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Connecting Symbolic Fractions to Their Underlying Proportions Using Iterative Partitioning

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Fractions are a challenging mathematics topic for many elementary and middle school students, and even for adults. However, a growing body of developmental research suggests that young children can reason about visually presented proportions, well before fraction instruction, providing insight into how fractions might be introduced to improve learning. We designed a card game to teach first and second grade children (N = 195, including a racially and economically diverse sample from the United States) about fractions in one of three ways. In the Actively Divided condition we iteratively divided an area model into equal-sized units, in the Predivided condition we used an area model with the end-state of the Actively Divided condition, and in the Nondivided condition we used a continuous representation of the fraction magnitude that was not divided into unit-sized parts. Children in the actively divided condition demonstrated larger improvements matching symbolic fractions and visual fractions (i.e., pie charts) than children in the other two conditions. Posthoc analyses of children's gameplay revealed that the actively divided condition may have provided a more optimal level of difficulty for young children than the predivided condition, which was particularly difficult, and the nondivided condition, which was trivially easy. These differences in gameplay performance provide insights into possible mechanisms for our results. We discuss open research questions highlighted by this work and implications of these findings for both the development of proportional reasoning and fraction learning.

Keywords: proportional reasoning, fractions, card games, nonsymbolic fractions, area models

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Fraction knowledge is an important gatekeeper for success on more advanced math topics, particularly algebra, which in turn is a gatekeeper for further math education (Booth & Newton, 2012; Siegler et al., 2012; Smith, 1996; Spielhagen, 2006). Fractions are also important for the workforce, with over two thirds of adults in the United States using rational numbers for their work (Handel, 2016). Unfortunately, despite the extensive time dedicated to fraction instruction throughout grade school, many students demonstrate a poor understanding of fractions that persists into adulthood (Ciosek & Samborska, 2016; Hecht & Vagi, 2010; National Mathematics Advisory Panel, 2008; Siegler & Lortie-Forgues, 2015; Siegler & Pyke, 2013). Thus, students' poor proficiency with fractions can have lasting consequences on future school success and career trajectories. In contrast to these fraction difficulties, infants and young children have sophisticated intuitions about nonsymbolic proportional quantities (Boyer & Levine, 2012; Boyer et al., 2008; Denison et al., 2013; Denison & Xu, 2010; Hurst & Cordes, 2018a; Jeong et al., 2007; Kushnir et al., 2010; McCrink & Wynn, 2007). This contrast between early nonsymbolic intuitions and later difficulty during formal fraction education has led researchers to suggest that children's nonsymbolic understanding of proportion can, and should, be leveraged to support their symbolic fraction learning (e.g., Boyer & Levine, 2015; Matthews & Hubbard, 2017). However, it is not clear what kind of nonsymbolic representations should be used or how they might support symbolic fraction learning. In the current study we address these questions using a fraction card game that varies the features of visual fraction representations used to help students compare symbolic fraction magnitudes. Importantly, we draw upon prior research investigating the development of children's reasoning with nonsymbolic visual proportions to generate clear predictions about the features of nonsymbolic area model representations that are likely to best support symbolic fraction learning. Our results have implications for both theories that characterize symbol learning and fraction education, as learning symbolic fractions is particularly difficult.

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Understanding how people connect symbols to nonsymbolic representations, often referred to as the symbol grounding problem, is a central question of human cognition across many different domains (e.g., Harnad, 1990). In the case of proportion, recent evidence suggests that humans have an analogue nonsymbolic representation for proportions and can map that analogue nonsymbolic representation to symbols (e.g., Binzak & Hubbard, 2020; Kalra et al., 2020; Lewis et al., 2016). However, understanding how people reason about proportion is further complicated by the variability in how it can be represented, both symbolically (e.g., fractions, decimals) and nonsymbolically (e.g., subset of items, part of a whole object). For example, people are better able to compare the magnitudes of decimals than fractions (DeWolf et al., 2014; Hurst & Cordes, 2016, 2018b) and are better able to compare the magnitudes of nonsymbolic fractions when the representations are continuous (e.g., line lengths, undivided shapes) rather than discrete (e.g., sets of dots, divided shapes; Boyer et al., 2008; Hurst & Cordes, 2018a; Park et al., 2021). Furthermore, when asked to directly map between symbolic and nonsymbolic representations, adults prefer and are better at mapping fractions with discrete rather than continuous representations but are better able at mapping decimals with continuous rather than discrete representations (DeWolf et al., 2015). Together, these findings suggest that although people can map symbolic and nonsymbolic representations of proportions to one another, the type of representation impacts this mapping. The current study adds to this growing literature by exploring how specific features of nonsymbolic fraction representations, namely how discrete information is presented in an area model, can impact children's ability to learn the meaning of fraction symbols.

In addition to being an important component of theories of proportional reasoning, visual representations have been used in the classroom to support fraction learning for decades and are a recommended component of fraction instruction (Matthews & Hubbard, 2017; National Governors Association Center for Best Practices, 2010; Rau & Matthews, 2017). However, the variability across different representations has resulted in substantial disagreement about the specific type of visual representation that best supports fraction learning. Area models, such as pie charts or rectangles that are divided into units to reflect the denominator and partially colored to reflect the numerator, are likely the most common representation of fraction values. In the United States, partitioning shapes into parts is the first introduction to fractions for many students (National Governors Association Center for Best Practices, 2010). Furthermore, area models are heavily used in U.S. mathematics textbooks (Alajmi, 2012) and both children and preservice teachers are better able to use area models than other kinds of representations (Kaminski, 2018; Luo et al., 2011). These representations have the benefit of clearly defining the whole (i.e., the entire shape) and conveying the specific part–whole values (i.e., the number of colored units out of the total number of units), but they are limited in their ability to continuously represent improper fractions (i.e., fractions that are larger than 1, such as 5/2) as these numbers necessarily require multiple wholes. As an alternative visual model, researchers and educators have increasingly focused on number lines as a tool for representing numerical magnitudes (e.g., Hamdan & Gunderson, 2017; Siegler et al., 2011; Siegler & Opfer, 2003; Siegler & Ramani, 2009). Some experimental intervention and curriculum studies have found that teaching children to represent fractions on number lines improves their ability to compare fraction magnitudes more so than area models (Gunderson et al., 2019; Hamdan & Gunderson, 2017; Saxe et al., 2013). Furthermore, when fourth to sixth grade children were asked to map a fraction onto a number line, they were significantly less likely to partition or divide up the number line than they were to partition a pie chart (Hurst et al., 2020), suggesting that number lines may be particularly useful for eliciting continuous magnitude-based thinking. However, the question of which visual model is better for learning may not have an all-or-nothing answer and instead depend heavily on specific features of the model, how that model is used, and what concept we are expecting children to learn (e.g., Ainsworth, 1999; Rau & Matthews, 2017). Thus, in the current study, rather than comparing different types of visual representation (e.g., pie charts vs. number lines), we instead focus on a specific feature of visual representations: the availability and salience of relative units.

