Utrecht University

Introduction to Complex Systems

Final Project

Spontaneous synchronisation for the Kuramoto model

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January 28, 2022

Contents

1 Introduction

Systems of coupled nonlinear oscillators exhibit remarkable phenomena of collective synchronization [3]. The Kuramoto model describes a system of globally coupled phase oscillators with distributed natural frequencies. The model exhibits a phase transition between a low-coupling incoherent phase in which the oscillators oscillate independently and a high-coupling synchronized phase [2].

The aim of Part I is to dive into the various concepts of the Kuramoto model and study synchronization in terms of the order parameters. The starting point is a theoretical study of the corresponding Lyapunov function and the stationary points, which ends up to be useful for the study of the ground state of mean-field Heisenberg XY ferromagnetic model. The discussion is expanded to a model with N identical oscillators, where certain properties are analysed, such as the fact that the equation of motion for each oscillator can be written as a gradient system, the relation of the gradient function and the coherence parameter and the conservation of the mean phase.

Next, a numerical study is implemented for the Kuramoto model. In contrast to the theoretical studies, where the frequencies of the oscillators were known, here they pick up values from a normal (initially) and a uniform (finally) distribution. Of particular interest is the study of the "infinite-time" values of the coherent parameter for different values of the coupling constant, as well as its time evolution. The results are expected to indicate a specific behaviour (synchronisation), which is apparent for values of the coupling constant higher than a critical value. A comparison with the theoretical predictions also takes place.

As a final and independent of the Kuramoto model task, one of the exercises met throughout the course is solved in Part II. More specifically, the strong solution of a linear SDE with white noise (Lecture 10) is found and its covariance is calculated explicitly.

2 PART I: Kuramoto model, theory and simulations

2.1 Theoretical studies

We consider a large population of interacting oscillators living in the unit circle, under the dynamics of the so called mean-field Kuramoto model, defined by

$$\frac{d\theta_i}{dt}(t) = \omega_i + \frac{K}{N}\sum_{j=1}^{N} sin(\theta_j(t) - \theta_i(t)) \tag{1}$$

with $i = 1, ..., N$ and K the coupling constant. As we have seen in the lecture, for small coupling K each oscillator rotates with its natural frequency ω_i, whereas for large coupling K almost all angles θ_i will be entrained by the mean field and the oscillators synchronize. In this project we study first a simple case where the natural frequencies of the rotators are known and in the second part, when they are sampled from a probability distribution $g(\omega)$.

2.1.1 Lyapunov function for studying the stability of the equilibria

If we define

$$\bar{\omega} := \frac{1}{N}\sum_{i=1}^{N}\omega_i \tag{2}$$

the mean natural frequency, it is very reasonable to think that it is the common frequency of the phase locked component.

We initially want to prove that indeed

$$\frac{d}{dt}\left(\frac{1}{N}\sum_{i=1}^{N}\theta_i(t)\right) = \bar{\omega} \tag{3}$$

This is straightforward. Starting from (2) and substituting ω_i's by virtue of (1) we obtain

$$\bar{\omega} := \frac{1}{N}\sum_{i=1}^{N}\omega_i = \frac{1}{N}\sum_{i=1}^{N}(\frac{d\theta_i(t)}{dt} - \frac{K}{N}\sum_{j=1}^{N} sin(\theta_j(t) - \theta_i(t))) \tag{4}$$

Then, we notice that the sum $\sum_{i=1}^{N}\sum_{j=1}^{N} sin(\theta_j(t) - \theta_i(t)))$ is actually vanishing, due to the fact that the sine function is odd. In particular, the term $sin(\theta_j(t) - \theta_i(t)$ is cancelled by the term $sin(\theta_i(t) - \theta_j(t)$ and this happens for all $i, j = 1, \ldots, N$, with $i \neq j$. Finally, the sine function automatically vanishes for the terms with $i = j$. Thus, the second term in (4) vanishes and (3) is immediately obtained.

We consider now the new variables ϕ_i in the reference frame rotating with $\omega = \bar{\omega}$, i.e.:

$$\phi_i(t) := \theta_i(t) - \bar{\omega}t$$

so that the Kuramoto model becomes:

$$\frac{d\phi_i}{dt}(t) = (\omega_i - \bar{\omega}) + \frac{K}{N}\sum_{j=1}^{N} sin(\phi_j(t) - \phi_i(t)) \tag{5}$$

We will next show that the function $\mathcal{H}$:

$$\mathcal{H} := -\frac{K}{N}\sum_{i,j} cos(\phi_i - \phi_j) - \sum_{i=1}^{N}(\omega_i - \bar{\omega})\phi_i \tag{6}$$

is a Lyapunov function,that is,it satisfies

$$\dot{\mathcal{H}} = \sum_{i=1}^{N} \frac{\partial \mathcal{H}}{\partial \phi_i}\frac{d\phi_i}{dt} \leq 0$$

This is also straightforward. First of all, we notice that

$$\begin{aligned}\frac{\partial \mathcal{H}}{\partial \phi_k} &= -((\omega_k - \bar{\omega}) - \frac{K}{2N}\sum_{i=1}^{N} sin(\phi_k - \phi_i) + \frac{K}{2N}\sum_{i=1}^{N} sin(\phi_i - \phi_k)) \\ &= -((\omega_k - \bar{\omega}) + \frac{K}{N}\sum_{i=1}^{N} sin(\phi_i - \phi_k)) = -\frac{d\phi_k}{dt} = -\dot{\phi}_k\end{aligned}$$

and thus

$$\dot{\mathcal{H}} = \sum_{k=1}^{N} \frac{\partial \mathcal{H}}{\partial \phi_k}\frac{d\phi_k}{dt} = -\sum_{k=1}^{N}(\dot{\phi}_k)^2 \leq 0 \tag{7}$$

Therefore, by the properties of the Lyapunov function, the dynamics in (5) converges to a minimum of $\mathcal{H}$.

