Review - Chapter 5: Limits and Continuity of Functions

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5.2 Continuous Functions

Definition 1. *Let f be a real-valued function with domain $E \subset R$. Fix an accumulation point of E, $p \in E$. Then f is continuous at p if*

$$\lim_{x \to p} f(x) = f(p)$$

Notice that we need p in the domain E to even consider continuity.

Example 1. *The function*

$$h(x) = \begin{cases} sin(\frac{1}{x}) & x \neq 0 \\ 1 & x = 0 \end{cases}$$

is discontinuous at $x = 0$ because $\lim_{x \to 0} h(x)$ doesn't exist. Thus it can't possibly equal $h(p)$.

Proof. proof by Proposition 2 from 5.1. We end up with two sequences whos images converge to different values. This can't happen since every sequence image should converge to the same limit. □

Theorem 1. *Let f and g be functions with domain E and let $p \in E$ be an accumulation point of E. If f and g are continuous at p then so are*

$$(f + g), (f * g), \text{ and } \left(\frac{f}{g}\right) \text{ (provided } g(p) \neq 0).$$

Proof. $\lim_{x \to p}(f + g)(x) = \lim_{x \to p} f(x) + \lim_{x \to p} g(x) = f(p) + g(p) = (f + g)(p)$. □

The other two proofs are essentially identical. Here's another extension of a result given in section 5.1.

Proposition 1. *f is a function with domain E and $p \in E$ is an accumulation point of E. Then*

$$\lim_{x \to p} f(x) = f(p)$$

if and only if

$$a_j \subset E \text{ and } \lim_{j \to \infty} a_j = p \implies \lim_{j \to \infty} f(a_j) = f(p).$$

We knew this was true for any limit point, ℓ, from section 5.1 proposition 2. This is just the special case where $\ell = f(p)$.

Restating this fact in the following form will be useful in a second.

Corollary 1. *If f is continuous at p and $\lim_{j \to \infty} a_j = p$, then*

$$\lim_{j \to \infty} f(a_j) = f\left(\lim_{j \to \infty} a_j\right)$$

Cool, now we can apply it in the next proposition.

Proposition 2. *Let $g : D \to E$ and $f : E \to F$ and suppose $p \in D$ is an accumulation point of D. Assume g is continuous at p and f is continuous at $g(p)$. Then $f \circ g$ is continuous at p.*

Proof. Let a_j be a sequence such that $\lim_{j\to\infty} a_j = p$. Then

$$\begin{aligned}
\lim_{j\to\infty} f \circ g(a_j) &= \lim_{j\to\infty} f(g(a_j)) \\
&= f(\lim_{j\to\infty} g(a_j)) \\
&= f(g(\lim_{j\to\infty} a_j)) \\
&= f(g(p)) \\
&= f \circ g(p)
\end{aligned}$$

□

This proof is probably best read from bottom to top. We need to be careful when assuming f is continuous at $\lim_{j\to\infty} g(a_j)$.

Remark 1. *to-do. Discuss limits of composite functions. piecewise and constant function counterexample*

Definition 2 (inverse image)**.** *f is a function with domain E and W is any set of real numbers. Then*

$$f^{-1}(W) = \{x \in E : f(x) \in W\}$$

*is the **inverse image** of W under f.*

Theorem 2. *f with domain E is continuous $\iff$ the inverse image of any open set under f is the intersection of E with an open set. If E is open, then f is continuous $\iff$ the inverse image of every open set under f is open.*

Proof. to-do. □

Corollary 2. *f with domain E is continuous $\iff$ the inverse image of any closed set under f is the intersection of E with a closed set. If E is closed, then f is continuous $\iff$ the inverse image of every closed set under f is closed.*

Proof. to-do. □