# Nondimensionalization of Haseloff et al. 2018

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## 1 Nondimensionalization of Velocity Equations

Nondimensionalizing the equations in terms of dimensional scales (this section will not yet try write the scales in terms of parameters)
Here I use:

$$
\begin{gathered}
y = [z]\,Y \qquad z = [z]\,Z \\
u = [u]\,U \qquad v = [v]\,V \qquad w = [v]\,W \\
\eta = [\eta]\,\mu \qquad p = [p]\,P \qquad s' = [s']\,S'
\end{gathered}
$$

The convention I used here is to provide the equation number when it matches the same equation in Haseloff's paper. If the equation doesn't match an equation in Haseloff, but I plan to resolve the differences later in the notes I will denote that with primes.

### 1.1 $2 \rightarrow 22$

$$
\frac{\partial}{\partial y}\left(\eta \frac{\partial u}{\partial y}\right) + \frac{\partial}{\partial z}\left(\eta \frac{\partial u}{\partial z}\right) = 0 \tag{2}
$$

$$
\frac{1}{[z]}\frac{\partial}{\partial Y}\left([\eta]\,\mu \frac{[u]}{[z]}\frac{\partial U}{\partial Y}\right) + \frac{1}{[z]}\frac{\partial}{\partial Z}\left([\eta]\,\mu \frac{[u]}{[z]}\frac{\partial U}{\partial Z}\right) = 0
$$

$$
\frac{[\eta]\,[u]}{[z]^2}\left[\frac{\partial}{\partial Y}\left(\mu \frac{\partial U}{\partial Y}\right) + \frac{\partial}{\partial Z}\left(\mu \frac{\partial U}{\partial Z}\right)\right] = 0
$$

$$
\frac{\partial}{\partial Y}\left(\mu \frac{\partial U}{\partial Y}\right) + \frac{\partial}{\partial Z}\left(\mu \frac{\partial U}{\partial Z}\right) = 0 \tag{22}
$$

### 1.2 $3\text{a} \rightarrow 23\text{a}$

$$
\frac{\partial}{\partial y}\left(2\eta \frac{\partial v}{\partial y}\right) + \frac{\partial}{\partial z}\left[\eta\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right)\right] - \frac{\partial p}{\partial y} = 0 \tag{3a}
$$

$$
\frac{1}{[z]}\frac{\partial}{\partial Y}\left(2\,[\eta]\,\mu \frac{[v]}{[z]}\frac{\partial V}{\partial Y}\right) + \frac{1}{[z]}\frac{\partial}{\partial Z}\left[[\eta]\,\mu \frac{[v]}{[z]}\left(\frac{\partial V}{\partial Z} + \frac{\partial W}{\partial Y}\right)\right] - \frac{[p]}{[z]}\frac{\partial P}{\partial Y} = 0
$$

$$
\frac{[\eta]\,[v]}{[z]^2}\frac{\partial}{\partial Y}\left(2\mu \frac{\partial V}{\partial Y}\right) + \frac{[\eta]\,[v]}{[z]^2}\frac{\partial}{\partial Z}\left[\mu\left(\frac{\partial V}{\partial Z} + \frac{\partial W}{\partial Y}\right)\right] - \frac{[p]}{[z]}\frac{\partial P}{\partial Y} = 0 \tag{3a$'$}
$$

### 1.3 3b → 23b

$$\frac{\partial}{\partial y}\left[\eta\left(\frac{\partial v}{\partial z}+\frac{\partial w}{\partial y}\right)\right]+\frac{\partial}{\partial z}\left(2\eta\frac{\partial w}{\partial z}\right)-\frac{\partial p}{\partial z}=0 \tag{3b}$$

$$\frac{1}{[z]}\frac{\partial}{\partial Y}\left[[\eta]\,\mu\frac{[v]}{[z]}\left(\frac{\partial V}{\partial Z}+\frac{\partial W}{\partial Y}\right)\right]+\frac{1}{[z]}\frac{\partial}{\partial Z}\left(2\,[\eta]\,\mu\frac{[v]}{[z]}\frac{\partial W}{\partial Z}\right)-\frac{[p]}{[z]}\frac{\partial P}{\partial Z}=0$$

$$\frac{[\eta]\,[v]}{[z]^2}\frac{\partial}{\partial Y}\left[\mu\left(\frac{\partial V}{\partial Z}+\frac{\partial W}{\partial Y}\right)\right]+\frac{[\eta]\,[v]}{[z]^2}\frac{\partial}{\partial Z}\left(2\mu\frac{\partial W}{\partial Z}\right)-\frac{[p]}{[z]}\frac{\partial P}{\partial Z}=0 \tag{3b$'$}$$