Let's now find the equations for the extreme points of $\mathcal{H}$, that is, the ones for which $\nabla\mathcal{H} = 0$. We have

$$\nabla\mathcal{H} = 0 \Rightarrow -\frac{K}{N}\sum_{j=1}^{N} sin(\phi_i - \phi_j) + (\omega_i - \bar{\omega}) = 0$$

and solving for ω_i we get

$$\omega_i = \bar{\omega} + \frac{K}{N}\sum_{j=1}^{N} sin(\phi_i - \phi_j) \tag{8}$$

For the configuration with $\omega_i = \bar{\omega}$, the above condition gives

$$\frac{K}{N}\sum_{j=1}^{N}(\phi_i(t) - \phi_j(t)) = 0 \tag{9}$$

Note however that

$$\nabla\mathcal{H} = \sum_{i=1}^{N}\frac{K}{N}\sum_{j=1}^{N}(\phi_i(t) - \phi_j(t)) = 0 \tag{10}$$

since sine is an odd function. This does not allow one to conclude that an asymptotically stable fixed point was found. One needs to involve further analysis in order to claim that.
We introduce the order parameter:

$$re^{i\Psi} := \frac{1}{N}\sum_{j=1}^{N} e^{i\phi_j} \tag{11}$$

We want to show that the stationary solutions of the Kuramoto model satisfy:

$$\Delta(\omega_i) = Krsin(\phi_i - \Psi) \tag{12}$$

with $\Delta(\omega_i) = \omega_i - \bar{\omega}$. For the stationary solutions we have

$$\frac{d\phi_i}{dt}(t) = 0 \Rightarrow (\omega_i - \bar{\omega}) = -\frac{K}{N}\sum_{j=1}^{N} sin(\phi_j(t) - \phi_i(t))$$

and defining $\Delta(\omega_i) = \omega_i - \bar{\omega}$ we obtain

$$\Delta(\omega_i) = -K\frac{1}{N}\sum_{j=1}^{N}\mathfrak{Im}e^{i(\phi_j(t)-\phi_i(t))} = -K\mathfrak{Im}\frac{1}{N}\sum_{j=1}^{N}e^{-\phi_i}e^{+i\phi_j} = -K\mathfrak{Im}e^{-\phi_i}\frac{1}{N}\sum_{j=1}^{N}e^{i\phi_j}$$
$$= -K\mathfrak{Im}e^{-\phi_i}re^{i\Psi} = -Kr\mathfrak{Im}e^{-i(\phi_i-\Psi)} = Krsin(\phi_i - \Psi)$$

The Lyapunov function defined by (6) can be interpreted as the Hamiltonian for the mean-field Heisenberg XY ferromagnet with coupling strength K. The first term aims at keeping all spins parallel. The second term represents a kind of random field: the two terms counteract each other (frustration). We want to show that the gound state of $\mathcal{H}$ is rotationally invariant also in the presence of the random field. To this end, consider the following phase translation transformation

$$\delta\phi_i = R \tag{13}$$

Under this transformation,

$$\begin{aligned}
\tilde{\mathcal{H}} &= -\frac{K}{N}\sum_{i,j} cos(\phi_i + \cancel{R} - \phi_j - \cancel{R}) - \sum_{i=1}^{N}(\omega_i - \bar{\omega})(\phi_i + R) \\
&= -\frac{K}{N}\sum_{i,j} cos(\phi_i - \phi_j) - \sum_{i=1}^{N}(\omega_i - \bar{\omega})\tilde{\phi}_i = \mathcal{H}
\end{aligned}$$

where we defined $\tilde{\phi}_i = \phi_i + R$. The reason why we are allowed to do this and conclude why finally the ground state remains invariant under the phase translation is because ϕ_i in this term is a random field and so is the transformed $\tilde{\phi}_i$. In order to reach a conclusion about the Jacobian matrix of the system, we consider the following matrix representing the phase-translations of all oscillators collectively

$$\Phi(t) = \begin{pmatrix} e^{i\phi_1(t)} \\ \dots \\ e^{i\phi_N(t)} \end{pmatrix} \rightarrow \begin{pmatrix} e^{i(\phi_1(t)+R)} \\ \dots \\ e^{i(\phi_N(t)+R)} \end{pmatrix} = e^{iR}\Phi(t) = \tilde{\Phi}(t) \tag{14}$$

Since the ground state of the system remains invariant under the above transformations, we can deduce that

$$\mathcal{J}\Phi(t) = \mathcal{J}e^{iR}\Phi(t) \Rightarrow \mathcal{J}\Phi(t)(1 - e^{iR}) = 0 \Rightarrow e^{iR} = 1 \tag{15}$$

which means that $(1, 1, \dots, 1)$ is an eigenvector.

2.1.2 Model with N identical oscillators

We consider now a set of identical oscillators, in the sense:

$$\frac{d\theta_i}{dt}(t) = \omega + \frac{K}{N}\sum_{j=1}^{N} sin(\theta_j(t) - \theta_i(t)) \tag{16}$$

i.e. the natural frequencies ω_i are all the same.

For simplicity we set $K = 1$ and $\omega = 0$ (or equivalently, we consider the rotating frame with angular velocity ω). We first want to show that (16) can be written in term of the order parameter in the form:

$$\frac{d\theta_i}{dt}(t) = r(t)sin(\Psi(t) - \theta_i(t)) \tag{17}$$

Indeed, one can write,

$$\begin{aligned}
\frac{1}{N}\sum_{j=1}^{N} sin(\theta_j(t) - \theta_i(t)) = \frac{1}{N}\sum_{j=1}^{N} \mathfrak{Im} e^{i(\theta_j(t)-\theta_i(t))} &= \frac{1}{N}\mathfrak{Im}\sum_{j=1}^{N} e^{-i\theta_i(t)}e^{i\theta_j(t)} \\
&= \mathfrak{Im} e^{-i\theta_i(t)}\frac{1}{N}\sum_{j=1}^{N} e^{i\theta_j(t)} = \mathfrak{Im} e^{-i\theta_i(t)} r(t)e^{i\Psi(t)} = r(t)\mathfrak{Im} e^{i(\Psi(t)-\theta_i(t))} \\
&= r(t)sin(\Psi(t) - \theta_i(t))
\end{aligned}$$

Next, we show that (16) is a gradient system (again for $\omega = 0, K = 1$), that is, it can be written in the form $\dot{\theta}_i(t) = \frac{\partial U}{\partial \theta_i}$, for a function U. Consider

$$U(\theta_1, \ldots, \theta_N) := \frac{1}{2N}\sum_{i,j} cos(\theta_i - \theta_j) \tag{18}$$

Then,

$$\begin{aligned}
\frac{\partial U}{\partial \theta_k} &= \frac{\partial}{\partial \theta_k}(\frac{1}{2N}\sum_{i,j} cos(\theta_i - \theta_j)) = \frac{1}{2N}\sum_{j=1}^{N} -sin(\theta_k - \theta_j) + \frac{1}{2N}\sum_{i=1}^{N} -sin(\theta_i - \theta_k)(-1) \\
&= \frac{1}{2N}\sum_{j=1}^{N}(sin(\theta_j - \theta_k) - sin(\theta_k - \theta_j)) = \frac{1}{2N}\sum_{j=1}^{N}(sin(\theta_j - \theta_k) + sin(\theta_j - \theta_k)) \\
&= \frac{1}{N}\sum_{j=1}^{N} sin(\theta_j - \theta_k) = \frac{d\theta_k(t)}{dt}
\end{aligned}$$

