Gauss on Gaussian Quadrature

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March 8th, 2022

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Introduction - general quadrature

A quadrature rule allows you to write an integral of some function $f(x)$ over a closed interval, as a weighted sum of the function evaluated at quadrature nodes.

Quadrature Rule

$\int_a^b f(x)dx \approx \sum_{i=1}^n w_i f(x_i)$

Examples

Simpsons rule

$\int_0^1 e^{\frac{-x^2}{2}} dx \approx \frac{1}{6}[e^{\frac{0^2}{2}} + 4e^{\frac{-0.5^2}{2}} + e^{\frac{-1^2}{2}}] \approx 0.856086378341836$

$w_0 = \frac{1}{6}, w_1 = \frac{2}{3}, w_2 = \frac{1}{6}, x_0 = 0, x_1 = 0.5, x_2 = 1$

Substitution rule

$\int_a^b f(t)dt = \int_{-1}^1 f(u(t))u'(t)dt$

Introduction - general quadrature

Newton-Cotes rules

$\int_{-1}^{1} f(x)dx \approx \sum_{i=1}^{n} w_i f(x_i)$, with $\Delta x_i = h$

Solving Newton-Cotes rule

$$\begin{bmatrix} 1 & 1 & \dots & 1 \\ x_1^1 & x_2^1 & \dots & x_n^1 \\ x_1^2 & x_2^2 & \dots & x_n^2 \\ \dots & & & \\ x_1^{n-1} & x_2^{n-1} & \dots & x_n^{n-1} \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ \dots \\ w_n \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ \dots \\ \frac{1-(-1)^n}{n} \end{bmatrix} = \begin{bmatrix} \int_{-1}^{1} X^0 dx \\ \int_{-1}^{1} X^1 dx \\ \dots \\ \int_{-1}^{1} X^{n-1} dx \end{bmatrix}$$

Motivation

Exact polynomials

$$\int_{-1}^{1}(a_0+a_1x+...+a_{2n-1}x^{2n-1})dx = \sum_{i=1}^{n} w_i(a_0+a_1x_i+...+a_{2n-1}x_i^{2n-1})$$

$$\sum_{i=1}^{n} w_i(a_0 + a_1x + ... + a_{2n-1}x^{2n-1}) = \sum_{i=1}^{n}\sum_{k=1}^{2n}(a_{k-1}x_i^{k-1}w_i)$$

Solving Gauss Quadrature rule

$$\begin{bmatrix} a_0 & \dots & a_{2n-1} \end{bmatrix} \begin{bmatrix} \int_{-1}^{1} x^0 dx \\ \int_{-1}^{1} x^1 dx \\ \dots \\ \int_{-1}^{1} x^{2n-1} dx \end{bmatrix} = \begin{bmatrix} a_0 \\ \dots \\ a_{2n-1} \end{bmatrix}^T \begin{bmatrix} 1 & \dots & 1 \\ x_1^1 & \dots & x_n^1 \\ x_1^2 & \dots & x_n^2 \\ \dots & & \\ x_1^{2n-1} & \dots & x_n^{2n-1} \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ \dots \\ w_n \end{bmatrix}$$

Gaussian quadrature

Gaussian Quadrature allows weights and nodes to vary giving us $2n$ degrees of freedom to work with

Exact polynomials

$$\int_{-1}^{1}(a_{2n-1}x^{2n-1}+...+a_1x+a_0)dx = \sum_{i=1}^{n} w_i(a_{2n-1}x_i^{2n-1}+...+a_1x_i+a_0)$$

Solving Gauss Quadrature rule

$$\begin{bmatrix} 1 & 1 & \dots & 1 \\ x_1^1 & x_2^1 & \dots & x_n^1 \\ x_1^2 & x_2^2 & \dots & x_n^2 \\ \dots & & & \\ x_1^{2n-1} & x_2^{2n-1} & \dots & x_n^{2n-1} \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ \dots \\ w_n \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ \dots \\ \frac{1-(-1)^{2n}}{2n} \end{bmatrix}$$

! Danger !

It is not obvious that a general solution would exist, except possibly to Gauss

Gaussian quadrature

Solving Gauss Quadrature rule

$$\begin{bmatrix} \frac{1}{x} & \frac{1}{x} & \dots & \frac{1}{x} \\ \frac{x_1}{x^2} & \frac{x_2}{x^2} & \dots & \frac{x_n}{x^2} \\ \frac{x_1^2}{x^3} & \frac{x_2^2}{x^3} & \dots & \frac{x_n^2}{x^3} \\ \dots \\ \frac{x_1^{2n-1}}{x^{2n}} & \frac{x_2^{2n-1}}{x^{2n}} & \dots & \frac{x_n^{2n-1}}{x^{2n}} \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ \dots \\ w_n \end{bmatrix} = \begin{bmatrix} 2/x \\ 0/x^2 \\ \dots \\ (\frac{1-(-1)^{2n}}{2n})/x^{2n} \end{bmatrix} \iff$$

