Review - Chapter 4: Basic Topology

Parker Hyde

February 20, 2022

4.5 Connected and Disconnected Sets

Different books give different definitions for connected and disconnected sets. Abbott and Rudin both formulate a definition in terms of *'separated'* sets and Krantz gives a 3 part definition in terms of open sets. As usual we'll give these definitions and then feel the need to prove their equivalence.

First, the Abbott/Rudin defition. We'll use the notation $\overline{A}$ to denote the closure of A.

Definition 1 (Disconnected Sets - A/R)**.** *Two nonempty sets $A, B \subset R$ are separated if $\overline{A} \cap B = \emptyset$ and $A \cap \overline{B} = \emptyset$. A set $E \subset R$ is **disconnected** if it can be written as the union $E = A \cup B$ of separated sets A and B.*

Krantz is harder to parse.

Definition 2 (Disconnected Sets - Krantz)**.** *A set $S \subset R$ is **disconnected** if it's possible to find open sets U and V such that*

$$U \cap S \neq \emptyset, V \cap S \neq \emptyset \tag{1}$$
$$(U \cap S) \cap (V \cap S) = \emptyset \tag{2}$$
$$S = (U \cap S) \cup (V \cap S) \tag{3}$$

Intuitively, the Krantz definition is a bit more transparent if we substitute $A = U \cap S$ and $B = V \cap S$. This gives the simpler looking requirements

$$A \neq \emptyset, B \neq \emptyset$$
$$A \cap B = \emptyset$$
$$S = A \cup B.$$

So basically we need disjoint nonempty sets $A, B \subset S$ whose union gives S. But we must also recall that $A = U \cap S$ and $B = V \cap S$ for open sets U and V. Thus, there must be at least one point 'between' A and B. This is clear if A and B end up both being open sets. If one of them is closed, say B, then it was formed by an open set V with $B \subset V$. A and B cannot share some boundary point $b \in B$. This would require that $A \cap (B = V \cap S) \neq \emptyset$ because V would necessarily contain points in A.

Proof. □