Master's Thesis

The correspondence between the Ruijsenaars–Schneider model and the sine-Gordon field theory

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Abstract

In this thesis we study the Ruijsenaars–Schneider model and the sine-Gordon field theory both in the classical context and in infinite volume, paying close attention to the 2-particle solutions and highlighting the relations that provide a bridge between the two theories.

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1 Introduction

Relativistic particle physics in the context of relativistic many-body systems of point particles is facing the famous "no-interaction theorem", which states that in the presence of interparticle action-at-a-distance interaction the canonical behaviour of the coordinates cannot be obtained, due to the Poincaré invariant property of the theory. All the non-trivial potentials are excluded (Balog, 2014b; D. G. Currie, Jordan, & Sudarshan, 1963). In the following paragraphs we will briefly introduce the techniques and formalisms that provide the opportunity to study nevertheless nontrivial systems. These are 1) Predictive relativistic mechanics (Bel, 1970), 2) Canonical approach (Komar, 1978), 3) Covariant approach (Droz-Vincent, 1975).

Predictive relativistic mechanics is working in such a way that for the equation of motion it uses the form of Newton's equation, meaning that the accelerations μ depend on the instantaneous positions x and velocities v of the particles indexed by $a \in \{1, ..., N\}$.

$$\ddot{x}_a = \mu_a(\{x\}, \{\dot{x}\}) \tag{1}$$

A consequence of relativistic invariance is that these quantities must fulfil the so-called Currie–Hill equations (D. Currie, 1966; Hill, 1967), that are quadratic equations for the above mentioned quantities. Unfortunately, no explicit solutions of the 3+1 dimensional Currie–Hill equations are known, but Balog (Balog, 2014a) was able to give a class of explicit solutions in 1+1 dimensions resulting a breakthrough and providing deeper understanding on the equations.

An other approach to study relativistic many-body systems is the canonical approach, which is based on the fact that one would like to represent the Poincaré group's Lie algebra on a symplectic phase space and identify the generators of the Poincaré group as the generators of space-time translations, rotations and boosts. The downside of this approach is the reconstruction of the trajectories, which in most cases is a really hard and non-trivial task, if even executable. The Poisson brackets of the coordinates do not vanish in the presence of interaction

$$\{x_a^i, x_b^j\} \neq 0, \tag{2}$$

where $a, b \in \{1, ..., N\}$ are the particle indices and $i, j \in \{1, 2, 3\}$ the space indices. But the advantage is that only the trajectories have to be constructed. The accelerations can be calculated from the trajectories and the velocities must satisfy the Currie–Hill equations.

We only mention that the third approach is the covariant one, because we will not use it in this thesis, rather we pay our attention on the canonical one, which will be in the center of our focus.

In this thesis we study two models, the Ruijsenaars–Schneider model and the sine-Gordon field theory, both in the classical context and in infinite volume. Both models are integrable. Integrability in the Liouville sense means that we have as many independent conserved charges as the number of degrees of freedom in the system (for the Ruijsenaars–Schneider model it is the number of particles and for the sine-Gordon theory it is infinite). Our goal in this thesis is to show the connection of the mentioned theories and have a deeper understanding of the origin of the common characteristics.

In section 2 we provide an overview of relativistic particle mechanics on a symplectic phase space using the canonical approach. Section 3 and 4 are used to introduce the models, have words on their properties and calculate most importantly the $2 \to 2$ particle scatterings in the hyperbolic limit of the Ruijsenaars–Schneider model and in the solitonic subspace of the sine-Gordon theory. Section 5 will be our place to compare these theories and finally we summarize the results in section 6.

2 Relativistic many-body theory

In this section we define the Poincaré algebra for a 1+1 dimensional relativistic system equipped with a symplectic phase space, introduce the canonical variables and discuss the time evolution, the solution of the Hamiltonian equations on the phase space, and last but not least we have a couple words on the trajectory reconstrucion. In this section we use the results of (Balog, 2014b) to introduce the mechanics.

2.1 Poincaré algebra

In the theory of 1+1 dimensional canonical relativistic systems in Minkowski space-time the starting point is that the theory has a phase space equipped with a symplectic structure and the generators of the Poincaré algebra $\{\mathcal{H}, \mathcal{P}, \mathcal{K}\}$, where these quantities generate the time translations, space translations and boosts, respectively. They represent the Poincaré algebra in this phase space, therefore they satisfy the Poisson bracket relations listed below, where c denotes the speed of light.

$$\{\mathcal{H}, \mathcal{P}\} = 0 \tag{3}$$

$$\{\mathcal{H}, \mathcal{K}\} = \mathcal{P} \tag{4}$$

$$\{\mathcal{P}, \mathcal{K}\} = \frac{1}{c^2}\mathcal{H} \tag{5}$$

For functions A defined on the phase space we associate differential operators $\hat{A}$ in such a way that for any function $\mathcal{F}$ on the phase space

$$\hat{A}\mathcal{F} = \{A, \mathcal{F}\} \tag{6}$$

should hold. For instance, for the time and space translation generators (with the usual dot and prime notations) it means

$$\hat{\mathcal{H}}\mathcal{F} := \dot{\mathcal{F}}, \qquad \hat{\mathcal{P}}\mathcal{F} := \mathcal{F}'. \tag{7}$$

Our canonical variables, the phase space coordinates q_i and rapidities θ_i are related to the physical coordinates x_i and momenta p_i in a free theory as

$$p_i = mc \sinh\theta_i = \sinh\theta_i, \tag{8}$$

$$x_i = \frac{q_i}{mc\cosh\theta_i} = \frac{q_i}{\cosh\theta_i}, \tag{9}$$

where m denotes the mass of the particles. (For the sake of simplicity, later on we will use $m = c = 1$.) The canonical commutation relations for x and p read in a free theory

$$\{x_i, p_j\} = \delta_{ij}, \qquad \{x_i, x_j\} = 0, \qquad \{p_i, p_j\} = 0. \tag{10}$$

Due to the no-go theorem these relations are prohibited in the presence of interaction but the canonical variables (q, θ) can satisfy the canonical Poisson commutation relations even in the case of the interaction

$$\{q_i, \theta_j\} = \delta_{ij}, \qquad \{q_i, q_j\} = 0, \qquad \{\theta_i, \theta_j\} = 0. \tag{11}$$

2.2 Phase space solutions

In the following we study the phase space solutions and their space-time evolution. We start at $t = 0$ with a set of initial canonical variables $\{q_a, \theta_a\}$ (stated in (14-15)) and evolve the system by the Hamiltonian to time t. The solutions of Hamilton's equations of motion will be Q_a and T_b and they satisfy

$$\partial_t Q_a(t) = \dot{q}_a(Q, T), \tag{12}$$

$$\partial_t T_b(t) = \dot{\theta}_b(Q, T). \tag{13}$$

Let's have a few words on the notations! $Q_a(t)$ is the phase space trajectory of the ath particle at time t if we started the time evolution with q_a. $\dot{q}_a(Q, T)$ also tells us that the equations are coupled because during the time evolution these time-dependent quantities start to affect each other, therefore they will depend on each other. Similar argumentation holds for the rapidities in (13).

For having a well defined problem, we have to add initial conditions that are

$$Q_a(0) = q_a, \tag{14}$$

$$T_b(0) = \theta_b. \tag{15}$$

And the formula for the time evolution, of any function $\mathcal{F}$, which is driven by the Hamiltonian $\hat{\mathcal{H}}$ is

$$\left(\mathrm{e}^{t\hat{\mathcal{H}}}\mathcal{F}\right)(q, \theta) = \mathcal{F}(Q, T). \tag{16}$$

This formula expresses the fact that for any function on the phase space its evolution is identical of its argument's evolution, which is related to the fact that there is no explicit time dependence of $\mathcal{F}$ (and similarly for the space dependence in (17)).

Similarly, we are able to define the space evolution of these quantities generated by the momentum $\hat{\mathcal{P}}$, but in this case the phase space solutions will be different quantities than Q and T and they are denoted by $\bar{Q}$ and $\bar{T}$. So the evolution is

$$\left(\mathrm{e}^{x\hat{\mathcal{P}}}\mathcal{F}\right)(q,\theta) = \mathcal{F}(\bar{Q},\bar{T}), \tag{17}$$

where the solutions satisfy

$$\begin{aligned} \partial_x \bar{Q}_a(x) &= q'_a(\bar{Q},\bar{T}), &(18)\\ \partial_x \bar{T}_b(x) &= \theta'_b(\bar{Q},\bar{T}) &(19) \end{aligned}$$

and the initial conditions are

$$\begin{aligned} \bar{Q}_a(0) &= q_a, &(20)\\ \bar{T}_b(0) &= \theta_b. &(21) \end{aligned}$$

The particle trajectories x_a, velocities v_a and accelerations μ_a in a chosen inertial frame can be expressed by the phase space quantities as

$$\begin{aligned} x_a(t) &= x_a(Q,T), &(22)\\ v_a(t) &= \dot{x}_a(t), &(23)\\ \mu_a(t) &= \dot{v}_a(t) = \ddot{x}_a(t). &(24) \end{aligned}$$

Obtaining physical coordinates, velocities and accelerations we have to study the transformation properties of these quantities under a Poincaré transformation on the phase space. The trajectories have to transform in a Lorentz covariant way. This condition results restrictions for the trajectories and they are called world-line conditions. They are for each particle a at $t=0$

$$\begin{aligned} x'_a &= -1, &(25)\\ \hat{\mathcal{K}}x_a &= -x_a\dot{x}_a. &(26) \end{aligned}$$

The proof of the latter is the following. The space-time coordinates have to transform by a Lorentz transformation as

$$x' = (x-vt)\gamma, \qquad t' = (t-vx)\gamma, \qquad \gamma = 1/\sqrt{1-v^2}, \tag{27}$$

where v denotes the parameter of the boost transformation. At leading order the space-time coordinates transform as

$$x \to x - vt, \qquad t \to t - vx. \tag{28}$$

The transformation of the trajectory, however, at leading order around $t = 0$

$$x(t) \to x'(t') = x(t') - vt' = x(t - vx) - v(t - vx) \approx x(t) - vx\dot{x} - vt \approx x(t) - vx\dot{x}. \quad (29)$$

Using the fact that the parameter of a boost transformation was v, we get for the transformation itself the statement in (26).

2.3 Trajectory reconstruction

Let us introduce a Lorentz invariant quantity for each particle, for instance for particle a introduce ρ_a, to which

$$\hat{\mathcal{K}}\rho_a = 0 \quad (30)$$

holds. Similarly to (17), we can introduce the space evolution of ρ_a as

$$R_a(x) = \mathrm{e}^{x\hat{\mathcal{P}}}\rho_a = \rho_a(\bar{Q}, \bar{T}). \quad (31)$$

The physical trajectory will be the solution of the following implicit equation (if it exists and unique)

$$R_a(x_a) = 0. \quad (32)$$

The proof follows. We have to show that this construction is in agreement with the world-line conditions. For the proof we use an identity between the Poincaré generators

$$\left[\left(\hat{\mathcal{K}} + x\hat{\mathcal{H}}\right) \mathrm{e}^{x\hat{\mathcal{P}}}\right] \mathcal{F} = \left[\mathrm{e}^{x\hat{\mathcal{P}}}\hat{\mathcal{K}}\right] \mathcal{F}. \quad (33)$$

Let's choose $\mathcal{F} = \rho_a$ and use (30, 32). We get

$$\hat{\mathcal{K}}R_a(x_a) + x_a\dot{R}_a(x) = 0. \quad (34)$$

Now, let's act with an operator $\hat{\mathcal{L}}$ on (32)

$$\hat{\mathcal{L}}\left(R_a(x_a)\right) = \left(\hat{\mathcal{L}}R_a\right)(x_a) + R_a'(x_a)\hat{\mathcal{L}}x_a = 0 \quad (35)$$

and specify this for the cases, where $\hat{\mathcal{L}}$ is one of the Poincaré generators.

$$\hat{\mathcal{L}} = \hat{\mathcal{H}} \implies \dot{R}_a(x_a) + R_a'(x_a)\dot{x}_a = 0 \quad (36)$$

$$\hat{\mathcal{L}} = \hat{\mathcal{P}} \implies x_a' = -1 \quad (37)$$

$$\hat{\mathcal{L}} = \hat{\mathcal{K}} \implies \left(\hat{\mathcal{K}}R_a\right)(x_a) + R_a'(x_a)\hat{\mathcal{K}}x_a = 0 \quad (38)$$

From (37) we got back one world-line condition and the other one is coming from the insertion of (36) and (38) into (34).

3 The Ruijsenaars–Schneider model

The Ruijsenaars–Schneider model was first discovered in (S. N. Ruijsenaars & Schneider, 1986). Ruijsenaars and Schneider studied a finite dimensional integrable system of point particles in a relativistic framework and classical context.