Prior work investigating young children's informal reasoning with visual proportions suggests that the availability of discrete countable units does impact their nonsymbolic proportional reasoning. Young children who do not yet understand symbolic fractions (e.g., as young as first grade) are able to reason about proportional information with continuous area models that are not divided (e.g., 75% red and 25% blue; Hurst & Cordes, 2018a; Jeong et al., 2007). However, when the area model is predivided into countable pieces, these same children tend to focus on the number of salient numerator pieces and make systematic errors, such as deciding a spinner with 4/9 red pieces is more likely to land on red than a spinner with 2/3 red pieces because $4 > 2$ even though $4/9 < 2/3$ (e.g., Boyer et al., 2008; Hurst & Cordes, 2018a; Jeong et al., 2007). Furthermore, this overattention to the numerator, at the expense of the relational fraction information, is commonly referred to as the *whole number bias* and continues when children learn symbolic fractions in later grades (e.g., Ni & Zhou, 2005). Taken at face value, this prior work might lead to the hypothesis that continuous undivided area models would also be more effective for symbolic fraction instruction. However, these continuous representations do not convey the specific components of fraction symbols (e.g., both 3/4 and 6/8 would have the same continuous representation), and people prefer to map fractions with discrete representations rather than with continuous ones (DeWolf et al., 2015). Thus, although continuous area models may be useful for thinking about proportion magnitude, the utility of these representations for teaching children how to interpret discrete fraction components (such as "3/4" being 3 parts out of 4 parts) may be limited.

Thus, given the potential shortcomings of both entirely continuous models, which do not have visible units at all, and entirely discrete models, which highlight the number of units in a way that leads to errors, we investigated a third alternative. We developed an area model that makes discrete countable units available but uses iteration to highlight relative unit size and connect the discrete countable units to continuous magnitude. Prior work has emphasized the importance of building on children's understanding of measurement, and specifically iterative units, to support fraction learning (e.g., Lamon, 1993; Pitkethly & Hunting, 1996; Sophian, 2007; Tzur & Hunt, 2015), including helping children bridge their understanding of whole numbers and fractions (Boyce & Norton, 2016; Sophian, 2007) and improve their understanding of fraction arithmetic (Braithwaite & Siegler, 2021). Furthermore, the idea of iterating units might be particularly important for helping children move from an immature part–whole conception of fractions to a more advanced measurement conception of fractions (Norton & Wilkins, 2009; Wilkins & Norton, 2018). However, given that children also make systematic errors with the part–whole interpretation of fractions (Miura et al., 1999; Paik & Mix, 2003) and visual proportions (e.g., Boyer et al., 2008), it may be that highlighting relative unit size through iteration can also improve this earlier developing skill of mapping a single symbolic fraction to a part–whole visual proportion. Moreover, this approach allows us to compare a specific feature (the saliency of fraction units) across one type of visual representation (i.e., area models), in order to formulate and test specific theoretical predictions about children's proportional reasoning.

The Current Study

We designed an instructional fraction game, based on the card game "War". Although previous studies have shown success in improving fraction learning through modifications of the card game War (Gabriel et al., 2012; Leutzinger & Nelson, 1980), we were specifically interested in using the basic game structure to investigate how the availability and salience of units impacts children's fraction reasoning by comparing different area models: entirely continuous, entirely discrete, and a hybrid model that actively highlights the unit-based components of fractions.

We focused on children in first and second grade—who have not yet received formal fraction instruction—for three reasons. First, we were interested in measuring different aspects of children's fraction knowledge, including their ability to compare symbolic and visual fractions, their understanding of equivalent fractions, and their more basic knowledge of mapping fractions to part–whole visual representations. There has been a recent surge in research focused on understanding how people reason about fraction magnitudes, both symbolically and nonsymbolically, much of which suggests that these skills are critically important components of fraction learning (e.g., Boyer et al., 2008; DeWolf et al., 2014, 2015; DeWolf & Vosniadou, 2015; Hurst & Cordes, 2016, 2018a; Kalra et al., 2020; Möhring et al., 2016; Siegler et al., 2011, 2012). However, even before being able to reason about the relative size of fraction magnitudes, children must learn how to interpret a single fraction, X/Y, as corresponding to X parts out of a whole with Y parts (Norton & Wilkins, 2009; Wilkins & Norton, 2018). The Common Core State Standards describe this skill as a fraction benchmark in third grade, but it is introduced for some simple fractions in second grade (National Governors Association Center for Best Practices, 2010). Thus, we targeted children who were unlikely to have formally been taught symbolic fractions, or at most had been introduced to some simple unit fractions (e.g., "one fourth" or "one half"), so that we could measure their basic understanding of the part–whole interpretation of fractions, as well as their understanding of more advanced fraction concepts, such as relative magnitude and equivalent fractions. Second, prior work has found that first and second graders make whole-number-based errors when identifying fraction symbols, such as erroneously mapping 2/3 to an image of two black dots and three gray dots (equivalent to 2/5 rather than 2/3; Miura et al., 1999; Paik & Mix, 2003), but that these errors are malleable (Mix & Paik, 2008). Thus, although iterative units are typically thought to be important for moving beyond the part–whole conception of fractions (e.g., Norton & Wilkins, 2009; Wilkins & Norton, 2018), this age group allows us to investigate whether area models with iterative units are also helpful for preventing whole number based errors when identifying the part–whole interpretation of fraction symbols. Third, and lastly, prior work with nonsymbolic proportional reasoning suggests that 6-year-olds, who are typically in first grade in the United States, can reason well with visually presented continuous proportions, but struggle with visually presented discrete proportions (e.g., Hurst & Cordes, 2018a; Jeong et al., 2007), making this a critical developmental age group for investigating the utility of visual proportions for scaffolding symbolic fraction learning in a way that aligns with the proportional reasoning literature more broadly.

Therefore, in the current article we investigated how training with area models that vary the availability and saliency of discrete units improves children's part–whole interpretation of fractions, measured by their ability to map symbolic fractions onto nonsymbolic representations. We predicted that drawing children's attention to the meaning of the numerator and denominator through iterative unit partitioning (actively divided condition) would best support children's learning of the mapping between symbolic fractions and visual fractions, in line with other work highlighting the role of partitive unit-based reasoning in early fraction learning (e.g., Boyce & Norton, 2016; Sophian, 2007; Wilkins & Norton, 2018). In contrast, we predicted that continuous, nonpartitioned models (nondivided condition) would be suboptimal because continuous representations do not convey information about how to connect the visual representation to symbolic fractions and thus may be insufficient for supporting the mapping between these representations. Thus, we predicted that children in the actively divided condition would outperform children in the nondivided condition. We further predicted that using static and predivided models (predivided condition), which are most similar to those used in fraction instruction, would result in intermediate performance relative to the other conditions as this condition provides the discrete information necessary to make the mapping, but in a way that might lead to whole number-based errors.

In addition to measuring children's ability to map between symbolic and nonsymbolic representations of fractions, we also included additional tasks measuring children's ability to compare and equate symbolic and visual fractions. These tasks were included for two reasons: (a) to investigate the generalizability of the benefit of unit iteration for supporting different fraction concepts, and (b) to address separate research questions about children's fraction knowledge. Specifically, we were interested in how children's ability to work with fractions differs based on the format of the fraction (symbol vs. visual spatial fraction) and the concept being measured (fraction equivalence vs. magnitude comparison), as well as how these more advanced skills relate to children's ability to map a symbolic fraction to a nonsymbolic representation, as a measure of their more basic part–whole knowledge of fractions. We chose to measure these more advanced skills because magnitude comparison is often used as a measure of children's fraction knowledge, allowing us to compare our findings with the broader literature, and equivalent fractions are reliant on the multiplicative relations of numerators and denominators (e.g., 2/4 = 4/8). However, these additional tasks had low reliability estimates with children performing around chance, suggesting that children in the present study found these tasks very difficult, either because of their immature understanding of fractions and/or general task demands. This is likely due to the young age of these children, which was a deliberate decision based on our developmental hypotheses of when the availability and salience of units is likely to have the largest impact on children's initial learning of fractions. Nevertheless, the age of children in the current study may have undermined our ability to effectively measure these more advanced skills. Therefore, although these tasks and additional research questions were included as part of our preregistration, we do not report them in the current article. For full transparency, a complete write up of the method and results for these tasks is included in the online supplementary materials and all data is accessible via the OSF project page.

