Nondimensionalization of Haseloff et al. 2018

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1 Nondimensionalization of Velocity Equations

Nondimensionalizing the equations in terms of dimensional scales (this section will not yet try write the scales in terms of parameters)
Here I use:

$$\begin{gathered} y = [z]\, Y \qquad z = [z]\, Z \\ u = [u]\, U \qquad v = [v]\, V \qquad w = [v]\, W \\ \eta = [\eta]\, \mu \qquad p = [p]\, P \qquad s' = [s']\, S' \end{gathered}$$

The convention I used here is to provide the equation number when it matches the same equation in Haseloff's paper. If the equation doesn't match an equation in Haseloff, but I plan to resolve the differences later in the notes I will denote that with primes.

1.1 2 → 22

$$\frac{\partial}{\partial y}\left(\eta \frac{\partial u}{\partial y}\right) + \frac{\partial}{\partial z}\left(\eta \frac{\partial u}{\partial z}\right) = 0 \tag{2}$$

$$\frac{1}{[z]}\frac{\partial}{\partial Y}\left([\eta]\,\mu \frac{[u]}{[z]}\frac{\partial U}{\partial Y}\right) + \frac{1}{[z]}\frac{\partial}{\partial Z}\left([\eta]\,\mu \frac{[u]}{[z]}\frac{\partial U}{\partial Z}\right) = 0$$

$$\frac{[\eta]\,[u]}{[z]^2}\left[\frac{\partial}{\partial Y}\left(\mu \frac{\partial U}{\partial Y}\right) + \frac{\partial}{\partial Z}\left(\mu \frac{\partial U}{\partial Z}\right)\right] = 0$$

$$\frac{\partial}{\partial Y}\left(\mu \frac{\partial U}{\partial Y}\right) + \frac{\partial}{\partial Z}\left(\mu \frac{\partial U}{\partial Z}\right) = 0 \tag{22}$$

1.2 3a → 23a

$$\frac{\partial}{\partial y}\left(2\eta \frac{\partial v}{\partial y}\right) + \frac{\partial}{\partial z}\left[\eta\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right)\right] - \frac{\partial p}{\partial y} = 0 \tag{3a}$$

$$\frac{1}{[z]}\frac{\partial}{\partial Y}\left(2\,[\eta]\,\mu \frac{[v]}{[z]}\frac{\partial V}{\partial Y}\right) + \frac{1}{[z]}\frac{\partial}{\partial Z}\left[[\eta]\,\mu \frac{[v]}{[z]}\left(\frac{\partial V}{\partial Z} + \frac{\partial W}{\partial Y}\right)\right] - \frac{[p]}{[z]}\frac{\partial P}{\partial Y} = 0$$

$$\frac{[\eta]\,[v]}{[z]^2}\frac{\partial}{\partial Y}\left(2\mu \frac{\partial V}{\partial Y}\right) + \frac{[\eta]\,[v]}{[z]^2}\frac{\partial}{\partial Z}\left[\mu\left(\frac{\partial V}{\partial Z} + \frac{\partial W}{\partial Y}\right)\right] - \frac{[p]}{[z]}\frac{\partial P}{\partial Y} = 0 \tag{3a$'$}$$

1.3 3b → 23b

$$\frac{\partial}{\partial y}\left[\eta\left(\frac{\partial v}{\partial z}+\frac{\partial w}{\partial y}\right)\right]+\frac{\partial}{\partial z}\left(2\eta\frac{\partial w}{\partial z}\right)-\frac{\partial p}{\partial z}=0 \tag{3b}$$

$$\frac{1}{[z]}\frac{\partial}{\partial Y}\left[[\eta]\,\mu\frac{[v]}{[z]}\left(\frac{\partial V}{\partial Z}+\frac{\partial W}{\partial Y}\right)\right]+\frac{1}{[z]}\frac{\partial}{\partial Z}\left(2\,[\eta]\,\mu\frac{[v]}{[z]}\frac{\partial W}{\partial Z}\right)-\frac{[p]}{[z]}\frac{\partial P}{\partial Z}=0$$

$$\frac{[\eta]\,[v]}{[z]^2}\frac{\partial}{\partial Y}\left[\mu\left(\frac{\partial V}{\partial Z}+\frac{\partial W}{\partial Y}\right)\right]+\frac{[\eta]\,[v]}{[z]^2}\frac{\partial}{\partial Z}\left(2\mu\frac{\partial W}{\partial Z}\right)-\frac{[p]}{[z]}\frac{\partial P}{\partial Z}=0 \tag{3b$'$}$$