### 1.4 3c → 23c

$$\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0 \tag{3c}$$

$$\frac{[v]}{[z]}\frac{\partial V}{\partial Y}+\frac{[v]}{[z]}\frac{\partial W}{\partial Z}=0$$

$$\frac{\partial V}{\partial Y}+\frac{\partial W}{\partial Z}=0 \tag{23c}$$

### 1.5 4 → 24

$$\eta=\frac{A^{-1/n}}{2^{1/n}}\left[\left|\frac{\partial u}{\partial y}\right|^2+\left|\frac{\partial u}{\partial z}\right|^2+\left|\frac{\partial v}{\partial z}+\frac{\partial w}{\partial y}\right|^2+2\left|\frac{\partial v}{\partial y}\right|^2+2\left|\frac{\partial w}{\partial z}\right|^2\right]^{\frac{1-n}{2n}} \tag{4}$$

$$\frac{A^{-1/n}}{2^{1/n}}\left[\frac{[u]^2}{[z]^2}\left(\left|\frac{\partial U}{\partial Y}\right|^2+\left|\frac{\partial U}{\partial Z}\right|^2\right)+\frac{[v]^2}{[z]^2}\left(\left|\frac{\partial V}{\partial Z}+\frac{\partial W}{\partial Y}\right|^2+2\left|\frac{\partial V}{\partial Y}\right|^2+2\left|\frac{\partial W}{\partial Z}\right|^2\right)\right]^{\frac{1-n}{2n}}$$

$$\frac{A^{-1/n}}{2^{1/n}}\left(\frac{[u]}{[z]}\right)^{\frac{1-n}{n}}\left[\left|\frac{\partial U}{\partial Y}\right|^2+\left|\frac{\partial U}{\partial Z}\right|^2+\frac{[v]^2}{[u]^2}\left(\left|\frac{\partial V}{\partial Z}+\frac{\partial W}{\partial Y}\right|^2+2\left|\frac{\partial V}{\partial Y}\right|^2+2\left|\frac{\partial W}{\partial Z}\right|^2\right)\right]^{\frac{1-n}{2n}} \tag{4$'$}$$

### 1.6 5 → 25

Boundary condition for: $y\to\infty$
Becomes the boundary condition for: $Y\to\infty$
This equation can be broken into three parts:

#### 1.6.1 u

$$\eta\frac{\partial u}{\partial y}\to\tau_s \tag{5a}$$

$$[\eta]\,\mu\frac{[u]}{[z]}\frac{\partial U}{\partial Y}\to\tau_s$$

$$\mu\frac{\partial U}{\partial Y}\to\frac{\tau_s\,[z]}{[u]\,[\eta]} \tag{5a$'$}$$

#### 1.6.2 v

$$\frac{\partial v}{\partial z}\to 0 \tag{5b}$$

$$\frac{\partial V}{\partial Z}\to 0 \tag{25b}$$

### 1.6.3 w

$$w \to 0 \tag{5c}$$
$$W \to 0 \tag{25c}$$

## 1.7 $6 \to 26$

Boundary condition for: $y \to -\infty$
Becomes the boundary condition for: $Y \to -\infty$
This equation can be broken into three parts:

### 1.7.1 u

$$u \to 0 \tag{6a}$$
$$U \to 0 \tag{26a}$$

### 1.7.2 v

$$v \to \frac{n+2}{n+1}\frac{q_r}{h_s}\left[1-\left(1-\frac{z}{h_s}\right)^{n+1}\right] \tag{6b}$$
$$v \to \frac{n+2}{n+1}\frac{q_r}{h_s}\left[1-\left(1-\frac{[z]\,Z}{h_s}\right)^{n+1}\right] \tag{6b'}$$

### 1.7.3 w

$$w \to 0 \tag{6c}$$
$$W \to 0 \tag{26c}$$

## 1.8 $7 \to 28$

Boundary condition for: $z = 0$
Becomes the boundary condition for: $Z = 0$

$$w = 0 \text{ at } z = 0 \tag{7}$$
$$W = 0 \text{ at } Z = 0 \tag{28}$$

## 1.9 $8 \to 29$

Boundary condition for: $y > 0,\ z = 0$
Becomes the boundary condition for: $Y > 0,\ Z = 0$
This equation can be broken into two parts:

### 1.9.1 u

$$\eta\frac{\partial u}{\partial z} = 0 \tag{8a}$$
$$[\eta]\,\mu\frac{[u]}{[z]}\frac{\partial U}{\partial Z} = 0$$
$$\mu\frac{\partial U}{\partial Z} = 0 \tag{29a}$$