Method

Participants

As preregistered, our final sample included 195 first and second grade children, randomly assigned to one of three between-subject conditions: nondivided area model condition (n = 65; M_{age} = 89.7 months, range = 77–111 months; 35 first graders and 30 second graders; 21 boys, 44 girls, 0 undisclosed/other); predivided area model condition (n = 65; M_{age} = 90.2 months, range = 76–107 months (one missing age information); 35 first graders and 30 second graders; 31 boys, 34 girls, 0 undisclosed/other); and actively divided area model condition (n = 65; M_{age} = 89.9 months; range = 77–103 months; 35 first graders and 30 second graders; 31 boys, 34 girls, 0 undisclosed/other). This sample size was chosen a priori (using the *pwr* package, Champely, 2018) to provide 80% power to detect condition differences as small as f^2 = .04 (considered a small effect; Cohen, 1992) in the regression analyses with alpha of .05. Four additional children (2 first graders and 2 second graders) completed the pretest but not the posttest (due to absences or refusal), and therefore, in line with our preregistered plan, are not included in the primary analysis of pre- to posttest learning but are included in analyses that involve pretest only. One additional child started pretest but did not finish the session and is therefore not included in any analyses.

Children were randomly assigned to condition within their grade (first or second grade; to ensure a similar grade distribution across conditions) and in smaller sets (e.g., a balanced sequence of 12 or 30 assignments at a time) based on the number of children recruited from each testing location and the timing of recruitment and testing across sites. When there was imbalance in condition assignments caused by participant exclusion or experimenter error, those conditions were replaced in subsequent blocks. This process ensured we were maintaining approximate balance across conditions throughout the data collection process and in the overall sample.

Children were recruited from the Chicagoland area from February 2019 to February 2020 through local schools, as well as other community organizations, summer camps, and our university database. We did not collect information about the fraction instruction received by the children for any individual or school. However, based on the Common Core State Standards (National Governors Association Center for Best Practices, 2010), it is unlikely that children had much, if any, formal experience with fraction symbols. As discussed in the introduction, this young age group was targeted in order to investigate the effectiveness of area models with different unit-based features on children's initial knowledge of fraction symbols, prior to formal instruction.

Based on the 84–89% of our sample (depending on the question) that completed the demographic survey,[1] 54% of parents reported their children as Black or African American, 31% as White, < 1% as Native American, <1% as Asian, 0% as Native Hawaiian or Pacific Islander, 10% as other or more than one race, and 23% identified as Hispanic or Latino/a/x. The sample also came from economically diverse backgrounds, with 50% reporting an annual family income below the Chicago median (which was $58,247 based on the 2019 U.S. Census; U.S. Census Bureau, n.d.) and 50% reporting the category at or above the Chicago median. More specifically, our sample included: 18% reporting an annual family income of $15,000 or less, 19% between $15,000 and $34,999, 13% between $35,000 and $49,999, 7% between $50,000 and $74,999, 12% between $75,000 and $99,999, and 30% reporting $100,000 or more. Lastly, the education levels of the parent or guardian who completed the demographic form (84% mothers, 13% fathers, 2% unidentified or nonbinary parent, 1% other guardian) also showed variability: 31% high school or less, 39% associate's or bachelor's degree (or equivalent), and 30% graduate degree.

Parents or legal guardians provided written consent prior to children's participation, and children 7 years old or older provided formal verbal assent. Parents of children tested in our lab were given $10/session as travel compensation and schools and community organizations were given a $50 gift card for supplies. All participating children were offered a small prize (e.g., sticker or small toy) after each testing session. All testing procedures were approved by the University of Chicago Institutional Review Board under protocol IRB17-1599 "Relational Math Reasoning."

Design

Children participated in two testing sessions one-on-one with an experimenter, each lasting approximately 30 minutes and occurring approximately 7 days apart (M = 7 days; Range: 5 to 10 days; 173 of 195 children had sessions 7 days apart). During Session 1, children completed a pretest battery of fraction assessments presented on a laptop computer and then played the fraction card game with the experimenter. In Session 2, children first played the fraction card game with the experimenter and then completed the posttest battery. The same set of tasks was used at both pre- and posttest across all conditions. These tasks included (in this order): (a) Symbolic-to-Nonsymbolic Mapping task (which of these pictures shows this fraction?), (b) Symbolic Comparison task (which is bigger?), (c) Symbolic Equality task (are these equal or not equal?), (d) Spatial Fraction Comparison task, and (e) Spatial Fraction Equality task. As mentioned in the introduction, only the Symbolic-to-Nonsymbolic Mapping task will be included in the article and all other tasks are reported in the online supplemental material because of low reliability estimates (Cronbach's αs < .53).

[1] Individual-level data from the demographic survey is not made publicly available because some combinations of information apply to only one individual, making the data identifiable. To provide a general description of the sample, however, the information is presented here for the entire group and separated by condition in the online supplemental materials Table S6.

The Fraction Card Game was identical in Session 1 and Session 2. Children in all three between-subjects conditions completed the same fraction comparisons within the game but the area models differed according to the assigned condition. Given the young age of the target sample and the focus on part–whole area models, all fractions used in the card game and the pre- and posttest measures were proper fractions between 0 and 1 that could be depicted as parts of a single-whole (but potentially presented in unreduced form, e.g., 2/4).

Procedure and Stimuli

Pretest Warm-Up

Prior to beginning the pretest, children were asked what they thought a fraction was and their response was written and recorded in summarized form by the experimenter. This task was primarily used as a warm-up question and was not asked at posttest. Children's responses to this pretest question were double coded to gain better insight into children's fraction knowledge prior to the intervention. The coders classified children's responses as either demonstrating no prior knowledge about fractions (i.e., the child said they did not know or gave an answer that was unrelated to fractions) or as demonstrating some fraction knowledge (i.e., the child's answer reflected at least a partial definition of fractions).

Symbolic to Nonsymbolic Mapping Task (Modeled After Miura et al., 1999)

The task was administered at both pre- and posttest and was conducted on a 13-in. laptop computer using PsychoPy2 Version 1.85.4 (Peirce et al., 2019). There were two sets of stimuli, one presented at pretest and the other presented at posttest, with the set of stimuli assigned to pre- versus posttest counterbalanced across individuals. Each set of stimuli included eight unique trials with good reliability (Cronbach's alpha = .86 and .87 for each set).