3.1 The Ansatz

The discovery of Ruijsenaars and Schneider lied on the fact that they found such an Ansatz that fulfils the Poisson brackets (3-5), realizing the representation of the Poincaré group in a presence of interparticle potential, which is the function of the phase space coordinates. Note that relativistic invariance does not fit with such nontrivial (non-constant) potential that depends on *physical space* coordinates. But, in this Ansatz it is possible to introduce such function, which is compatible with relativistic invariance (moreover, it is actually integrable) and determined by the *phase space* coordinates of the individual particles q_js.

$$\mathcal{H} = \sum_{j=1}^{N} \cosh\theta_j \, V_j, \tag{39}$$

$$\mathcal{P} = \sum_{j=1}^{N} \sinh\theta_j \, V_j, \tag{40}$$

$$\mathcal{K} = -\sum_{j=1}^{N} q_j, \tag{41}$$

where

$$V_j = \prod_{1\leq k\leq N,\, k\neq j} f(q_j - q_k), \tag{42}$$

with a positive and even function f. It turns out that the most general form of f (which we will call later on "interaction") for at least 3 particles is built by the Weierstrass $\wp$-function and takes the form

$$f(q) = \sqrt{a + b\wp(q)}, \tag{43}$$

where a and b are positive numbers. (For $N \geq 3$ the most general form of the interaction is presented above, but for $N = 2$ the interacion can be an arbitrary even function of the difference of the phase space coordinates.) From now on, we use the $a = 1$ convention, or in other words, it can be scaled out in the Hamiltonian and the momentum generators. The Weierstrass $\wp$-function is a doubly periodic function on the complex plane, and it provides three special cases for the interaction depending on which period goes to infinity (hyperbolic and trigonometric) or both (rational).

$$
\begin{aligned}
f_{\text{rational}}(q) &= \sqrt{1+\frac{g^2}{q^2}} && (44)\\
f_{\text{hyperbolic}}(q) &= \sqrt{1+\frac{\alpha^2}{\sinh^2(\nu q)}} && (45)\\
f_{\text{trigonometric}}(q) &= \sqrt{1+\frac{\alpha^2}{\sin^2(\nu q)}} && (46)
\end{aligned}
$$

Further properties of the $\wp$-function can be found in Appendix A.1.

Using the Ansatz (39-41) it is easy to prove the Poisson commutativity relations for the generators (3-5), the only nontrivial one is (3), which results in a functional differential equation for f and this is the N particle generalization of the special case displayed in Appendix A.1 in (125) (S. N. Ruijsenaars & Schneider, 1986).

$$
\{\mathcal{H},\mathcal{P}\} \sim \sum_{j=1}^{N} \partial_j \prod_{1\le k\le N,\, k\neq j} f^2(q_j-q_k) = 0 \tag{47}
$$

It is important to mention the fact that the mathematical property for the Weierstrass $\wp$-function in (125) is exactly the condition for the $N=3$ case, therefore for 3 particles the Weierstrass $\wp$-function automatically fulfills this requirement.

3.1.1 Further properties of the Ansatz

We can introduce other further functions on the phase space built by the rapidities and the interaction:

$$
S_{\pm l} = \sum_{\substack{I\subset\{1,\ldots,N\}\\ |I|=l}} \exp\left(\pm\sum_{j\in I}\theta_j\right) \prod_{\substack{j\in I\\ k\notin I}} f(q_j-q_k), \tag{48}
$$

that in the $l=N$ case we get

$$
S_{\pm N} = \exp\left(\pm\sum_{j=1}^{N}\theta_j\right). \tag{49}
$$

They show some useful Poisson commutativity features that we will demonstrate in a minute, after studying another important characteristic, the index shifting property of the S functions.

We can derive some nice properties of these newly introduced S quantities. For instance, there is a relation between one with negative index and one with such an index that shifted by N, the number of particles. Namely,

$$S_{-l} = S_{-N} S_{N-l} \qquad \text{for} \qquad l \in \{1, ..., N-1\}. \tag{50}$$

The proof needs two properties to use: (i) if we multiply the exponential factor coming from S_N consisting N rapidities with the one which appears in S_{N-l} that has only $N-l$ with opposite signs, we are left with only those rapidities that appear in a set of l particles on the other side, (ii) the interaction f is an even function.

Moreover, we are able to express the Hamiltonian and the momentum generators of the N particle system only with the S_1 and S_{-1} functions.

$$\mathcal{H} = \frac{S_1 + S_{-1}}{2} \tag{51}$$

$$\mathcal{P} = \frac{S_1 - S_{-1}}{2} \tag{52}$$

The proofs are trivial, the only thing that needed is the exponential function expression of the hyperbolic ones. Applying these, we immediately get (39) and (40). We did not emphasize the values of the phase space variables where we take the above mentioned quantities, but we note them here: $(q, \theta) \equiv (q_1, ..., q_N; \theta_1, ..., \theta_N)$, where in this shorthand notation q and θ contains all the information of the particles. It is important to note that the dependence on the phase space coordinates is only in the interaction, which is even, therefore the dependence on the phase space coordinates of the Hamiltonian and momentum generators and the S functions are even. If we change the sign of the rapidities, it is equivalent to the sign change of the S function's index, so we only need S_1 to express $\mathcal{H}$ and $\mathcal{P}$ (no such property is applied to the dependence on the phase space coordinates due to the evenness of the interaction) (S. Ruijsenaars, 2001).

$$S_{-l}(q, \theta) = S_l(q, -\theta) \qquad \text{and} \qquad S_l(q, \theta) = S_l(-q, \theta) \qquad \text{for} \qquad l \in \{1, ..., N\}. \tag{53}$$

Right now, we can get back to the previously promised relations. We do not present the proof, instead highlight the importance of them (S. N. Ruijsenaars, 2009).

$$\{S_a(q, \theta), S_b(q, \theta)\} = 0, \tag{54}$$

$$\{\mathcal{H}, S_a(q, \theta)\} = 0, \qquad \{\mathcal{P}, S_a(q, \theta)\} = 0 \qquad \text{for} \qquad a, b \in \{\pm 1, ..., \pm N\}. \tag{55}$$

The first relation states that in our N particle system with degrees of freedom N there are $2N$ Poisson commuting quantity (one can call them conserved charges, as well), but the property stated in (53) reduces this number to N independent ones, which means that the theory is integrable. (Actually, the relations in (55) are not independent from (54), because $\mathcal{H}$ and $\mathcal{P}$ are just the linear combinations of $S_{\pm 1}$ according to (51-52).)

In the following we study 2-particle motion and determine the time shift (for the Ruijsenaars–Schneider model with hyperbolic interaction it will be time advance). It is crucial to mention that the study of the 2-particle case is enough, because the integrable nature of the theory manifests in the sense that the multiparticle scattering process factorizes into $2 \to 2$ particle scattering processes. To determine the time shift corresponding to particle j in a multiparticle system, this particular particle has to scatter on all the other particles. Let us denote the time shift aquired by particle j from the interaction with particle k with Δt_2^{jk}. This quantity can be obtained by the asymptotic behaviour of the 2-particle solution. The total time shift for particle j is the sum of the individual time shifts as

$$\Delta t_N^j = \sum_{k=1, k\neq j}^{N} \Delta t_2^{jk}. \tag{56}$$

3.2 Motion and time shift in the $N = 2$ case with repulsive hyperbolic type interaction

Let us apply the knowledge we gained so far to investigate the $N = 2$ case. First, we study the 2-particle scattering in the Ruijsenaars–Schneider framework with the general form of interacion. The Hamiltonian and momentum are now

$$\mathcal{H}(q_1, q_2, \theta_1, \theta_2) = \quad \mathcal{H} \quad = (\cosh\theta_1 + \cosh\theta_2) f(q_1 - q_2), \tag{57}$$

$$\mathcal{P}(q_1, q_2, \theta_1, \theta_2) = \quad \mathcal{P} \quad = (\sinh\theta_1 + \sinh\theta_2) f(q_1 - q_2), \tag{58}$$

where we used the evenness of the potential. The Hamiltonian equations of motion are

$$\dot{q}_l = \frac{\partial \mathcal{H}}{\partial \theta_l}, \qquad \dot{\theta}_l = -\frac{\partial \mathcal{H}}{\partial q_l} \quad \text{for} \quad l \in \{1, 2\}. \tag{59}$$

Recall, that our goal is to try to determine the phase space trajectories, therefore our obvious choice for changing the coordinate system to reach a symmetric motion (center of mass (COM) frame)

$$q_1 = -q_2 := q/2, \qquad \theta_1 = -\theta_2 := \theta, \tag{60}$$

from which we get the following equations of motion:

$$\dot{q} = 2\sinh(\theta)\, f(q), \tag{61}$$

$$\dot{\theta} = -2\cosh(\theta)\, f'(q). \tag{62}$$

These give a vector field in the phase space (q, θ) whose constant energy surfaces, the phase space trajectories are

$$\mathcal{H} = \text{constant} := H_0 = 2\cosh(\theta) f(q) \Longrightarrow \theta(q) = \pm\, \text{arcosh}\left[\frac{H_0}{2f(q)}\right]. \tag{63}$$

Note that the domain of H_0 is bounded from below, for instance in the hyperbolic case $H_0 > 2$ and $\lim_{q\to\pm\infty} H_0 \equiv 2\cosh\theta_\infty$.

Let us turn our attention to the repulsive[1] hyperbolic case, where the interaction is of the form

$$f(q) = \left(1 + \frac{\alpha^2}{\sinh^2(\nu q)}\right)^{1/2} \tag{64}$$

and visualized on Figure 1. The potential is singular at $q = 0$, therefore the particles are localized only in one half of the phase space (in the COM frame), they are not allowed to "scatter through" each other.

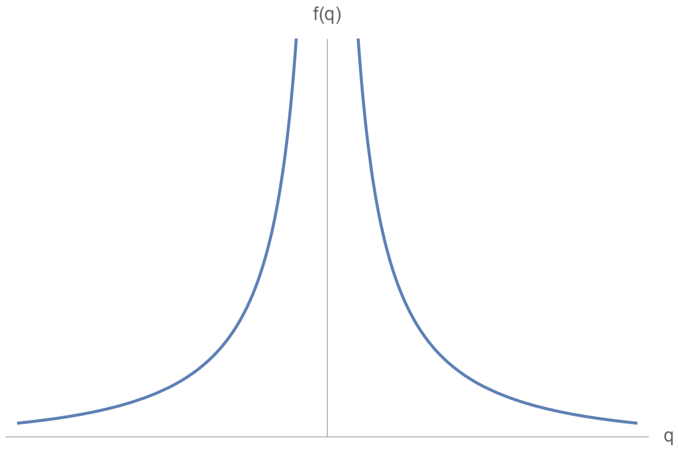


Figure 1: The hyperbolic potential.

The equations of motion look in the COM frame

$$\dot{q}(t) = 2\sinh(\theta(t))\sqrt{1 + \frac{\alpha^2}{\sinh^2(\nu q(t))}}, \tag{65}$$

$$\dot{\theta}(t) = 2\nu\alpha^2\cosh(\theta(t))\frac{\cosh(\nu q(t))}{\sinh^2(\nu q(t))\sqrt{\sinh^2(\nu q(t)) + \alpha^2}}. \tag{66}$$

[1]There is an analogous potential with which attractive behaviour can be studied. We study it in Appendix B. Later on we omit the word "repulsive" and refer only with the word "hyperbolic" case.

The phase space trajectories according to (63) of the function of H_0, α and ν are visualized on Figure 2. and 3., respectively.

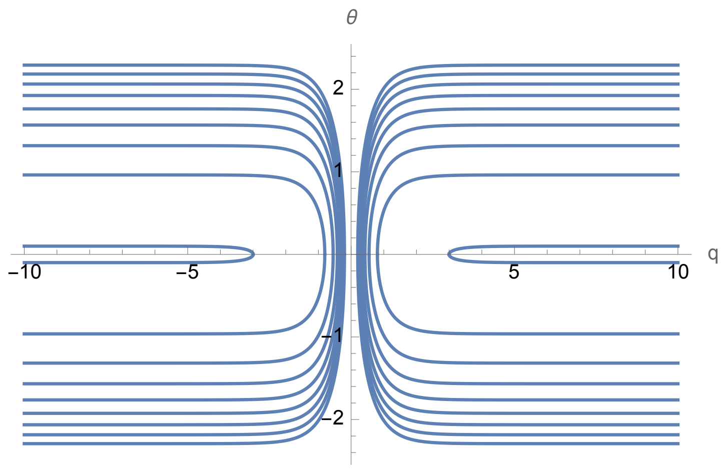


Figure 2: The phase space trajectories (constant energy surfaces) for the values of $H_0 \in \{2.01, 3, 4, 5, 6, 7, 8, 9, 10\}$ with $\alpha = 1$ and $\nu = 1$. (The bigger the value of H_0 the bigger the maximum of the modulus of the rapidity.)