By virtue of (18) and (11), we can write

$$\begin{aligned}
U &= \frac{1}{2N}\sum_{i,j=1}^{N} cos(\theta_i - \theta_j) = \frac{1}{2N}\sum_{i,j=1}^{N} \mathfrak{Re} e^{i(\theta_i-\theta_j)} = \frac{1}{2}\mathfrak{Re}\frac{1}{N}\sum_{i=1}^{N}\sum_{j=1}^{N} e^{i\theta_i}e^{-i\theta_j} = \frac{1}{2}\mathfrak{Re}\sum_{j=1}^{N} re^{i\Psi}e^{-i\theta_j} \\
&= \frac{1}{2}\mathfrak{Re} rN(\frac{1}{N}\sum_{j=1}^{N} e^{-i\theta_j})e^{i\Psi} = \frac{1}{2}Nrr\mathfrak{Re} e^{-i\Psi}e^{i\Psi} = \frac{1}{2}Nr^2
\end{aligned} \tag{19}$$

Consider the function

$$F(t) = \frac{1}{N}\sum_{i=1}^{N}\theta_i(t) \tag{20}$$

Taking a time derivative we get

$$\frac{dF}{dt} = \frac{1}{N}\sum_{i=1}^{N}\frac{d\theta_i(t)}{dt} = \frac{1}{N}\sum_{i=1}^{N}\frac{\partial U}{\partial\theta_i} = -\frac{1}{N}\sum_{i=1}^{N}\frac{1}{2N}\sum_{j=1}^{N}sin(\theta_i-\theta_j)$$
$$= -\frac{1}{2N^2}\sum_{i,j}sin(\theta_i-\theta_j) = 0$$

for the same argument as earlier. This means that $F = const$ and therefore we conclude that the mean phase is conserved:

$$\frac{1}{N}\sum_{i=1}^{N}\theta_i(t) = \frac{1}{N}\sum_{i=1}^{N}\theta_i(0) \tag{21}$$

Let us now show that for a non-stationary solution $\{\theta_(t)\}_i$, it is always $\frac{dr}{dt} > 0$. Starting from (19) and taking a time derivative, one can write

$$\frac{N}{2}2r\frac{dr}{dt} = \frac{dU}{dt} = \frac{\partial U}{\partial\theta_i}\frac{d\theta_i}{dt} = \Big(\frac{d\theta_i}{dt}\Big)^2$$

and thus

$$\frac{dr}{dt} = \frac{1}{rN}\Big(\frac{d\theta_i}{dt}\Big)^2 > 0 \tag{22}$$

since by definition $r \geq 0$.

Closing the theoretical studies, it is useful to give a characterization of the stationary solutions in terms of coherent and incoherent states. Recall that for stationary solutions ($\Delta(\omega_i) = 0$ for the case at hand), we have:

$$rsin(\phi_i - \Psi) = 0 \tag{23}$$

The equation admits two different scenarios:

1. $r = 0$ and $sin(\phi_i - \Psi)$:arbitrary or
2. r: arbitrary and $\phi_i = \Psi + n\pi$, $n \in \mathbb{Z}$

The first scenario is characterized by fully incoherence, in the sense that each stationary solution oscillates with its own phase. The second case is characterized by full coherence, since all stationary solutions have phases differing by integer multiples of π.

2.2 Numerical studies for the Kuramoto Model

In this last part of the section, we aimed at a numerical solution of 1, using Euler's method. The simulations for the first part of the study were performed for a uniform distribution of the initial phases, while the natural frequencies ω_i's were chosen from a normal distribution. The code, used to perform the simulations, was conducted in Python and can be found in the Appendix section. A slight modification of this code will also be used in the next part, where the initial phases are still picked up from a uniform distribution $\mathcal{U}[0, 2\pi]$, while the natural frequencies were chosen to take values from a uniform distribution $\mathcal{U}[-\frac{1}{2}, \frac{1}{2}]$.

2.2.1 Normal distribution of the natural frequencies

Before going through the results obtained by the simulations, it is important to note what expressions were used for the order parameters. In particular, Ψ was defined as the mean phase, given by (24)

$$\psi = \frac{1}{N}\sum_{i=1}^{N}\theta_i(t) \tag{24}$$

Then, independent on Ψ, an expression for r is found as follows:

$$re^{i\Psi} := \frac{1}{N}\sum_{j=1}^{N} e^{i\phi_j} = \frac{1}{N}\sum_{j=1}^{N}(cos\phi_j + i\frac{1}{N}\sum_{j=1}^{N} sin(\phi_j) = \langle cos(\phi_i)\rangle + i\langle sin(\phi_i)\rangle$$

Then, multiplying by its complex conjugate, one obtains

$$\begin{aligned} r^2 &= (\langle cos(\phi_i)\rangle + i\langle sin(\phi_i)\rangle)(\langle cos(\phi_i)\rangle - i\langle sin(\phi_i)\rangle) \\ &= (\langle cos(\phi_i)\rangle)^2 + (\langle sin(\phi_i)\rangle)^2 \end{aligned}$$

from which follows that (since $r > 0$)

$$r = \sqrt{(\langle cos(\phi_i)\rangle)^2 + (\langle sin(\phi_i)\rangle)^2} \tag{25}$$

From the theory, we know the order parameter r satisfies the consistency equation:

$$1 = K \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} cos(\theta)^2 e^{-\frac{1}{2}(Krsin(\theta))^2} \tag{26}$$

Moreover, the theoretical value of the critical coupling constant is given by

$$K_C = \frac{2}{\pi g(0)} = \frac{2}{\pi\sqrt{\frac{1}{2\pi}}} = \sqrt{\frac{8}{\pi}} \approx 1.5957 \tag{27}$$

In Figure 1 one can observe the behaviour of the "infinite-time" valued coherence parameter with various values of K. The dotted blue line represents the theoretically computed critical coupling constant. Indeed one can verify that beyond the critical value the system enters a coherence phase.