On each trial, children were presented with a symbolic fraction centered in the upper half of the screen and four divided pie charts equally spaced along the bottom half of the screen (see Figure 1A). The fractions (6.5 cm tall, 5.4 cm wide, presented upright with a horizontal bar) only included values between zero and one and the numerators and denominators were single-digit integers. The nonsymbolic pie charts (4.7 cm diameter) included four randomly ordered options, with the numerator pieces colored red and the remaining pieces colored light yellow. On each trial, there were four options: (a) the correct answer (e.g., if the target is 2/5, then 2 red pieces and 3 yellow pieces; Figure 1A, third option); (b) the numerator + denominator answer (e.g., 2 red pieces and 5 yellow pieces; Figure 1A, fourth option), (c) the correct number of pieces, but the pieces are not equally sized (e.g., 2 red pieces and 3 yellow pieces, but the pieces are different sizes; Figure 1A, second option), and (d) a different fraction with the same denominator (e.g., 4 red pieces and 1 yellow piece; Figure 1A, first option).

Children were told they would see a symbolic fraction at the top of the screen and pictures of fractions at the bottom of the screen and their job was to choose which of the pictures showed the fraction at the top. On each trial, children pointed to one of the four options and the experimenter recorded the child's response by pressing 1, 2, 3, or 4 on the keyboard (corresponding to the position of each option). The trial ended as soon as the experimenter pressed the key. After each trial, a fixation point appeared for 1,000 ms before the next trial started. Children had as much time as they needed to respond and if they responded that they did not know, they were told to make their best guess. The eight trials were presented in a random order and children were scored based on proportion correct.

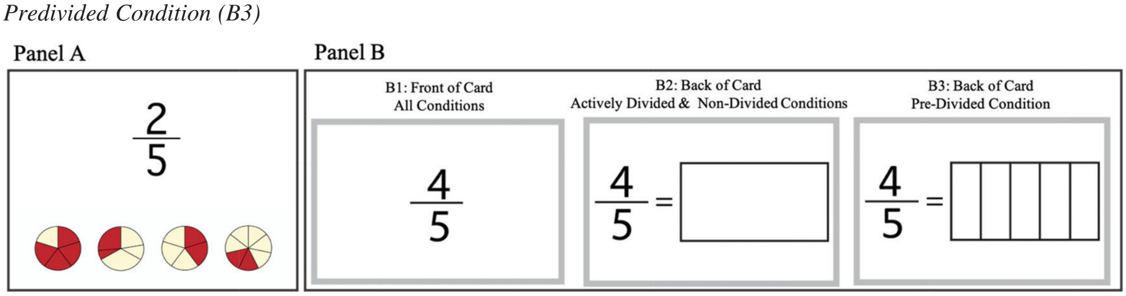
Fraction Card Game

Each card (8.75 cm wide by 5.75 cm tall) had a symbolic fraction on the front and that same symbolic fraction, an equal sign, and a rectangle on the back (see Figure 1B). To distinguish each player's cards, the experimenter's cards had a gray border and the child's cards had a green border. The symbolic fractions (2.5 cm tall by 1.8 cm wide) were presented in upright notation with a horizontal line between the numerator and denominator and included unit fractions and nonunit fractions between zero and one. The rectangle (5 cm wide by 2.5 cm tall) was blank for the nondivided and actively divided conditions (Figure 1, Panel B2) and was divided into the denominator number of pieces for the predivided condition (Figure 1, Panel B3).

Children were told they would play a card game with the experimenter where each card had a fraction on it and that on each turn whoever had the card with the bigger fraction would get to keep both cards. At the end of the game whoever had the most cards won. Children were also told that before they decided who would

Figure 1

Example Stimuli From the Symbolic to Nonsymbolic Mapping Task (A) and the Card Game (B), Including the "Front" of the Card (B1), Which Was Identical for All Conditions, the "Back" of the Card in the Actively Divided and Nondivided Conditions (B2), Before the Experimenter Applied the Stencil, and the "Back" of the Card in the Predivided Condition (B3)



Note. See the online article for the color version of this figure.

keep the cards, they should put the cards on the designated game board (referred to as the "checking board") and figure out which fraction was bigger. The checking board was a laminated piece of cardstock (15 cm tall by 10 cm wide) with card-sized rectangles on the top and bottom labeled "bigger" and "smaller," respectively. The game consisted of two sections. Section 1 included three trials involving only unit fractions and Section 2 included six trials involving unit and nonunit fractions. Between the two sections there was a transition trial on which children were introduced to nonunit fractions.

On each trial, the experimenter pulled a card from their envelope and read the fraction label aloud: "my fraction is [fraction label, for example, *one fifth*]." The experimenter then helped the child pull a card out of their envelop and read the fraction label aloud: "your fraction is [fraction label, for example, *one half*]." The cards were prearranged in their respective envelopes so that children would progress through a set order of trials. After both cards were on the table (with only the symbolic fractions visible), the experimenter asked the child to decide which fraction was bigger and to place the card with the bigger fraction in the spot labeled "bigger" and the card with the smaller fraction in the spot labeled "smaller." The experimenter then recorded the child's initial response. If the child said they did not know or that the fractions were the same, they were told to make their best guess. After the child's initial response, the cards were flipped over to reveal the rectangles and the experimenter used a condition-specific approach to fill them in.

In the nondivided condition, the experimenter said "my fraction is [fraction label, for example, *one fifth*], so let's draw [fraction label, for example, *one fifth*]" while taking a premeasured stencil and using a fine-point sharpie to draw a line indicating the fraction. Then, the experimenter said: "now I need to color in this much out of the whole" pointing to the fraction part of the rectangle. Next, the experimenter used a blue colored pencil to color that section and then said: "so that shows [fraction label, for example, *one fifth*]." This was then repeated for the child's card. On the child's card, the experimenter used the stencil and drew the fraction line in the appropriate position (to ensure consistency), but the child colored in their own fraction with the colored pencil.

In the predivided condition, the experimenter said "my fraction is [fraction label, for example, *one fifth*]. So my shape is divided into [denominator, for example, *five*] equal pieces and I need to color in [numerator, for example, *one*] out of [denominator, for example, *five*] pieces." The experimenter then colored in the appropriate number of pieces one at a time with the blue pencil and then said, "so that shows [fraction label, for example, *one fifth*]." This process was then repeated for the child's card, but the child colored in their own fraction.

In the actively divided condition, the experimenter said "my fraction is [fraction label, for example, *one fifth*]. So my shape needs to be divided into [denominator, for example, *five*] equal pieces." The experimenter then took a premade stencil the size of the denominator unit and drew the appropriate number of divisions to fully divide the rectangle (e.g., for 1/5, divided the rectangle into fifths). After the division the experimenter said: "and I need to color in [numerator, for example, *one*] out of [denominator, for example, *five*] pieces" and colored in the appropriate number one at a time with the blue pencil, then saying "so that shows [fraction label, for example, *one fifth*]." As in the other conditions, this was then repeated for the child's card, on which the experimenter used the stencil and drew the dividing lines, but the child colored in the fraction.

In all conditions, after both rectangles were filled in, the experimenter then asked: "Now that we can see the fraction pictures, which fraction do you think is bigger?" After the child responded, the experimenter recorded the child's response and provided verbal feedback (i.e., "that's right!" if they were correct or "actually, this fraction is bigger"). Lastly, the experimenter said: "that means that [fraction label] is bigger than [fraction label], so [my/your] fraction is biggest and [I/you] get to keep both cards!" and the cards were added to either their own or the child's pile to count later.

