Nondimensionalization of Haseloff et al. 2018

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1 Nondimensionalization of Velocity Equations

Nondimensionalizing the equations in terms of dimensional scales (this section will not yet try write the scales in terms of parameters)
Here I use:

$$\begin{gathered} y = [z]\,Y \qquad z = [z]\,Z \\ u = [u]\,U \qquad v = [v]\,V \qquad w = [v]\,W \\ \eta = [\eta]\,\mu \qquad p = [p]\,P \qquad s' = [s']\,S' \end{gathered}$$

The convention I used here is to provide the equation number when it matches the same equation in Haseloff's paper. If the equation doesn't match an equation in Haseloff, but I plan to resolve the differences later in the notes I will denote that with primes.

1.1 $2 \rightarrow 22$

$$\frac{\partial}{\partial y}\left(\eta\frac{\partial u}{\partial y}\right) + \frac{\partial}{\partial z}\left(\eta\frac{\partial u}{\partial z}\right) = 0 \tag{2}$$

$$\frac{1}{[z]}\frac{\partial}{\partial Y}\left([\eta]\,\mu\frac{[u]}{[z]}\frac{\partial U}{\partial Y}\right) + \frac{1}{[z]}\frac{\partial}{\partial Z}\left([\eta]\,\mu\frac{[u]}{[z]}\frac{\partial U}{\partial Z}\right) = 0$$

$$\frac{[\eta]\,[u]}{[z]^2}\left[\frac{\partial}{\partial Y}\left(\mu\frac{\partial U}{\partial Y}\right) + \frac{\partial}{\partial Z}\left(\mu\frac{\partial U}{\partial Z}\right)\right] = 0$$

$$\frac{\partial}{\partial Y}\left(\mu\frac{\partial U}{\partial Y}\right) + \frac{\partial}{\partial Z}\left(\mu\frac{\partial U}{\partial Z}\right) = 0 \tag{22}$$

1.2 3a $\rightarrow$ 23a

$$\frac{\partial}{\partial y}\left(2\eta\frac{\partial v}{\partial y}\right) + \frac{\partial}{\partial z}\left[\eta\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right)\right] - \frac{\partial p}{\partial y} = 0 \tag{3a}$$

$$\frac{1}{[z]}\frac{\partial}{\partial Y}\left(2\,[\eta]\,\mu\frac{[v]}{[z]}\frac{\partial V}{\partial Y}\right) + \frac{1}{[z]}\frac{\partial}{\partial Z}\left[[\eta]\,\mu\frac{[v]}{[z]}\left(\frac{\partial V}{\partial Z} + \frac{\partial W}{\partial Y}\right)\right] - \frac{[p]}{[z]}\frac{\partial P}{\partial Y} = 0$$

$$\frac{[\eta]\,[v]}{[z]^2}\frac{\partial}{\partial Y}\left(2\mu\frac{\partial V}{\partial Y}\right) + \frac{[\eta]\,[v]}{[z]^2}\frac{\partial}{\partial Z}\left[\mu\left(\frac{\partial V}{\partial Z} + \frac{\partial W}{\partial Y}\right)\right] - \frac{[p]}{[z]}\frac{\partial P}{\partial Y} = 0 \tag{3a$'$}$$

1.3 3b → 23b

$$\frac{\partial}{\partial y}\left[\eta\left(\frac{\partial v}{\partial z}+\frac{\partial w}{\partial y}\right)\right]+\frac{\partial}{\partial z}\left(2\eta\frac{\partial w}{\partial z}\right)-\frac{\partial p}{\partial z}=0 \tag{3b}$$

$$\frac{1}{[z]}\frac{\partial}{\partial Y}\left[[\eta]\,\mu\frac{[v]}{[z]}\left(\frac{\partial V}{\partial Z}+\frac{\partial W}{\partial Y}\right)\right]+\frac{1}{[z]}\frac{\partial}{\partial Z}\left(2\,[\eta]\,\mu\frac{[v]}{[z]}\frac{\partial W}{\partial Z}\right)-\frac{[p]}{[z]}\frac{\partial P}{\partial Z}=0$$

