Modified Cosmology - Exercises for Cosmology course

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In this work we will examine how the Friedmann equations change by generalizing the metric tensor by introducing a time-dependent lapse function L for the 00-th component and making the curvature k time-dependent.

For the sake of simplicity we will not write down all the algebraic steps to reach the results, instead using our own Mathematica code to do the calculations and we will only write down the results from there. We provide the code alongside this report. In the code the indices are not the conventionally used Lorentz indices 0, 1, 2, 3 but 1, 2, 3, 4 because of the indexing characteristic of the program.

Metric tensor:

$$g_{\mu\nu}(t) = \begin{pmatrix} -L(t) & 0 & 0 & 0 \\ 0 & \frac{a^2(t)}{1-k(t)r^2} & 0 & 0 \\ 0 & 0 & a^2(t)r^2 & 0 \\ 0 & 0 & 0 & a^2(t)r^2\sin^2\theta \end{pmatrix} \quad g^{\mu\nu}(t) = \begin{pmatrix} -\frac{1}{L(t)} & 0 & 0 & 0 \\ 0 & \frac{1-k(t)r^2}{a^2(t)} & 0 & 0 \\ 0 & 0 & \frac{1}{a^2(t)r^2} & 0 \\ 0 & 0 & 0 & \frac{1}{a^2(t)r^2\sin^2\theta} \end{pmatrix} \tag{1}$$

Energy-momentum tensor:

$$T^{\mu\nu} = \left(\rho(t) + \frac{p(t)}{c^2}\right) u^\mu u^\nu + p(t) g^{\mu\nu}, \tag{2}$$

where u^μ is the 4-velocity from the comoving frame, which has to be normalized to 1 (in our case -1 due to the signature), namely $g_{\mu\nu}u^\mu u^\nu = -1$ must hold, therefore

$$u^\mu = \left(\frac{c}{\sqrt{L(t)}}, 0, 0, 0\right). \tag{3}$$

The energy-momentum tensor is diagonal, because the metric tensor is so and the product $u^\mu u^\nu$ has only one nonzero element. Its form follows as by using (2) and (3) and for the lower index one we use $T_{\mu\nu} = g_{\mu\alpha}g_{\nu\beta}T^{\alpha\beta} = g_{\mu\mu}\delta_{\mu,\alpha}g_{\nu\nu}\delta_{\nu,\beta}T^{\alpha\beta} = g_{\mu\mu}g_{\nu\nu}T^{\mu\nu}$.

$$T_{\mu\nu}(t) = \begin{pmatrix} \rho c^2 L(t) & 0 & 0 & 0 \\ 0 & p(t)\frac{a^2(t)}{1-k(t)r^2} & 0 & 0 \\ 0 & 0 & p(t)a^2(t)r^2 & 0 \\ 0 & 0 & 0 & p(t)a^2(t)r^2\sin^2\theta \end{pmatrix} \tag{4}$$

$$T^{\mu\nu}(t) = \begin{pmatrix} \frac{\rho c^2}{L(t)} & 0 & 0 & 0 \\ 0 & p(t)\frac{1-k(t)r^2}{a^2(t)} & 0 & 0 \\ 0 & 0 & p(t)\frac{1}{a^2(t)r^2} & 0 \\ 0 & 0 & 0 & p(t)\frac{1}{a^2(t)r^2\sin^2\theta} \end{pmatrix} \tag{5}$$

In the code we calculated with $T^{\mu\nu}$, but the lower index type is easily computable from the upper one.

Now, we have to calculate the Christoffel symbols ($\Gamma^{\mu}_{\alpha\beta}$) and with them the Ricci tensor ($R_{\mu\nu}$), Ricci scalar (R) and Einstein tensor ($G_{\mu\nu}$). Let's dive into this. The following formulas are needed.

