Trying to Understand the B.C.'s in Comsol

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1 Deriving $2\mu D$

$$
\begin{aligned}
D &= \frac{1}{2}\left[\nabla \underline{v} + (\nabla \underline{v})^{\mathrm{T}}\right] \\
\underline{v} &= (V, W) \\
\nabla \underline{v} &= \begin{bmatrix} \dfrac{\partial V}{\partial x} & \dfrac{\partial V}{\partial y} \\ \dfrac{\partial W}{\partial x} & \dfrac{\partial W}{\partial y} \end{bmatrix} \\
(\nabla \underline{v})^{\mathrm{T}} &= \begin{bmatrix} \dfrac{\partial V}{\partial x} & \dfrac{\partial W}{\partial x} \\ \dfrac{\partial V}{\partial y} & \dfrac{\partial W}{\partial y} \end{bmatrix} \\
\mu\left(\nabla \underline{v} + (\nabla \underline{v})^{\mathrm{T}}\right) &= \begin{bmatrix} 2\mu \dfrac{\partial V}{\partial x} & \mu\left(\dfrac{\partial V}{\partial y} + \dfrac{\partial W}{\partial x}\right) \\ \mu\left(\dfrac{\partial V}{\partial y} + \dfrac{\partial W}{\partial x}\right) & 2\mu \dfrac{\partial W}{\partial y} \end{bmatrix} = 2\mu D
\end{aligned}
$$

2 A Table of Normals and Boundary Conditions

	$\underline{n}$	U	V	W
Left	$(-1,0)$	$U=0$	$V=1-(1-Z)^4$	$W=0$
Right	$(1,0)$	$\mu\dfrac{\partial U}{\partial Y}=1$	$\dfrac{\partial V}{\partial Z}=0$	$W=0$
Top	$(0,1)$	$\mu\dfrac{\partial U}{\partial Z}=0$	$\mu\dfrac{\partial V}{\partial Z}=0$	$W=0$ $2\mu\dfrac{\partial W}{\partial Z}-P+S'=0$
Ridge (No Slip)	$(0,-1)$	$U=0$	$V=0$	$W=0$
Ridge (Slip)	$(0,-1)$	$\begin{cases}\mu\frac{\partial U}{\partial Z}=\tau\frac{U}{\sqrt{U^2+\varepsilon^2V^2}} & \sqrt{U^2+\varepsilon^2V^2}>0\\ \sqrt{\left(\mu\frac{\partial U}{\partial Z}\right)^2+\varepsilon^2\left(\mu\frac{\partial V}{\partial Z}\right)^2}\leq\tau & \sqrt{U^2+\varepsilon^2V^2}=0\end{cases}$	$\begin{cases}\mu\frac{\partial V}{\partial Z}=\tau\frac{V}{\sqrt{U^2+\varepsilon^2V^2}} & \sqrt{U^2+\varepsilon^2V^2}>0\\ \sqrt{\left(\mu\frac{\partial U}{\partial Z}\right)^2+\varepsilon^2\left(\mu\frac{\partial V}{\partial Z}\right)^2}\leq\tau & \sqrt{U^2+\varepsilon^2V^2}=0\end{cases}$	$W=0$
Stream	$(0,-1)$	$\mu\dfrac{\partial U}{\partial Z}=0$	$\mu\dfrac{\partial V}{\partial Z}=0$	$W=0$

For conversion to comsol need $Y\rightarrow x,\ Z\rightarrow y$

3 Downstream Velocity Boundary Conditions in Comsol

3.1 Left Boundary

Dirichlet Boundary Condition:

$$U = r$$
$$r = 0 \implies U = 0$$

3.2 Right Boundary

Flux/Source Boundary Condition:

$$-\underline{n} \cdot \underline{\Gamma} = g - qu$$
$$\underline{n} = (1, 0)$$
$$\underline{\Gamma} = \left(\mu \frac{\partial U}{\partial x}, \mu \frac{\partial U}{\partial y} \right)$$
$$-\underline{n} \cdot \underline{\Gamma} = -\mu \frac{\partial U}{\partial x}$$
$$\text{Set: } g = -1, \; q = 0 \implies -\mu \frac{\partial U}{\partial x} = -1$$
$$\mu \frac{\partial U}{\partial x} = 1$$

3.3 Top Boundary

Zero Flux Boundary Condition:

$$-\underline{n} \cdot \underline{\Gamma} = 0$$
$$-(0, 1) \cdot \left(\mu \frac{\partial U}{\partial x}, \mu \frac{\partial U}{\partial y} \right) = 0$$
$$-\mu \frac{\partial U}{\partial y} = 0$$
$$\mu \frac{\partial U}{\partial y} = 0$$

3.4 Bottom Boundary - Stream

Zero Flux Boundary Condition:

$$-\underline{n} \cdot \underline{\Gamma} = 0$$
$$-(0, -1) \cdot \left(\mu \frac{\partial U}{\partial x}, \mu \frac{\partial U}{\partial y} \right) = 0$$
$$\mu \frac{\partial U}{\partial y} = 0$$

3.5 Bottom Boundary - Ridge (No Slip)

Dirichlet Boundary Condition:

$$U = r$$
$$r = 0 \implies U = 0$$

3.6 Bottom Boundary - Ridge (Slip)

Flux/Source Boundary Condition:

$$-\underline{n} \cdot \underline{\Gamma} = g - qu$$
$$\underline{n} = (0, -1)$$
$$\underline{\Gamma} = \left(\mu \frac{\partial U}{\partial x}, \mu \frac{\partial U}{\partial y} \right)$$
$$-\underline{n} \cdot \underline{\Gamma} = \mu \frac{\partial U}{\partial y}$$
$$\text{Set: } g = \tau \frac{U}{\sqrt{U^2 + \varepsilon^2 V^2}}, \; q = 0 \implies \mu \frac{\partial U}{\partial y} = \tau \frac{U}{\sqrt{U^2 + \varepsilon^2 V^2}}$$
$$\text{As: } \sqrt{U^2 + \varepsilon^2 V^2} \to 0, \; \mu \frac{\partial U}{\partial y} \to \infty$$