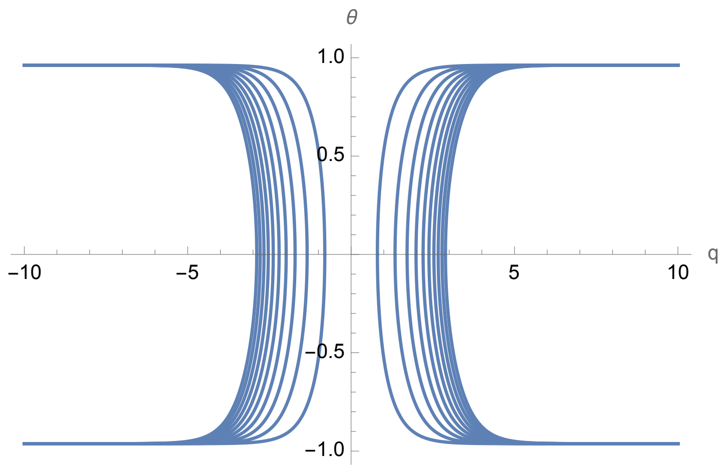


Figure 3: The phase space trajectories (constant energy surfaces) for the values of $\alpha \in \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$ with $H_0 = 3$ and $\nu = 1$. The bigger the value of α the smaller the minimal distance between the particles.

The analytic solution of the equations of motion (65-66) is not easy to get in a closed form, therefore we do not focus on solving these equations immediately, instead we are aiming to calculate the time delay in this particular system, and during the calculation the time dependence of the canonical phase space coordinates of the particles will surprisingly be determined. After this the rapidity follows from (65) immediately.

The Hamiltonian gives the energy which has to be constant during the motion (2ε)

$$\frac{\mathcal{H}}{2} = \cosh(\theta)\, f(q) = \varepsilon = \cosh\theta_\infty, \tag{67}$$

where $\cosh\theta_\infty$ is a reparametrisation of the energy. Let's see 2 special cases!

- $q \to \infty$: $f(q) \to 1$ and $\cosh\theta$ remains only, so the meaning of $\cosh\theta_\infty$ is the asymptotic energy corresponding to the free motion and $\pm\theta_\infty$ denotes the asymptotic/free rapidities of the particles.
- $q = q_\varepsilon$ is the turning point. Here $\theta = 0$, therefore we get a relation for the minimal distance of the 2 particles in terms of the phase space coordinate ($2q_\varepsilon$) as the function of the energy:

$$\varepsilon = f(q_\varepsilon) = \sqrt{1 + \frac{\alpha^2}{\sinh^2(\nu q_\varepsilon)}} \Longrightarrow \sinh(\nu q_\varepsilon) = \frac{\alpha}{\sqrt{\varepsilon^2 - 1}} \Longrightarrow q_\varepsilon(\varepsilon) = \frac{1}{\nu}\text{arsinh}\left(\frac{\alpha}{\sqrt{\varepsilon^2 - 1}}\right). \tag{68}$$

Our goal is to determine the time dependent coordinates $x(t)$ in a(n arbitrarily) chosen inertial frame (COM) to be able to to determine the time shift. To do that we need to figure out the time dependence (in this particular frame) of the phase space variables. In order to perform this, let's use the equations of motion. For calculating the time shift it is enough to use only one of them, therefore we present only the one we use for our calculation.

$$\frac{\mathrm{d}q}{\mathrm{d}t} = \dot{q} = \frac{\partial \mathcal{H}}{\partial \theta} = 2\sinh(\theta)\, f(q) = 2\sqrt{\cosh^2(\theta) - 1} f(q) = 2\sqrt{\varepsilon^2 - f^2(q)} \tag{69}$$

Let's integrate this separable differential equation with the following initial condition: q_ε occurs at $t = 0$ and the particle is in q at time t

$$g_\varepsilon(q) := \int_{q_\varepsilon}^{q} \frac{\mathrm{d}q'}{2\sqrt{\varepsilon^2 - f^2(q')}} = \int_0^t \mathrm{d}t' = t, \tag{70}$$

where we introduced a shorthand notation for the q-dependent integral. This integral on the left cannot be computed for the general Weierstrass case analitically, but for the hyperbolic one we present the process.

$$\begin{aligned} g_\varepsilon(q) &= \int_{q_\varepsilon}^{q} \frac{\mathrm{d}q'}{2\sqrt{\varepsilon^2 - f^2(q')}} = \frac{1}{2}\frac{\sinh(\nu q_\varepsilon)}{\alpha} \int_{q_\varepsilon}^{q} \frac{\sinh(\nu q')\mathrm{d}q'}{\sqrt{\sinh^2(\nu q') - \sinh^2(\nu q_\varepsilon)}} \\ &= \frac{1}{2}\frac{\sinh(\nu q_\varepsilon)}{\nu\alpha} \int_{\cosh(\nu q_\varepsilon)}^{\cosh(\nu q)} \frac{\mathrm{d}y}{\sqrt{y^2 - 1 - \sinh^2(\nu q_\varepsilon)}} = \frac{1}{2}\frac{\sinh(\nu q_\varepsilon)}{\nu\alpha} \int_1^{\frac{\cosh(\nu q)}{\cosh(\nu q_\varepsilon)}} \frac{\mathrm{d}z}{\sqrt{z^2 - 1}} \\ &= \frac{1}{2\nu\sqrt{\varepsilon^2 - 1}}\text{arcosh}\left(\frac{\cosh(\nu q)}{\cosh(\nu q_\varepsilon)}\right) \end{aligned} \tag{71}$$

Combining the last 2 equations, we are able to determine the time dependence of the canonical coordinate.

$$\operatorname{arcosh}\left(\frac{\cosh(\nu q)}{\cosh(\nu q_\varepsilon)}\right) = 2\nu\sqrt{\varepsilon^2-1}t \qquad \Longrightarrow \qquad \cosh(\nu q) = \cosh(\nu q_\varepsilon)\cosh\left(2\nu\sqrt{\varepsilon^2-1}t\right) \tag{72}$$

The time dependence of the canonical coordinate is according to (72) and (68)

$$\begin{aligned} q(t;\varepsilon,\alpha,\nu) &= \frac{1}{\nu}\operatorname{arcosh}\left[\cosh\left(\operatorname{arsinh}\left(\frac{\alpha}{\sqrt{\varepsilon^2-1}}\right)\right)\cosh\left(2\nu\sqrt{\varepsilon^2-1}t\right)\right] \\ &= \frac{1}{\nu}\operatorname{arcosh}\left[\sqrt{\frac{\alpha^2+\varepsilon^2-1}{\varepsilon^2-1}}\cosh\left(2\nu\sqrt{\varepsilon^2-1}t\right)\right], \end{aligned} \tag{73}$$

and the rapidity according to (73) and (65)

$$\begin{aligned} \sinh\theta(t;\varepsilon,\alpha,\nu) = &\sqrt{\frac{(\varepsilon^2-1)(\alpha^2+\varepsilon^2-1)}{(\alpha^2-1)(\varepsilon^2-1)+(\alpha^2+\varepsilon^2-1)\cosh^2\left(2\nu\sqrt{\varepsilon^2-1}t\right)}} \\ &\times \sinh\left(2\nu\sqrt{\varepsilon^2-1}t\right). \end{aligned} \tag{74}$$

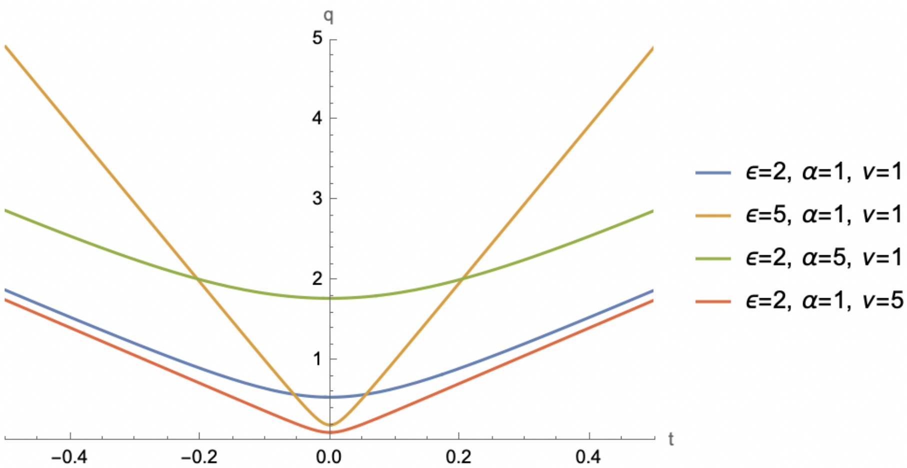


Figure 4: We present the time evolution of the phase space coordinate in the COM inertial frame. The used values of the parameters are shown in the right.

3.2.1 Coordinate reconstruction

Naively, we immediately try to use (9) to determine the time dependence of the coordinate, but due to the interaction it is prohibited, we need to use the trajectory reconstruction method introduced in subsections 2.2 and 2.3. We have to determine the space-time evolution of the q_a functions. The space and time evolution commute, therefore we can separately apply the evolution operators in terms of space and time. Let's see space

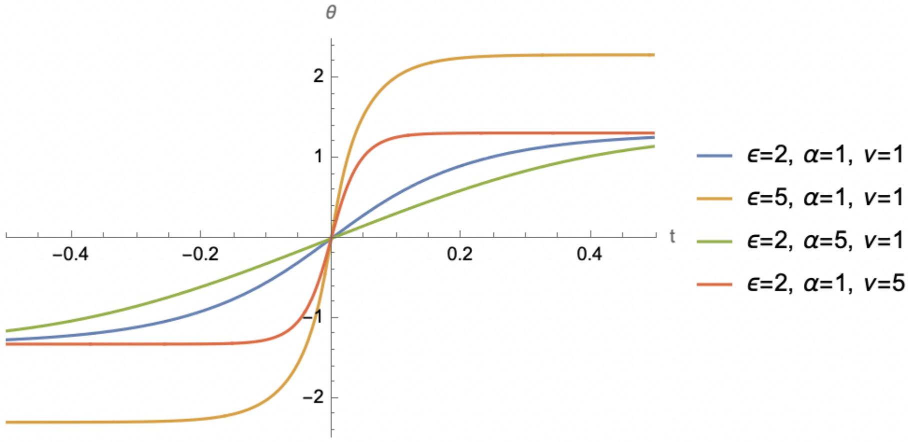


Figure 5: We present the time evolution of the rapidity of one of the 2 particles (the other one's is the same but with opposite sign) in the COM inertial frame. The used values of the parameters are shown in the right. The rapidity converges to an only energy dependent constant stated in (83).

evolution. We start with (18).

$$\partial_x \bar{Q}_a(x) = q_a'(x) = \{q_a, \mathcal{P}\} \overset{(58)}{=} \cosh\theta_a\, f(q_1 - q_2) \overset{(60)}{=} \cosh\theta\, f(q) \overset{(67)}{=} \varepsilon = \cosh\theta_\infty \tag{75}$$

After performing the integration and using the initial condition (20) we get for the space evolution

$$\bar{Q}_a(x) = \cosh\theta_\infty \cdot x + q_a = \varepsilon x + q_a. \tag{76}$$

The time evolution is simply follows from (73). To summarize the space-time evolution, the space-time dependent q functions are

$$q_1(x,t) = \varepsilon x + q(t;\varepsilon,\alpha,\nu)/2 \tag{77}$$

$$q_2(x,t) = \varepsilon x - q(t;\varepsilon,\alpha,\nu)/2, \tag{78}$$

where we used the initial condition that requires the particles to be in $\pm q_\varepsilon$ at $t = 0$. Finally, the space-time trajectories follows as

$$q_1(x,t) = 0 \implies x_1(t) = -\frac{1}{2\varepsilon} q(t;\varepsilon,\alpha,\nu), \tag{79}$$

$$q_2(x,t) = 0 \implies x_2(t) = \frac{1}{2\varepsilon} q(t;\varepsilon,\alpha,\nu). \tag{80}$$

We got the motion $x_1(t) = -x_2(t)$, as we expected.

3.2.2 Time shift

From these formulas it is clear that the turning point happens at $t = 0$ because $\theta(0;\varepsilon,\alpha,\nu) = 0$ and the minimal distance of the particles in terms of the phase space coordinate and

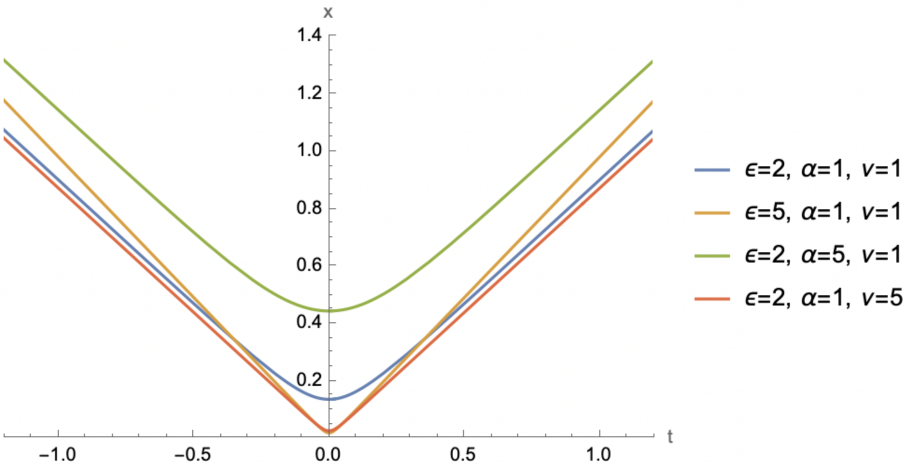


Figure 6: Time evolution of the physical coordinate of one particle (the other one is the same but with opposite sign) in the COM frame.