#### 1.9.2 v

$$\eta \frac{\partial v}{\partial z} = 0 \tag{8b}$$

$$[\eta]\,\mu \frac{[v]}{[z]} \frac{\partial V}{\partial Z} = 0$$

$$\mu \frac{\partial V}{\partial Z} = 0 \tag{29b}$$

## 1.10 9a → 30a

No slip boundary condition for: $y < 0,\ z = 0$
Becomes the no slip boundary condition for: $Y < 0,\ Z = 0$

$$u = v = 0 \tag{9a}$$

$$U = V = 0 \tag{30a}$$

## 1.11 9b → 30b

Subtemperate slip boundary condition for: $y < 0,\ z = 0$
Becomes the subtemperate slip boundary condition for: $Y < 0,\ Z = 0$ This equation can be broken into three parts:

#### 1.11.1 $\sqrt{u^2 + v^2} > 0$, u

$$\eta \frac{\partial u}{\partial z} = \tau_c \frac{u}{\sqrt{u^2 + v^2}} \tag{9b,I}$$

$$[\eta]\,\mu \frac{[u]}{[z]} \frac{\partial U}{\partial Z} = \tau_c \frac{U}{\sqrt{U^2 + \frac{[v]^2}{[u]^2} V^2}}$$

$$\mu \frac{\partial U}{\partial Z} = \frac{[z]\,\tau_c}{[u]\,[\eta]} \frac{U}{\sqrt{U^2 + \frac{[v]^2}{[u]^2} V^2}} \tag{9b',I}$$

#### 1.11.2 $\sqrt{u^2 + v^2} > 0$, v

$$\eta \frac{\partial v}{\partial z} = \tau_c \frac{v}{\sqrt{u^2 + v^2}} \tag{9b,II}$$

$$[\eta]\,\mu \frac{[v]}{[z]} \frac{\partial V}{\partial Z} = \tau_c \frac{[v]}{[u]} \frac{V}{\sqrt{U^2 + \frac{[v]^2}{[u]^2} V^2}}$$

$$\mu \frac{\partial V}{\partial Z} = \frac{[z]\,\tau_c}{[u]\,[\eta]} \frac{V}{\sqrt{U^2 + \frac{[v]^2}{[u]^2} V^2}} \tag{9b',II}$$

#### 1.11.3 $\sqrt{u^2+v^2}=0$

$$\sqrt{\left(\eta\frac{\partial u}{\partial z}\right)^2+\left(\eta\frac{\partial v}{\partial z}\right)^2}\leq\tau_c \tag{9b,III}$$

$$\frac{[\eta]\,[u]}{[z]}\sqrt{\left(\mu\frac{\partial U}{\partial Z}\right)^2+\left(\mu\frac{[v]}{[u]}\frac{\partial V}{\partial Z}\right)^2}\leq\tau_c$$

$$\sqrt{\left(\mu\frac{\partial U}{\partial Z}\right)^2+\frac{[v]^2}{[u]^2}\left(\mu\frac{\partial V}{\partial Z}\right)^2}\leq\frac{[z]\,\tau_c}{[u]\,[\eta]} \tag{9b',III}$$

### 1.12 $10 \to 27$

Boundary condition for: $z=h_s$
Becomes the boundary condition for: $Z=1$
This equation can be broken into four parts:

#### 1.12.1 u

$$\eta\frac{\partial u}{\partial z}=0 \tag{10a}$$

$$\mu\frac{\partial U}{\partial Z}=0 \tag{27a}$$

#### 1.12.2 v

$$\eta\frac{\partial v}{\partial z}=0 \tag{10b}$$

$$\mu\frac{\partial V}{\partial Z}=0 \tag{27b}$$

#### 1.12.3 w

$$w=0 \tag{10c}$$

$$W=0 \tag{27c}$$

#### 1.12.4 p

$$2\eta\frac{\partial w}{\partial z}-p+\rho g s'=0 \tag{10d}$$

$$2\,[\eta]\,\mu\frac{[v]}{[z]}\frac{\partial W}{\partial Z}-[p]\,P+\rho g\,[s']\,S'=0$$

$$2\mu\frac{\partial W}{\partial Z}-\frac{[z]\,[p]}{[\eta]\,[v]}P+\frac{\rho g\,[s']\,[z]}{[\eta]\,[v]}S'=0 \tag{10d'}$$

## 2 Finding Scales and Resolving the Primes

There is not yet enough information to resolve equations 3 and 4, so here I skip to 5.

