Review - Chapter 4: Basic Topology

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4.2 Points in Open and Closed Sets

The Krantz book uses the terms *'limit point'* and *'accumulation point'* interchangebly. Let's define these terms and prove they're the same thing.

Definition 1 (limit point)**.** *A point $x \in R$ is a **limit point** of a set $S \subset R$ if $\forall \epsilon > 0$, $N_\epsilon(x)$ contains an element of S other than x.*

Definition 2 (accumulation point)**.** *A point $x \in R$ is a **accumulation point** of a set $S \subset R$ if $\forall \epsilon > 0$, $N_\epsilon(x)$ contains infinitely many elements of S.*

Basically we consider every $N_\epsilon(x)$ to determine what kind of point x is for a set $S \subset R$.

1) if every $N_\epsilon(x)$ contains a point other than x in $S \rightarrow x$ is a limit point
2) if every $N_\epsilon(x)$ contains an infinite number of points in $S \rightarrow x$ is an accumulation point

Proposition 1. *A point $x \in S \subset R$ is a limit point of S iff it's an accumulation point of S.*

Proving accumulation point $\implies$ limit point seems pretty simple. Any $N_\epsilon(x)$ of an accumulation point contains infinitely many points in S. In particular it contains at least 2 points in S. So it contains a point in S other than x.

Alright lets try the other direction. We have some $N_\epsilon(x)$ for a limit point x and we need to show it satisfies the requirements for an accumulation point. x is a limit point so $N_\epsilon(x)$ contains some $s_1 \neq x \in S$. Cool, we have one point. Infinitely many to go. But this is actually pretty easy right? We can just choose $s_2 \neq x \in S$ from a smaller $N_{\epsilon'}(x)$ where ϵ' is set to $|s_1 - x|$. We can do this infinitely many times and always get a new s_n becaues x is a limit point. Thus we get an infinite number of $s_1 \neq x, s_2 \neq x, ... \in S$ so x is an accumulation point.

Other sources have a variety of definitions for these terms. So this result just follows from our particular definitions. Moving on...

boundary points, interior points, isolated points

Definition 3 (boundary point)**.** *$b \in R$ is a **boundary point** of $S \subset R$ if every $N_\epsilon(b)$ contains points in S and points in $R \setminus S$. We denote the set of boundary points for S as ∂S.*

For the most part, boundary points are what we expect them to be. You can perturb a boundary point in the appropriate direction and it will no longer be in S.

Boundary points may or may not be in the set S. We'll see in a second that

1) closed sets - contain all their boundary points
2) open sets - contain none of their boundary points

Oh this is pretty interesting. The boundary set of Q, ∂Q, is the entire real line. This makes sense because any neighborhood around a rational contains infinitely many rational and irrational numbers.

Definition 4 (interior point)**.** *A point $s \in S \subset R$ is an **interior point** of S if there is an $N_\epsilon(s) \subset S$.*

From this definition we see that open sets require *all* points to interior points.

Definition 5 (isolated point)**.** *A point $t \in S \subset R$ is an **isolated point** if there is an $N_\epsilon(t)$ such that $N_\epsilon(t) \cap S = \{t\}$*

Proposition 2. *Each point of $S \subset R$ is either an interior point or a boundary point.*

Proof. let $x \in S$. If x is an interior point then we're done. Suppose x is not an interior point. Then all $N_\epsilon(x)$ contain points in $R \setminus S$. Also $N_\epsilon(x)$ contains $x \in S$. So x contains points in both S and $R \setminus S$. x is a boundary point. □

Quick remark. Isolated points are a special class of boundary points. Also accumulation point can either be interior or boundary points but never isolated.

Proposition 3. *The boundary ∂S of a set $S \subset R$ is also the boundary of $R \setminus S$.*

Proof.

$$\begin{aligned} b \in \partial S &\iff \text{every } N_\epsilon(b) \text{ contains points of } S \text{ and } R \setminus S \\ &\iff \text{every } N_\epsilon(b) \text{ contains points of } R \setminus S \text{ and } S \\ &\iff \text{every } N_\epsilon(b) \text{ contains points of } R \setminus S \text{ and } R \setminus (R \setminus S) \\ &\iff b \in \partial R \setminus S \end{aligned}$$

□

This proof is trivial if you think about it. Let two sets A and B be complements in the universe of R. Then a point b is a boundary point of A if all $N_\epsilon(b)$ contain points in A and B. But this is the same criteria for boundary points in B.

