Review - Chapter 4: Basic Topology

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February 20, 2022

4.3 Compact Sets

In section 4.2 we established that every infinte sequence in a bounded sets S has a convergent subsequence. In particular, we used this to show that bounded infinite sets, which 'do' have infinite sequences, must have accumulation points. This was the theorem of Bolzano-Weierstrass.

It turns out that the converse is also true. We verify this in the following lemma.

Lemma 1. *If a set $S \subset R$ has the property that every sequence a_j in S has a convergent subsequence a_{j_k}, then S is bounded.*

Proof. Suppose for a contradiction that S is not bounded. Then we can construct a sequnce $\{a_j\}$ where each $a_j > j\ \forall j \in \mathbb{N}$. Now consider any subsequence a_{j_k}. Let's take the long road and show that this subsequence cannot be Cauchy.
Let $\epsilon = 1 > 0$. Now let $N \in \mathbb{N}$ and consider $a_{j_k}, j > N$. Then $m = \lceil a_{j_k} \rceil + 1 \in \mathbb{N}$ so there is some subsequence element $a_{j'_k} > m$. Thus $|a_{j'_k} - a_{j_k}| > 1 = \epsilon$. Hence we can't find an N satisfying the $\epsilon - N$ definition for $\{a_{j_k}\}$. a_{j_k} is not Cauchy.
This is a contradiction so S must be bounded. $\square$

We now have two independent identities concerning bounded and closed sets respectively.

1) A set S is **bounded** if and only if every sequence a_j in S has a convergent subsequence a_{j_k}.
2) A set S is **closed** if and only if every Cauchy sequence in S converges to a limit point $\alpha \in S$.

It follows that a set is closed and bounded if and only if the two RHS statments in the identities above are true. But we can get a simpler identity.

Lemma 2. *Let $S \subset R$ be a set with the property that every sequence $a_j \in S$ has a subsequence that converges to a limit point $\alpha \in S$. Then S satisfies the RHS statements of (1) and (2).*

Proof. The RHS of (1) is trivial. To prove (2), let a_j be a Cauchy sequence in S. Then a_j converges to a limit point x. By hypothesis, a_j has a subsequence a_{j_k} that converges to $\alpha \in S$. But this means $x = \alpha \in S$. $\square$

It's pretty easy to see that the converse of lemma 2 holds so we can skip it. Just notice that applying (2) to the subsequence produced by (1) yields the desired property.

Thus we get

Theorem 1 (Heine-Borel)**.** *A set $S \subset R$ is **closed** and **bounded** if and only if every sequence in S has a subsequence that converges to a limit that is also in S.*

This theorem motivates the following defintion.

Definition 1. *A set $S \subset R$ is called **compact** if every sequence in S has a subsequence that converges to a limit that is also in S.*

From Theroem 1 (Heine-Borel), we get for free that a set is *compact* if and only if it is closed and bounded.