During the transition trial between the two sections, children were asked to think about what would happen if there was more than one piece. The two cards from the previous comparison remained visible and a third card with a "1/6" fraction on the front and a removable "3" in the upper corner was introduced, saying "let's say we had three one-sixths." The experimenter continued: "when that happens, we can write it like this [move the three to be on top of the one, so it now says 3/6] and we can call it three sixths." The experimenter then filled in the rectangle and prompted the child to color in the correct amount to show three-sixths using the condition-specific process described earlier. The child was then prompted to compare three-sixths to one fourth (one of the fractions used in the previous comparison, which was still visible) and the experimenter recorded the child's response and provided verbal feedback ("you're right, three sixths is bigger than one fourth" or if the child was incorrect, "actually, three sixths is bigger than one fourth"). Regardless of the child's accuracy, the child was then prompted to add all three cards (the two from the previous comparison and the new transition card) to their pile.

At the end of the game, the experimenter and child counted the cards they collected (i.e., the ones they won from having the biggest fraction). The game was set up so that the child was always declared the winner, having collected 11 cards (they had the bigger fraction on 5 out of 9 trials + the transition card) versus the experimenter's eight cards (they had the bigger fraction on 4 out of 9 trials).

Transparency and Openness

The study design, hypotheses, and analysis plan were preregistered on aspredicted.org (#19466; https://aspredicted.org/c7hc6.pdf). We follow the Journal Article Reporting Standards for Quantitative Designs (Kazak, 2018), including reporting how we determined our sample size, all data exclusions, all manipulations, and all measures (although methodological details and results of some measures and preregistered research questions are reported only in the online supplemental materials due to low task reliability, as discussed above). Analyses that were not preregistered are clearly indicated when introduced in the Results section. All data, analysis code, preregistration documents, and research materials are available on the Open Science Framework (Hurst et al., 2022; https://osf.io/ths6m/).

All data processing and analysis was performed in R Version 4.1 (R Core Team, 2020) using RStudio (Version 1.2.5001; R Studio Team, 2016), using packages from the tidyverse (Wickham et al., 2019), as well as *ggpubr* v0.4.0 (Kassambara, 2020a), *rstatix* v0.6.0

(Kassambara, 2020b), *effsize* v0.8.0 (Torchiano, 2018), *psych* v2.09 (Revelle, 2020), *apaTables* v2.5 (Stanley, 2018), *gt* v0.2.2 (Iannone et al., 2020), and *car* v3.0–9 (Fox & Weisberg, 2019).

Results

Pretest Warm-Up Question

Two coders classified children's responses to the initial "What is a fraction?" question as either demonstrating no prior knowledge about fractions (i.e., the child said they did not know or gave an answer that was unrelated to fractions) or as demonstrating some understanding of fractions (i.e., the child's answer reflected at least a partial definition of fractions). The two coders agreed on 97.5% of responses, and a third coder resolved disagreements (n = 5). Most children's responses (71.7%) indicated they had no knowledge about fractions prior to our intervention (e.g., "not sure," "subtracting," "food") and the remaining children (28.3%) demonstrated at least some understanding of fractions (e.g., "part of a whole," "pieces of a circle"). Thus, prior to the intervention, the children in our sample demonstrated very limited understanding of what the word *fraction* means.

How Do Different Area Models Impact Children's Understanding of Fraction Symbols?

To address our first research question, and as preregistered, we used linear regression predicting posttest performance on the symbolic-to-nonsymbolic mapping task with pretest score (continuous covariate), grade (first vs. second grade), and condition (dummy coded) as predictors. Although not preregistered, we also compared performance at each time point to chance (.25) and tested for significant learning in each condition separately using a mixed ANOVA on proportion correct with session (2: pre and post) as a within-subjects factor and grade (2: first, second) as a between-subjects factor. Pre- and posttest means and standard deviations, as well as the within condition statistical analyses (i.e., one-sample *t*-tests comparing performance to chance and pre- vs. posttest comparisons within condition) are provided in Table 1.

The overall regression model was significant, R^2 = .526, Adjusted R^2 = .516, $F(4, 190)$ = 52.7, $p < .001$, with pretest score a significant predictor B = .75, 95% CI [.65, .86], $t(190)$ = 13.95, $p < .001$, and grade not significant B = .04, 95% CI [−.03, .12], $t(190)$ = 1.22, p = .22. The critical test is the dummy coded predictors for condition (see Figure 2). When using the actively divided condition as the reference group, children in this condition scored significantly higher at posttest than children in the predivided condition, B = −.11, SE = .04, 95% CI [−.20, −.02], $t(190)$ = −2.41, p = .02, and children in the nondivided condition, B = −.12, SE = .04, 95% CI [−.20, −.03], $t(190)$ = −2.57, p = .01. Children in the predivided and nondivided conditions (reference = predivided) did not score significantly differently, B = −.01, SE = .04, $t(190)$ = −.17, p = .87.

Next, given our condition differences at posttest, we looked at learning in each of the conditions separately (using Bonferroni adjusted α = 0.0167 for the main effect of session in each of the three conditions). In the actively divided condition, there was a main effect of session, $F(1,63)$ = 9.26, p = .003, $\eta^2_{partial}$ = 0.13, with children scoring significantly higher at posttest than at pretest. There was also a small main effect of grade, $F(1,63)$ = 4.47, p = .04, $\eta^2_{partial}$ = 0.07, with second graders, M = 0.45, scoring higher than first graders, M = 0.40. However, there was not a significant interaction between session and grade, p = .24, $\eta^2_{partial}$ = 0.02. There were not significant main effects of session in the predivided condition, $F(1,63)$ = 0.14, p = .71, $\eta^2_{partial}$ = 0.002, or the nondivided condition, $F(1,63)$ = 0.12, p = .73, $\eta^2_{partial}$ = 0.002. Neither grade nor the interaction between grade and session were significant in either of these conditions, all $ps > .4$, $\eta^2_{partial} < 0.02$.

To further investigate the change in children's fraction knowledge in the actively divided condition, we analyzed the data in two additional ways: change at the individual child level and the pattern of children's errors. Although these analyses are posthoc and were not preregistered, they provide additional insight that can generate testable hypotheses for future research.

First, beyond the magnitude of improvement from pre- to posttest, we also calculated the number of children who showed an improvement from pre- to posttest in each condition: 29/65 (45%) in the actively divided condition, 21/65 (32%) in the nondivided condition, and 24/65 (37%) in the predivided condition. Although there are numerically more children who improved in the actively divided condition than in the other two conditions, this pattern is not statistically significant, $\chi^2(2)$ = 2.13, p = .34. Thus, the actively divided condition may have led to larger improvements rather than impacting more children. Consistent with this interpretation, the children who improved in the actively divided condition did improve by a larger amount, M = .38, relative to the nondivided condition, M = .23, $t(44.14)$ = 2.71, p = .01, and to the predivided condition, M = .29, although this comparison was not statistically significant, $t(49.84)$ = 1.44, p = .16.