$$\frac{[\eta]\,[v]}{[z]^2}\frac{\partial}{\partial Y}\left[\mu\left(\frac{\partial V}{\partial Z}+\frac{\partial W}{\partial Y}\right)\right]+\frac{[\eta]\,[v]}{[z]^2}\frac{\partial}{\partial Z}\left(2\mu\frac{\partial W}{\partial Z}\right)-\frac{[p]}{[z]}\frac{\partial P}{\partial Z}=0 \tag{3b$'$}$$

1.4 3c → 23c

$$\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0 \tag{3c}$$

$$\frac{[v]}{[z]}\frac{\partial V}{\partial Y}+\frac{[v]}{[z]}\frac{\partial W}{\partial Z}=0$$

$$\frac{\partial V}{\partial Y}+\frac{\partial W}{\partial Z}=0 \tag{23c}$$

1.5 4 → 24

$$\eta=\frac{A^{-1/n}}{2^{1/n}}\left[\left|\frac{\partial u}{\partial y}\right|^2+\left|\frac{\partial u}{\partial z}\right|^2+\left|\frac{\partial v}{\partial z}+\frac{\partial w}{\partial y}\right|^2+2\left|\frac{\partial v}{\partial y}\right|^2+2\left|\frac{\partial w}{\partial z}\right|^2\right]^{\frac{1-n}{2n}} \tag{4}$$

$$\frac{A^{-1/n}}{2^{1/n}}\left[\frac{[u]^2}{[z]^2}\left(\left|\frac{\partial U}{\partial Y}\right|^2+\left|\frac{\partial U}{\partial Z}\right|^2\right)+\frac{[v]^2}{[z]^2}\left(\left|\frac{\partial V}{\partial Z}+\frac{\partial W}{\partial Y}\right|^2+2\left|\frac{\partial V}{\partial Y}\right|^2+2\left|\frac{\partial W}{\partial Z}\right|^2\right)\right]^{\frac{1-n}{2n}}$$

$$\frac{A^{-1/n}}{2^{1/n}}\left(\frac{[u]}{[z]}\right)^{\frac{1-n}{n}}\left[\left|\frac{\partial U}{\partial Y}\right|^2+\left|\frac{\partial U}{\partial Z}\right|^2+\frac{[v]^2}{[u]^2}\left(\left|\frac{\partial V}{\partial Z}+\frac{\partial W}{\partial Y}\right|^2+2\left|\frac{\partial V}{\partial Y}\right|^2+2\left|\frac{\partial W}{\partial Z}\right|^2\right)\right]^{\frac{1-n}{2n}} \tag{4$'$}$$

1.6 5 → 25

Boundary condition for: $y \to \infty$
Becomes the boundary condition for: $Y \to \infty$
This equation can be broken into three parts:

1.6.1 u

$$\eta\frac{\partial u}{\partial y}\to\tau_s \tag{5a}$$

$$[\eta]\,\mu\frac{[u]}{[z]}\frac{\partial U}{\partial Y}\to\tau_s$$

$$\mu\frac{\partial U}{\partial Y}\to\frac{\tau_s\,[z]}{[u]\,[\eta]} \tag{5a$'$}$$

1.6.2 v

$$\frac{\partial v}{\partial z}\to 0 \tag{5b}$$

$$\frac{\partial V}{\partial Z}\to 0 \tag{25b}$$

1.6.3 w

$$w \to 0 \tag{5c}$$
$$W \to 0 \tag{25c}$$

1.7 $6 \to 26$

Boundary condition for: $y \to -\infty$
Becomes the boundary condition for: $Y \to -\infty$
This equation can be broken into three parts:

1.7.1 u

$$u \to 0 \tag{6a}$$
$$U \to 0 \tag{26a}$$

1.7.2 v

$$v \to \frac{n+2}{n+1}\frac{q_r}{h_s}\left[1-\left(1-\frac{z}{h_s}\right)^{n+1}\right] \tag{6b}$$
$$v \to \frac{n+2}{n+1}\frac{q_r}{h_s}\left[1-\left(1-\frac{[z]\,Z}{h_s}\right)^{n+1}\right] \tag{6b'}$$

1.7.3 w

$$w \to 0 \tag{6c}$$
$$W \to 0 \tag{26c}$$

1.8 $7 \to 28$

Boundary condition for: $z = 0$
Becomes the boundary condition for: $Z = 0$

$$w = 0 \text{ at } z = 0 \tag{7}$$
$$W = 0 \text{ at } Z = 0 \tag{28}$$

1.9 $8 \to 29$

Boundary condition for: $y > 0,\ z = 0$
Becomes the boundary condition for: $Y > 0,\ Z = 0$
This equation can be broken into two parts:

1.9.1 u

$$\eta\frac{\partial u}{\partial z} = 0 \tag{8a}$$
$$[\eta]\,\mu\frac{[u]}{[z]}\frac{\partial U}{\partial Z} = 0$$
$$\mu\frac{\partial U}{\partial Z} = 0 \tag{29a}$$

1.9.2 v

$$\eta \frac{\partial v}{\partial z} = 0 \tag{8b}$$

$$[\eta]\, \mu \frac{[v]}{[z]} \frac{\partial V}{\partial Z} = 0$$

$$\mu \frac{\partial V}{\partial Z} = 0 \tag{29b}$$

1.10 9a $\rightarrow$ 30a

No slip boundary condition for: $y < 0,\ z = 0$
Becomes the no slip boundary condition for: $Y < 0,\ Z = 0$

$$u = v = 0 \tag{9a}$$

$$U = V = 0 \tag{30a}$$

1.11 9b $\rightarrow$ 30b

Subtemperate slip boundary condition for: $y < 0,\ z = 0$
Becomes the subtemperate slip boundary condition for: $Y < 0,\ Z = 0$ This equation can be broken into three parts:

1.11.1 $\sqrt{u^2 + v^2} > 0$, u

$$\eta \frac{\partial u}{\partial z} = \tau_c \frac{u}{\sqrt{u^2 + v^2}} \tag{9b,I}$$

$$[\eta]\, \mu \frac{[u]}{[z]} \frac{\partial U}{\partial Z} = \tau_c \frac{U}{\sqrt{U^2 + \frac{[v]^2}{[u]^2} V^2}}$$

$$\mu \frac{\partial U}{\partial Z} = \frac{[z]\, \tau_c}{[u]\, [\eta]} \frac{U}{\sqrt{U^2 + \frac{[v]^2}{[u]^2} V^2}} \tag{9b',I}$$

1.11.2 $\sqrt{u^2 + v^2} > 0$, v

$$\eta \frac{\partial v}{\partial z} = \tau_c \frac{v}{\sqrt{u^2 + v^2}} \tag{9b,II}$$

$$[\eta]\, \mu \frac{[v]}{[z]} \frac{\partial V}{\partial Z} = \tau_c \frac{[v]}{[u]} \frac{V}{\sqrt{U^2 + \frac{[v]^2}{[u]^2} V^2}}$$

$$\mu \frac{\partial V}{\partial Z} = \frac{[z]\, \tau_c}{[u]\, [\eta]} \frac{V}{\sqrt{U^2 + \frac{[v]^2}{[u]^2} V^2}} \tag{9b',II}$$

1.11.3 $\sqrt{u^2+v^2}=0$

$$\sqrt{\left(\eta\frac{\partial u}{\partial z}\right)^2+\left(\eta\frac{\partial v}{\partial z}\right)^2}\leq\tau_c \tag{9b,III}$$

$$\frac{[\eta]\,[u]}{[z]}\sqrt{\left(\mu\frac{\partial U}{\partial Z}\right)^2+\left(\mu\frac{[v]}{[u]}\frac{\partial V}{\partial Z}\right)^2}\leq\tau_c$$

$$\sqrt{\left(\mu\frac{\partial U}{\partial Z}\right)^2+\frac{[v]^2}{[u]^2}\left(\mu\frac{\partial V}{\partial Z}\right)^2}\leq\frac{[z]\,\tau_c}{[u]\,[\eta]} \tag{9b',III}$$

