Nondimensionalization of Haseloff et al. 2018

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1 Definitions

A few of the scales jump out right away:

$$[z] = h_s \qquad [T] = T_m - T_s$$

These will be used in the following:

$$y = [z]\,Y \qquad z = [z]\,Z$$
$$T = [T]\,\mathcal{T} + T_s$$

Note: Haseloff et al. reads: $T = [T]\,\mathcal{T} + T_m$ which is incorrect.

The three velocity scales can also be defined:

$$u = [u]\,U \qquad v = [v]\,V \qquad w = [v]\,W$$

I will, however, make no assumptions in regards to the values of these scales.

$$[s'] = \frac{n+2}{n+1}\frac{q_r}{A\tau_s^n h_s^2}\frac{\tau_s}{\rho g} \qquad [u] = Ah_s\tau_s^n$$
$$[v] = \frac{n+2}{n+1}\frac{q_r}{h_s}$$
$$s' = [s']\,S' \qquad p = \rho g\,[s']\,P$$
$$\alpha = \frac{A\tau_s^{n+1}h_s^2}{k\,(T_m - T_b)} \qquad Pe = \frac{n+2}{n+1}\frac{\rho c_p q_r}{k} \qquad \nu = \frac{T_b - T_s}{T_m - T_s}$$
$$\tau = \frac{\tau_c}{\tau_s} \qquad \varepsilon = \frac{n+2}{n+1}\frac{q_r}{A\tau_s^n h_s^2}$$

2 Nondimensionalization of Velocity Equations

Nondimensionalizing the equations in terms of dimensional scales (this section will not yet try write the scales in terms of parameters)
Here I use:

$$[z] = h_s \qquad y = [z]\,Y \qquad z = [z]\,Z$$
$$u = [u]\,U \qquad v = [v]\,V \qquad w = [v]\,W$$
$$\eta = [\eta]\,\mu \qquad p = [p]\,P$$

The convention I used here is to provide the equation number when it matches the same equation in Haseloff's paper. If the equation doesn't match an equation in Haseloff, but I plan to resolve the differences later in the notes I will denote that with primes.

2.1 $2 \to 22$

$$\frac{\partial}{\partial y}\left(\eta\frac{\partial u}{\partial y}\right)+\frac{\partial}{\partial z}\left(\eta\frac{\partial u}{\partial z}\right)=0 \tag{2}$$

$$\frac{1}{[z]}\frac{\partial}{\partial Y}\left([\eta]\,\mu\frac{[u]}{[z]}\frac{\partial U}{\partial Y}\right)+\frac{1}{[z]}\frac{\partial}{\partial Z}\left([\eta]\,\mu\frac{[u]}{[z]}\frac{\partial U}{\partial Z}\right)=0$$

$$\frac{[\eta]\,[u]}{[z]^2}\left[\frac{\partial}{\partial Y}\left(\mu\frac{\partial U}{\partial Y}\right)+\frac{\partial}{\partial Z}\left(\mu\frac{\partial U}{\partial Z}\right)\right]=0$$

$$\frac{\partial}{\partial Y}\left(\mu\frac{\partial U}{\partial Y}\right)+\frac{\partial}{\partial Z}\left(\mu\frac{\partial U}{\partial Z}\right)=0 \tag{22}$$

2.2 3a $\to$ 23a

$$\frac{\partial}{\partial y}\left(2\eta\frac{\partial v}{\partial y}\right)+\frac{\partial}{\partial z}\left[\eta\left(\frac{\partial v}{\partial z}+\frac{\partial w}{\partial y}\right)\right]-\frac{\partial p}{\partial y}=0 \tag{3a}$$

$$\frac{1}{[z]}\frac{\partial}{\partial Y}\left(2\,[\eta]\,\mu\frac{[v]}{[z]}\frac{\partial V}{\partial Y}\right)+\frac{1}{[z]}\frac{\partial}{\partial Z}\left[[\eta]\,\mu\frac{[v]}{[z]}\left(\frac{\partial V}{\partial Z}+\frac{\partial W}{\partial Y}\right)\right]-\frac{[p]}{[z]}\frac{\partial P}{\partial Y}=0$$