$$\Gamma^{\mu}_{\alpha\beta} = \frac{1}{2} g^{\mu\nu} \left(\frac{\partial g_{\alpha\nu}}{\partial x^{\beta}} + \frac{\partial g_{\beta\nu}}{\partial x^{\alpha}} - \frac{\partial g_{\alpha\beta}}{\partial x^{\nu}} \right) \tag{6}$$

$$R_{\mu\nu} = \Gamma^{\alpha}_{\mu\nu,\alpha} - \Gamma^{\alpha}_{\mu\alpha,\nu} + \Gamma^{\alpha}_{\beta\alpha}\Gamma^{\beta}_{\mu\nu} - \Gamma^{\alpha}_{\beta\nu}\Gamma^{\beta}_{\mu\alpha}, \quad \text{where} \quad \Gamma^{\alpha}_{\mu\nu,\alpha} = \frac{\partial \Gamma^{\alpha}_{\mu\nu}}{\partial x^{\alpha}}. \tag{7}$$

$$R = g^{\mu\nu} R_{\mu\nu} \tag{8}$$

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} \tag{9}$$

Before we start to calculate the concrete values of the above mentioned quantities, we shall think about whether there are useful characteristics of the Christoffel symbols or not, which we can use to simplyfy the calculation. (i) Symmetry of Christoffel symbols: $\Gamma^{\mu}_{\alpha\beta} = \Gamma^{\mu}_{\beta\alpha}$. (ii) $g^{\mu\nu}$ is diagonal, therefore it is enough to consider $\nu = \mu$. (iii) $g^{\mu\nu}$ is diagonal, therefore the Christoffel symbol is zero if α, β and ν are all different.

We list the Christoffel symbols with at least 2 identical indices, where we denote the derivation with respect to ct by $'$ (and we do not write out the time-dependence). For further usage we mention that this derivative is the temporal derivative (denoted by upper dot in a usual way) divided by c.

$\Gamma^0_{00} = \frac{L'}{2L}$	$\Gamma^1_{00} = 0$	$\Gamma^2_{00} = 0$	$\Gamma^3_{00} = 0$
$\Gamma^0_{11} = \frac{\left[2(1-k^2)a' + k' r^2 a\right] a}{2(1-kr^2)^2 L}$	$\Gamma^1_{11} = \frac{kr}{1-kr^2}$	$\Gamma^2_{11} = 0$	$\Gamma^3_{11} = 0$
$\Gamma^0_{22} = \frac{r^2 a a'}{L}$	$\Gamma^1_{22} = -r(1-kr^2)$	$\Gamma^2_{22} = 0$	$\Gamma^3_{22} = 0$
$\Gamma^0_{33} = \frac{r^2 a a' \sin^2\theta}{L}$	$\Gamma^1_{33} = -r(1-kr^2)\sin^2\theta$	$\Gamma^2_{33} = -\sin\theta\cos\theta$	$\Gamma^3_{33} = 0$
$\Gamma^0_{01} = 0 = \Gamma^0_{10}$	$\Gamma^1_{10} = \frac{a'}{a} + \frac{k' r^2}{2(1-k^2)} = \Gamma^1_{01}$	$\Gamma^2_{20} = \frac{a'}{a} = \Gamma^2_{02}$	$\Gamma^3_{30} = \frac{a'}{a} = \Gamma^3_{03}$
$\Gamma^0_{02} = 0 = \Gamma^0_{20}$	$\Gamma^1_{12} = 0 = \Gamma^1_{21}$	$\Gamma^2_{21} = \frac{1}{r} = \Gamma^2_{12}$	$\Gamma^3_{31} = \frac{1}{r} = \Gamma^3_{13}$
$\Gamma^0_{03} = 0 = \Gamma^0_{30}$	$\Gamma^1_{13} = 0 = \Gamma^1_{31}$	$\Gamma^2_{23} = 0 = \Gamma^2_{32}$	$\Gamma^3_{32} = \cot\theta = \Gamma^3_{23}$

Now, let's calculate the element of the Ricci tensor using our Mathematica code.