1.12 10 $\rightarrow$ 27

Boundary condition for: $z=h_s$
Becomes the boundary condition for: $Z=1$
This equation can be broken into four parts:

1.12.1 u

$$\eta\frac{\partial u}{\partial z}=0 \tag{10a}$$

$$\mu\frac{\partial U}{\partial Z}=0 \tag{27a}$$

1.12.2 v

$$\eta\frac{\partial v}{\partial z}=0 \tag{10b}$$

$$\mu\frac{\partial V}{\partial Z}=0 \tag{27b}$$

1.12.3 w

$$w=0 \tag{10c}$$

$$W=0 \tag{27c}$$

1.12.4 p

$$2\eta\frac{\partial w}{\partial z}-p+\rho g s'=0 \tag{10d}$$

$$2\,[\eta]\,\mu\frac{[v]}{[z]}\frac{\partial W}{\partial Z}-[p]\,P+\rho g\,[s']\,S'=0$$

$$2\mu\frac{\partial W}{\partial Z}-\frac{[z]\,[p]}{[\eta]\,[v]}P+\frac{\rho g\,[s']\,[z]}{[\eta]\,[v]}S'=0 \tag{10d'}$$

2 Finding Scales and Resolving the Primes

There is not yet enough information to resolve equations 3 and 4, so here I skip to 5.

2.1 5a'

This equation gives the first scale:

$$\tau_s = \frac{[u]\,[\eta]}{[z]} \tag{S1}$$

Which gives the final form:

$$\mu \frac{\partial U}{\partial Y} \to 1 \tag{25a}$$

2.2 6b'

This equation provides the following scales:

$$[z] = h_s \tag{S2}$$

$$[v] = \frac{n+2}{n+1}\frac{q_r}{h_s} \tag{S3}$$

Which gives:

$$v \to \frac{n+2}{n+1}\frac{q_r}{h_s}\left[1 - \left(1 - \frac{[z]\,Z}{h_s}\right)^{n+1}\right]$$

$$[v]\,V \to [v]\left[1 - \left(1 - \frac{h_s}{h_s}Z\right)^{n+1}\right]$$

$$V \to \left[1 - (1-Z)^{n+1}\right] \tag{26b}$$

2.3 9b',I

This equation requires the relations:

$$\tau = \frac{\tau_c}{\tau_s} \tag{N1}$$

$$\varepsilon = \frac{[v]}{[u]} = \frac{n+2}{n+1}\frac{q_r}{h_s}\frac{1}{[u]} \tag{N2}$$

And the boundary condition criterion becomes:

$$\sqrt{u^2+v^2} > 0$$

$$[u]\sqrt{U^2 + \frac{[v]}{[u]}V^2} > 0$$

$$\sqrt{U^2 + \varepsilon^2 V^2} > 0$$

Which gives the final form:

$$\mu \frac{\partial U}{\partial Z} = \tau \frac{U}{\sqrt{U^2+\varepsilon^2 V^2}} \tag{30b,I}$$

2.4 9b',II

Similarly:

$$\mu \frac{\partial V}{\partial Z} = \tau \frac{V}{\sqrt{U^2+\varepsilon^2 V^2}} \tag{30b,II}$$

2.5 9b',III

The criterion here is:

$$\sqrt{U^2 + \varepsilon^2 V^2} = 0$$

And the equation becomes:

$$\sqrt{\left(\mu \frac{\partial U}{\partial Z}\right)^2 + \varepsilon^2 \left(\mu \frac{\partial V}{\partial Z}\right)^2} \leq \tau \tag{30b,III}$$

2.6 10d'

