity of integer Gaussian elimination. *Proceedings of the 1997 international symposium on Symbolic and algebraic computation*. 28-31. http://perso.ens-lyon.fr/gilles.villard/BIBLIOGRAPHIE/PDF/ft_gateway.cfm.pdf

Leinster, T. and Willerton, S. 2009. On the asymptotic magnitude of subsets of Euclidean space. *ArXiv e-prints*. 1-42. arXiv:0908.1582v1 [math.MG]. http://arxiv.org/abs/0908.1582

Llorente, M., and Winter, S. 2006. A notion of Euler characteristic for fractals. *Mathematische Nachrichten*. 280(1-2):152–170.

Tyler, S., and Wheatcraft, S. 1989. Application of fractal mathematics to soil water retention estimation. *Soil Science Society of America*. 53(4):987–996.

Wen-Li, L., Yung-Chang, C., Ying-Cheng, C., and Kai-Sheng, H. 2005. Unsupervised segmentation of ultrasonic liver images by multiresolution fractal feature vector. *Information Sciences*. 175(3):177-199.

Acknowledgements

We extend our thanks to Dr. Dorette Pronk for her direction and findings, which were absolutely indispensible to this work. Additionally, to Professor Josh MacArthur for his ongoing help and support.

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