$$R_{00} = \frac{(1-kr^2)\left[6(1-kr^2)a' + k'r^2 a\right] L' + L\left\{-4r^2(1-kr^2)a'k' + 12(1-kr^2)^2 a'' - a\left[3k'^2 r^4 - 2r^2(1-kr^2)k''\right]\right\}}{4a(1-kr^2)^2 L} \tag{10}$$

$$R_{01} = \frac{k' r}{1-kr^2} \tag{11}$$

$$R_{02} = 0 \tag{12}$$

$$R_{03} = 0 \tag{13}$$

$$R_{10} = \frac{k' r}{1-kr^2} \tag{14}$$

$$\begin{aligned} R_{11} = {} & -\frac{1}{4(1-kr^2)^3 L^2}\Big(-8k^3 r^4 L^2 + a(2a' + k' r^2 a)L' + 2k^2 r^2\left[8L^2 + r^2 a a' L' - 2r^2 L(2a'^2 + a a'')\right] - \\ & -L\left\{8a'^2 + 8k' r^2 a a' + a\left[4a'' + a(3k'^2 r^4 + 2k'' r^2)\right]\right\} + \\ & +k\left\{-8L^2 - r^2 a L'(4a' + k' r^2 a) + 2r^2 L\left[8a'^2 + 4k' r^2 a a' + a(4a'' + k'' r^2 a)\right]\right\}\Big) \end{aligned} \tag{15}$$

$$R_{12} = 0 \tag{16}$$

$$R_{13} = 0 \tag{17}$$

$$R_{20} = 0 \tag{18}$$

$$R_{21} = 0 \tag{19}$$

$$R_{22} = \frac{r^4\left\{4k^2 r^2 L^2 + a'[-L'(4a' + k' r^2 a) + aL'] - 2aa''L + k[-4L^2 - r^2 a a' L' + 2r^2 L(2a'^2 + aa'')]\right\}}{2(1-kr^2)L^2} \tag{20}$$

$$R_{23} = 0 \tag{21}$$

$$R_{30} = 0 \tag{22}$$

$$R_{31} = 0 \tag{23}$$

$$R_{32} = 0 \tag{24}$$

$$R_{33} = \frac{r^4\left\{4k^2 r^2 L^2 + a'[-L'(4a' + k' r^2 a) + aL'] - 2aa''L + k[-4L^2 - r^2 a a' L' + 2r^2 L(2a'^2 + aa'')]\right\}}{2(1-kr^2)L^2}\sin^2\theta \tag{25}$$

The Ricci scalar using (8):

$$\begin{aligned} R &= \frac{1}{2(1-kr^2)a^2L^2}\Big\{12k^3r^4L^2 - aL'(6a'+k'r^2a) + 6k^2[-4r^2L^2 - r^4L'aa' + 2r^4L(a'^2+aa'')] + \\ &\quad + L[12a'^2 + 8k'r^2aa' + a(3k'^2r^4a + 12a'' + 2k''r^2a)] + \\ &\quad + k[12L^2 + r^2aL'(12a'+k'r^2a) - 2r^2L(12a'^2 + 4k'r^2aa' + 12aa'' + k''r^2a^2)]\Big\}. \end{aligned} \tag{26}$$

The components of the Einstein tensor.

$$G_{00} = \frac{3kL + a'(3a' + \frac{k'r^2a}{1-kr^2})}{a^2} \tag{27}$$

$$G_{01} = \frac{k'r}{1-kr^2} \tag{28}$$

$$G_{02} = 0 \tag{29}$$

$$G_{03} = 0 \tag{30}$$

$$G_{10} = \frac{k'r}{1-kr^2} \tag{31}$$

$$G_{11} = -\frac{kL^2 - aa'L' + L(a'^2+2aa'')}{(1-kr^2)L^2} \tag{32}$$

$$G_{12} = 0 \tag{33}$$

$$G_{13} = 0 \tag{34}$$

$$G_{20} = 0 \tag{35}$$

$$G_{21} = 0 \tag{36}$$

$$\begin{aligned} G_{22} &= -\frac{r^2}{4(1-kr^2)^2L^2}\Big\{4k^3r^4L^2 - aL'(4a'+k'r^2a) + 4k^2[-2r^2L^2 - r^4L'aa' + r^4L(a'^2+2aa'')] + \\ &\quad + L[4a'^2 + 6k'r^2aa' + a(3k'^2r^4a + 8a'' + 2k''r^2a)] + \\ &\quad + k[4L^2 + r^2L'a(8a'+k'r^2a) - 2r^2L(4a'^2 + 3k'r^2aa' + 8aa'' + k''r^2a^2)]\Big\} \end{aligned} \tag{37}$$