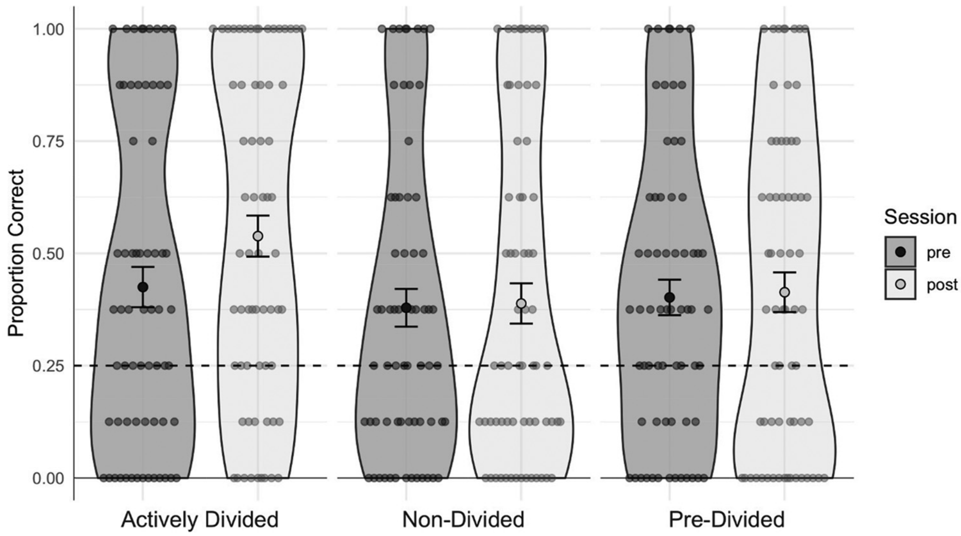
Table 1
Descriptive Statistics on the Symbolic to Nonsymbolic Mapping Task in Each Condition at Pre-and Posttest

	Pretest				Posttest				Learning
			One sample vs .25				One sample vs .25		
Condition	M	SD	t	p	M	SD	t	p	η^2 partial
Nondivided	0.38	0.34	3.07	.003	0.39	0.36	3.09	.003	0.002
Predivided	0.40	0.32	3.84	<.001	0.41	0.36	3.69	<.001	0.002
Actively divided	0.42	0.36	3.90	<.001	0.54	0.37	6.31	<.001	0.13*

Note. One-sample *t*-test (df = 64) compares children's performance to chance (.25; 4 options). Although not preregistered, learning compares pre- vs. posttest performance within each condition, with α levels adjusted for multiple comparisons.
* $p < .0167$.

Figure 2

Performance on the Symbolic to Nonsymbolic Mapping Task Across Condition (x-axis) at Pretest (Dark Gray, Left) and Posttest (Light Gray, Right)

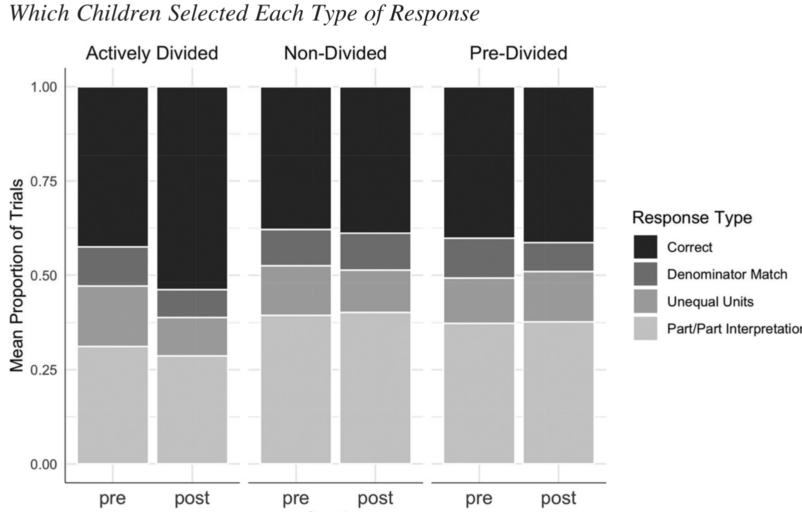


Note. Plots are violin plots with the mean proportion correct and standard error as error bars. Chance performance (0.25) is indicated with a dotted line.

Second, the task included carefully selected incorrect responses that correspond to common errors, allowing us to better understand the specific kinds of errors children made and whether children's increase in correct responding corresponded with a decrease in a specific type of error: Unequal Units (correct denominator and numerator, but unequal sized units; Option 2 in Figure 1A), Part/Part Interpretation (numerator + denominator, such as 2 red and 3 yellow pieces for 2/5; Option 4 in Figure 1A), or Denominator Match (same denominator but different numerator, such as 4 red out of 5 for a target of 2/5; Option 1 in Figure 1A). First, the most common error for all three conditions at both pre- and posttest was the Part/Part response (see Figure 3), in line with prior work on children's whole number component bias in this task (Miura et al., 1999; Paik & Mix, 2003). Second, to provide some insight into what aspects of children's fraction knowledge might be shifting in the actively divided condition, we analyzed the proportion of trials selecting each type of incorrect response at pre- and posttest. Numerically, the proportion of trials on which children in the actively divided condition selected each type of incorrect response decreased from pre- to posttest, but this decrease was only significant for the Unequal Units response, $M_{pre} = .16$, $M_{post} = .10$, $t(64) = -2.31$, $p = .02$, $d = -.29$, and was not significant for either the Part/Part response, $M_{pre} = .31$, $M_{post} =$

Figure 3

Mean Proportion of Trials on the Symbolic to Nonsymbolic Mapping Task on Which Children Selected Each Type of Response



.29, $t(64) = -.71$, $p = .48$, $d = -.09$, or the Denominator Match response, $M_{\text{pre}} = .10$, $M_{\text{post}} = .07$, $t(64) = -1.29$, $p = .20$, $d = -.16$. However, the change from pre- to posttest for the various error types did not significantly differ (all $ps > .40$). Thus, this posthoc and exploratory analysis reveals that although there is a numerically larger increase in children's attention to equal sized units, the increase in correct responses is not attributable to a decrease in a single type of error, but rather a decrease in errors across all error types.

How Did Children's Performance Vary on the Card Game (Additional RQ)?

To better understand the potential mechanisms supporting children's learning, we explored differences during the actual game play between conditions. Thus, although not preregistered, we performed a posthoc analysis of children's performance during the card game intervention itself. Specifically, we compared children's ability to compare symbolic fractions in the three training conditions before and after the condition-specific area models were available.

We isolated analyses to only the first session to assess baseline differences in the use of the three distinct models, before substantial learning could occur. Given that children's judgements about which fraction was bigger were recorded twice, once when only the symbols were visible (the same across conditions) and again after the area models were constructed for each fraction (which differed by condition), we can investigate how performance on the game differed across conditions as a direct result of the different area models. To see the effect of the specific kind of area model on children's comparison ability, we used an ANOVA with condition (3: actively divided, predivided, nondivided) as a between-subjects factor and Judgment type (2: symbols only, with area models) as a within-subjects factor on proportion correct (combined across the unit and nonunit fractions within Session 1). Twenty-seven children (out of the 199 who completed Session 1) were excluded from these analyses because they had missing game-play data resulting from the experimenter erroneously not recording at least one of the child's responses during the game. Although the children were excluded for experimenter error, it's possible that the experimenter error was caused by the child's behavior during the game (e.g., more chaotic game play, requiring more behavioral management from the experimenter), resulting in systematic missingness. Notably, both included and excluded children scored similarly on the pretest symbolic ($M_{\text{included}} = .42$, $M_{\text{excluded}} = .45$) and nonsymbolic fraction comparisons ($M_{\text{included}} = .55$, $M_{\text{excluded}} = .64$), the two pretest tasks most like the game play. Nonetheless, this limitation should be kept in mind when interpreting game play performance from the subset of children with complete game play data. A total of 172 children are included in these analyses: actively divided $n = 57$; predivided $n = 59$; nondivided $n = 56$. There was a significant main effect of condition, $F(2, 169) = 19.03$, $p < .001$, $\eta^2_{\text{partial}} = .18$, judgment type, $F(1, 169) = 235.13$, $p < .001$, $\eta^2_{\text{partial}} = .58$, and a condition by judgment type interaction, $F(2, 169) = 23.9$, $p < .001$, $\eta^2_{\text{partial}} = .22$ (see Figure 4).