Theorem 1. *A closed set contains all it's boundary points.*

So a given a boundary point b in a closed set S, we need to show $b \in S$. Closed sets are defined in terms of open sets so let's consider the alternative. Suppose b is in the open set $R \setminus S$. Ok well clearly that can't happen because then there would exist some $N_\epsilon(b)$ containing only points in $R \setminus S$. This contradicts the definition of a boundary point so b has to be in the closed set S.

The krantz book gives a direct proof using a fact about accumulation points. We'll state that fact as a lemma and then give their proof below.

Lemma 1. *A closed set $S \subset R$ contains all it's accumulation points.*

The proof for this should sound familiar.
We know that every neighborhood of an accumulation point $s \in R$ for a set S contains infinitely many points in S. Placing s in the open set $R \setminus S$ produces a neighborhood that contradicts this.
Now for the main proof.

proof (Theorem 1). Let $b \in R$ be a boundary point for a set $S \subset R$. If b is an accumulation point then $b \in S$ by Lemma 1. If it's not, then it's not a limit point so there's an $N_\epsilon(b)$ that doesn't contain any points of S distinct from b. In other words, $N_\epsilon(b) \cap S = \{b\}$ or $N_\epsilon(b) \cap S = \emptyset$. b is a boundary point so $N_\epsilon(b)$ must contain at least one element of S. Thus $N_\epsilon(b) \cap S = \{b\}$ and we conclude $b \in S$. □

In short, if b is an accumulation point, then it's in S. If not, then it's isolated by a neighborhood of points in $R \setminus S$. But b is a boundary so we have to have some $x \in S$ in that neighborhood. That can only happen if $x = b$. This also tells us that b is an isolated point.

Corollary 1. *An open set contains none of it's boundary points.*

Proof. b is the boundary of an open set $S \implies$ it's the boundary of the closed set S^c.
Thus $b \in S^c \implies b \notin S$. □

Theorem 2. *If b is a boundary for $S \subset R$, then b is either exclusively*
1) an accumulation point
2) an isolated point

Definition 6. *$S \subset R$ is bounded if $\exists M > 0$ such that $|s| \leq M \ \forall s \in S$.*

Theorem 3 (Bolzano Weierstrass)**.** *Every bounded, infinite $S \subset R$ has an accumulation point.*

It's critical that S is infinite. This gaurantees the exsitence of an infinite sequence of distinct elements from S. If the sequence converges, then it's limit will be the accumulation point we're looking for.

proof. Let $S \subset R$ be bounded and infinite. Then there is an infinite sequence a_j of distince elements of S. By another B-W theorem, this means there is a subsequence a_{j_k} converging to limit α (because a_j is bounded). Thus S has an accumulation point α. □

Proposition 4. *Let $S \subset R$. Then*

$$S \text{ is closed} \iff \text{every Cauchy sequence } \{a_j\} \subset S \text{ converges to an element of } S.$$

I'm too lazy to write the proof. Basically if S is closed and a_j is a Cauchy (convergent) sequence, then a_j can't converge to something in $R \setminus S$. We can produce an ϵ-neighborhood which isolates the limit point.
On the other hand, if S is open then there's a Cauchy sequence that converges outside of S. Its any sequece that converges to a boundary point b of S.

Remark 1. *Proposition 4 is vacuously true if S is finite.*

Corollary 2. *Let $S \subset R$ be nonempty, closed, and bounded. If a_j is any sequence in S, then there is a cauchy subsequence a_{j_k} that converges to $\alpha \in S$.*

Proof. Suppose there is a sequence a_j in S. Then there is a Cauchy subsequence a_{j_k} converging to $\alpha \in R$ because S is bounded. By Proposition 4, we conclude $\alpha \in S$ because S is closed. □