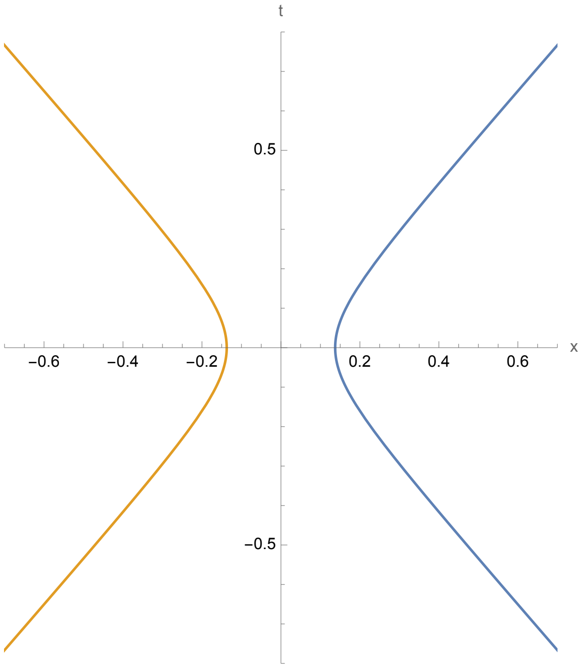


Figure 7: We present here a typical space-time diagram for a $2 \to 2$ particle scattering process. (Used parameter values: $\varepsilon = 2$, $\alpha = 1$, $\nu = 1$.)

the physical space coordinate are

$$q_{\min}(\varepsilon,\alpha,\nu) \;:=\; 2q_\varepsilon = 2q(0;\varepsilon,\alpha,\nu) = \frac{2}{\nu}\operatorname{arsinh}\left(\frac{\alpha}{\sqrt{\varepsilon^2-1}}\right), \tag{81}$$

$$x_{\min}(\varepsilon,\alpha,\nu) \;:=\; 2x_\varepsilon = 2x(0;\varepsilon,\alpha,\nu) = \frac{1}{\nu\varepsilon}\operatorname{arsinh}\left(\frac{\alpha}{\sqrt{\varepsilon^2-1}}\right). \tag{82}$$

And the asymptotic rapidity is only the function of the energy of the particle as it should be.

$$\theta_\infty = \lim_{t\to\infty}\theta(t;\varepsilon,\alpha,\nu) = \ln\left(\varepsilon+\sqrt{\varepsilon^2-1}\right) \tag{83}$$

For determining the time shift the asymptotic expansion of $x(t)$ at leading order in t is enough.

For $t \to +\infty$ (and $x \to +\infty$, as well) at leading order we get using (72) and (79-80)

$$\mathrm{e}^{2\nu\varepsilon x} = \mathrm{e}^{\ln\cosh(2\nu\varepsilon x_\varepsilon)}\mathrm{e}^{2\nu\sqrt{\varepsilon^2-1}t} \Longrightarrow x(t) = \frac{1}{2\nu\varepsilon}\ln\cosh(2\nu\varepsilon x_\varepsilon) + \frac{1}{\varepsilon}\sqrt{\varepsilon^2-1}t. \tag{84}$$

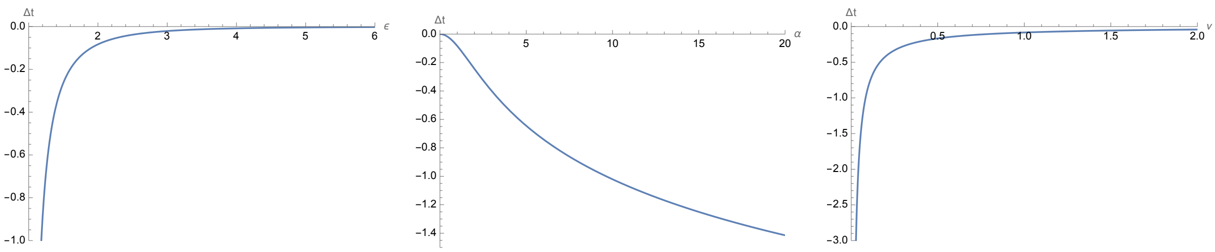
For $t \to -\infty$ (and $x \to -\infty$ the other particle) at leading order we get

$$x(t) = -\frac{1}{2\nu\varepsilon}\ln\cosh(2\nu\varepsilon x_\varepsilon) + \frac{1}{\varepsilon}\sqrt{\varepsilon^2-1}t. \tag{85}$$

From there we are able to identify the time shift Δt in this particular inertial frame, which is the difference of the intersection of these linear functions

$$\Delta t(\varepsilon;\alpha,\nu) = -\frac{1}{2\nu\sqrt{\varepsilon^2-1}}\ln\left(\frac{\alpha^2}{\varepsilon^2-1}+1\right) \tag{86}$$

in agreement with the result (4.31) in (Balog, 2014b) and the minus sign refers to the fact that the time shift is a time advance. It is qualitatively seen that for harder potential (bigger ν), as well as higher energy the time shift is smaller, as we intuitively expected.



(a) $\alpha = 1$, $\nu = 1$ (b) $\varepsilon = 2$, $\nu = 1$ (c) $\varepsilon = 2$, $\alpha = 1$

Figure 8: Time shift as the function of the parameters.

We show a space-time diagram with time shifts for a $2 \to 2$ particle scattering in Figure 9. To summarize our results (73), (74), (79-80) we visualize these quantities as the functions of their parameters in Figure 10.

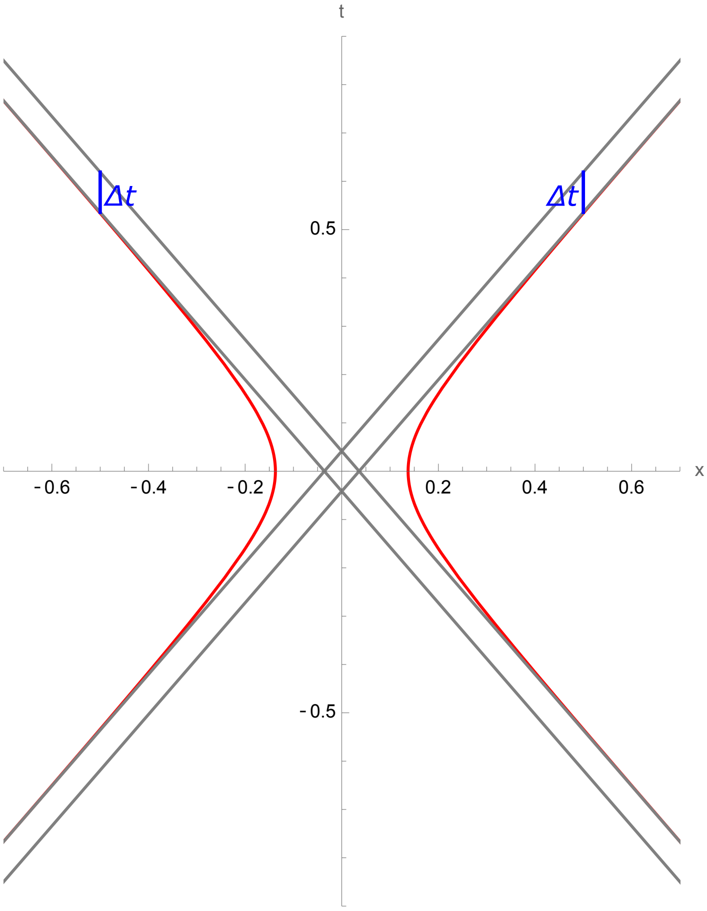
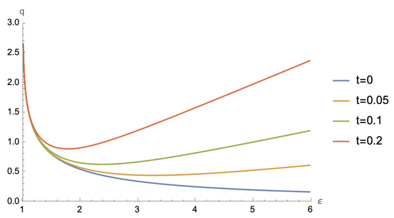
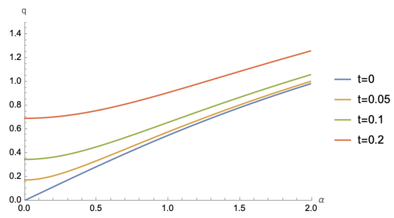


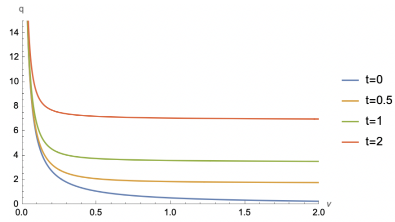
Figure 9: Space-time trajectories of a 2-particle scattering in the presence of repulsive interaction with the asymptotes, that correspond to free motions with slope of the inverse of the asymptotic velocity. The time advance Δt suffered by the particles is shown in the figure, as well. (Used parameter values: $\varepsilon = 2$, $\alpha = 1$, $\nu = 1$.)



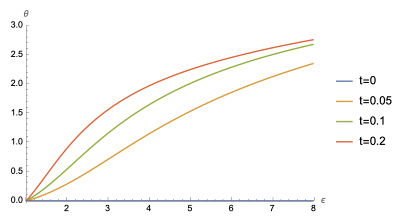
(a) $\alpha = 1,\ \nu = 1$



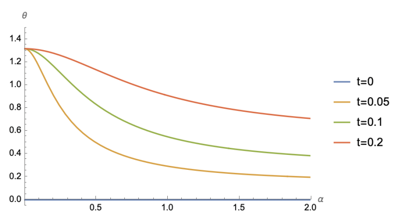
(b) $\varepsilon = 2,\ \nu = 1$



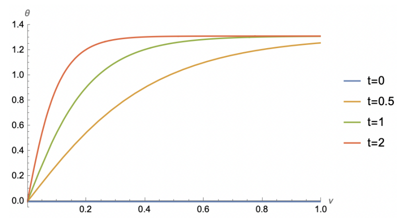
(c) $\varepsilon = 2,\ \alpha = 1$



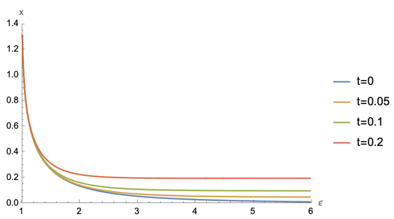
(d) $\alpha = 1,\ \nu = 1$



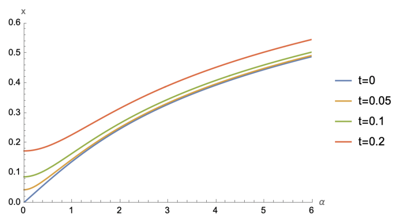
(e) $\varepsilon = 2,\ \nu = 1$



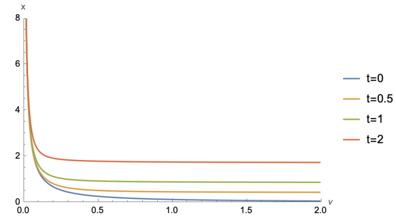
(f) $\varepsilon = 2,\ \alpha = 1$



(g) $\alpha = 1,\ \nu = 1$



(h) $\varepsilon = 2,\ \nu = 1$



(i) $\varepsilon = 2,\ \alpha = 1$

Figure 10: The parameter dependence of the studied quantities q, θ and x in the COM inertial frame, they were evaluated for different elapsed times after the scattering process. (The values of the applied parameters are shown in the captions of the individual subfigures.)

4 The sine-Gordon model

The sine-Gordon model is a widely studied field theory in various fields of theoretical physics, including classical field theory, quantum field theory, conformal field theory. It is also one of the foundational building block of integrable systems. Its appearance comes from the following arising question: what potentials can be integrable in a 1+1 dimensional (classical) field theory? Starting with the Lagrangian density ($c = 1$)

$$\mathcal{L} = \frac{1}{2}(\partial_t \phi)^2 - \frac{1}{2}(\partial_x \phi)^2 - V(\phi) \tag{87}$$

and recall Noether's theorem, which states that for every continuous symmetry of the system there is a conserved current and charge and conserved charge generates the symmetry. It turns out that the condition of having a conserved spin-3 charge can only be fulfilled by the potential, which has the most general form of

$$V(\phi) = a\mathrm{e}^{\sqrt{\alpha}\phi} + b\mathrm{e}^{-\sqrt{\alpha}\phi}, \tag{88}$$

where $a, b \in \mathbb{R}$ and $\alpha \in \mathbb{C}$.[2] The proof can be found in Appendix A.2. By shifting the field we obtain the sine-Gordon (sG) and sinh-Gordon (shG) potentials with appropriate new parameters.

$$V_{\mathrm{sG}} = -\frac{m^2}{\beta^2}\cos(\beta\phi)^3, \qquad V_{\mathrm{shG}} = -\frac{m^2}{b^2}\cosh(b\phi) \tag{89}$$

These potentials are related to each other by an analytical continuation ($\beta \longleftrightarrow ib$) and the appearing parameter m is related to the mass of the particle in the particular theory. Later, we focus on the sine-Gordon theory whose equation of motion reads ($m = c = 1$)

$$(\partial_t^2 - \partial_x^2)\phi + \frac{1}{\beta}\sin(\beta\phi) = 0. \tag{90}$$

As we mentioned earlier, the sine-Gordon theory is an integrable one, but here the construction of the infinitely many conserved charges is not a trivial question, therefore we will not mathematically formulate the process, instead present the general idea of the so-called Bäcklund transformation and the origin of the conserved charges.

