Review - Chapter 4: Basic Topology

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4.1 Open and Closed Sets

Interval notation

For real numbers $a \leq b$, we define

$$\begin{aligned}
(a,b) &= \{x \in R : a < x < b\} \\
[a,b] &= \{x \in R : a \leq x \leq b\} \\
[a,b) &= \{x \in R : a \leq x < b\} \\
(a,b] &= ...
\end{aligned}$$

These are simplest open, closed, and half-open (half-closed, clopen) sets everyone knows.

Definition 1 (open sets)**.** *A set U is **open** if $\forall x \in U$ we can find an $\epsilon > 0$ so that $N_\epsilon(x) \subset U$*

Basically the set is open if all points are "padded" by other points in set. For any $x \in (0,1)$ there's always an infintude of points surrounding that x.

How to prove a set U is open

The general strategy is to fix some $x \in U$ and then find an ϵ so that $N_\epsilon(x) \subset U$.

Example 1. *Show that the set $U = \{x \in R : |x - m| < d\}$ is open*

We need to find the appropriate ϵ.
Start by fixing $x \in U$. If $|x - m| < d$ then $d - |x - m| > 0$. This just says that the maximum allowed distance d is greater than whatever distance x actually is from m.
So lets pick this difference for ϵ. We set $\epsilon = d - |x - m|$.
Whatever distance x is from m, adding less than ϵ distance will never take x more than distance d from m. So now we just use this info to prove it.

Proof. Fix $x \in U$ and set $\epsilon = d - |x - m| > 0$. Let $t \in N_\epsilon(x)$. Then $|t - x| < \epsilon$ and

$$\begin{aligned}
|t - m| &= |(t - x) + (x - m)| \\
&\leq |t - x| + |x - m| \\
&< d - |x - m| + |x - m| \\
&= d \\
&\implies t \in U
\end{aligned}$$

Hence $N_\epsilon(x) \subset U$. Get fucked, we done. $\square$

Moving on..
Things we are about to cover:

1) The (possibly infinite) union of open sets is $\rightarrow$ open
2) The finite intersection of open sets is $\rightarrow$ open. Not necessarily true for infinite (think $(-\frac{1}{j}, \frac{1}{j})$).
3) The (possible infinite) intersection of closed sets is $\rightarrow$ closed
4) The finite union of of closed sets is $\rightarrow$ closed.

Infinite intersections do weird things to open sets.
Infinite unions do weird things to closed sets.

Theorem 1. *If U_α are open sets (possibly denumerable or uncountable), then*

$$U = \bigcup_{\alpha \in A} U_\alpha$$

is open.

How do we go about proving this. Again for any $x \in U$ we need to produce $N_\epsilon(x) \subset U$. In this case we can just borrow an N_ϵ whatever U_α our x comes from.

Proof. Fix $x \in U$. Then $x \in U_\alpha$ for some α. U_α is open so there exists an $N_\epsilon(x) \subset U_\alpha \subset U$.
U is open. come get some little bitch. □

Theorem 2. *$U_1, U_2, U_3, ..., U_k$ are open sets, then*

$$V = \bigcap_{j=1}^{k} U_j$$

is open.

This time we can't just take some $N_\epsilon(x)$ from U_j, assume it'll be a subset of V and call it a day. The intersection with other sets may subtract away some of our chosen $N_\epsilon(x)$. We have to pick the 'right' $N_\epsilon(x)$.

Proof. Fix $x \in V$. Then $x \in U_j$, $\forall j = 1, 2, ..., k$. Each of these sets are open so they all have corresponding neighborhoods $N_{\epsilon_j}(x)$. Let $\epsilon = \min\{\epsilon_j : j = 1, 2, ..., k\}$.
Then $N_\epsilon(x) \subset U_j$ for $j = 1, 2, ..., k$

$$\implies N_\epsilon(x) \subset V$$

. □

Again this is not necessarily true. For a situation like

$$\bigcap_{j=1}^{\infty} = (-\frac{1}{j}, \frac{1}{j}) \quad \text{or even} \quad \bigcap_{j=1}^{\infty} = (0, 1 + \frac{1}{j})$$

we get closed and clopen sets respectively. 1 is in the second set but a suitable $N_\epsilon(1)$ doesn't exist.

Theorem 3. *$U \subset R$ is a nonempty open set. Then for either finitiely many or countably many ($k = \infty$) pairwise disjoint open intervals I_j.*

$$U = \bigcup_{j=1}^{k} I_j$$

Proof. This is the proof by equivalence relation. Setting $a \cong b$ if $(a, b) \subset U$ creates a partion with equivalence classes I_α. Being equivalence classes means their union equals U. We just need to show the classes are open intervals. Clearly they're intervals. Let $x \in I_\alpha$. Then $x \in U$ so we have a $N_\epsilon(x)$. We constructed the equivalence relation so that $N_\epsilon(x) \subset I_\alpha$. All the points in $N_\epsilon(x)$ would have to be in I_α. □

Basically, open intervals are the building blocks of open sets. If you have an open set, you can build it from a union of open intervals. This is trivial for open sets that are intervals themselves.

The contrapositive is interesting here. If you can't set U equal to a union of open intervals then U isn't an open set.

Closed Sets

Definition 2. *$F \subset R$ is closed if the complement $R \setminus F$ is open.*

Immediately we should notice the potential contradiction. The sets $\emptyset$ and R are both open by definition (vacuously true for $\emptyset$). They are mutual complements so they sould both also be closed. In fact they're both closed and open.. At least that's what I read on math stack.

Example 2. *The set $B = \{\frac{1}{j} : j = 1, 2, 3, ...\} \cup \{0\}$ is closed because it's complement is the union of open sets given by*

$$(-\infty, 0) \ \cup \ \bigcup_{j=1}^{\infty} (\frac{1}{j+1}, \frac{1}{j}) \ \cup \ (1, \infty)$$

Theorem 4. *If E_α are closed sets (possibly uncountably many), then*

$$E = \bigcap_{\alpha \in A} E_\alpha$$

is closed.

Translating this using the definition of closed sets, we get a pretty quick proof:

Proof. If E_α are closed sets, then $(E_\alpha)^c$ are open sets and

$$E^c = (\bigcap_{\alpha \in A} E_\alpha)^c = \bigcup_{\alpha \in A} (E_\alpha)^c \text{ is open } \implies E \text{ is closed} \tag{1}$$

□

Here's another way. And pay attention, the subtle difference here is a little pedantic. In the proof above we showed that E^c is open so that E has to be closed. This time we show that E is the complement of something that is open.. So yeah E has to be closed.

Proof.

$$E = \bigcap_{\alpha \in A} E_\alpha = \bigcap_{\alpha \in A} (E_\alpha^c)^c = (\bigcup_{\alpha \in A} E_\alpha^c)^c \implies E \text{ is closed}$$

because $\bigcup_{\alpha \in A} E_\alpha^c$ is open. □