$$G_{23} = 0 \tag{38}$$

$$G_{30} = 0 \tag{39}$$

$$G_{31} = 0 \tag{40}$$

$$G_{32} = 0 \tag{41}$$

$$\begin{aligned} G_{33} &= -\frac{r^2\sin^2\theta}{4(1-kr^2)^2L^2}\Big\{4k^3r^4L^2 - aL'(4a'+k'r^2a) + 4k^2[-2r^2L^2 - r^4L'aa' + r^4L(a'^2+2aa'')] + \\ &\quad + L[4a'^2 + 6k'r^2aa' + a(3k'^2r^4a + 8a'' + 2k''r^2a)] + \\ &\quad + k[4L^2 + r^2L'a(8a'+k'r^2a) - 2r^2L(4a'^2 + 3k'r^2aa' + 8aa'' + k''r^2a^2)]\Big\} \end{aligned} \tag{42}$$

Friedmann's equations are derived from Einstein's equation

$$G_{\mu\nu} = \frac{8\pi G}{c^4}T_{\mu\nu}. \tag{43}$$

For FI. we need to solve this equation for the 00-th component and for FII. we have to multiply the equation by $g^{\mu\nu}$.

FI.

$$\begin{aligned} G_{00} &= \frac{8\pi G}{c^4}T_{00} \\ \frac{3kL + a'(3a' + \frac{k'r^2a}{1-kr^2})}{a^2} &= \frac{8\pi G}{c^4}\rho c^2 L \\ \frac{3kL}{a^2} + \frac{3}{c^2}\left(\frac{\dot{a}}{a}\right)^2 + \frac{1}{c^2}\frac{\dot{a}}{a}\frac{\dot{k}\,r^2}{1-kr^2} &= \frac{8\pi G}{c^2}\rho L \end{aligned} \tag{44}$$

$$\boxed{\left(\frac{\dot{a}}{a}\right)^2 = \left(\frac{8\pi G}{3}\rho - \frac{kc^2}{a^2}\right)L + \frac{\dot{a}}{a}\frac{\dot{k}\,r^2}{3(1-kr^2)}} \tag{45}$$

If we take $L(t) = 1$ and $k(t) = k \longrightarrow \dot{k} = 0$, we get back our well-known FI. equation. The lapse function only scales the equation with a time-dependent factor, but the time-dependence in the curvature cause a new term which is proportional to the Hubble parameter $\dot{a}\,/a$ and the temporal derivative of the curvature itself.

FII.

$$
\begin{aligned}
g^{\mu\nu}G_{\mu\nu} &= \frac{8\pi G}{c^4} g^{\mu\nu}T_{\mu\nu} && (46)\\
-R &= \frac{8\pi G}{c^4}(-\rho c^2 + 3p)\\
\frac{8\pi G}{c^4}(\rho c^2 - 3p) &= \frac{1}{2(1-kr^2)a^2L^2}\Big\{12k^3r^4L^2 - aL'(6a' + k'r^2a) + 6k^2[-4r^2L^2 - r^4L'aa' + 2r^4L(a'^2 + aa'')] +\\
&\quad +L[12a'^2 + 8k'r^2aa' + a(3k'^2r^4a + 12a'' + 2k''r^2a)] +\\
&\quad +k[12L^2 + r^2aL'(12a' + k'r^2a) - 2r^2L(12a'^2 + 4k'r^2aa' + 12aa'' + k''r^2a^2)]\Big\} && (47)
\end{aligned}
$$

This is FII., it is really complicated in this case where we took into account all two generalizations. For subsituting $k(t) = k$, $k'(t) = k''(t) = 0$, $L(t) = 1$ and $L'(t) = L''(t) = 0$ (with the help of the code), we get the following equation

$$\frac{6}{a^2}\left(k + a'^2 + aa''\right) = \frac{8\pi G}{c^4}(\rho c^2 - 3p) \tag{48}$$

and using the relation between the $'$ and temporal derivatives and the corresponding FI. equation (after the substitutions), we reach the well-known FII.