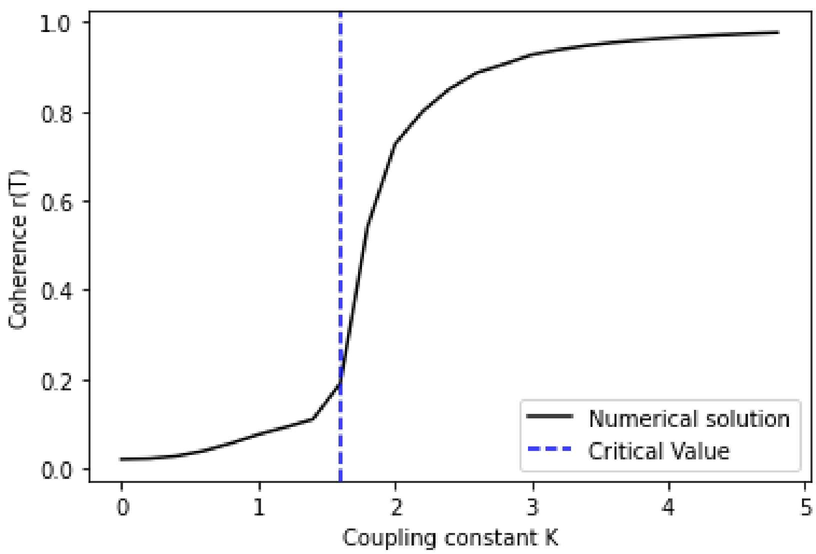

Figure 1: Numerical solution r(T) for various values of K. The dotted blue line represents the critical coupling constant.

For the set up of the simulations here, the number of oscillators was chosen to be $N = 1000$ while the coupling constant took values in the interval $[0, K_{max} = 5]$ with step $dk = 0.2$. The time step was set to be $dt = 0.01$ letting the system evolve from $T = 0$ to $T = 100$. As already mentioned, the initial angles $\theta_i(0)$ were picked up from a uniform distribution $\mathcal{U}[0, 2\pi]$. Finally, the natural frequencies were chosen to take values from a normal distribution $\mathcal{N}[0, 1]$.

In Figure 2 the comparison of the integration of (26) for the numerical values of (r, K) with the line $y = 1$ (dotted red line) is displayed. The critical value of the coupling constant is also apparent (dotted blue line). The consistency above the critical value is remarkable. Below the critical value, the integrated values decline from 1, but this is expected as well.

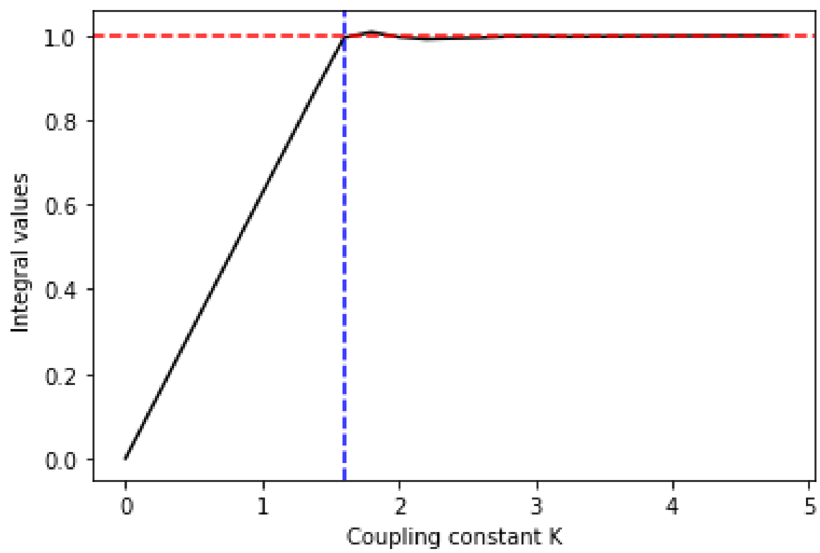

Figure 2: Comparison of the integral values for the pairs $(r(T), K)$ we found (black solid line) with the Kuramoto constraint (red dotted line). The critical coupling K_C is also included (blue dotted line).

Let us now focus on the specific cases $K = 1$ and $K = 2$. We want to study the time evolution of the coherence parameter. Figure 3 indeed verifies what is expected. In particular, the blue line representing the values of the coherence parameter for $K = 1$ indicates that the coherence will never appear. Since the coupling constant small (below the critical value) this is expected. In contrast, the green line corresponding to $K = 2$ shows that after a certain time (around $t = 15$) coherence is achieved. This is fully consistent with what the theory predicts.

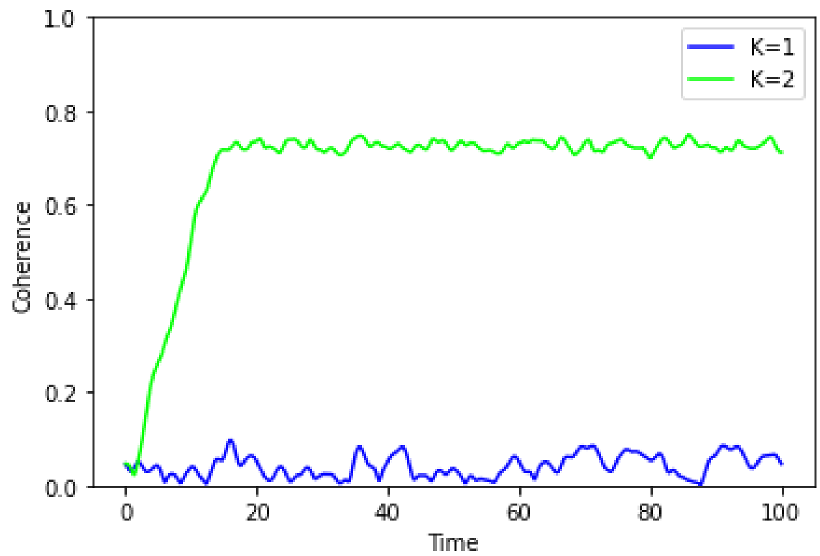

Figure 3: Time evolution of the coherence parameter for the values $K = 1$ and $K = 2$

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As a closing remark here, it would be interesting to make the above comparison among different cases of K. It is expected that for increasing values of K, the system will enter the coherent phase more quickly. This is indeed verified by Figure 4.