## 2.1 5a'

This equation gives the first scale:

$$\tau_s = \frac{[u]\,[\eta]}{[z]} \tag{S1}$$

Which gives the final form:

$$\mu \frac{\partial U}{\partial Y} \to 1 \tag{25a}$$

## 2.2 6b'

This equation provides the following scales:

$$[z] = h_s \tag{S2}$$

$$[v] = \frac{n+2}{n+1} \frac{q_r}{h_s} \tag{S3}$$

Which gives:

$$\begin{aligned} v &\to \frac{n+2}{n+1} \frac{q_r}{h_s} \left[ 1 - \left( 1 - \frac{[z]\,Z}{h_s} \right)^{n+1} \right] \\ [v]\,V &\to [v] \left[ 1 - \left( 1 - \frac{h_s}{h_s} Z \right)^{n+1} \right] \\ V &\to \left[ 1 - (1-Z)^{n+1} \right] \end{aligned} \tag{26b}$$

## 2.3 9b',I

This equation requires the relations:

$$\tau = \frac{\tau_c}{\tau_s} \tag{N1}$$

$$\varepsilon = \frac{[v]}{[u]} = \frac{n+2}{n+1} \frac{q_r}{h_s} \frac{1}{[u]} \tag{N2}$$

And the boundary condition criterion becomes:

$$\begin{aligned} \sqrt{u^2 + v^2} &> 0 \\ [u] \sqrt{U^2 + \frac{[v]}{[u]} V^2} &> 0 \\ \sqrt{U^2 + \varepsilon^2 V^2} &> 0 \end{aligned}$$

Which gives the final form:

$$\mu \frac{\partial U}{\partial Z} = \tau \frac{U}{\sqrt{U^2 + \varepsilon^2 V^2}} \tag{30b,I}$$

## 2.4 9b',II

Similarly:

$$\mu \frac{\partial V}{\partial Z} = \tau \frac{V}{\sqrt{U^2 + \varepsilon^2 V^2}} \tag{30b,II}$$

## 2.5 9b',III

The criterion here is:

$$\sqrt{U^2 + \varepsilon^2 V^2} = 0$$

And the equation becomes:

$$\sqrt{\left(\mu \frac{\partial U}{\partial Z}\right)^2 + \varepsilon^2 \left(\mu \frac{\partial V}{\partial Z}\right)^2} \leq \tau \tag{30b,III}$$

## 2.6 10d'

Here the following relations can be used:

$$[p] = \rho g \, [s'] \tag{S4}$$

$$[s'] = \frac{[\eta] \, [v]}{\rho g \, [z]} = \frac{n+2}{n+1} \frac{q_r}{h_s} \frac{[\eta]}{\rho g h_s} = \frac{n+2}{n+1} \frac{q_r}{\rho g h_s^2} \, [\eta] \tag{S5}$$

This gives the relation:

$$\frac{[z] \, [p]}{[\eta] \, [v]} = \frac{\rho g \, [s'] \, [z]}{[\eta] \, [v]} = 1$$

And gives the final form:

$$2\mu \frac{\partial W}{\partial Z} - P + S' = 0 \tag{27d}$$

We now have enough information to go back to equation 4.

## 2.7 4'

Here the following will be utilized:

$$[\eta] = A^{-1/n} \left(\frac{[u]}{[z]}\right)^{\frac{1-n}{n}} \tag{S6}$$

So the equation for $\eta$ becomes:

$$\eta = \frac{A^{-1/n}}{2^{1/n}} \left(\frac{[u]}{[z]}\right)^{\frac{1-n}{n}} \left[ \left|\frac{\partial U}{\partial Y}\right|^2 + \left|\frac{\partial U}{\partial Z}\right|^2 + \frac{[v]^2}{[u]^2} \left( \left|\frac{\partial V}{\partial Z} + \frac{\partial W}{\partial Y}\right|^2 + 2\left|\frac{\partial V}{\partial Y}\right|^2 + 2\left|\frac{\partial W}{\partial Z}\right|^2 \right) \right]^{\frac{1-n}{2n}}$$

$$[\eta] \, \mu = \frac{[\eta]}{2^{1/n}} \left[ \left|\frac{\partial U}{\partial Y}\right|^2 + \left|\frac{\partial U}{\partial Z}\right|^2 + \varepsilon^2 \left( \left|\frac{\partial V}{\partial Z} + \frac{\partial W}{\partial Y}\right|^2 + 2\left|\frac{\partial V}{\partial Y}\right|^2 + 2\left|\frac{\partial W}{\partial Z}\right|^2 \right) \right]^{\frac{1-n}{2n}}$$

$$\mu = \frac{1}{2^{1/n}} \left[ \left|\frac{\partial U}{\partial Y}\right|^2 + \left|\frac{\partial U}{\partial Z}\right|^2 + \varepsilon^2 \left( \left|\frac{\partial V}{\partial Z} + \frac{\partial W}{\partial Y}\right|^2 + 2\left|\frac{\partial V}{\partial Y}\right|^2 + 2\left|\frac{\partial W}{\partial Z}\right|^2 \right) \right]^{\frac{1-n}{2n}} \tag{24}$$