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}\left(\rho + \frac{3p}{c^2}\right). \tag{49}$$

In the case of $L(t)$ and $k(t) = k$, it is easier to get the Ricci scalar and FII., in the other case, $L(t) = 1$ and $k(t)$, it is algebraically much more complicated, similar to the one written above, therefore we only focus on the first case, where we will provide the FII. in a similar form as the original one is.

$$
\begin{aligned}
\frac{6kL^2 - 3aa'L' + 6L(a'^2 + aa'')}{a^2L^2} &= \frac{8\pi G}{c^4}(\rho c^2 - 3p)\\
\frac{6k}{a^2} - \frac{3}{c^2L}\frac{\dot{a}}{a}\frac{\dot{L}}{L} + \frac{6}{c^2L}\left[\left(\frac{\dot{a}}{a}\right)^2 + \frac{\ddot{a}}{a}\right] &= \frac{8\pi G}{c^4}(\rho c^2 - 3p)
\end{aligned}
$$

Using the fact that $k(t) = k$ and FI. with this condition (the last term vanishes), we will get FII.

$$\boxed{\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}\left(\rho + \frac{3p}{c^2}\right)L + \frac{1}{2}\frac{\dot{a}}{a}\frac{\dot{L}}{L}} \tag{50}$$

In this equation we can see 2 things. Firstly, the well-known FII. is scaled by the lapse function at every time and secondly we have a new term which contains the derivative of the lapse function as well as itself. The newly appearing terms in each equations are proportional to the Hubble parameter!

Last, but not least we have to examine the energy-momentum conservation, namely whether the equation of continuity is fulfilled or not. It must be fulfilled. (One can find the corresponding calculation at the end of the code with name "Tmodified".)

$$T^{\mu\nu}{}_{;\nu} = \frac{\partial T^{\mu\nu}}{\partial x^\nu} + \Gamma^\mu_{\nu\alpha}T^{\alpha\nu} + \Gamma^\nu_{\nu\alpha}T^{\mu\alpha} \stackrel{!}{=} (0,0,0,0) \tag{51}$$

After performing the algebra by using the code, we get

$$T^{\mu\nu}{}_{;\nu} = \left(\frac{(p + c^2\rho)\left(\frac{6a'}{a} + \frac{k'r^2}{1-kr^2}\right) + 2c^2\rho'}{2L}, 0, 0, 0\right). \tag{52}$$

This form says that if $L(t) = 0$ at some time t, the energy diverges, which requires some explanation, obviously. In this case $g_{00} = 0$, which means that the time does not go by. It is such as the photon, it does not ages, but have finite energy with zero mass.

Let's get back to the conservation of the 0-th component, the energy. Using the relation between the $'$ and temporal derivatives and using the EOS of perfect fluid $p = w\rho c^2$ we are able to get the density as the function of the scale factor and the curvature (the laps function does not appear there).

$$\begin{aligned}
(p + c^2\rho)\left(\frac{6a'}{a} + \frac{k' r^2}{1 - kr^2}\right) + 2c^2\rho' &= 0 \\
(w+1)c^2\rho\left(\frac{6}{c}\frac{\dot{a}}{a} + \frac{1}{c}\frac{\dot{k}\,r^2}{1-kr^2}\right) &= -2c\,\dot{\rho} \\
\frac{\dot{\rho}}{\rho} &= -3(1+w)\frac{\dot{a}}{a} - \frac{1}{2}(1+w)\frac{\dot{k}\,r^2}{1-kr^2} \\
\partial_t \ln\rho &= \partial_t \ln a^{-3(1+w)} + \partial_t \ln\bigl(1 - kr^2\bigr)^{(1+w)/2} \\
\ln\rho &= \ln\left[a^{-3(1+w)} \cdot (1-kr^2)^{(1+w)/2}\right] + \ln\rho_0
\end{aligned}$$

$$\boxed{\rho = \rho_0 \cdot a^{-3(1+w)} \cdot (1-kr^2)^{(1+w)/2} \equiv \rho(a, kr^2)} \tag{53}$$

We got a beautiful result, it tells us that the density is not only determined by the scale factor and the type of energy we are speaking about (w), but the curvature is also appears and effects it in every point. The power dependence of these two parameters are not the same.

It can not be clearly seen from this result why can we get back the well-known result (our result without the last factor), but if we look a little bit before, we immediately see the reason, which is the temporal derivative of k and it is zero, therefore it does not appears further in the calculation.