Review - Chapter 5: Limits and Continuity of Functions

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5.1 Limit of a Function

Definition 1 ($\epsilon - \delta$ limit definition)**.** *Let f be a real-valued function with domain $E \in R$ and fix a point $p \in R$ that is an accumulation point of E and let $\ell \in R$. We say*

$$\lim_{x \to p} f(x) = \ell$$

if $\forall \epsilon > 0$, $\exists \delta > 0$ *s.t.*

$$x \in E \text{ and } 0 < |x - p| < \delta \implies |f(x) - \ell| < \epsilon$$

.

In other words, we can always make $f(x)$ arbitrarily close to ℓ by making x sufficiently close to p. Also, the condition $0 < |x - p|$ says that we do not care about the value of $f(x)$ at $x = p$. If we did, then the definition would require funtions that are defined at the point p to satisfy $f(p) = \ell$. Limits only address the behavior of a function at points near $x = p$.

Example 1.

$$\lim_{x \to 0} x \cdot sin\left(\frac{1}{x}\right) = 0$$

Using the fact that $|x \cdot sin\left(\frac{1}{x}\right)| \leq |x|$ we see that our function $f(x)$ will be within a given distance from 0 so long as x is within that distance from 0. This motivates the following proof.

Proof. Fix $\epsilon > 0$ and let $\delta = \epsilon$. Then

$$0 < |x - 0| < \delta \implies |f(x) - 0| = |x \cdot sin\left(\frac{1}{x}\right)| \leq |x| \leq \delta = \epsilon.$$

$\square$

Example 2. *Let $E = R$. Then $\lim_{x \to p} g(x)$ doesnt exist for any $p \in E$ where $g(x)$ is defined*

$$g(x) = \begin{cases} 1 & x \text{ is rational} \\ 0 & x \text{ is irrational} \end{cases}$$

Proof. Suppose for a contradiction that $\lim_{x \to p} g(x) = \ell$ and let $\epsilon = \frac{1}{2} > 0$. Then there is δ such that $0 < |x - p| < \delta \implies |g(x) - \ell| < \epsilon = \frac{1}{2}$. For any $\delta > 0$ we have rational and irrational values of x satisfying $0 < |x - p| < \delta$. Thus $|1 - \ell| < \frac{1}{2}$ and $|0 - \ell| < \frac{1}{2}$.

$$\implies |1| = |1 - 0| = |1 - \ell + \ell - 0| \leq |1 - \ell| + |\ell - 0| < \frac{1}{2} + \frac{1}{2} = 1.$$

This is a contradiction so the limit does not exist. The limit does not exist! $\square$

Proposition 1. *f is a function definded on domain E and p is an accumulation point. If $\lim_{x\to p} f(x) = \ell$ and $\lim_{x\to p} f(x) = m$, then $\ell = m$.*

Proof. Let $\epsilon > 0$. Then there is δ_1 such that $0 < |x-p| < \delta_1 \implies |f(x) - \ell| < \frac{\epsilon}{2}$ and δ_2 such that $0 < |x-p| < \delta_2 \implies |f(x) - m| < \frac{\epsilon}{2}$. Let $\delta = \min(\delta_1, \delta_1)$. Then we have $0 < |x-p| < \delta$ such that

$$\begin{aligned} |\ell - m| &= |\ell - f(x) + f(x) - m| \\ &\leq |\ell - f(x)| + |f(x) - m| \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\ &= \epsilon \end{aligned}$$

Hence we can make ℓ and m as close as we like. $\ell = m$. □

This proof essentially zips together ℓ and m using $f(x)$ as $x \to p$. ℓ and m both get arbitrarily close to the same value $f(x)$ as $x \to p$.

Corollary 1. *$\lim_{x\to p} f(x) = \lim_{h\to 0} f(p+h)$ if either limit is defined.*

Theorem 1 (Elementary Properties of Limits)**.** *Let f and g be functions defined on domain E and let p be and accumulation point of E. Assume that*

$$\begin{aligned} &i)\lim_{x\to p} f(x) = \ell \\ &ii)\lim_{x\to p} g(x) = m \end{aligned}$$

Then

$$\begin{aligned} &a)\lim_{x\to p}(f+g)(x) = \ell + m \\ &b)\lim_{x\to p}(f\cdot g)(x) = \ell \cdot m \\ &c)\lim_{x\to p}(f/g)(x) = \ell/m \qquad \textit{provided that } m \neq 0 \end{aligned}$$

Proof. to-do □

We can pretty quickly verify that $\forall p \in R$
1) $\lim_{x\to p} x = p$
2) $\lim_{x\to p} \alpha = \alpha$

Using the elementary properties above this gives the following for any polynomial F and any rational function R.

$$1)\lim_{x\to p} F(x) = F(p) \tag{1}$$

$$2)\lim_{x\to p} R(x) = R(p) \tag{2}$$

Example 3. $\lim_{x\to 0} sin(x) = 0$, $\lim_{x\to 0} cos(x) = 1$.

Proof. For small values of $x > 0$, $sin(x) < x$. On the other hand, small values of $x < 0$ give $sin(x) > -x$. In either case $|sin(x)| < |x|$. Hence for $\epsilon > 0$, we set $\delta = \epsilon$ so that

$$0 < |x - 0| < \delta \implies |sin(x) - 0| < |x| < \delta = \epsilon.$$

It's reasonable to then conclude $\lim_{x\to 0} cos(x) = 1$ because $cos(x) = \sqrt{1 - sin^2(x)}$ near $x = 0$. □

Remark 1. *We didn't really show that radicals preserve limits. I should probably prove this to myself later.*

This is all we need to find *sin* and *cos* limits at any point p.

$$\begin{aligned}
\lim_{x\to p} sin(x) &= \lim_{h\to 0} sin(p+h) \\
&= lim_{h\to 0} sin(P)cos(h) + sin(h)cos(P) \\
&= sin(p)\cdot 1 + 0\cdot cos(P) \\
&= sin(p)
\end{aligned}$$

For completeness, we should also do

$$\begin{aligned}
\lim_{x\to p} cos(x) &= \lim_{h\to 0} cos(p+h) \\
&= lim_{h\to 0} cos(P)cos(h) - sin(h)sin(P) \\
&= cos(p)\cdot 1 - 0\cdot sin(P) \\
&= cos(p)
\end{aligned}$$

Proposition 2. *Let f be a function with domain E and p be an accumulation point of E. Then*

$$\lim_{x\to p} f(x) = \ell$$

if and only if

$$a_j \subset E\setminus p \text{ and } \lim_{j\to\infty} a_j = p \implies \lim_{j\to\infty} f(a_j) = \ell.$$

In other words every sequence in $E\setminus p$ converging to p must have its image converging to ℓ.

Proof. to-do □