Review - Chapter 4: Basic Topology

Parker Hyde

February 18, 2022

4.2 Points in Open and Closed Sets

The Krantz book uses the terms *'limit point'* and *'accumulation point'* interchangebly. Let's define these terms and prove they're the same thing.

Definition 1 (limit point)**.** *A point $x \in R$ is a **limit point** of a set $S \subset R$ if $\forall \epsilon > 0$, $N_\epsilon(x)$ contains an element of S other than x.*

Definition 2 (accumulation point)**.** *A point $x \in R$ is a **accumulation point** of a set $S \subset R$ if $\forall \epsilon > 0$, $N_\epsilon(x)$ contains infinitely many elements of S.*

Basically we consider every $N_\epsilon(x)$ to determine what kind of point x is for a set $S \subset R$.

1) if every $N_\epsilon(x)$ contains a point other than x in $S \to x$ is a limit point
2) if every $N_\epsilon(x)$ contains an infinite number of points in $S \to x$ is an accumulation point

Proposition 1. *A point $x \in S \subset R$ is a limit point of S iff it's an accumulation point of S.*

Proving accumulation point $\implies$ limit point seems pretty simple. Any $N_\epsilon(x)$ of an accumulation point contains infinitely many points in S. In particular it contains at least 2 points in S. So it contains a point in S other than x.

Alright lets try the other direction. We have some $N_\epsilon(x)$ for a limit point x and we need to show it satisfies the requirements for an accumulation point. x is a limit point so $N_\epsilon(x)$ contains some $s_1 \neq x \in S$. Cool, we have one point. Infinitely many to go. But this is actually pretty easy right? We can just choose $s_2 \neq x \in S$ from a smaller $N_{\epsilon'}(x)$ where ϵ' is set to $|s_1 - x|$. We can do this infinitely many times and always get a new s_n becaues x is a limit point. Thus we get an infinite number of $s_1 \neq x, s_2 \neq x, ... \in S$ so x is an accumulation point.

Other sources have a variety of definitions for these terms. So this result just follows from our particular definitions. Moving on...

boundary points, interior points, isolated points

Definition 3 (boundary point)**.** *$b \in R$ is a **boundary point** of $S \subset R$ if every $N_\epsilon(b)$ contains points in S and points in $R \setminus S$. We denote the set of boundary points for S as ∂S.*

For the most part, boundary points are what we expect them to be. You can perturb a boundary point in the appropriate direction and it will no longer be in S.

Boundary points may or may not be in the set S. We'll see in a second that

1) closed sets - contain all their boundary points
2) open sets - contain none of their boundary points

Oh this is pretty interesting. The boundary set of Q, ∂Q, is the entire real line. This makes sense because any neighborhood around a rational contains infinitely many rational and irrational numbers.

Definition 4 (interior point)**.** *A point $s \in S \subset R$ is an **interior point** of S if there is an $N_\epsilon(s) \subset S$.*

From this definition we see that open sets require *all* points to interior points.

Definition 5 (isolated point)**.** *A point $t \in S \subset R$ is an **isolated point** if there is an $N_\epsilon(t)$ such that $N_\epsilon(t) \cap S = \{t\}$*

Proposition 2. *Each point of $S \subset R$ is either an interior point or a boundary point.*

Proof. let $x \in S$. If x is an interior point then we're done. Suppose x is not an interior point. Then all $N_\epsilon(x)$ contain points in $R \setminus S$. Also $N_\epsilon(x)$ contains $x \in S$. So x contains points in both S and $R \setminus S$. x is a boundary point. $\square$

Quick remark. interior points are a special class of boundary points. Also accumulation point can either be interior or boundary points but never isolated.

Proposition 3. *The boundary ∂S of a set $S \subset R$ is also the boundary of $R \setminus S$.*

Proof.

$$\begin{aligned} b \in \partial S &\iff \text{there exsits } N_\epsilon(b) \text{ containing points in } S \text{ and } R \setminus S \\ &\iff \text{there exsits } N_\epsilon(b) \text{ containing points in } R \setminus S \text{ and } S \\ &\iff b \in \partial R \setminus S \end{aligned}$$

$\square$

This is trivial if you think about it. The definition of ∂A and ∂A^c is identical no matter the set A.