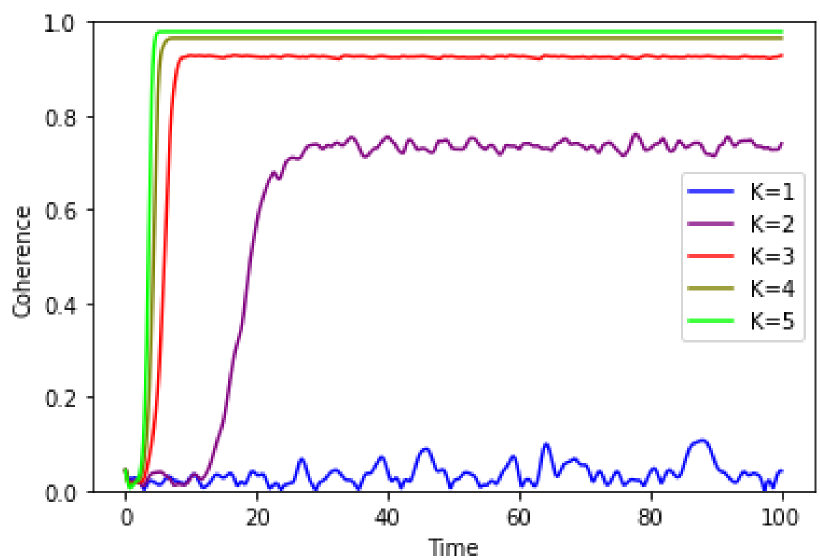

Figure 4: Time evolution of the coherence parameter for various values of K.

2.2.2 Natural frequencies uniformly distributed

The same process was repeated for the natural frequencies taking values from a uniform distribution $\mathcal{U}[-\frac{1}{2}, \frac{1}{2}]$:

$$g(\omega) = \begin{cases} \frac{1}{2\gamma} & , |\omega| \leq \gamma \\ 0 & , |\omega| > \gamma \end{cases} \tag{28}$$

where $\gamma = \frac{1}{2}$. For the set up of the simulations, the number of oscillators was chosen to be $N = 2000$.Moreover, The values of the coupling constant were chosen to be in the interval $[0, K_{max} = 1.5]$ with step $dK = 0.03$. Finally,the time step was altered to $dt = 0.05$ and the final time to $T = 200$ such that in total 40.000 steps were done for each value of K.

Again we aim to study the behaviour of the coherence parameter obtained for $T = 200$, for these values of the coupling constant. This is displayed in 5. Again, the behaviour is the same, The only difference from the case of the unimodal distribution is that now the coherence takes place for much lower values of K. In particular, the theory predicts that

$$K_C = \frac{2}{\pi g(0)} = \frac{2}{\pi} \approx 0.63366 \tag{29}$$

This theoretical value is represented by the dotted blue line in Figure 5 and is in complete agreement with what is numerically observed.

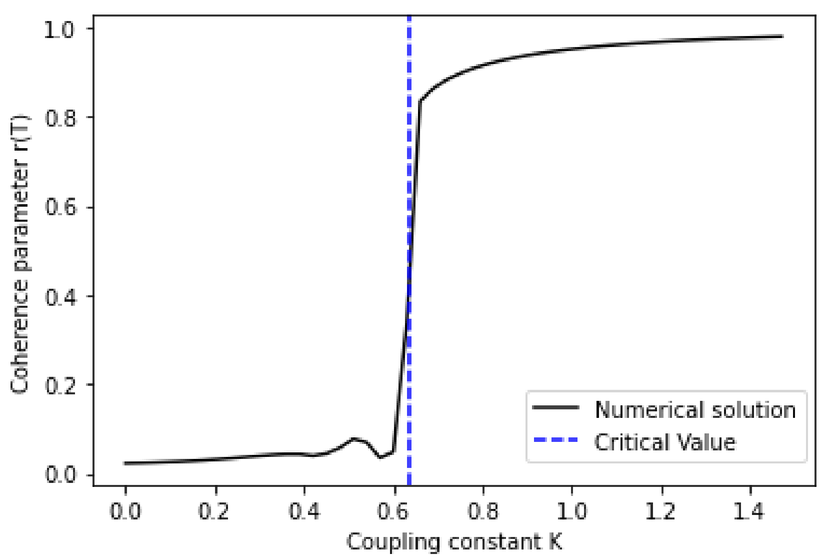

Figure 5: Numerical solution r(T) for various values of K. The dotted blue line represents the critical coupling constant.

For the rest of the study, the coupling constant was set to $K = 1$. We want to observe the time evolution of the coherence parameter for 10 different simulations, obtained by sampling different initial conditions $\theta_i(0)$, while keeping the natural frequencies ω_i's fixed. The results are represented in Figure 6. Notice that instead of running until $T = 200$, a cut-off time $T = 50$ was chosen. This was done in order to make the discrepancies in the region $0 < T < 30$ more apparent.

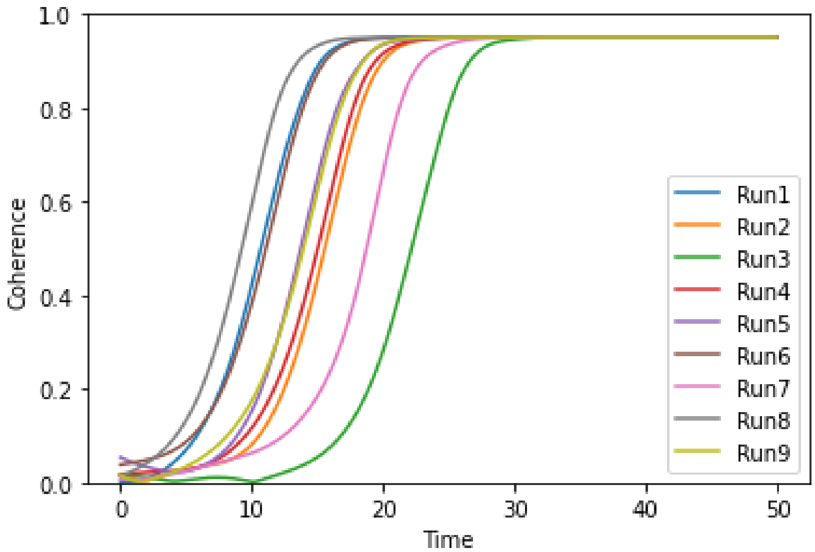

Figure 6: Time evolution of the coherence parameter for 10 sample runs of different initial conditions and fixed natural frequencies.

What one observes is that the behaviour is more less the same. The important feature to note here is that the time that is required for the system to enter the coherence phase is strongly dependent on the initial conditions. Despite that the initial conditions are picked randomly (nevertheless from the same uniform distribution), it is apparent that among all possible configurations one can select the one which brings the system to coherence faster. Of course, this is not something that can be done easily.

Finally, we want to study the same question as above, but from a slightly different perspective. This time we want to run 10 different simulations by keeping the initial conditions $\theta_(0)$ fixed and sampling different natural frequencies. This is displayed is Figure 7.