[2]There are actually other kinds of theories with integrable potentials, for instance the Bullough–Dodd model with $V_{\mathrm{BD}} \sim (2\mathrm{e}^{g\phi} + \mathrm{e}^{-2g\phi})$, where g is the coupling constant and ϕ is a real scalar field (*Bullough–Dodd model*, 2022).

[3]Sometimes it is more convenient to use $V_{\mathrm{sG}} = \frac{m^2}{\beta^2}(1 - \cos(\beta\phi))$ because for small ϕ it gives back the Klein–Gordon equation of motion with the $m^2\phi$ term.

Let us see take a system where we have two sine-Gordon theories, one on the positive part of the coordinate frame, and the other on the negative and introduce a field dependent defect (potential) at the origin in such a way that the whole system is integrable. From matching the two solutions of the individual equations of motion of the bulks we get a restriction for the defect, providing a one parameter family of solutions for it. The relation that connects the two bulk solutions with the defect is called the Bäcklund transformation, which makes it possible, in general, to generate a new solution from an already existing one. Last but not least, the source of the infinitely many conserved charges is actually the Bäcklund transformation, because expanding the solution on one side around the other side's field configuration in powers of the newly appearing parameter and plugging this expansion back to the defect condition we obtain infinitely many independent conservation laws (Fring, Mussardo, & Simonetti, 1993). One can find a more detailed argumentation about the Bäcklund transformation in Appendix A.3.

In the following parts of this section we study the classical solutions of the sine-Gordon equation, starting with the 1-particle solution, after that we pay attention to the 2-particle scattering solutions and time shifts, and finally we have a couple words on the general N-particle solution with Hirota's formula.

4.1 1-particle solutions

Due to the relativistically invariant nature of the theory it is convenient the determine the static solution first and after acting with a Lorentz transformation we can get the general form of the 1-particle solutions. The equation of motion reads according to (90)

$$-\partial_x^2\phi + \frac{1}{\beta}\sin(\beta\phi) = 0 \Longleftrightarrow \frac{\mathrm{d}^2\phi}{\mathrm{d}x^2} = V'(\phi), \tag{91}$$

which is analogous to Newton's equation but the difference is the sign of the potential. Therefore we are able to express the coordinate x as a function of the field ϕ by using the analogous classical mechanical property, the conservation of the energy ε. From this we obtain the solution

$$x = x_0 \pm \int \frac{\mathrm{d}\phi}{\sqrt{2(\varepsilon + V(\phi))}}, \tag{92}$$

where x_0 is an integration constant. In our theory the energy of the field configuration is finite and therefore the field should decay at spatial infinities, meaning it should be on top of the potential at times equal $\pm\infty$, therefore $\varepsilon = 0$. Now, we are able to perform the integral and after inverting the result we reach the field configuration for a static solution that reads

$$\phi(x) = \pm\frac{4}{\beta}\arctan\left[\mathrm{e}^{-(x-x_0)}\right]. \tag{93}$$

From the inertial system of a moving observer the field configuration will be time dependent, as well, so let us apply a boost transformation with velocity v and Lorentz factor of γ.

$$\phi(x,t) = \pm\frac{4}{\beta}\arctan\left[\mathrm{e}^{-\gamma(x-vt-x_0)}\right] \tag{94}$$

We got two different but almost identical solutions of the sine-Gordon equation of motion, the only difference is the sign of the field configuration. The two solutions are called soliton (+) and anti-soliton (−), see Figure 11. Actually, the difference between the two solutions is not the sign, instead, the so-called topological charge Q_{top}, which is a conserved quantity.

$$Q_{\text{top}} = \frac{\beta}{2\pi}\int_{-\infty}^{\infty}\partial_x\phi(x,t)\mathrm{d}x = \frac{\beta}{2\pi}[\phi(\infty,t)-\phi(-\infty,t)] = \pm 1, \tag{95}$$

which is 1 for the soliton and −1 for the anti-soliton. These solutions are dispersionless and have localized energy densities. Their energy and momentum are $E(v) = M_0\gamma$, $P(v) = M_0 v\gamma$, respectively, where $M_0 = 8/\beta^2$ (Rajaraman, 1982).

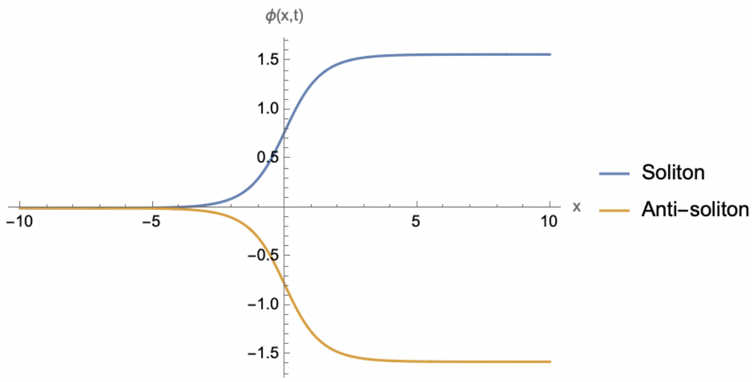


Figure 11: A soliton and an anti-soliton solution ($\beta = 4$, $x_0 = 0$, $t = 0$, $v = 0.5$).

4.2 2-particle solutions

It is not a trivial question to answer that there are any multiparticle solutions of the sine-Gordon equation from using only the equation itself. But we are able to point to the fact that we can easily construct them using the 1-particle solutions and the previously introduced Bäcklund transformation. For the case of 2 particles we present the possible 2-particle solutions (s, and $\bar{\text{s}}$ will denote soliton and anti-soliton, respectively) (Rajaraman, 1982).

$$\phi_{\text{ss}}(x,t) = \frac{4}{\beta}\arctan\left[\frac{v\sinh(x\gamma)}{\cosh(vt\gamma)}\right] = -\phi_{\bar{\text{s}}\bar{\text{s}}}(x,t) \tag{96}$$

$$\phi_{\text{s}\bar{\text{s}}}(x,t) = \frac{4}{\beta}\arctan\left[\frac{\sinh(vt\gamma)}{v\cosh(x\gamma)}\right] \tag{97}$$

These solutions describe $2 \to 2$ scatterings and are visualized in Figure 12. and 13. In the soliton – soliton scattering they reflect on each other, however in the soliton – anti-soliton scattering they go through each other.

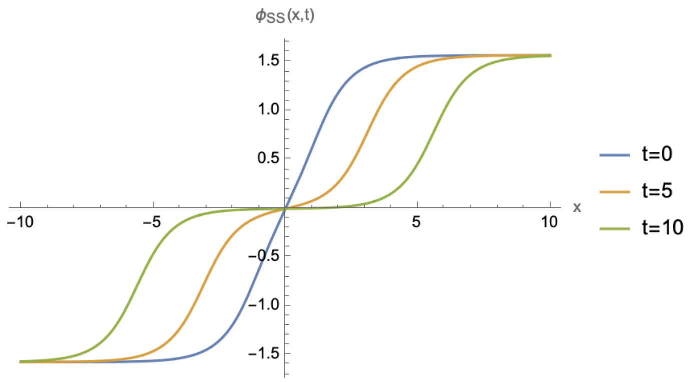


Figure 12: Time evolution of a soliton – soliton solution in the COM frame. Evaluation the field configuration for $-t$ and t we get the same due to the COM frame ($\beta = 4$, $v = 0.5$).

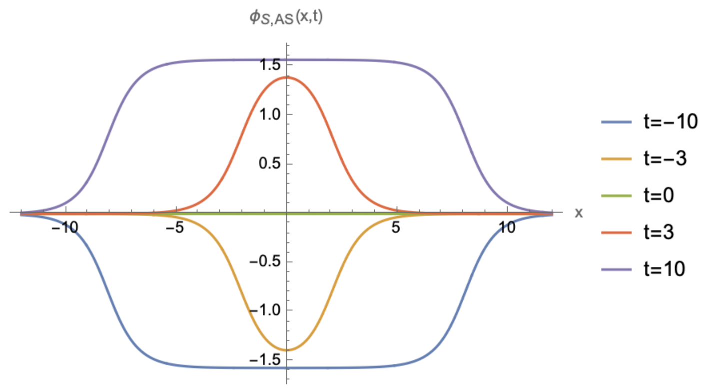


Figure 13: Time evolution of a soliton – anti-soliton solution in the COM frame. Evaluation for the field configurations for $-t$ and t are the opposite due to the fact that the particles cross each other and we are in the COM frame ($\beta = 4$, $v = 0.5$).

Let us see the asymptotic behaviour of the soliton – soliton scattering process and determine the time shift. To do that, it is easier to work with the exponential expression of the hyperbolic functions.

$$\phi_{\mathrm{ss}}(x,t) = \frac{4}{\beta} \arctan\left[\frac{\mathrm{e}^{\gamma x+\ln v} - \mathrm{e}^{-\gamma x+\ln v}}{\mathrm{e}^{\gamma v t} + \mathrm{e}^{-\gamma v t}}\right] \tag{98}$$

Let's analyze the asymptotic limits in the remote past and remote future, where the particles are well-separated, therefore we will only keep the leading order terms in x. The obtained solutions are two solitons but with different time arguments. The result for the free cases at leading order follows as

$$\lim_{t\to-\infty} \phi_{\mathrm{ss}}(x,t) = \phi_{\mathrm{s}}(\gamma[x+v(t-\Delta t/2)]) + \phi_{\mathrm{s}}(\gamma[x-v(t-\Delta t/2)]), \tag{99}$$

$$\lim_{t\to+\infty} \phi_{\mathrm{ss}}(x,t) = \phi_{\mathrm{s}}(\gamma[x+v(t+\Delta t/2)]) + \phi_{\mathrm{s}}(\gamma[x-v(t+\Delta t/2)]). \tag{100}$$

The time shift Δt occured is

$$\Delta t = 2\frac{\ln v}{v\gamma}, \tag{101}$$

which is actually negative (time advance).

We would like to highlight an important characteristic of the sine-Gordon theory here, which is also present in the hyperbolic case of the Ruijsenaars–Schneider model, the presence of bound states. (For the Ruijsenaars–Schneider model the potential needs to be in the form of $f(q) = [1-\alpha^2/\cosh^2(\nu q)]^{1/2}$.) Bound states occur when the interaction is attractive. We call the bound states in this theory breathers, and they are formed by a soliton and an anti-soliton. We obtain the breather by analytically continuing the velocity to $v \longrightarrow iu$ and the time-periodic field configuration is

$$\phi_{\mathrm{B}}(x,t) = \frac{4}{\beta} \arctan\left[\frac{\sin(ut\gamma)}{u\cosh(x\gamma)}\right]. \tag{102}$$

4.3 N-particle solutions

Last, but not least, we present the result of Hirota (Hirota, 1980) which provides an analytic and exact formula for the N-particle solution of the sine-Gordon equation of motion. The derivation is the following: one has to apply the Bäcklund transformation to create a solution with one more particle. The field configuration looks

$$\phi(x,t) = \frac{4}{\beta} \arctan\left[\frac{\mathrm{Im}(\tau(x,t))}{\mathrm{Re}(\tau(x,t))}\right], \tag{103}$$

where

$$\tau(x,t) = \sum_{\mu_j=0,1} \mathrm{e}^{-\sum_{j=1}^{N}\mu_j\left(x\cosh\theta_j + t\sinh\theta_j - x_j - \frac{i\pi}{2}\varepsilon_j\right) + 2\sum_{i<j}\mu_i\mu_j \ln\tanh\left[\frac{\theta_i-\theta_j}{2}\right]}. \tag{104}$$

[4]

[4] The formula is not correct in Ref. (Bajnok & Šamaj, 2011), the proper one stated here. The problem is the argument of the logarithm in the mentioned reference, which stated as $[\tanh(\theta_i - \theta_j)]/2$.

The θ_j parameters are the physical rapidities of the individual particles and the appearing ε_j quantities are 1 for solitons and -1 for anti-solitons. (One can incorporate breather modes, as well, as a moving bound state of a soliton with an anti-soliton, with velocities $v + iu$ and $v - iu$, respectively (Rajaraman, 1982)). Qualitatively the first sum in the exponent describes free motion of the particles with a phase factor and the second term describes the interaction, which is only such a type that appears between two particles.