1.4 3c → 23c

$$\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0 \tag{3c}$$

$$\frac{[v]}{[z]}\frac{\partial V}{\partial Y}+\frac{[v]}{[z]}\frac{\partial W}{\partial Z}=0$$

$$\frac{\partial V}{\partial Y}+\frac{\partial W}{\partial Z}=0 \tag{23c}$$

1.5 4 → 24

$$\eta=\frac{A^{-1/n}}{2^{1/n}}\left[\left|\frac{\partial u}{\partial y}\right|^2+\left|\frac{\partial u}{\partial z}\right|^2+\left|\frac{\partial v}{\partial z}+\frac{\partial w}{\partial y}\right|^2+2\left|\frac{\partial v}{\partial y}\right|^2+2\left|\frac{\partial w}{\partial z}\right|^2\right]^{\frac{1-n}{2n}} \tag{4}$$

$$\frac{A^{-1/n}}{2^{1/n}}\left[\frac{[u]^2}{[z]^2}\left(\left|\frac{\partial U}{\partial Y}\right|^2+\left|\frac{\partial U}{\partial Z}\right|^2\right)+\frac{[v]^2}{[z]^2}\left(\left|\frac{\partial V}{\partial Z}+\frac{\partial W}{\partial Y}\right|^2+2\left|\frac{\partial V}{\partial Y}\right|^2+2\left|\frac{\partial W}{\partial Z}\right|^2\right)\right]^{\frac{1-n}{2n}}$$

$$\frac{A^{-1/n}}{2^{1/n}}\left(\frac{[u]}{[z]}\right)^{\frac{1-n}{n}}\left[\left|\frac{\partial U}{\partial Y}\right|^2+\left|\frac{\partial U}{\partial Z}\right|^2+\frac{[v]^2}{[u]^2}\left(\left|\frac{\partial V}{\partial Z}+\frac{\partial W}{\partial Y}\right|^2+2\left|\frac{\partial V}{\partial Y}\right|^2+2\left|\frac{\partial W}{\partial Z}\right|^2\right)\right]^{\frac{1-n}{2n}} \tag{4$'$}$$

1.6 5 → 25

Boundary condition for: $y \to \infty$
Becomes the boundary condition for: $Y \to \infty$
This equation can be broken into three parts:

1.6.1 u

$$\eta\frac{\partial u}{\partial y}\to\tau_s \tag{5a}$$

$$[\eta]\,\mu\frac{[u]}{[z]}\frac{\partial U}{\partial Y}\to\tau_s$$

$$\mu\frac{\partial U}{\partial Y}\to\frac{\tau_s\,[z]}{[u]\,[\eta]} \tag{5a$'$}$$

1.6.2 v

$$\frac{\partial v}{\partial z}\to 0 \tag{5b}$$

$$\frac{\partial V}{\partial Z}\to 0 \tag{25b}$$

1.6.3 w

$$w \to 0 \tag{5c}$$
$$W \to 0 \tag{25c}$$

1.7 6 → 26

Boundary condition for: $y \to -\infty$
Becomes the boundary condition for: $Y \to -\infty$
This equation can be broken into three parts:

1.7.1 u

$$u \to 0 \tag{6a}$$
$$U \to 0 \tag{26a}$$

1.7.2 v

$$v \to \frac{n+2}{n+1}\frac{q_r}{h_s}\left[1-\left(1-\frac{z}{h_s}\right)^{n+1}\right] \tag{6b}$$
$$v \to \frac{n+2}{n+1}\frac{q_r}{h_s}\left[1-\left(1-\frac{[z]\,Z}{h_s}\right)^{n+1}\right] \tag{6b'}$$

1.7.3 w

$$w \to 0 \tag{6c}$$
$$W \to 0 \tag{26c}$$

1.8 7 → 28

Boundary condition for: $z = 0$
Becomes the boundary condition for: $Z = 0$

$$w = 0 \text{ at } z = 0 \tag{7}$$
$$W = 0 \text{ at } Z = 0 \tag{28}$$

1.9 8 → 29

Boundary condition for: $y > 0,\ z = 0$
Becomes the boundary condition for: $Y > 0,\ Z = 0$
This equation can be broken into two parts:

1.9.1 u

$$\eta\frac{\partial u}{\partial z} = 0 \tag{8a}$$
$$[\eta]\,\mu\frac{[u]}{[z]}\frac{\partial U}{\partial Z} = 0$$
$$\mu\frac{\partial U}{\partial Z} = 0 \tag{29a}$$