Given the interaction between condition and judgment type, we conducted two one-way ANOVAs across conditions on the symbolic only judgements and the judgements with area models separately. As would be expected, children's scores did not significantly differ across conditions when making judgements about the fraction symbols alone, $M(SD)_{\text{Actively-Divided}} = 0.50(0.23)$, $M(SD)_{\text{Nondivided}} = 0.53(0.22)$, $M(SD)_{\text{Predivided}} = 0.49(0.21)$, $F(2,169) = 0.514$, $p = .60$, $\eta^2_{\text{partial}} < 0.01$. However, when making judgements with the area models available, there were significant condition differences, $F(2, 169) = 48.82$, $p < .001$, $\eta^2_{\text{partial}} = 0.37$. In particular, children scored highest in the nondivided condition, $M(SD)_{\text{Nondivided}} = 0.97(0.09)$, followed by the actively divided condition, $M(SD)_{\text{Actively-Divided}} = 0.76$ (0.21), and then the predivided condition, $M(SD)_{\text{Predivided}} = 0.62$ (0.23). All pairwise comparisons were significant using $\alpha = .0167$ to adjust for three pair-wise comparisons and Welch's t-tests when there was a significant difference in variance: nondivided vs. actively divided: $t(74.6) = -6.34$, $p < .001$, Cohen's $d = -1.30$; nondivided vs. predivided: $t(74.5) = 10.70$, $p < .001$, Cohen's $d = 1.96$; actively divided vs. predivided: $t(114) = 3.31$, $p = .001$, Cohen's $d = 0.62$. Thus, replicating work in the proportional reasoning literature more broadly (Boyer et al., 2008; Hurst & Cordes, 2018a), children were more accurate comparing continuous, undivided area models than comparing static, divided area models. However, a novel finding that emerges from this analysis is that comparing models that were actively divided is at an intermediate level, falling between these other two models. We discuss the potential importance of this intermediate difficulty on children's performance in the Discussion.

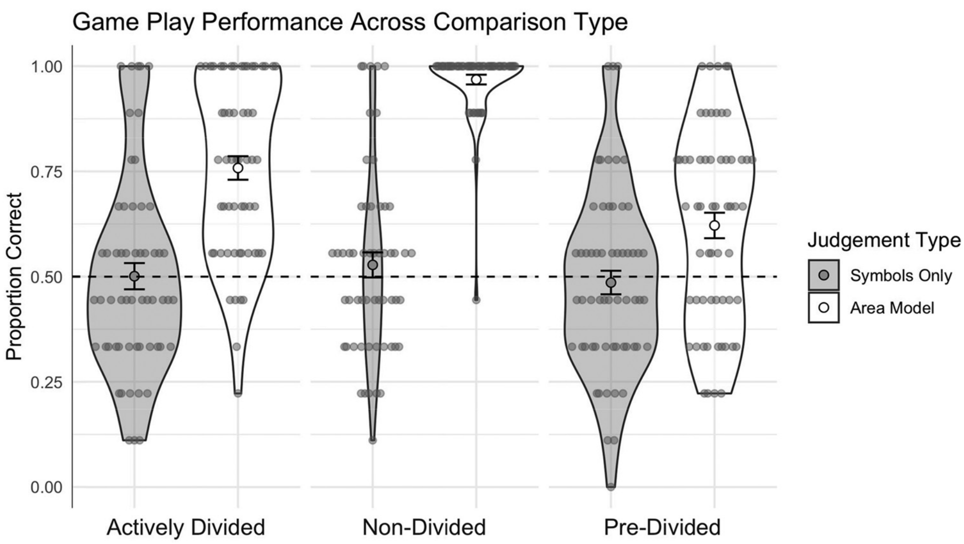
Discussion

In the current experiment, we report the results of a brief fraction card game that compared the effectiveness of using three different area models that varied in the availability and salience of fraction units to teach first and second graders about fractions. As hypothesized, we found that actively divided area models are more effective than strictly discrete or strictly continuous models for improving children's part/whole interpretation of fraction symbols. However, we also hypothesized that discrete area models would result in intermediate performance, between the actively divided and strictly continuous area models. Yet, contrary to our hypothesis, the strictly discrete and strictly continuous area models resulted in similarly low learning. That is, despite the discrete area model providing explicit information about the fraction components and being similar to the kinds of area models commonly used in fraction instruction, playing the game with discrete predivided area models did not help children learn the part–whole interpretation of fractions. In fact, the actively divided area model was the only card game intervention to result in significantly higher fraction symbol mapping at posttest. Taken together, our findings align with the hypothesis that highlighting the denominator unit through iteration can support young children's fraction learning, consistent with prior research highlighting the benefits of measurement models for fraction instruction (e.g., Boyce & Norton, 2016; Sophian, 2007). Our findings extend this research by showing that even relative to other discrete part–whole models, actively divided area models support young children's initial part–whole interpretation of symbolic fractions, a precursor skill to understanding relative fraction magnitudes.

Differences in performance during the card game itself provide further insight into children's differential use of area models. During gameplay, children in the actively divided condition scored between children in the nondivided condition and children in the predivided condition, highlighting the potential limitations of both entirely continuous and entirely discrete area models. Children who received the continuous area model in the nondivided condition

Figure 4

Performance During the Game Play Across the Three Conditions (x-axis), Comparing Performance With Symbols Only (Dark Gray, Left) With the Condition-Specific Area Model (Light Gray, Right) During Session 1 Only



Note. Plots are violin plots with the mean proportion correct and standard error as error bars. There are not condition differences on symbolic comparisons, but when the areas models are available, there are significant pairwise differences between all conditions.

were able to complete the card game trivially easily and did not have to confront their errors, and as a result did not receive any feedback or relevant information related to erroneous whole-number strategies. In contrast, children who received the discrete area models in the predivided condition struggled to compare the fractions, even visually, likely because of their erroneous whole-number-based strategies (as has been shown in prior work; Boyer et al., 2008; Ni & Zhou, 2005). Like in the predivided condition, and unlike in the nondivided condition, children in the actively divided condition were forced to confront their errors when reasoning about discrete numerical information in the context of fractions (e.g., Ni & Zhou, 2005). Yet, in contrast to the more difficult predivided condition, the active iteration of units in the actively divided condition provided children with the support they needed to confront their errors successfully (e.g., rely on a unit fraction; attend to number of units and size of units). Thus, one benefit of the actively divided area model may be that it provided the optimal level of difficulty and scaffolding, in line with desirable difficulty accounts of learning and development (e.g., Bjork, 2018; Schmidt & Bjork, 1992). Together, these findings suggest that the actively divided condition provided children with discrete information (rather than omitting it entirely, as in the nondivided area models) in a way that highlighted the size of the denominator units and the relation between unit size, number of units, and fraction size (rather than statically, as in the predivided area model), thus preventing children's default attention to countable numerical information only. This pattern of results underscores the importance of considering children's intuitions, namely their whole number-based heuristics and errors, and how best to mitigate them when designing educational materials.