In terms of the initial and final states the set of the final momenta will exactly be the set of initial momenta and they attached to the same particles, while the order of the particles is the opposite after the scattering processes. (In the final state the initially leftmost is the rightmost, the initially rightmost is the leftmost and similarly for the particles in the middle.)

We would like to have a few words on the time shift concerning the N-particle system. The sine-Gordon theory is also an integrable one, alongside the Ruijsenaars–Schneider model, therefore the time shift felt by a particle in a many-body system can be summed up by the ones caused by the individual 2-particle interactions that the chosen particle is involved in, (56) is also applicable here.

5 Correspondence

In this section we would like to prove that there is a map between the 2-particle solution of the sine-Gordon theory and the 2-particle case of the Ruijsenaars–Schneider model with hyperbolic interaction, with special parameter values.

Ruijsenaars and Schneider claim (S. N. Ruijsenaars & Schneider, 1986) that

$$\phi(x,t) = 4\sum_{j=1}^{N=2} \arctan \mathrm{e}^{u_j(x,t)} \tag{105}$$

satisfy the sine-Gordon equation of motion, where

$$u_{1,2}(x,t) = 2\nu q_{1,2}(x,t) = 2\nu\varepsilon x \pm \nu q(t). \tag{106}$$

In the proof we show that the Ruijsenaars–Schneider Ansatz for 2-particles (105) has the same form as the Hirota formula (103-104) for 2 particles and additionally we show that the result checks up with the soliton – soliton field configuration (96).

For proving the correspondence we need to use specific values for the parameters of the hyperbolic potential (64): $\alpha = 1$, $\nu = 1/2$, resulting the new form of

$$f(q) = |\coth(q/2)|. \tag{107}$$

Here we have for the 2-particle solution according to the Ansatz (105)

$$\begin{aligned}
\phi_{2,\mathrm{RS}}(x,t) &= 4\left[\arctan \mathrm{e}^{2\nu q_1(x,t)} + \arctan \mathrm{e}^{2\nu q_2(x,t)}\right] = 4\arctan\left[\frac{\mathrm{e}^{2\nu q_1(x,t)} + \mathrm{e}^{2\nu q_2(x,t)}}{1-\mathrm{e}^{2\nu q_1(x,t)+2\nu q_2(x,t)}}\right] \\
&\overset{(106)}{=} -4\arctan\left[\frac{\cosh(\nu q(t))}{\sinh(2\nu\varepsilon x)}\right] \overset{(73)}{=} -4\arctan\left[\sqrt{1+\frac{\alpha^2}{\varepsilon^2-1}}\,\frac{\cosh\left(2\nu\sqrt{\varepsilon^2-1}t\right)}{\sinh(2\nu\varepsilon x)}\right] \\
&= -4\arctan\left[\sqrt{\frac{\varepsilon^2}{\varepsilon^2-1}}\,\frac{\cosh\left(\sqrt{\varepsilon^2-1}t\right)}{\sinh(\varepsilon x)}\right] \\
&= -4\arctan\left[\frac{1}{\tanh\theta_\infty}\,\frac{\cosh(\sinh\theta_\infty\cdot t)}{\sinh(\cosh\theta_\infty\cdot x)}\right]
\end{aligned} \tag{108}$$

Let's see the $N=2$ case of Hirota's form (103-104) with $\beta = 1$.

$$\phi_2(x,t) = 4\arctan\left[\frac{\mathrm{Im}(\tau_2(x,t))}{\mathrm{Re}(\tau_2(x,t))}\right], \tag{109}$$

where

$$\tau_2^{\mathrm{ss}}(x,t) = 1 + iE_1 + iE_2 - E_1E_2\tanh^2\left(\frac{\theta_1-\theta_2}{2}\right) \tag{110}$$

for the solitonic case with

$$E_k = \exp\left(x\cosh\theta_k + t\sinh\theta_k - x_k\right) \equiv \mathrm{e}^{s_k(x,t)}, \qquad k = 1, 2. \tag{111}$$

The soliton – soliton solution is

$$\phi^{\mathrm{ss}}_{2,\mathrm{Hirota}}(x,t) \;=\; 4\arctan\left[\frac{\mathrm{e}^{s_1(x,t)} + \mathrm{e}^{s_2(x,t)}}{1 - \mathrm{e}^{s_1(x,t)+s_2(x,t)+\Delta s}}\right], \tag{112}$$

where

$$\Delta s = \ln\tanh^2\left(\frac{\theta_1-\theta_2}{2}\right) = \ln\tanh^2\theta \tag{113}$$

with the rapidities are $\theta_1 = -\theta_2 := \theta$. For the calculation we will use the following choice

$$x_1 = x_2 = \Delta s/2 = \ln\tanh\theta. \tag{114}$$

$$\begin{aligned}
\phi^{\mathrm{ss}}_{2,\mathrm{Hirota}}(x,t) \;&=\; 4\arctan\left[\frac{\mathrm{e}^{x\cosh\theta+t\sinh\theta-x_1} + \mathrm{e}^{x\cosh\theta-t\sinh\theta-x_2}}{1-\mathrm{e}^{2x\cosh\theta-x_1-x_2+\Delta s}}\right]\\
&\overset{(114)}{=}\; -4\arctan\left[\mathrm{e}^{-\Delta s/2}\frac{\cosh(\sinh\theta\cdot t)}{\sinh(\cosh\theta\cdot x)}\right]\\
&=\; -4\arctan\left[\frac{1}{\tanh\theta}\frac{\cosh(\sinh\theta\cdot t)}{\sinh(\cosh\theta\cdot x)}\right]\\
&=\; -4\arctan\left[\frac{1}{\tanh\theta_\infty}\frac{\cosh(\sinh\theta_\infty\cdot t)}{\sinh(\cosh\theta_\infty\cdot x)}\right] = \phi_{2,\mathrm{RS}}(x,t)
\end{aligned} \tag{115}$$

In the last step we used the consequence of the energy conservation $E = \sum_i \cosh\theta_i$ that the asymptotic rapidities can be applied here.

Now, we do the calculation for the soliton – soliton solution (96) for $\beta = 1$ ($\gamma = \varepsilon = \cosh\theta_\infty = 1/\sqrt{1-v^2}$).

$$\phi_{\mathrm{ss}}(x,t) = 4\arctan\left[\frac{v\sinh(x\gamma)}{\cosh(vt\gamma)}\right] = 4\arctan\left[\tanh\theta_\infty\frac{\sinh(\cosh\theta_\infty\cdot x)}{\cosh(\sinh\theta_\infty\cdot t)}\right], \tag{116}$$

which is exactly the aformentioned solution (up to a shift with with a period in the argument of the arctan) due to the following indentity

$$\arctan(x) + \arctan(1/x) = \frac{\pi}{2}\mathrm{sign}(x). \tag{117}$$

For visualizing our results, we plotted (108) and (116) in Figure 14. One can identify, that those two graphs describe the same field configuration, but we have to mention two aspects. 1) The 2-particle Ruijsenaars–Schneider curve (orange) is shifted upwards with

2π with respect to the soliton – soliton sine-Gordon solution and 2) it also has a cut (at least visually). The explanation of the 1) is that if $\phi(x,t)$ is a solution, then $\phi(x,t)+2\pi n$ is also a solution of the sine-Gordon equation of motion ($n \in \mathbb{Z}$), therefore the plotted graphs are equivalent up to $2\pi n$. The reason for 2) is actually coming from the periodicity of the tan function because the 2-particle Ruijsenaars–Schneider curve actually crosses $\pi/2$ at $x = 0$, therefore for $x > 0$ it is in the new period. (We used Wolfram Mathematica, where the periodicity behaviour is implemented in this way into it, otherwise we would be able to get a nice continuous function for the 2-particle Ruijsenaars–Schneider curve as well.)

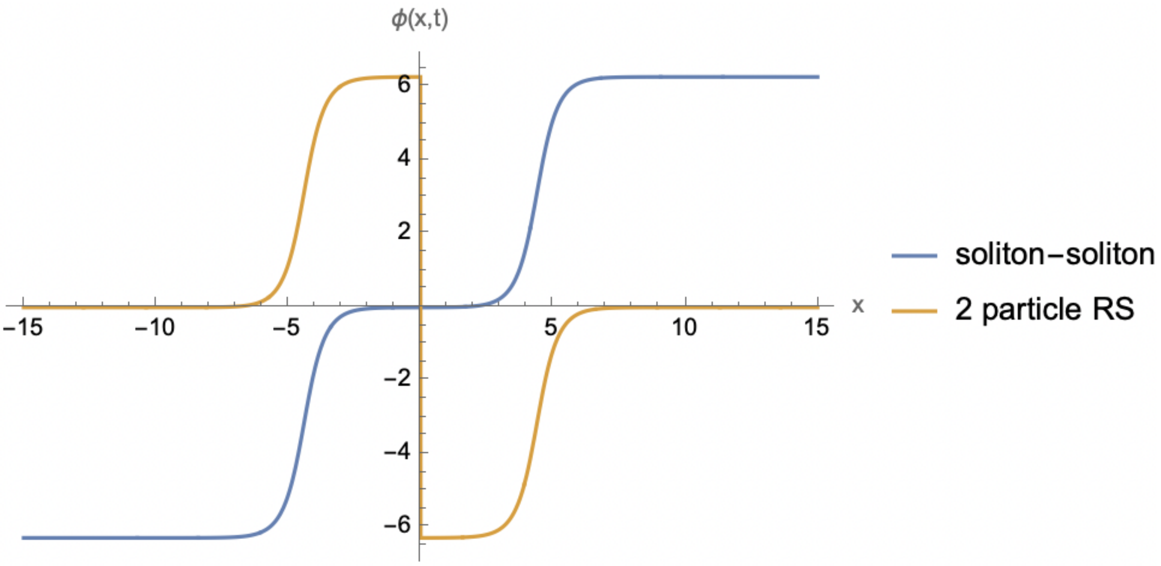


Figure 14: 2-particle Ruijsenaars–Schneider and the soliton – soliton sine-Gordon solutions ($\varepsilon = 2$, $t = 5$).

Moreover, we prove here the equality of the time advances.

$$\Delta t_{\mathrm{RS}} = -\frac{1}{2\nu\sqrt{\varepsilon^2-1}}\ln\left(\frac{\alpha^2}{\varepsilon^2-1}+1\right) = -\frac{1}{\sqrt{\varepsilon^2-1}}\ln\left(\frac{\varepsilon^2}{\varepsilon^2-1}\right) \tag{118}$$

$$\Delta t_{\mathrm{ss}} = 2\frac{\ln v}{v\gamma} = \frac{\ln v^2}{v\varepsilon} = \frac{\ln\left(\frac{\varepsilon^2-1}{\varepsilon^2}\right)}{\sqrt{\varepsilon^2-1}} = -\frac{1}{\sqrt{\varepsilon^2-1}}\ln\left(\frac{\varepsilon^2}{\varepsilon^2-1}\right) \tag{119}$$

6 Conclusion

In this thesis we studied the integrable Ruijsenaars–Schneider model and the integrable sine-Gordon field theory both in the classical context and in infinite volume. In a $1+1$ dimensional relativistically invariant theory there are 3 generators of the Poincaré group that are identified as the generators of time translations, space translations and boosts, respectively. They satisfy certain Poisson commutativity relations. In order to use the most general form for these generators we used the Ansatz proposed by Ruijsenaars and Schneider, which incorporates interparticle interaction on the phase space, therefore the usual transformation from the physical space coordinates and momenta to the appropriate canonical ones is not a trivial task to carry out. It needs to be treated by studying the trajectories on the phase space and ultimately to reconstruct them in the physical space.

We studied the proposed Ansatz generally by looking at its limits, properties and integrable nature, as well. For 2 particles, the exact solution of the theory was obtained in the hyperbolic limit by providing analytical formulas for the canonical variables' time and energy dependence and reconstructing the trajectory in the physical space. The latter made possible to calculate the time advance for the $2 \to 2$ scattering process.

As a next step we paid our attention to the classical sine-Gordon theory starting from the perspective of integrability. The equation of motion of the theory is such that the spectrum can be a 1-particle spectrum or multiparticle spectrum as well, and there are 2 topologically inequivalent solutions (soliton and anti-soliton) and their bound states (breathers). We calculated the time advance for the "soliton, soliton $\to$ soliton, soliton" $2 \to 2$ scattering process. Lastly we mentioned the general multiparticle solution for the sine-Gordon equation of motion through Hirota's formula and the time shift in a multiparticle system.

Providing general meaning to the thesis we showed that these two theories are strongly related and we highlighted this connection in the 2-particle sectors by showing that an appropriately chosen combination of the Ruijsenaars–Schneider phase space trajectories satisfy the sine-Gordon equation of motion, more precisely it equals to the field configuration of its soliton – soliton solution. As a last step we proved that the time advances in the two cases are the same.