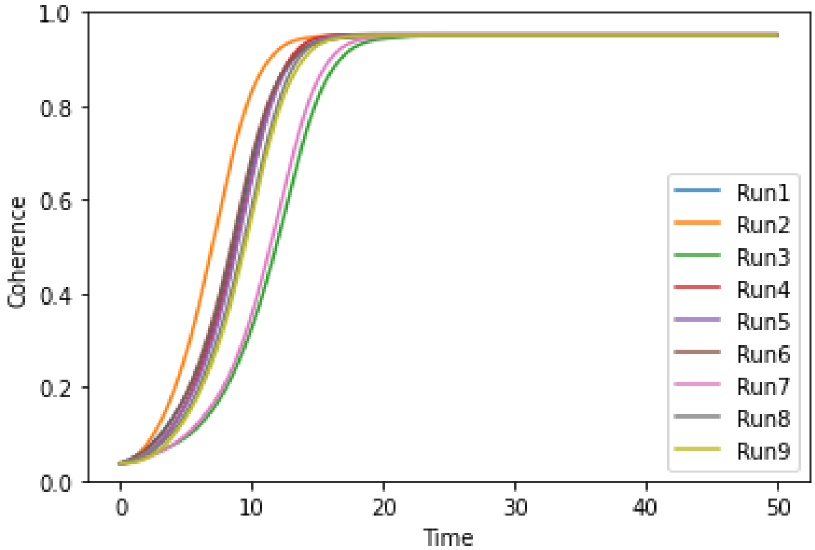

Figure 7: Time evolution of the coherence parameter for 10 sample runs of different natural frequencies initial conditions and fixed initial conditions .

The behaviour now is more less the same as in the previous case, but one could notice two differences. First, the value of the coherence parameter starts from zero no matter what the values of the natural frequencies are, in contrast to the previous case where the corresponding coherence parameter values at time zero are not the same. The other important feature here is that the relaxation time lies (approximately) in the interval $(15, 20)$, whereas in the previous case in the interval $(15, 28)$.

A plausible explanation for this lies in the way we actually solved the equations of motion, as well well as the ranges of the uniform distributions that feed $\theta_i(0)$ and ω_i. The second one is important, because the initial conditions take values in $(0, 2\pi)$ whereas the natural frequencies in $(-\frac{1}{2}, \frac{1}{2})$. But this alone is not enough. When applied the Euler method to solve the system numerically, the dependence of $\theta_i(t)$ on $\theta_i(t - dt)$ is greater than that on the natural frequency, since the later was multiplied by a very small time-step. Taking these two facts in combination could give a good explanation on why this difference occurs.

3 PART II: exercises from the course

In this part, a certain exercise from the course is discussed and analysed. In particular, we consider the following Stochastic Differential Equation (SDE) discussed in Lecture 10:

$$dT_t = -\gamma T_t dt + \sigma dW_t \tag{30}$$

First, we want to show that a strong solution of the above SDE with initial condition can be written as:

$$T_t = e^{-\gamma t}(T_0 + \sigma \int_0^t e^{\gamma x} dW_x) \tag{31}$$

Before going through the calculation it is useful to recall some basic elements for SDE's [1].

Definition 1. A SDE is an equation of the form

$$dX_t = f(X_t, t)dt + g(X_t, t)dW_t \tag{32}$$

where $f, g : \mathbb{R} \times [0, T] \to \mathbb{R}$ are deterministic measurable functions. Then a strong solution in this equation is by definition an adapted process satisfying

$$X_t = X_0 + \int_0^t f(X_s, s)ds + \int_0^t g(X_s, s)dW_s \tag{33}$$

almost surely for all $t \in [0, T]$, as well as the regularity conditions

$$\mathbb{P}\{\int_0^T |f(X_s, s)|ds < \infty\} = \mathbb{P}\{\int_0^T g(X_s, s)^2 ds < \infty\} = 1 \tag{34}$$

Lemma 3.1. *Ito's isometry rule*

If $\int_0^t \mathbb{E}[e_s^2]ds < \infty$, *then* $\mathbb{E}[\int_0^t (e_s dW_s)^2] = \int_0^t \mathbb{E}[e_s^2]ds$

Lemma 3.2. *Ito's formula in differential form*

Let $u : [0, \infty) \times \mathbb{R} \to \mathbb{R}, (t, X) \mapsto u(t, X)$ *be continuously differentiable with respect to* t *and twice continuously differentiable with respect to* X*. Then the stochastic process* $Y_t = u(t, X_t)$ *satisfies the equation*

$$dY_t = \frac{\partial u}{\partial t}(t, X_t) + \frac{\partial u}{\partial X}(t, X_t)[f_t dt + g_t dW_t] + \frac{1}{2}\frac{\partial^2 u}{\partial X^2}(t, X_t) g_t^2 dt \tag{35}$$

In the current case, we have $X_t \equiv T_t, f = -\gamma T_t$ and $g = \sigma$,that is, we are dealing with a linear SDE with additive noise (γ and σ are constants). The solution for the paricular case $\sigma = 0$ is simply

$$dT_t = -\gamma T_t dt \Rightarrow \frac{dT_T}{T_t} = -\gamma dt \Rightarrow lnT_t = -\int \gamma dt + C$$
$$\Rightarrow T_t = T_0 e^{-\gamma(t)}$$

where $T_0 = e^C$ is an initial condition and $\gamma(t) = \int_0^t \gamma ds$. To find a strong solution, we apply the method of variation of the constant, that is, we look for a solution of the form

$$T_t = e^{-\gamma(t)} Y_t \tag{36}$$

Then, Ito's formula applied to $Y_t = u(T_t, t) = e^{\gamma(t)} X_t$ gives

$$dY_t = \gamma e^{\gamma(t)} T_t dt + e^{\gamma(t)} dT_t = e^{\gamma(t)} \sigma dW_t$$

Integrating and using that $Y_0 = T_0$ gives

$$Y_t = T_0 + \int_0^t e^{\gamma(s)} \sigma dW_s \tag{37}$$

Substituting in (36), one immediately obtains the strong solution in the form of (31).