$$\frac{[\eta]\,[v]}{[z]^2}\frac{\partial}{\partial Y}\left(2\mu\frac{\partial V}{\partial Y}\right)+\frac{[\eta]\,[v]}{[z]^2}\frac{\partial}{\partial Z}\left[\mu\left(\frac{\partial V}{\partial Z}+\frac{\partial W}{\partial Y}\right)\right]-\frac{[p]}{[z]}\frac{\partial P}{\partial Y}=0 \tag{3a$'$}$$

2.3 3b $\to$ 23b

$$\frac{\partial}{\partial y}\left[\eta\left(\frac{\partial v}{\partial z}+\frac{\partial w}{\partial y}\right)\right]+\frac{\partial}{\partial z}\left(2\eta\frac{\partial w}{\partial z}\right)-\frac{\partial p}{\partial z}=0 \tag{3b}$$

$$\frac{1}{[z]}\frac{\partial}{\partial Y}\left[[\eta]\,\mu\frac{[v]}{[z]}\left(\frac{\partial V}{\partial Z}+\frac{\partial W}{\partial Y}\right)\right]+\frac{1}{[z]}\frac{\partial}{\partial Z}\left(2\,[\eta]\,\mu\frac{[v]}{[z]}\frac{\partial W}{\partial Z}\right)-\frac{[p]}{[z]}\frac{\partial P}{\partial Z}=0$$

$$\frac{[\eta]\,[v]}{[z]^2}\frac{\partial}{\partial Y}\left[\mu\left(\frac{\partial V}{\partial Z}+\frac{\partial W}{\partial Y}\right)\right]+\frac{[\eta]\,[v]}{[z]^2}\frac{\partial}{\partial Z}\left(2\mu\frac{\partial W}{\partial Z}\right)-\frac{[p]}{[z]}\frac{\partial P}{\partial Z}=0 \tag{3b$'$}$$

2.4 3c $\to$ 23c

$$\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0 \tag{3c}$$

$$\frac{[v]}{[z]}\frac{\partial V}{\partial Y}+\frac{[v]}{[z]}\frac{\partial W}{\partial Z}=0$$

$$\frac{\partial V}{\partial Y}+\frac{\partial W}{\partial Z}=0 \tag{23c}$$

2.5 $4 \to 24$

$$\eta=\frac{A^{-1/n}}{2^{1/n}}\left[\left|\frac{\partial u}{\partial y}\right|^2+\left|\frac{\partial u}{\partial z}\right|^2+\left|\frac{\partial v}{\partial z}+\frac{\partial w}{\partial y}\right|^2+2\left|\frac{\partial v}{\partial y}\right|^2+2\left|\frac{\partial w}{\partial z}\right|^2\right]^{\frac{1-n}{2n}} \tag{4}$$

$$\frac{A^{-1/n}}{2^{1/n}}\left[\frac{[u]^2}{[z]^2}\left(\left|\frac{\partial U}{\partial Y}\right|^2+\left|\frac{\partial U}{\partial Z}\right|^2\right)+\frac{[v]^2}{[z]^2}\left(\left|\frac{\partial V}{\partial Z}+\frac{\partial W}{\partial Y}\right|^2+2\left|\frac{\partial V}{\partial Y}\right|^2+2\left|\frac{\partial W}{\partial Z}\right|^2\right)\right]^{\frac{1-n}{2n}}$$

$$\frac{A^{-1/n}}{2^{1/n}}\left(\frac{[u]}{[z]}\right)^{\frac{1-n}{n}}\left[\left|\frac{\partial U}{\partial Y}\right|^2+\left|\frac{\partial U}{\partial Z}\right|^2+\frac{[v]^2}{[u]^2}\left(\left|\frac{\partial V}{\partial Z}+\frac{\partial W}{\partial Y}\right|^2+2\left|\frac{\partial V}{\partial Y}\right|^2+2\left|\frac{\partial W}{\partial Z}\right|^2\right)\right]^{\frac{1-n}{2n}} \tag{4$'$}$$