There is also a correspondence for the soliton – anti-soliton subspace of the 2-particle sine-Gordon solutions, but the attractive nature of the interaction requires a same type in the Ruijsenaars–Schneider case. Its study can be found in Appendix B.

As an outlook we plan to study the correspondence in the quantum level and in finite volume as well. There are further questions about these directions, most importantly the relation between the finite size energy corrections, and the correspondence between them if it even exist.

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A Appendix

A.1 Weierstrass $\wp$-function

The $\wp$-function and its derivative is a building block in the theory of elliptic curves because we can parametrize those ones with them in a given lattice. We define the $\wp$-function on $\mathbb{C}$ as

$$\wp(z, \omega_1, \omega_2) \equiv \wp(z, \Lambda) = \frac{1}{z^2} + \sum_{\lambda \in \Lambda \setminus \{0\}} \left(\frac{1}{(z-\lambda)^2} - \frac{1}{\lambda^2} \right), \tag{120}$$

where $\omega_1, \omega_2 \in \mathbb{C}$ complex numbers that are linearly independent over $\mathbb{R}$ and generate the lattice $\Lambda := \omega_1\mathbb{Z} + \omega_2\mathbb{Z} := \{m\omega_1 + n\omega_2 | m, n \in \mathbb{Z}\}$. The ω_1, ω_2 numbers are the periods of this doubly periodic function. One can see that this function has poles, each of them placed in the lattice points and have order of two. Let's see its properties (*Weierstrass elliptic function*, 2023)!

Evenness:

$$\wp(-z) = \wp(z). \tag{121}$$

Double periodicity:

$$\wp(z + \omega_1) = \wp(z) = \wp(z + \omega_2). \tag{122}$$

Laurent expansion for $0 < |z| < r = \min\{|\lambda| : \lambda \in \Lambda \setminus \{0\}\}$:

$$\wp(z) = \frac{1}{z^2} + \sum_{n=1}^{\infty} (2n+1) G_{2n+2} z^{2n}, \quad \text{where} \quad G_n = \sum_{\lambda \in \Lambda \setminus \{0\}} \lambda^{-n}. \tag{123}$$

Differential equation:

$$\wp'^2(z) = 4\wp^3(z) - g_2\wp(z) - g_3, \quad \text{where} \quad g_2 = 60G_4 \quad \text{and} \quad g_3 = 140G_6. \tag{124}$$

Functional (differential) equation (S. N. Ruijsenaars & Schneider, 1986):

$$\begin{vmatrix} f^2(u) & f(u)f'(u) & 1 \\ f^2(v) & f(v)f'(v) & 1 \\ f^2(u+v) & -f(u+v)f'(u+v) & 1 \end{vmatrix} = 0 \text{ is satisfied when } f^2(x) = A + B\wp(x), \tag{125}$$

with $A, B \in \mathbb{C}$.

Let us set $A = 1$ and our lattice in such a way that its directions be parallel to the real and imaginary axis, therefore one of the ω parameters is real, and the other is purely imaginary: $(\omega_1,\omega_2) = (\Omega_1, i\Omega_2)$, where $\Omega_1,\Omega_2 \in \mathbb{R}$. In this setup, the following limits are achieved, where the appearing $\pm\nu^2/3$ constants can be omitted to A with a redefinition.

$$\lim_{\Omega_2\to\infty} \wp(z,\pi/\nu, i\Omega_2) = \frac{\nu^2}{\sin^2(\nu z)} - \frac{\nu^2}{3} \tag{126}$$

$$\lim_{\Omega_1\to\infty} \wp(z,\Omega_1, i\pi/\nu) = \frac{\nu^2}{\sinh^2(\nu z)} + \frac{\nu^2}{3} \tag{127}$$

$$\lim_{\Omega_1,\Omega_2\to\infty} \wp(z,\Omega_1, i\Omega_2) = \frac{1}{z^2} \tag{128}$$

And the ν parameter comes from the periodicity of the $\wp$-function, in other words the denseness of the lattice what we are working on.

A.2 Integrable potentials

In a relativistically invariant field theory (in classical context and 1+1 dimensions) with Lagrangian density

$$\mathcal{L} = \frac{1}{2}(\partial_t\phi)^2 - \frac{1}{2}(\partial_x\phi)^2 - V(\phi) = \frac{1}{2}(\partial_\mu\phi)^2 - V(\phi) \tag{129}$$

it is an important question that what are the potentials V that correspond to an integrable theory. For integrability we need infinitely many functionally independent conserved charges, therefore we aim for the conserved charges and this will give a restriction for the potential.

For providing manifestly Lorentz covariant formulas, it is convenient to work with light cone coordinates, instead of space and time.

$$x_\pm = \frac{1}{2}(t \pm x) \tag{130}$$

$$\partial_\pm = \partial_t \pm \partial_x \tag{131}$$

We define the spin of a conserved charge via its transformation as (for a spin s conserved charge)

$$Q_s \to \mathrm{e}^{s\Theta} Q_s, \tag{132}$$

where Θ is the rapidity of the transformation. The sign of the spin is basically determined by the sign of the light cone derivative we use. Noether's theorem implies conserved currents J^μ

$$\partial_\mu J^\mu = \partial_t J^t - \partial_x J^x = 0. \tag{133}$$

In terms of the energy (E) and momentum (P) as conserved quantities we can define spin ± 1 conserved charges as

$$Q_{\pm 1} = \frac{1}{2}(E \pm P). \tag{134}$$

Let's start to construct the infinitely many conserved charges. For $V(\phi) = 0$, the equation of motion looks

$$-\partial_t^2\phi + \partial_x^2\phi - V'(\phi) = -\partial_t^2\phi + \partial_x^2\phi = -\partial_+\partial_-\phi = 0. \tag{135}$$

From this form we can immeadiately read infinitely many adequate conserved currents, because the lightcone derivatives are independent

$$J_- = (\partial_-^{a_1}\phi)^{b_1}...(\partial_-^{a_n}\phi)^{b_n}, \qquad J_+ = 0, \tag{136}$$

and any differential polynomial of ϕ is an adequate choice ($a_i, b_j \in \mathbb{N}\ \forall i, j$). Our conserved charges are

$$Q = \int J_t \mathrm{d}x. \tag{137}$$

For swapping the role of $\pm$ we also have infinitely many conserved currents and charges. (The boundary condition we used was that the fields should decay at spatial infinities.)

Let's see the case, where the potential is nonzero! Here, the aforementioned strategy does not work because $\partial_\pm$ are not independent due to the equation of motion (if the potential is non-constant). In this case one should go through the following protocol: one has to separately study whether there are any conserved currents built by a given number of derivatives or not. We use the nomenclature level n for the case of $n+1$ derivatives in terms of the current, meaning n derivatives related to the charge.

It turns out that at level 1 there are two conserved charges, the energy and the momentum as we already mentioned via (134). At level 2 there are no conserved charges, however at level 3 there are two candidates $(\partial_\pm\phi)^4$ and $(\partial_\pm^2\phi)^2$, whose linear combination with a parameter $\alpha \in \mathbb{R}$ provides a restriction only for the potential itself (Bajnok & Šamaj, 2011).

$$V''(\phi) = \alpha V(\phi) \tag{138}$$

This restriction is equivalent to the one we mentioned for the most general form for the potential, when we have a spin-3 conserved charge (88). The spin-3 conserved charge provides the sine-Gordon, sinh-Gordon and Klein-Gordon theories, that are integrable. As a conclusion, the existence of a spin-3 conserved charge makes the theory integrable.

One can study higher spin cases, there are also integrable theories present, for instance the aformentioned Bullough–Dodd model.

A.3 Bäcklund transformation

The Bäcklund transformation is a really useful tool to generate new solution of the sine-Gordon equation of motion in the line. In section 4 we briefly discussed the qualitative meaning of the transformation but we would like to give a more detailed discussion on that with its mathematical formulation.

Generally speaking, Bäcklund transformation is typically a system of first order partial differential equations for two functions and the equations can contain an additional parameter (*Bäcklund transform*, 2022).

For the Bäcklund transformation in the sine-Gordon theory we have to have a solution of the sine-Gordon equation of motion ϕ_l, meaning ($m = 1$)

$$-\partial_+\partial_-\phi_l = \frac{1}{\beta}\sin(\beta\phi_l), \tag{139}$$

where the light cone derivatives are introduced in (131). The transformation states that if a field ϕ_r satisfies the conditions

$$\partial_+\phi_l + \partial_+\phi_r = \frac{2\sigma}{\beta}\sin\left[\frac{\beta}{2}(\phi_l - \phi_r)\right], \tag{140}$$

$$\partial_-\phi_l - \partial_-\phi_r = -\frac{2}{\beta\sigma}\sin\left[\frac{\beta}{2}(\phi_l + \phi_r)\right], \tag{141}$$

then this field also solves the sine-Gordon equation of motion (Fring et al., 1993). The appearing quantity σ is a free real parameter. For providing visual meaning we have to define what an integrable defect is. Defect is such an internal point of the physical space of the theory that separates bulk parts and there fields might have jumps but certain properties can remain, for example the integrable nature of the theory can be such.

One can visualize the process as there is a solution on the left side ($x < 0$ part of the line) and an integrable defect sits at $x = 0$, then the defect makes it possible, due to its integrable nature, to have also a solution on the right part of the line ($x > 0$) and the defect conditions are (140-141).

The advantage of the transformation is that it gives two inhomogenous but first order differential equations instead of one inhomogenous second order one (equation of motion).

Further property of the transformation is that we can prove the integrability of the theory, namely the existence of infinitely many conserved charges. Let us use the mentioned setup with the integrable defect sitting at $x = 0$. The construction requires the expansion of ϕ_r around ϕ_l in powers of the free parameter σ as

$$\phi_r = \phi_l + \sum_{n=1}^{\infty} \sigma^n \phi^{(n)}. \tag{142}$$

Plugging this expression back to the defect conditions and expanding in powers of σ we obtain infinitely many independent conservation laws.

B Appendix: Soliton – anti-soliton case

We did not want to be too lengthy and more or less redundant in the core of this thesis, therefore we present the examination of the soliton – anti-soliton case, or in the Ruijsenaars–Schneider language the hyperbolic case with attractive potential here. The philosophy of the calculations is almost identical to the one presented before, therefore we only highlight the points where the differences appear. The used potential is locally attractive, therefore bound states (breathers) can be formed. Its analysis is similar to the one we presented before but the difference is that the energy of the bound state is below the potential and the state is confined into it, manifesting through a periodic oscillation in time between the classical turning points.

B.1 Ruijsenaars–Schneider solution

The attractive potential takes the form

$$f(q) = \left(1 - \frac{\alpha^2}{\cosh^2(\nu q)}\right)^{1/2}. \tag{143}$$

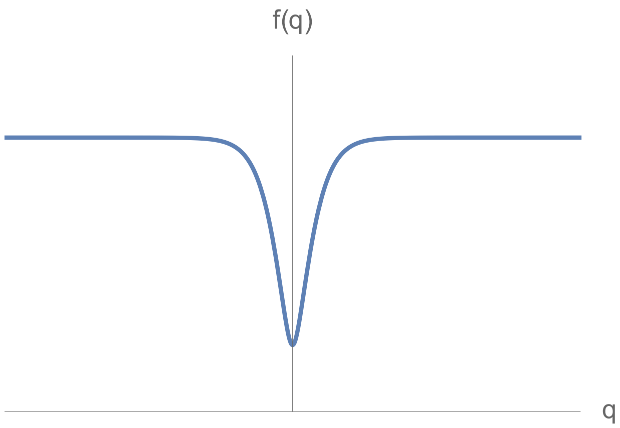

Figure 15: The attractive hyperbolic potential.

The equations of motion are

$$\dot{q} = 2\sinh(\theta)\, f(q) \implies \dot{q}(t) = 2\sinh(\theta(t))\sqrt{1 - \frac{\alpha^2}{\cosh^2(\nu q(t))}}, \tag{144}$$

$$\dot{\theta} = -2\cosh(\theta)\, f'(q) \implies -2\nu\alpha^2\cosh(\theta(t))\frac{\sinh(\nu q(t))}{\cosh^2(\nu q(t))\sqrt{\cosh^2(\nu q(t)) - \alpha^2}}. \tag{145}$$

The energy is conserved.

$$\frac{\mathcal{H}}{2} = \cosh(\theta)\, f(q) = \varepsilon = \cosh\theta_\infty \tag{146}$$

We show the phase space trajectories for various energy ($H_0 = 2\varepsilon$) and interaction strength (α) values in Figure 16. and 17. according to (63). It is clearly seen on the figures mentioned that for smaller phase space distance the rapidity is higher, so the interaction is attractive.