Finally, we want to find an expression for the covariance $Cov(T_s, T_t)$ of the process. To do so, we first have to find the average of the distribution we found earlier. We have

$$\mathbb{E}[T_t] = \mathbb{E}[T_0 e^{-\gamma t} + \sigma e^{-\gamma t} \int_0^t e^{\gamma x} dW_x] = T_0 e^{-\gamma t} \tag{38}$$

where we used that the integral over dW_x is a Gaussian distribution centered at zero and therefore its average vanishes. Next, we need to compute the variance. By definition,

$$\begin{aligned} Var(T_t) &= \mathbb{E}\Big[(T_t - \mathbb{E}[T_t])^2\Big] = \mathbb{E}\Big[\cancel{T_0 e^{-\gamma t}} + \sigma e^{-\gamma t} \int_0^t e^{\gamma x} dW_x - \cancel{T_0 e^{-\gamma t}})^2\Big] \\ &= \mathbb{E}\Big[(e^{-\gamma t} \sigma \int_0^t e^{\gamma x} dW_x)^2\Big] \end{aligned} \tag{39}$$

Then, using Ito's isometry rule, the variance can be equivalently written as

$$
\begin{aligned}
Var(T_t) &= \sigma^2 e^{-2\gamma t}\int_0^t e^{2\gamma x}dx = \sigma^2 e^{-2\gamma t}\frac{1}{2\gamma}e^{2\gamma x}|_0^t = \frac{\sigma^2}{2\gamma}e^{-2\gamma t}(e^{2\gamma t}-1)\\
&= \frac{\sigma^2}{2\gamma}(1-e^{-2\gamma t})
\end{aligned}
$$

Now we are ready to calculate the covariance of the process. By definition

$$
\begin{aligned}
Cov(T_s,T_t) &= \mathbb{E}\Big[(T_s-\mathbb{T}_{\mathbb{s}})(T_t-\mathbb{T}_{\mathbb{t}})\Big]\\
&= \mathbb{E}\Big[(\cancel{T_0e^{-\gamma s}}+\sigma e^{-\gamma s}\int_0^s e^{\gamma x}dW_x-\cancel{T_0e^{-\gamma s}})(\cancel{T_0e^{-\gamma t}}+\sigma e^{-\gamma t}\int_0^s e^{\gamma y}dW_y-\cancel{T_0e^{-\gamma t}})\Big]\\
&= \mathbb{E}\Big[(\sigma e^{-\gamma s}\int_0^s e^{\gamma x}dW_x)(\sigma e^{-\gamma t}\int_0^s e^{\gamma y}dW_y)\Big]
\end{aligned}
$$

Passing the deterministic parts out of the expectation and applying again Ito's isometry rule we immediately obtain

$$
\begin{aligned}
Cov(T_s,T_t) &= \sigma^2 e^{-\gamma(t+s)}\int_0^t e^{2\gamma y}dy = \frac{1}{2\gamma}\sigma^2 e^{-\gamma(s+t)}e^{2\gamma y}|_0^t = \frac{\sigma^2}{2\gamma}e^{-\gamma(s+t)}(e^{2\gamma t}-1)\\
&= \frac{\sigma^2}{2\gamma}(e^{2\gamma t-\gamma t-\gamma s}-e^{-\gamma(s+t)}) = \frac{\sigma^2}{2\gamma}(e^{-\gamma(t-s)}-e^{-\gamma(t+s)})
\end{aligned}
$$

Notice that when we applied Ito's isometry rule, we picked t to be smaller than s. We could have done the inverse and perform an integration involving s and x. The process is exactly the same and we can more generally write

$$
Cov(T_s,T_t) = \frac{\sigma^2}{2\gamma}(e^{-\gamma|t-s|}-e^{-\gamma(t+s)}) \tag{40}
$$

4 Appendix

4.1 Normal distribution of the natural frequencies

```
import numpy as np
import matplotlib.pyplot as plt

t0 = 0 #starting time
T = 100 #final time
dt = 0.01 #timestep
N = 1000 #number of oscilators
dK = 0.2 #coupling step

## Find omega through normal distribution
omegaa = np.random.normal(0,1,1000)
omega = omegaa.tolist()

## Find theta for t=0 through random uniform distribution
theta0 = np.random.uniform(0,2*np.pi,1000)

##Set values of array for t=0 equal to theta0
old_theta = np.array(theta0)

##Create an empty list and append theta0
theta_list = []
theta_list.append(old_theta)

## Create empty list to save the order parameters
order_par = []

for k in np.arange(0,5,dK):
    old_theta = theta0
    for t in np.arange(0,100,dt):
      cos = 0
      sin = 0
      r = 0
      cos = np.cos(old_theta)
      sin = np.sin(old_theta)
      mean_cos =np.mean(cos)
      mean_sin = np.mean(sin)
      r = np.sqrt(mean_cos**2 + mean_sin**2)
      order_par.append(r)
      old_theta =  old_theta + dt*(omegaa + k*r*np.sin(sum(old_theta)/N -
          old_theta))
      theta_list.append(old_theta)
      print(t)


## From the order parameter list, per 10000 steps --> k is changed
## We want r(T) --> last item (10000-1) from t=0 and then going up with step=
    10000
par = []
for i in range(9999,len(order_par),10000):
    par.append(order_par[i])


time = np.arange(0,100,0.01)

## Plot coupling constant K with the coherence r(T)
number_of_k = np.arange(0,5,0.2)
```

```
plt.plot(number_of_k,par, label = 'Numerical solution', color = 'black')
plt.xlabel('Coupling constant K')
plt.ylabel('Coherence r(T)')
plt.axvline(x=1.6, color = 'b', label = 'Critical Value', linestyle = '--')
plt.legend()

## Comparison with equation 6
##Solving for r(T) and K

number_of_k = np.arange(0,5,0.2)

##Convert the previous par list with r(T) to array, in order to compute
par_array = np.array(par)

##Import library to compute the quad
import scipy.integrate

## Save the integration values in a list
integration_list = []
for (i,j) in zip(number_of_k,par_array):
    f= lambda x: (1/np.sqrt(pi*2))*i * np.cos(x)**2 * np.exp(-0.5*(i**2)
    *(np.sin(x)**2)*(j**2))
    a = scipy.integrate.quad(f, -pi/2, pi/2)
    integration_list.append(a)

intergration_list = np.array
## Keep only the integration values and not the errors
integ = []
for i in range(0,25):
    integ.append(integration_list[i][0])

##Plot integ with coupling constants K
plt.plot(number_of_k,integ, color = 'black')
plt.xlabel('Coupling constant K')
plt.ylabel('Integral values ')
plt.axhline(y=1, color = 'r', linestyle = '--')
plt.axvline(x=1.6, color = 'b', linestyle = '--')

## Run now the model for K=1, and K=2
order_par_1 = []
order_par_2 = []

## Reset old_theta as theta0 to run the model for K=1 and K=2
for k in range(1,2):
    old_theta = theta0
    for t in np.arange(0,100,0.01):
      cos = 0
      sin = 0
      r = 0
      cos = np.cos(old_theta)
      sin = np.sin(old_theta)
      mean_cos =np.mean(cos)
      mean_sin = np.mean(sin)
      r = np.sqrt(mean_cos**2 + mean_sin**2)
      if k ==1:
          order_par_1.append(r)
      elif k==2:
          order_par_2.append(r)
      print(t)
      old_theta =  old_theta + dt*(omegaa + k*r*np.sin(sum(old_theta)/N -
          old_theta))
      theta_list.append(old_theta)
```

```
      print(t)

time = np.arange(0,100,0.01)

fig1 = plt.figure(1)
plt.plot(time,order_par_1)

fig2 = plt.figure(2)
plt.plot(time, order_par_2)

## Plot coherence parameters for K=1 and K=2 in the same graph
num_plots = 2
colormap = plt.cm.brg
plt.gca().set_prop_cycle(plt.cycler('color', plt.cm.brg(np.linspace(0, 1,
    num_plots))))
plt.plot(time, order_par_1, time, order_par_2)
plt.legend(['K=1', 'K=2'])
plt.ylim(0,1)
plt.xlabel('Time')
plt.ylabel('Coherence')
```

