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Cross-notation knowledge of fractions and decimals

David W. Braithwaite[a,*], Jake McMullen[b], Michelle A. Hurst[c]

[a] Department of Psychology, Florida State University, Tallahassee, FL 32306, USA
[b] Department of Teacher Education, University of Turku, FIN-20014 Turun Yliopisto, Finland
[c] Department of Psychology, University of Chicago, Chicago, IL 60637, USA

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ABSTRACT

Understanding fractions and decimals requires not only understanding each notation separately, or *within-notation knowledge,* but also understanding relations between notations, or *cross-notation knowledge.* Multiple notations pose a challenge for learners but could also present an opportunity, in that cross-notation knowledge could help learners to achieve a better understanding of rational numbers than could easily be achieved from within-notation knowledge alone. This hypothesis was tested by reanalyzing three published datasets involving fourth- to eighth-grade children from the United States and Finland. All datasets included measures of rational number arithmetic, within-notation magnitude knowledge (e.g., accuracy in comparing fractions vs. fractions and decimals vs. decimals), and cross-notation magnitude knowledge (e.g., accuracy in comparing fractions vs. decimals). Consistent with the hypothesis, cross-notation magnitude knowledge predicted fraction and decimal arithmetic when controlling for within-notation magnitude knowledge. Furthermore, relations between within-notation magnitude knowledge and arithmetic were not notation specific; fraction magnitude knowledge did not predict fraction arithmetic more than decimal arithmetic, and decimal magnitude knowledge did not predict decimal arithmetic more than fraction arithmetic. Implications of the findings for assessing rational number knowledge and learning and teaching about rational numbers are discussed.

* Corresponding author.
E-mail address: braithwaite@psy.fsu.edu (D.W. Braithwaite).

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Introduction

Rational numbers are among the most important and difficult topics children encounter in early math education. More than two thirds of a nationally representative sample of U.S. adults reported using rational numbers in their jobs (Handel, 2016). Furthermore, rational number knowledge predicts success in algebra (Booth, Newton, & Twiss-Garrity, 2014) and general math achievement in high school when controlling for whole number arithmetic skill, nonverbal IQ, working memory, and family socioeconomic status (Siegler et al., 2012). However, many individuals struggle with rational numbers even after years of instruction (Siegler, Thompson, & Schneider, 2011).

Unlike whole numbers, rational numbers are regularly represented using several different notations—fractions, decimals, and percentages. This fact poses a challenge for learners, in that understanding rational numbers requires not only understanding each notation on its own, or *within-notation knowledge,* but also understanding the relations between notations, or *cross-notation knowledge.* Even many high school students have trouble understanding that fractions and decimals can represent the same numbers (Vamvakoussi & Vosniadou, 2010). However, multiple rational number notations may also present an opportunity: making connections between fractions and decimals could enable learners to achieve better understanding of, and proficiency with, each notation than could easily be achieved otherwise.

The current study tested a specific version of the above hypothesis—that cross-notation knowledge of fraction and decimal magnitudes contributes to fraction and decimal arithmetic skill. To do so, we investigated relations between cross-notation magnitude understanding and fraction and decimal arithmetic skill, controlling for within-notation magnitude understanding. Within- and cross-notation magnitude understanding were measured using magnitude comparison and ordering tasks involving either one notation (e.g., comparing fractions to fractions) or multiple notations (e.g., comparing fractions to decimals).

Below, we briefly review previous research on children's knowledge of fractions and decimals, including knowledge of magnitudes and knowledge of arithmetic. Next, we elaborate the concept of cross-notation knowledge, discuss how such knowledge might facilitate learning about rational numbers, and review previous research relating to cross-notation knowledge. Then, we describe the goals and approach of the current study in more detail.

Children's knowledge of fractions and decimals

Understanding fractions and decimals requires understanding that they have magnitudes that can be compared, ordered, and placed on number lines (Siegler et al., 2011). Unfortunately, this understanding proves to be elusive for many children with respect to both fractions (Braithwaite & Siegler, 2018; Bright, Behr, Post, & Wachsmuth, 1988; Jordan, Resnick, Rodrigues, Hansen, & Dyson, 2017; Mazzocco & Devlin, 2008) and decimals (Desmet, Grégoire, & Mussolin, 2010; DeWolf, Bassok, & Holyoak, 2015; Malone, Loehr, & Fuchs, 2017; Resnick, Rinne, Barbieri, & Jordan, 2019; Rittle-Johnson, Siegler, & Alibali, 2001). For example, on the U.S. National Assessment of Educational Progress (NAEP), only 50% of eighth graders in 2007 correctly ordered 2/7, 5/9, and 1/12 from smallest to largest, and only 42% of eighth graders in 2005 correctly identified a point midway between .005 and .006 on a number line as .0055 (U.S. Department of Education, Institute of Education Sciences, National Center for Education Statistics, 2005, 2007).