Here the following relations can be used:

$$[p] = \rho g \left[s'\right] \tag{S4}$$

$$\left[s'\right] = \frac{[\eta]\,[v]}{\rho g\,[z]} = \frac{n+2}{n+1} \frac{q_r}{h_s} \frac{[\eta]}{\rho g h_s} = \frac{n+2}{n+1} \frac{q_r}{\rho g h_s^2} [\eta] \tag{S5}$$

This gives the relation:

$$\frac{[z]\,[p]}{[\eta]\,[v]} = \frac{\rho g \left[s'\right][z]}{[\eta]\,[v]} = 1$$

And gives the final form:

$$2\mu \frac{\partial W}{\partial Z} - P + S' = 0 \tag{27d}$$

We now have enough information to go back to equation 4.

2.7 4'

Here the following will be utilized:

$$[\eta] = A^{-1/n} \left(\frac{[u]}{[z]}\right)^{\frac{1-n}{n}} \tag{S6}$$

So the equation for η becomes:

$$\eta = \frac{A^{-1/n}}{2^{1/n}} \left(\frac{[u]}{[z]}\right)^{\frac{1-n}{n}} \left[\left|\frac{\partial U}{\partial Y}\right|^2 + \left|\frac{\partial U}{\partial Z}\right|^2 + \frac{[v]^2}{[u]^2}\left(\left|\frac{\partial V}{\partial Z} + \frac{\partial W}{\partial Y}\right|^2 + 2\left|\frac{\partial V}{\partial Y}\right|^2 + 2\left|\frac{\partial W}{\partial Z}\right|^2\right)\right]^{\frac{1-n}{2n}}$$

$$[\eta]\,\mu = \frac{[\eta]}{2^{1/n}} \left[\left|\frac{\partial U}{\partial Y}\right|^2 + \left|\frac{\partial U}{\partial Z}\right|^2 + \varepsilon^2\left(\left|\frac{\partial V}{\partial Z} + \frac{\partial W}{\partial Y}\right|^2 + 2\left|\frac{\partial V}{\partial Y}\right|^2 + 2\left|\frac{\partial W}{\partial Z}\right|^2\right)\right]^{\frac{1-n}{2n}}$$

$$\mu = \frac{1}{2^{1/n}} \left[\left|\frac{\partial U}{\partial Y}\right|^2 + \left|\frac{\partial U}{\partial Z}\right|^2 + \varepsilon^2\left(\left|\frac{\partial V}{\partial Z} + \frac{\partial W}{\partial Y}\right|^2 + 2\left|\frac{\partial V}{\partial Y}\right|^2 + 2\left|\frac{\partial W}{\partial Z}\right|^2\right)\right]^{\frac{1-n}{2n}} \tag{24}$$

2.8 3a'

The following relation will be useful here:

$$[p] = \rho g \left[s'\right] = \frac{\rho g\,[\eta]\,[v]}{\rho g\,[z]} = \frac{[\eta]\,[v]}{[z]} \tag{S7}$$

$$\frac{[\eta]\,[v]}{[z]^2} \frac{\partial}{\partial Y}\left(2\mu \frac{\partial V}{\partial Y}\right) + \frac{[\eta]\,[v]}{[z]^2} \frac{\partial}{\partial Z}\left[\mu\left(\frac{\partial V}{\partial Z} + \frac{\partial W}{\partial Y}\right)\right] - \frac{[p]}{[z]} \frac{\partial P}{\partial Y} = 0$$

$$\frac{[\eta]\,[v]}{[z]^2} \frac{\partial}{\partial Y}\left(2\mu \frac{\partial V}{\partial Y}\right) + \frac{[\eta]\,[v]}{[z]^2} \frac{\partial}{\partial Z}\left[\mu\left(\frac{\partial V}{\partial Z} + \frac{\partial W}{\partial Y}\right)\right] - \frac{[\eta]\,[v]}{[z]^2} \frac{\partial P}{\partial Y} = 0$$

$$\frac{\partial}{\partial Y}\left(2\mu \frac{\partial V}{\partial Y}\right) + \frac{\partial}{\partial Z}\left[\mu\left(\frac{\partial V}{\partial Z} + \frac{\partial W}{\partial Y}\right)\right] - \frac{\partial P}{\partial Y} = 0 \tag{23a}$$