2.6 $5 \rightarrow 25$

Boundary condition for: $y \rightarrow \infty$
Becomes the boundary condition for: $Y \rightarrow \infty$
This equation can be broken into three parts:

2.6.1 u

$$\eta \frac{\partial u}{\partial y} \rightarrow \tau_s \tag{5a}$$

$$[\eta]\,\mu \frac{[u]}{[z]} \frac{\partial U}{\partial Y} \rightarrow \tau_s$$

$$\mu \frac{\partial U}{\partial Y} \rightarrow \frac{\tau_s\,[z]}{[u]\,[\eta]} \tag{5a$'$}$$

2.6.2 v

$$\frac{\partial v}{\partial z} \rightarrow 0 \tag{5b}$$

$$\frac{\partial V}{\partial Z} \rightarrow 0 \tag{25b}$$

2.6.3 w

$$w \rightarrow 0 \tag{5c}$$

$$W \rightarrow 0 \tag{25c}$$

2.7 $6 \rightarrow 26$

Boundary condition for: $y \rightarrow -\infty$
Becomes the boundary condition for: $Y \rightarrow -\infty$
This equation can be broken into three parts:

2.7.1 u

$$u \rightarrow 0 \tag{6a}$$

$$U \rightarrow 0 \tag{26a}$$

2.7.2 v

$$v \rightarrow \frac{n+2}{n+1} \frac{q_r}{h_s} \left[1 - \left(1 - \frac{z}{h_s}\right)^{n+1}\right] \tag{6b}$$

$$v \rightarrow \frac{n+2}{n+1} \frac{q_r}{h_s} \left[1 - \left(1 - \frac{h_s Z}{h_s}\right)^{n+1}\right]$$

$$v \rightarrow \underbrace{\frac{n+2}{n+1} \frac{q_r}{h_s}}_{[v]} \left[1 - (1 - Z)^{n+1}\right]$$

$$V \rightarrow 1 - (1 - Z)^{n+1} \tag{26b}$$

2.7.3 w

$$w \to 0 \tag{6c}$$
$$W \to 0 \tag{26c}$$

2.8 $7 \to 28$

Boundary condition for: $z = 0$
Becomes the boundary condition for: $Z = 0$

$$w = 0 \text{ at } z = 0 \tag{7}$$
$$W = 0 \text{ at } Z = 0 \tag{28}$$

2.9 $8 \to 29$

Boundary condition for: $y > 0,\ z = 0$
Becomes the boundary condition for: $Y > 0,\ Z = 0$
This equation can be broken into two parts:

2.9.1 u

$$\eta \frac{\partial u}{\partial z} = 0 \tag{8a}$$
$$[\eta]\, \mu \frac{[u]}{[z]} \frac{\partial U}{\partial Z} = 0$$
$$\mu \frac{\partial U}{\partial Z} = 0 \tag{29a}$$

2.9.2 v

$$\eta \frac{\partial v}{\partial z} = 0 \tag{8b}$$
$$[\eta]\, \mu \frac{[v]}{[z]} \frac{\partial V}{\partial Z} = 0$$
$$\mu \frac{\partial V}{\partial Z} = 0 \tag{29b}$$

2.10 $9a \to 30a$

No slip boundary condition for: $y < 0,\ z = 0$
Becomes the no slip boundary condition for: $Y < 0,\ Z = 0$

$$u = v = 0 \tag{9a}$$
$$U = V = 0 \tag{30a}$$

2.11 $9b \to 30b$

Subtemperate slip boundary condition for: $y < 0,\ z = 0$
Becomes the subtemperate slip boundary condition for: $Y < 0,\ Z = 0$

2.11.1 *title*

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