4.2 Natural frequencies uniformly distributed

```
import numpy as np
import matplotlib.pyplot as plt

t0 = 0 #starting time
T2 = 200 #final time
dt2 = 0.05 #timestep
N2 = 2000 #number of oscilators
dK2 = 0.03 #coupling step


## Find omega through uniform distribution
omegaa2 = np.random.uniform(-1/2,1/2,2000)
omega2 = omegaa2.tolist()

## Find theta for t=0 through random uniform distribution
theta0 = np.random.uniform(0,2*np.pi,2000)

##Set values of array for t=0 equal to theta0
old_theta2 = np.array(theta0)

##Create an empty list and append theta0
theta_list2 = []
theta_list2.append(old_theta2)

## Create empty list to save the order parameters
order_par2 = []
for k in np.arange(0,1.5,0.03):
    old_theta2 = theta0
    for t in np.arange(0,200,0.05):
      cos2 = 0
      sin2 = 0
      r2 = 0
      cos2 = np.cos(old_theta2)
      sin2 = np.sin(old_theta2)
      mean_cos2 =np.mean(cos2)
```

```
    mean_sin2= np.mean(sin2)
    r2= np.sqrt(mean_cos2**2 + mean_sin2**2)
    order_par2.append(r2)
    old_theta2 =  old_theta2 + dt2*(omegaa2 + k*r2*np.sin(sum(old_theta2)/N2 -
        old_theta2))
    theta_list2.append(old_theta2)
    print(t)

## From the order parameter list, per 4000 steps --> k is changed
## We want r(T) --> last item (4000-1) from t=0 and then going up with step= 4000
par2 = []
for i in range(3999,len(order_par2),4000):
    par2.append(order_par2[i])

number_of_k2 = np.arange(0,1.5,0.03)
time2 = np.arange(0,200,dt2)
critical_constant = 2/np.pi


plt.plot(number_of_k2, par2, color = 'black', label = 'Numerical solution')
plt.xlabel('Coupling constant K')
plt.ylabel('Coherence parameter r(T)')
plt.axvline(x=critical_constant, color = 'b', label = 'Critical Value', linestyle
    = '--')
plt.legend(loc=(0.6, 0.05))
```

4.2.1 Fixed realisation of the natural frequencies

```
import numpy as np
import matplotlib.pyplot as plt

## Create empty list to store the new thetas'
theta_list_new = []
## Create empty list to store the new coherences
order_par_new = []

dt2 = 0.05 #timestep
N2 = 2000 #number of oscilators

## Keep omega the same for all the runs
omegaa2  = np.random.uniform(-1/2, 1/2, 2000)

## For 10 runs
for j in range(0,10):
    old_theta2 = np.random.uniform(0,2*np.pi,2000) #change theta in every run
        from a uniform distribution
    for t in np.arange(0,50,0.05):
      cos_new = 0
      sin_new = 0
      r_new = 0
      cos_new = np.cos(old_theta2)
      sin_new = np.sin(old_theta2)
      mean_cos_new =np.mean(cos_new)
      mean_sin_new= np.mean(sin_new)
      r_new= np.sqrt(mean_cos_new**2 + mean_sin_new**2)
      order_par_new.append(r_new)
      old_theta2 =  old_theta2 + dt2*(omegaa2 + r_new*np.sin(sum(old_theta2)/N2 -
          old_theta2))
      theta_list_new.append(old_theta2)
```

```
      print(t)

## Plot time with coherence
for (i,j) in zip(range(0,len(order_par_new), 1000), range(1,10)):
    time_new = np.arange(0,50,0.05)
    plt.plot(time_new, order_par_new[i:i+1000], label ='Run' +str(j))
    plt.ylim(0,1)
    plt.xlabel('Time')
    plt.ylabel('Coherence')

plt.legend()
```

4.2.2 Fixed realisation of the initial conditions

```
import numpy as np
import matplotlib.pyplot as plt

## Create empty list to store the new thetas'
theta_list_new = []
## Create empty list to store the new coherences
order_par_new = []

dt2 = 0.05 #timestep
N2 = 2000 #number of oscilators

## Theta0 taking values from a uniform distribution
theta0 = np.random.uniform(0,2*np.pi,2000)
old_theta2 = np.array(theta0)


## For 10 runs
for j in range(0,10):
    old_theta2 = theta0 ## Keep theta as theta0 for all the runs
    omegaa2  = np.random.uniform(-1/2, 1/2, 2000) #change omega in every run from
         a uniform distribution
    for t in np.arange(0,50,0.05):
      cos_new = 0
      sin_new = 0
      r_new = 0
      cos_new = np.cos(old_theta2)
      sin_new = np.sin(old_theta2)
      mean_cos_new =np.mean(cos_new)
      mean_sin_new= np.mean(sin_new)
      r_new= np.sqrt(mean_cos_new**2 + mean_sin_new**2)
      order_par_new.append(r_new)
      old_theta2 =  old_theta2 + dt2*(omegaa2 + r_new*np.sin(sum(old_theta2)/N2 -
           old_theta2))
      theta_list_new.append(old_theta2)
      print(t)

## Plot time with coherence
for (i,j) in zip(range(0,len(order_par_new), 1000), range(1,10)):
    time_new = np.arange(0,50,0.05)
    plt.plot(time_new, order_par_new[i:i+1000], label ='Run' +str(j))
    plt.ylim(0,1)
    plt.xlabel('Time')
    plt.ylabel('Coherence')

plt.legend()
```

References

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