2.9 3b'

$$\frac{[\eta]\,[v]}{[z]^2}\frac{\partial}{\partial Y}\left[\mu\left(\frac{\partial V}{\partial Z}+\frac{\partial W}{\partial Y}\right)\right]+\frac{[\eta]\,[v]}{[z]^2}\frac{\partial}{\partial Z}\left(2\mu\frac{\partial W}{\partial Z}\right)-\frac{[p]}{[z]}\frac{\partial P}{\partial Z}=0$$

$$\frac{\partial}{\partial Y}\left[\mu\left(\frac{\partial V}{\partial Z}+\frac{\partial W}{\partial Y}\right)\right]+\frac{\partial}{\partial Z}\left(2\mu\frac{\partial W}{\partial Z}\right)-\frac{\partial P}{\partial Z}=0 \tag{23b}$$

3 Scales in terms of Parameters

Now that all of the primed equations have been resolved, the scales need to be written in terms of parameters. The scales are:

$$[z]\,,[u]\,,[v]\,,[\eta]\,,[p]\,,[s']$$

From S2 have:

$$[z]=h_s$$

And from S3 have:

$$[v]=\frac{n+2}{n+1}\frac{q_r}{h_s}$$

S1 and S6 can be combined to find scales for u and η simultaneously:

$$\tau_s=\frac{[u]\,[\eta]}{[z]}$$

$$\tau_s=A^{-1/n}\left(\frac{[u]}{[z]}\right)^{\frac{1-n}{n}}\frac{[u]}{[z]}$$

$$\tau_s=A^{-1/n}\left(\frac{[u]}{h_s}\right)^{\frac{1}{n}}$$

$$\tau_s^n=\frac{1}{A}\frac{[u]}{h_s}$$

$$[u]=A\tau_s^n h_s$$

$$\implies [\eta]=\frac{\tau_s h_s}{A\tau_s^n h_s}=\frac{1}{A\tau_s^{n-1}}$$

Now can use $[\eta]$ to get an expression for $[s']$:

$$[s']=\frac{n+2}{n+1}\frac{q_r}{\rho g h_s^2}[\eta]=\frac{n+2}{n+1}\frac{q_r}{\rho g h_s^2}\frac{1}{A\tau_s^{n-1}}$$

Which in turn gives an expression for $[p]$:

$$[p]=\rho g\,[s']=\frac{n+2}{n+1}\frac{q_r}{h_s^2}\frac{1}{A\tau_s^{n-1}}$$

4 Nondimensionalization of Temperature Equations

For temperature the following are required:

$$y = [z]\,Y \qquad z = [z]\,Z$$
$$v_m = [v_m]\,V_m \qquad u = [u]\,U \qquad v = [v]\,V \qquad w = [v]\,W$$
$$a = [a]\,\mathcal{A} \qquad T = [T]\,\mathcal{T} + T_s$$

4.1 12a → 31a

Valid for: $0 < z < h_s$
Becomes valid for: $0 < Z < 1$

$$\rho c_p \left(v_m \frac{\partial T}{\partial y} + v \frac{\partial T}{\partial y} + w \frac{\partial T}{\partial z} \right) - k \left(\frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} \right) = a \tag{12a}$$

$$\rho c_p \left([v_m]\,V_m \frac{[T]}{[z]} \frac{\partial \mathcal{T}}{\partial Y} + \frac{[v]\,[T]}{[z]} V \frac{\partial \mathcal{T}}{\partial Y} + \frac{[v]\,[T]}{[z]} W \frac{\partial \mathcal{T}}{\partial Z} \right) - k \left(\frac{[T]}{[z]^2} \frac{\partial^2 \mathcal{T}}{\partial Y^2} + \frac{[T]}{[z]^2} \frac{\partial^2 \mathcal{T}}{\partial Z^2} \right) = [a]\,\mathcal{A}$$