1.9.2 v

$$\eta \frac{\partial v}{\partial z} = 0 \tag{8b}$$
$$[\eta]\,\mu \frac{[v]}{[z]} \frac{\partial V}{\partial Z} = 0$$
$$\mu \frac{\partial V}{\partial Z} = 0 \tag{29b}$$

1.10 9a → 30a

No slip boundary condition for: $y < 0,\ z = 0$
Becomes the no slip boundary condition for: $Y < 0,\ Z = 0$

$$u = v = 0 \tag{9a}$$
$$U = V = 0 \tag{30a}$$

1.11 9b → 30b

Subtemperate slip boundary condition for: $y < 0,\ z = 0$
Becomes the subtemperate slip boundary condition for: $Y < 0,\ Z = 0$ This equation can be broken into three parts:

1.11.1 $\sqrt{u^2 + v^2} > 0$, u

$$\eta \frac{\partial u}{\partial z} = \tau_c \frac{u}{\sqrt{u^2 + v^2}} \tag{9b,I}$$
$$[\eta]\,\mu \frac{[u]}{[z]} \frac{\partial U}{\partial Z} = \tau_c \frac{U}{\sqrt{U^2 + \frac{[v]^2}{[u]^2} V^2}}$$
$$\mu \frac{\partial U}{\partial Z} = \frac{[z]\,\tau_c}{[u]\,[\eta]} \frac{U}{\sqrt{U^2 + \frac{[v]^2}{[u]^2} V^2}} \tag{9b',I}$$

1.11.2 $\sqrt{u^2 + v^2} > 0$, v

$$\eta \frac{\partial v}{\partial z} = \tau_c \frac{v}{\sqrt{u^2 + v^2}} \tag{9b,II}$$
$$[\eta]\,\mu \frac{[v]}{[z]} \frac{\partial V}{\partial Z} = \tau_c \frac{[v]}{[u]} \frac{V}{\sqrt{U^2 + \frac{[v]^2}{[u]^2} V^2}}$$
$$\mu \frac{\partial V}{\partial Z} = \frac{[z]\,\tau_c}{[u]\,[\eta]} \frac{V}{\sqrt{U^2 + \frac{[v]^2}{[u]^2} V^2}} \tag{9b',II}$$

1.11.3 $\sqrt{u^2+v^2}=0$

$$\sqrt{\left(\eta\frac{\partial u}{\partial z}\right)^2+\left(\eta\frac{\partial v}{\partial z}\right)^2}\leq\tau_c \tag{9b,III}$$

$$\frac{[\eta]\,[u]}{[z]}\sqrt{\left(\mu\frac{\partial U}{\partial Z}\right)^2+\left(\mu\frac{[v]}{[u]}\frac{\partial V}{\partial Z}\right)^2}\leq\tau_c$$

$$\sqrt{\left(\mu\frac{\partial U}{\partial Z}\right)^2+\frac{[v]^2}{[u]^2}\left(\mu\frac{\partial V}{\partial Z}\right)^2}\leq\frac{[z]\,\tau_c}{[u]\,[\eta]} \tag{9b',III}$$

1.12 $10\rightarrow 27$

Boundary condition for: $z=h_s$
Becomes the boundary condition for: $Z=1$
This equation can be broken into four parts:

1.12.1 u

$$\eta\frac{\partial u}{\partial z}=0 \tag{10a}$$

$$\mu\frac{\partial U}{\partial Z}=0 \tag{27a}$$

1.12.2 v

$$\eta\frac{\partial v}{\partial z}=0 \tag{10b}$$

$$\mu\frac{\partial V}{\partial Z}=0 \tag{27b}$$

1.12.3 w

$$w=0 \tag{10c}$$

$$W=0 \tag{27c}$$

1.12.4 p

$$2\eta\frac{\partial w}{\partial z}-p+\rho g s'=0 \tag{10d}$$

$$2\,[\eta]\,\mu\frac{[v]}{[z]}\frac{\partial W}{\partial Z}-[p]\,P+\rho g\,[s']\,S'=0$$

$$2\mu\frac{\partial W}{\partial Z}-\frac{[z]\,[p]}{[\eta]\,[v]}P+\frac{\rho g\,[s']\,[z]}{[\eta]\,[v]}S'=0 \tag{10d'}$$

2 Finding Scales and Resolving the Primes

2.1 5a'

This equation gives the first two scales:

$$[z]=h_s$$

$$\tau_s=\frac{[u]\,[\eta]}{[z]}=\frac{[u]\,[\eta]}{h_s}$$

Which gives the final form:

$$\mu \frac{\partial U}{\partial Y} \to 1 \tag{25a}$$