## 2.8 3a'

The following relation will be useful here:

$$[p] = \rho g \, [s'] = \frac{\rho g \, [\eta] \, [v]}{\rho g \, [z]} = \frac{[\eta] \, [v]}{[z]} \tag{S7}$$

$$\frac{[\eta] \, [v]}{[z]^2} \frac{\partial}{\partial Y} \left(2\mu \frac{\partial V}{\partial Y}\right) + \frac{[\eta] \, [v]}{[z]^2} \frac{\partial}{\partial Z} \left[\mu \left(\frac{\partial V}{\partial Z} + \frac{\partial W}{\partial Y}\right)\right] - \frac{[p]}{[z]} \frac{\partial P}{\partial Y} = 0$$

$$\frac{[\eta] \, [v]}{[z]^2} \frac{\partial}{\partial Y} \left(2\mu \frac{\partial V}{\partial Y}\right) + \frac{[\eta] \, [v]}{[z]^2} \frac{\partial}{\partial Z} \left[\mu \left(\frac{\partial V}{\partial Z} + \frac{\partial W}{\partial Y}\right)\right] - \frac{[\eta] \, [v]}{[z]^2} \frac{\partial P}{\partial Y} = 0$$

$$\frac{\partial}{\partial Y} \left(2\mu \frac{\partial V}{\partial Y}\right) + \frac{\partial}{\partial Z} \left[\mu \left(\frac{\partial V}{\partial Z} + \frac{\partial W}{\partial Y}\right)\right] - \frac{\partial P}{\partial Y} = 0 \tag{23a}$$

### 2.9 3b'

$$\frac{[\eta]\,[v]}{[z]^2}\frac{\partial}{\partial Y}\left[\mu\left(\frac{\partial V}{\partial Z}+\frac{\partial W}{\partial Y}\right)\right]+\frac{[\eta]\,[v]}{[z]^2}\frac{\partial}{\partial Z}\left(2\mu\frac{\partial W}{\partial Z}\right)-\frac{[p]}{[z]}\frac{\partial P}{\partial Z}=0$$
$$\frac{\partial}{\partial Y}\left[\mu\left(\frac{\partial V}{\partial Z}+\frac{\partial W}{\partial Y}\right)\right]+\frac{\partial}{\partial Z}\left(2\mu\frac{\partial W}{\partial Z}\right)-\frac{\partial P}{\partial Z}=0 \tag{23b}$$

## 3 Scales in terms of Parameters

Now that all of the primed equations have been resolved, the scales need to be written in terms of parameters. The scales are:

$$[z]\,,[u]\,,[v]\,,[\eta]\,,[p]\,,[s']$$

From S2 have:

$$[z]=h_s$$

And from S3 have:

$$[v]=\frac{n+2}{n+1}\frac{q_r}{h_s}$$

S1 and S6 can be combined to find scales for $u$ and $\eta$ simultaneously:

$$\tau_s=\frac{[u]\,[\eta]}{[z]}$$
$$\tau_s=A^{-1/n}\left(\frac{[u]}{[z]}\right)^{\frac{1-n}{n}}\frac{[u]}{[z]}$$
$$\tau_s=A^{-1/n}\left(\frac{[u]}{h_s}\right)^{\frac{1}{n}}$$
$$\tau_s^n=\frac{1}{A}\frac{[u]}{h_s}$$
$$[u]=A\tau_s^n h_s$$
$$\implies [\eta]=\frac{\tau_s h_s}{A\tau_s^n h_s}=\frac{1}{A\tau_s^{n-1}}$$

Now can use $[\eta]$ to get an expression for $[s']$:

$$[s']=\frac{n+2}{n+1}\frac{q_r}{\rho g h_s^2}[\eta]=\frac{n+2}{n+1}\frac{q_r}{\rho g h_s^2}\frac{1}{A\tau_s^{n-1}}$$

Which in turn gives an expression for $[p]$:

$$[p]=\rho g\,[s']=\frac{n+2}{n+1}\frac{q_r}{h_s^2}\frac{1}{A\tau_s^{n-1}}$$