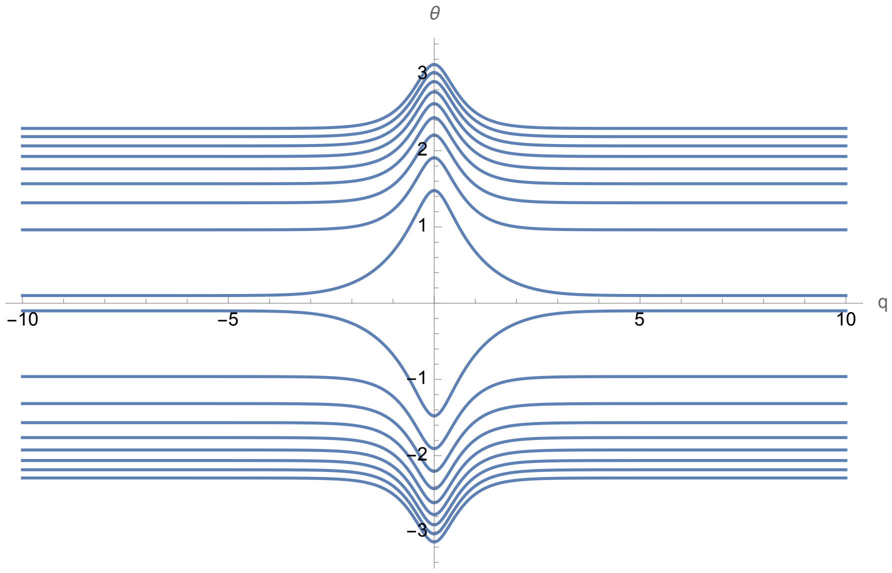

Figure 16: Phase space trajectories for $H_0 \in \{2.01, 3, 4, 5, 6, 7, 8, 9, 10\}$ with $\alpha = 1$ and $\nu = 1$.

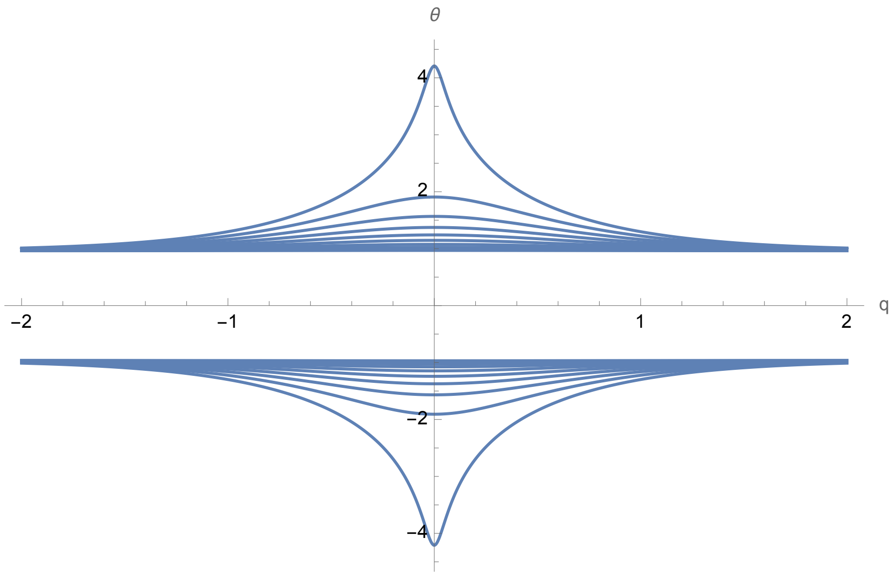

Figure 17: Phase space trajectories for $\alpha \in \{0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9, 0.999\}$ with $H_0 = 3$ and $\nu = 1$.

We would like to solve the equation of motion, which take the form (144). Similarly to (69)

$$\frac{\mathrm{d}q}{\mathrm{d}t} = 2\sqrt{\varepsilon^2 - f^2(q)}. \tag{147}$$

Integrating the equation we can also define here the function g_ε but now we have to apply different initial condition: let $q(t=0)=0$. (In this case the particle scatter through each other, therefore it is a convenient choice.)

$$g_\varepsilon(q) := \int_0^q \frac{\mathrm{d}q'}{2\sqrt{\varepsilon^2 - f^2(q)}} = \int_0^t \mathrm{d}t' = t \tag{148}$$

Let's calculate $g_\varepsilon(q)$!

$$\begin{aligned} g_\varepsilon(q) &= \frac{1}{2}\int_0^q \frac{\mathrm{d}q'}{\sqrt{\varepsilon^2 - 1 + \frac{\alpha^2}{\cosh^2(\nu q)}}} = \frac{1}{2\nu}\int_0^{\sinh(\nu q)} \frac{\mathrm{d}y}{\sqrt{(\alpha^2+\varepsilon^2-1)+(\varepsilon^2-1)y^2}} \\ &= \frac{1}{2\nu\sqrt{\varepsilon^2-1}}\int_0^{\sinh(\nu q)\sqrt{\frac{\varepsilon^2-1}{\alpha^2+\varepsilon^2-1}}} \frac{\mathrm{d}z}{\sqrt{1+z^2}} \\ &= \frac{1}{2\nu\sqrt{\varepsilon^2-1}}\mathrm{arsinh}\left[\sqrt{\frac{\varepsilon^2-1}{\alpha^2+\varepsilon^2-1}}\sinh(\nu q)\right] = t \end{aligned} \tag{149}$$

From there the time dependence of the phase space coordinate follows

$$q(t;\varepsilon,\alpha,\nu) = \frac{1}{\nu}\mathrm{arsinh}\left[\sqrt{\frac{\alpha^2+\varepsilon^2-1}{\varepsilon^2-1}}\sinh\left(2\nu\sqrt{\varepsilon^2-1}t\right)\right] \tag{150}$$

and the rapidity from the equation of motion (144) is

$$\begin{aligned} \sinh\theta(t;\varepsilon,\alpha,\nu) = & \sqrt{\frac{(\varepsilon^2-1)(\alpha^2+\varepsilon^2-1)}{(1-\alpha^2)(\varepsilon^2-1)+(\alpha^2+\varepsilon^2-1)\sinh^2\left(2\nu\sqrt{\varepsilon^2-1}t\right)}} \\ & \times \ \cosh\left(2\nu\sqrt{\varepsilon^2-1}t\right). \end{aligned} \tag{151}$$

The asymptotic/free rapidity is the same as in the previously studied case.

$$\theta_\infty = \lim_{t\to\infty}\theta(t;\varepsilon,\alpha,\nu) = \ln\left(\varepsilon + \sqrt{\varepsilon^2-1}\right) \tag{152}$$

It is worthwhile to compare these results to the ones in the repulsive case (73,74). There are two differences.

- The role of sinh and cosh is interchanged.
- From $\alpha^2 - 1$ the new rapidity formula contains $1-\alpha^2$ due to the fact that in this case $\alpha^2 \leq 1$.

Another difference is that in this case there is no turning point, the particles do not reflect from each other, they scatter through each other as we can see in Figure 13. for the soliton – anti-soliton scattering.

The physical space trajectories follow from (79,80).

The time shift is coming from the asymptotes, which can be calculated from the asymptotic expansion of (150) at leading order using the transformation between the physical space coordinates and the phase space coordinates (79,80).

For $t \to +\infty$ (and $x \to +\infty$, as well) at leading order we get

$$\mathrm{e}^{2\nu\varepsilon x} = \mathrm{e}^{\ln\sqrt{\frac{\alpha^2+\varepsilon^2-1}{\varepsilon^2-1}}}\mathrm{e}^{2\nu\sqrt{\varepsilon^2-1}t} \Longrightarrow x(t) = \frac{1}{2\nu\varepsilon}\ln\sqrt{\frac{\alpha^2+\varepsilon^2-1}{\varepsilon^2-1}} + \frac{1}{\varepsilon}\sqrt{\varepsilon^2-1}t. \tag{153}$$

For $t \to -\infty$ (and $x \to -\infty$ the other particle) at leading order we get

$$x(t) = -\frac{1}{2\nu\varepsilon}\ln\sqrt{\frac{\alpha^2+\varepsilon^2-1}{\varepsilon^2-1}} + \frac{1}{\varepsilon}\sqrt{\varepsilon^2-1}t. \tag{154}$$

From the $x(t) = 0$ intersections we can easily read the time shift (time advance in this case), which is in the COM frame

$$\Delta t(\varepsilon;\alpha,\nu) = -\frac{1}{2\nu\sqrt{\varepsilon^2-1}}\ln\left(\frac{\alpha^2}{\varepsilon^2-1}+1\right). \tag{155}$$

We get exactly same as in the soliton – soliton case. We visualize such scattering process in Figure 18.

If we define the field that corresponds to the Ruijsenaars–Schneider particles in the way Ruijsenaars and Schneider did with (105) the theory cannot be mapped to the sine-Gordon theory in the case of the soliton – anti-soliton scattering. The proper way to introduce this quantity incorporates the type of the interaction. Namely, for the 2-particle case it is

$$\phi(x,t) = 4\arctan\mathrm{e}^{u_1(x,t)} - 4\arctan\mathrm{e}^{u_2(x,t)}. \tag{156}$$

Plugging (106) into this form, taking $\alpha = 1$ and $\nu = 1/2$ and using (146, 150) we get

$$\phi_{2,\mathrm{RS}}(x,t) = 4\arctan\left[\frac{1}{\tanh\theta_\infty}\frac{\sinh(\sinh\theta_\infty\cdot t)}{\cosh(\cosh\theta_\infty\cdot x)}\right]. \tag{157}$$

B.2 sine-Gordon field configuration

The τ soliton – anti-soliton field according to (104) follows as

$$\tau_2^{\mathrm{s}\bar{\mathrm{s}}}(x,t) = 1 + iE_1 - iE_2 + E_1E_2\tanh^2\left(\frac{\theta_1-\theta_2}{2}\right) \tag{158}$$

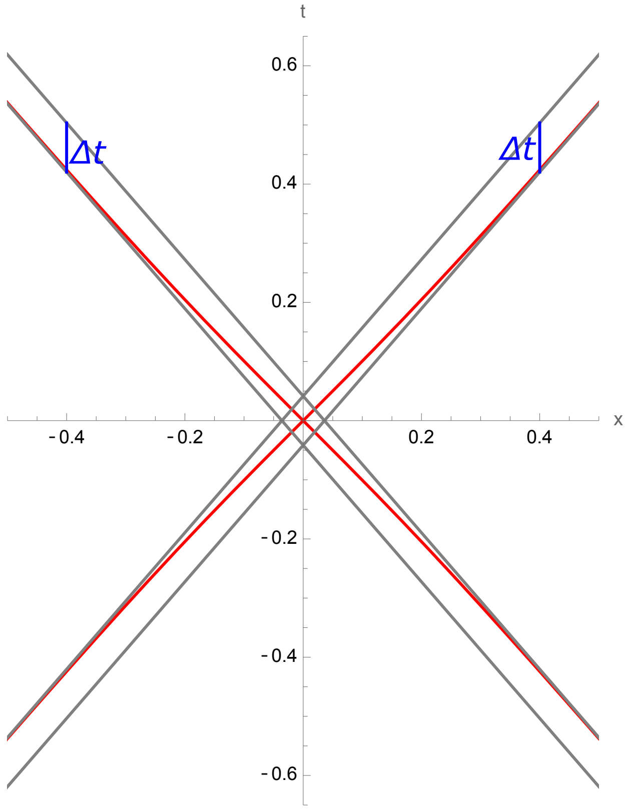

Figure 18: Space-time trajectories of 2 particle scattering in the presence of attractive interaction with the asymptotes, that correspond to free motions with slope of the inverse of the asymptotic velocity. The time advance Δt suffered by the particles is shown in the figure, as well. (Used parameter values: $\varepsilon = 2$, $\alpha = 1$, $\nu = 1$.)

and the Hirota field configuration (103) with $\beta = 1$ as

$$\phi^{\mathrm{ss}}_{2,\mathrm{Hirota}}(x,t) = 4 \arctan\left[\frac{\mathrm{e}^{s_1(x,t)} - \mathrm{e}^{s_2(x,t)}}{1 + \mathrm{e}^{s_1(x,t)+s_2(x,t)+\Delta s}}\right]. \tag{159}$$

Choosing $x_1 = x_2 = \Delta s/2 = \ln \tanh \theta$ in (111) we get the same result as in the Ruijsenaars–Schneider case in (157).

Finally we show that the soliton – anti-soliton 2-particle sine-Gordon solution gives the same exact result as the previous two using (97).

$$\phi_{\mathrm{s\bar{s}}}(x,t) = \frac{4}{\beta} \arctan\left[\frac{\sinh(vt\gamma)}{v\cosh(x\gamma)}\right] = 4 \arctan\left[\frac{1}{\tanh\theta_\infty}\frac{\sinh(\sinh\theta_\infty \cdot t)}{\cosh(\cosh\theta_\infty \cdot x)}\right], \tag{160}$$

where we just simply used $\gamma = \varepsilon = \cosh\theta_\infty = 1/\sqrt{1-v^2}$ and $\beta = 1$.

Last but not least, it is trivial to show that the time shift for the soliton – anti-soliton system is also $\Delta t = 2\frac{\ln v}{v\gamma}$ as for the soliton – soliton case in (101) and this is exactly the same as the Ruijsenaars–Schneider one.