$$\begin{bmatrix} \sum_{k=1}^{n} \frac{w_k}{x} \\ \sum_{k=1}^{n} \frac{w_k x_k}{x^2} \\ \sum_{k=1}^{n} \frac{w_k x_k^2}{x^3} \\ \dots \\ \sum_{k=1}^{n} \frac{w_k x_k^{2n-1}}{x^{2n}} \end{bmatrix} = \begin{bmatrix} 2/x \\ 0/x^2 \\ \dots \\ (\frac{1-(-1)^{2n}}{2n})/x^{2n} \end{bmatrix} \iff$$

$$\sum_{k=1}^{n} \frac{w_k}{x} + \sum_{k=1}^{n} \frac{w_k x_k}{x^2} + \dots + \sum_{k=1}^{n} \frac{w_k x_k^{2n-1}}{x^{2n}} = \frac{2}{x} + \frac{0}{x^2} + \frac{\frac{2}{3}}{x^3} + \dots + \frac{\frac{2}{2n-1}}{x^{2n-1}}$$

Gauss

$$\frac{w_k}{x-x_k} = \frac{\frac{w_k}{x}}{1-\frac{x_k}{x}} = (\frac{w_k}{x})(1+\frac{x_k}{x}+(\frac{x_k}{x})^2+\ldots) = \frac{w_k}{x}+\frac{w_k x_k}{x^2}+\frac{w_k x_k^2}{x^3}+\ldots$$

$$\sum_{k=1}^{n} \frac{w_k}{x-x_k} = \sum_{k=1}^{n} \frac{w_k}{x} + \sum_{k=1}^{n} \frac{w_k x_k}{x^2} + \sum_{k=1}^{n} \frac{w_k x_k^2}{x^3} + \ldots + \frac{w_k x_k^{2n-1}}{x^{2n}}$$

$$\sum_{k=1}^{n} \frac{w_k}{x-x_k} = \frac{2}{x} + \frac{0}{x^2} + \frac{\frac{2}{3}}{x^3} + \ldots + \frac{\frac{2}{2n-1}}{x^{2n}} \approx \log\frac{1+x}{1-x}$$

Gauss realized that the conditions that $x_1, x_2, \ldots, x_n, w_1, w_2, \ldots, w_n$ are subject to, are identical with the requirement that as many terms as possible in the expansion coincide with the expansion of the logarithm.

Gauss

The sum is a rational function of x, $\sum_{k=1}^{n} \frac{w_k}{x-x_k} = P_n(x)/Q_n(x)$, where P and Q are polynomials of degree $n-1$ and n respectively. The roots of the polynomial $Q_n(x)$ are the quadrature nodes.

Gauss Continued fractions and hypergeometric functions

$$\frac{1}{2}\ln\frac{x+1}{x-1} = \cfrac{1}{x - \cfrac{\frac{1*1}{1*3}}{x - \cfrac{\frac{2*2}{3*5}}{x - \cfrac{\frac{3*3}{5*7}}{x - \ldots}}}}$$

lemma 1 - There is a formula for computing $P_n(x), Q_n(x)$

$$P_0 = 0, P_1 = 1, Q_{-1} = 0, Q_0 = 1$$

$$P_n = q_n P_{n-1} + P_{n-2}, Q_n = q_n Q_{n-1} + Q_{n-2}$$

$$q_{2n} = -\frac{2^2 * 4^2 \ldots (2n-2)^2}{3^2 * 5^2 \ldots (2n-1)^2}(4n-1)x$$

$$q_{2n+1} = -\frac{1^2 * 3^2 \ldots (2n-1)^2}{2^2 * 4^2 \ldots (2n)^2}(4n+1)x$$

Theorem 1

Choose the nodes $x_1, ..., x_n$ as roots of the polynomial Q_n computed recursively, (lemma 1), and the corresponding weights $w_1, ..., w_n$ from the formula

$$w_k = 2\frac{P_n(x_k)}{Q_n' x_k}$$

$P_n(x_k)$ computed by (lemma 2). then $\int_{-1}^{1} f(x)dx \approx \sum_{i=1}^{n} w_i f(x_i)$ is the optimal quadrature formula.

Conclusion

quote

> *"Gauss himself wrote elegant, but highly compact, carefully polished papers with no hint of the motivation, meaning, or details of the steps. When criticized, he said that no architect leaves the scaffolding after completing the building. But the fact is that even excellent mathematicians found the reading of Gauss's papers very difficult".*

-Morris Kline, Mathematics and the Physical World (1959) Ch. 5: Numbers Known and Unknown, pp. 58-59; for the original of the building quote, see Wolfgang Sartorius von Waltershausen's 1856 memorial (quoted above)