$$\frac{\rho c_p\,[v_m]\,[T]}{[z]} V_m \frac{\partial \mathcal{T}}{\partial Y} + \frac{\rho c_p\,[v]\,[T]}{[z]} \left(V \frac{\partial \mathcal{T}}{\partial Y} + W \frac{\partial \mathcal{T}}{\partial Z} \right) - \frac{k\,[T]}{[z]^2} \left(\frac{\partial^2 \mathcal{T}}{\partial Y^2} + \frac{\partial^2 \mathcal{T}}{\partial Z^2} \right) = [a]\,\mathcal{A}$$

$$\frac{\rho c_p\,[z]\,[v_m]}{k} V_m \frac{\partial \mathcal{T}}{\partial Y} + \frac{\rho c_p\,[z]\,[v]}{k} \left(V \frac{\partial \mathcal{T}}{\partial Y} + W \frac{\partial \mathcal{T}}{\partial Z} \right) - \left(\frac{\partial^2 \mathcal{T}}{\partial Y^2} + \frac{\partial^2 \mathcal{T}}{\partial Z^2} \right) = \frac{[a]\,[z]^2}{k\,[T]} \mathcal{A} \tag{12a'}$$

4.2 12b → 31b

Valid for: $z < 0$
Becomes valid for: $Z < 0$

$$\rho_{\text{bed}} c_{p,\text{bed}} v_m \frac{\partial T}{\partial y} - k_{\text{bed}} \left(\frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} \right) = 0 \tag{12b}$$

$$\frac{\rho_{\text{bed}} c_{p,\text{bed}}\,[v_m]\,[T]}{[z]} V_m \frac{\partial \mathcal{T}}{\partial Y} - \frac{k_{\text{bed}}\,[T]}{[z]^2} \left(\frac{\partial^2 \mathcal{T}}{\partial Y^2} + \frac{\partial^2 \mathcal{T}}{\partial Z^2} \right) = 0$$

$$\frac{\rho_{\text{bed}} c_{p,\text{bed}}\,[z]\,[v_m]}{k_{\text{bed}}} V_m \frac{\partial \mathcal{T}}{\partial Y} - \left(\frac{\partial^2 \mathcal{T}}{\partial Y^2} + \frac{\partial^2 \mathcal{T}}{\partial Z^2} \right) = 0 \tag{12b'}$$

4.3 13 → 32

$$a = \frac{A^{-1/n}}{2^{1/n}} \left(\left|\frac{\partial u}{\partial y}\right|^2 + \left|\frac{\partial u}{\partial z}\right|^2 + \left|\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right|^2 + 2\left|\frac{\partial v}{\partial y}\right|^2 + 2\left|\frac{\partial w}{\partial z}\right|^2 \right)^{\frac{1+n}{2n}} \tag{13}$$

$$[a]\,\mathcal{A} = \frac{A^{-1/n}}{2^{1/n}} \left[\frac{[u]^2}{[z]^2} \left(\left|\frac{\partial U}{\partial Y}\right|^2 + \left|\frac{\partial U}{\partial Z}\right|^2 \right) + \frac{[v]^2}{[z]^2} \left(\left|\frac{\partial V}{\partial Z} + \frac{\partial W}{\partial Y}\right|^2 + 2\left|\frac{\partial V}{\partial Y}\right|^2 + 2\left|\frac{\partial W}{\partial Z}\right|^2 \right) \right]^{\frac{1+n}{2n}}$$

$$[a]\,\mathcal{A} = \frac{A^{-1/n}}{2^{1/n}} \left(\frac{[u]}{[z]} \right)^{\frac{1+n}{n}} \left[\left|\frac{\partial U}{\partial Y}\right|^2 + \left|\frac{\partial U}{\partial Z}\right|^2 + \frac{[v]^2}{[u]^2} \left(\left|\frac{\partial V}{\partial Z} + \frac{\partial W}{\partial Y}\right|^2 + 2\left|\frac{\partial V}{\partial Y}\right|^2 + 2\left|\frac{\partial W}{\partial Z}\right|^2 \right) \right]^{\frac{1+n}{2n}} \tag{13'}$$