Furthermore, although the importance of measurement models, and iteration in particular, is not new, prior work has primarily focused on the benefits of iteration for more advanced fraction concepts, such as moving beyond part–whole reasoning (e.g., Wilkins & Norton, 2018) and learning fraction arithmetic (e.g., Braithwaite & Siegler, 2021). Here, we extend this prior work to suggest that iteration may also play an important role in supporting even more basic part–whole conceptions of fractions. Taken together then, it may be that the use of measurement models that highlight unit iteration is an important instructional approach for many different aspects of fraction learning, from children's first introduction to part/whole fractions to their later learning of formal fraction arithmetic. However, the current study was limited in its ability to investigate how actively dividing area models might support fraction magnitude knowledge and understanding of equivalent fractions, fraction concepts that are typically taught before fraction arithmetic, because children in the current sample showed low performance on these tasks, resulting in low reliability. Thus, future work is necessary to better understand the role of iteration and units across fraction concepts and for children with different levels of fraction knowledge.

In addition to being a promising approach for supporting children's understanding of symbolic fractions, the current study also extends our understanding of the development of proportional reasoning more generally. Prior work investigating attention to discrete numerical information in both symbolic and nonsymbolic proportional reasoning tasks has primarily compared representations that either do or do not have discrete countable information available (e.g., Boyer et al., 2008; DeWolf et al., 2015; Hurst & Cordes, 2016, 2018a). However, the current study suggests that it is not the mere presence of discrete countable information that impacts children's strategy use and/or the effectiveness of the representation as a learning tool. In other words, even though both the predivided and actively divided area models provided the same nonsymbolic representation of the fraction, they resulted in different patterns of performance and different levels of learning. This suggests that the form of the nonsymbolic representation may not be enough to determine how children will interpret them and map them to their symbolic fraction system. Rather, how the form is constructed (in this case, with or without iterative units) may impact children's interpretation of the nonsymbolic form itself. Thus, discrete representations can be useful for grounding symbolic fractions, even more useful than continuous representations, but children need support to direct their attention to the relevant information within the discrete representation. Specifically, our study suggests that active iteration of discrete units positively impacts children's attention to relevant denominator information and enhances their ability to accurately map nonsymbolic proportions to fraction symbols.

This conclusion has implications for education because different instructional mediums offer different opportunities for using representations. For example, an area model presented in a textbook is necessarily a static image, limiting the use of iteration in this context. However, working with manipulatives or informal contexts that involve partitioning and iteration (e.g., equal sharing; Empson, 1999) may lead children to engage with area models in a way that highlights denominator as well as numerator units, while also aligning with children's informal mathematical reasoning. Overall, the current findings highlight the need for work that moves beyond gross level comparisons of discrete versus continuous area models or area models versus numbers lines, and instead focuses on the specific features of these representations that are helpful to children's proportional reasoning and symbolic fraction understanding, as well as which features of these representations can lead students astray. In the case of the current study, for example, future research is needed to better understand the mechanisms through which our actively divided condition supported learning, including which part(s) of the active division process (e.g., highlighting the denominator, seeing the area model transform, using an iterative division process) is most critical.

Although the actively divided version of the card game training did show success in improving children's symbolic fraction knowledge, there are some limitations, both of the training and for the study as a whole. First, the actively divided condition led to larger gains, but did not impact substantially more children than the other two conditions. Suggesting that, at least in its current form, the training might not be a very robust intervention across many diverse students. However, there are two related limitations of our design that might have contributed to this: our training sessions were short (only two 20- to 30-minute sessions, 1 week apart) and the children tested were very young. Working with young children was a deliberate decision so that we could investigate children's initial understanding of fractions before they have much (or any) formal fraction input. However, this may have limited our ability to reliably measure and impact more advanced components of children's fraction understanding, such as their ability to compare the magnitude of fractions or judge the equivalence of fractions. The low reliability of these measures and near chance performance of participants further highlight children's difficulty with these tasks. Furthermore, children who were particularly unfamiliar with fractions may have needed more time to benefit from the training. Thus, it may be that additional training and/or training that is threaded throughout the curriculum would have led to larger learning gains and allowed more children to benefit from the intervention. Together, the limited training we provided to children and the difficulty they had with most of the tasks weakened our ability to make conclusions about the effectiveness of our intervention in supporting more advanced skills (although, see online supplemental materials for details on these tasks) and may have restricted the benefit of the intervention to a subset of children. Additionally, we did not include a delayed posttest in the current study, making it unclear whether the learning gains we found are sustained over time. Future research is necessary to examine whether older children's fraction knowledge may also benefit from our actively divided area model for different and more advanced fraction concepts and whether the extended use of these models would lead to stronger learning gains.

Second, a potentially confounding explanation of our findings is that the actively divided training condition may have involved more game play time. Although we did not systematically time the sessions, it is possible that there were systematic condition differences in the time spent with the experimenter because drawing many lines and coloring (actively divided condition) likely took longer than either not drawing any lines and only coloring (predivided condition) or only drawing one line and coloring (nondivided condition). However, it is important to note that all sessions were fairly short (as discussed above), meaning that any systematic variation would likely be small. Furthermore, anecdotally, the length of the sessions varied based on many other factors, including individual differences in the child (e.g., how carefully they colored) and the experimenter, which likely overshadowed potential systematic time differences across conditions.

Lastly, the current study carefully compared three different methods of fraction representation but was limited in that only rectangular area models in the training tasks and proper fractions between zero and one were used. It remains an open question whether a similar dynamic and active division would be as effective, or potentially more effective, for other models, such as number lines or pie charts, and when incorporating a wider range of fraction values. One of the potential benefits of number lines over area models is representing fractions beyond one, which require multiple wholes (e.g., 4/3 requires one complete whole, 3/3, plus part of a whole, 1/3). Other work suggests that, in general, number lines are more effective than area models for teaching fraction magnitudes (Gunderson et al., 2019; Hamdan & Gunderson, 2017) and the number line representation may be even more likely to draw on children's measurement concepts (Siegal & Smith, 1997). Thus, it may be that number lines, when paired with the active division that highlights unit-size, may be a particularly effective and generalizable approach for incorporating measurement models into fraction instruction. Furthermore, it is worth noting that the rectangular area model used in the current training differed from typical pie chart representations because the fraction unit was iterated only across one dimension, from left to right, whereas pie charts are typically divided into segments around the center, resulting in units along two dimensions. Although children were able to generalize from the rectangular area models used during the training to circular pie charts used during posttest, whether the training itself would have also been effective with pie charts, or whether the unidirectional iteration is an important feature of these models, is an open question. Importantly, building competency with fractions involves the ability to flexibly and accurately use multiple representations (e.g., Rau & Matthews, 2017). Thus, more research is needed to understand whether these same mechanisms apply to other types of fraction representations and to develop instructional guidelines for how to best integrate the use of multiple representations to teach fractions.

In conclusion, we provided evidence that area models can be effective for children's initial learning of fraction symbols, but that dynamically and iteratively dividing area models is a more effective approach than presenting static divided area models or continuous nondivided area models. Directing children's attention to the relevant features within an area model, in this case through unit iteration, may be necessary for using them effectively. These findings highlight the importance of considering the development of children's intuitions about educationally relevant concepts, including their common incorrect heuristics or systematic errors, prior to formal instruction and how educational materials may interact with those intuitions by directing children's attention to different and more relevant features.

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