4.4 $14 \to$ 34c-d

This equation can be broken into three parts:

4.4.1 $y \to -\infty,\ z \geq 0$

$$T = T_s + \frac{q_{\text{geo}}}{k}\left(h_s - z\right) \tag{14a}$$

$$[T]\,\mathcal{T} + T_s = T_s + \frac{q_{\text{geo}}}{k}\left(h_s - [z]\,Z\right)$$

$$[T]\,\mathcal{T} = \frac{q_{\text{geo}}}{k}\left(h_s - h_s Z\right)$$

$$\mathcal{T} = \frac{q_{\text{geo}} h_s}{k\,[T]}\left(1 - Z\right) \tag{14a'}$$

4.4.2 $y \to -\infty,\ z < 0$

$$T = T_s + \frac{q_{\text{geo}}}{k}\left(h_s - \frac{k}{k_{\text{bed}}} z\right) \tag{14b}$$

$$[T]\,\mathcal{T} + T_s = T_s + \frac{q_{\text{geo}}}{k}\left(h_s - [z]\,\frac{k}{k_{\text{bed}}} Z\right)$$

$$[T]\,\mathcal{T} = \frac{q_{\text{geo}}}{k}\left(h_s - h_s \frac{k}{k_{\text{bed}}} Z\right)$$

$$\mathcal{T} = \frac{q_{\text{geo}} h_s}{k\,[T]}\left(1 - \frac{k}{k_{\text{bed}}} Z\right) \tag{14b'}$$

4.4.3 $y \to \infty$

$$\frac{\partial T}{\partial y} \to 0 \tag{14c}$$

$$\frac{\partial \mathcal{T}}{\partial Y} \to 0 \tag{14c'}$$

4.5 $16 \to$ 34a-b

This equation can be broken into two parts:

4.5.1 $z = h_s$

$$T = T_s \tag{16a}$$

$$[T]\,\mathcal{T} + T_s = T_s$$

$$[T]\,\mathcal{T} = 0$$

$$\mathcal{T} = 0 \tag{16a'}$$

4.5.2 $z \to -\infty$

$$-k_{\text{bed}} \frac{\partial T}{\partial z} \to q_{\text{geo}} \tag{16b}$$

$$\frac{[T]}{[z]} \frac{\partial \mathcal{T}}{\partial Z} \to -\frac{q_{\text{geo}}}{k_{\text{bed}}}$$

$$\frac{\partial \mathcal{T}}{\partial Z} \to -\frac{q_{\text{geo}} h_s}{k_{\text{bed}}\,[T]} \tag{16b'}$$

4.6 17a $\rightarrow$ 35a

The boundary condition: $y < 0,\ z = 0$
Becomes the boundary condition: $Y < 0,\ Z = 0$
The constraint: $T < T_m$
Becomes the constraint: $\mathcal{T} < 1$ The equation can be broken into two parts:

4.6.1 $\tau_c = \infty$

$$-k\left.\frac{\partial T}{\partial z}\right|^{+} + k_{\text{bed}}\left.\frac{\partial T}{\partial z}\right|^{-} = 0 \tag{17a,I}$$

$$-k\frac{[T]}{[z]}\left.\frac{\partial \mathcal{T}}{\partial Z}\right|^{+} + k_{\text{bed}}\frac{[T]}{[z]}\left.\frac{\partial \mathcal{T}}{\partial Z}\right|^{-} = 0$$

$$-\left.\frac{\partial \mathcal{T}}{\partial Z}\right|^{+} + \frac{k_{\text{bed}}}{k}\left.\frac{\partial \mathcal{T}}{\partial Z}\right|^{-} = 0 \tag{17a',I}$$

4.6.2 $\tau_c < \infty$

$$temp$$

4.7 17b $\rightarrow$ 35b