Many children also display poor proficiency with rational number arithmetic, again with both fractions (Byrnes & Wasik, 1991; Gabriel, Coché, et al., 2013; Hansen et al., 2015; Hecht & Vagi, 2012; Mack, 1995; Newton, Willard, & Teufel, 2014; Siegler et al., 2011) and decimals (Hurst & Cordes, 2018; Hiebert & Wearne, 1985; Kouba et al., 1988; Rittle-Johnson & Koedinger, 2009). For example, sixth graders correctly answered only 46% of fraction arithmetic problems in one study (Siegler & Pyke, 2013) and correctly answered only 57% of decimal arithmetic problems in another study (Tian, Braithwaite, & Siegler, 2021). Common arithmetic errors include adding fractions by separately adding their numerators and denominators (e.g., 3/5 + 1/4 = 4/9), passing through a common denominator when multiplying fractions (e.g., 3/5 × 1/5 = 3/5), adding digits with different place values when adding decimals (e.g., 4 + .3 = .7), and placing the decimal point incorrectly when multiplying decimals (e.g., .4 × .2 = .8).

According to the Integrated Theory of Numerical Development (Siegler & Braithwaite, 2017; Siegler et al., 2011), understanding numerical magnitudes is critical for learning arithmetic. The theory therefore implies that difficulties understanding the magnitudes of fractions and decimals contribute to difficulties with fraction and decimal arithmetic. Consistent with the theory, individual differences in magnitude understanding predict differences in arithmetic skill for both fractions (Bailey, Hansen, & Jordan, 2017; Siegler & Pyke, 2013; Siegler et al., 2011; Torbeyns, Schneider, Xin, & Siegler, 2015) and decimals (Rittle-Johnson & Koedinger, 2009). Further, interventions designed to improve fraction magnitude understanding have been shown to improve fraction arithmetic skill (Dyson, Jordan, Rodrigues, Barbieri, & Rinne, 2020; Fuchs et al., 2013). A possible mechanism underlying these phenomena is that understanding the magnitudes of individual numbers enables students to evaluate whether candidate answers to arithmetic problems are plausible and thereby to reject incorrect procedures that generate implausible answers (Braithwaite & Siegler, 2021; Siegler et al., 2011).

Cross-notation knowledge of rational numbers

Much prior research has treated fractions and decimals as separate topics. Similarly, many math curricula devote separate chapters to fractions and decimals and devote relatively little space to relations between them (e.g., Charles et al., 2012; Dixon, Adams, Larson, & Leiva, 2012). Knowledge of such relations, or cross-notation knowledge, is the focus of the current study. We argue that cross-notation knowledge merits greater attention than it has previously received, in part because it may confer benefits that are difficult to achieve through within-notation knowledge alone.

Potential benefits of cross-notation knowledge

First, cross-notation knowledge could enable learners to use knowledge about each notation to help solve problems involving the other notation. For example, to understand why .4 × .2 = .08 rather than .8, a child might reason that 4/10 × 2/10 = 8/100, or more generally that multiplying tenths by tenths yields hundredths. As another example, a child who has not learned or does not remember how to solve 3/5 + 1/4 but knows how to solve 0.6 + 0.25 might translate the former problem into the latter to solve it. Children could use such strategies once they have been taught to translate between fractions and decimals, which typically occurs in fourth grade in the United States, and once they have been taught—although not necessarily mastered—at least some fraction or decimal arithmetic procedures, which occurs mainly in fifth and sixth grades (Common Core State Standards Initiative [CCSSI], 2010). The above approach to understanding decimal multiplication is provided in some math textbooks (e.g., *Eureka Math;* Great Minds, 2015), and both of the above strategies were observed in a recent study of adults' fraction and decimal arithmetic among adults (Braithwaite & Sprague, 2021). Individuals with stronger cross-notation knowledge may be more likely to benefit from using such strategies.

Second, cross-notation knowledge could focus learners' attention on similarities between fractions and decimals, leading to a deeper understanding of rational numbers. Fractions and decimals are superficially different; for example, adding fractions with unlike denominators requires conversion to a common denominator, whereas adding decimals requires adding digits with the same place value. Yet, fractions and decimals have many similarities relating to general properties of rational numbers, such as the principle that a sum of positive numbers is larger than either addend. Making connections between fractions and decimals could reinforce learners' understanding of these general properties, which in turn could facilitate acquisition of within-notation knowledge. For instance, understanding the above principle could help learners to avoid errors that violate it, such as 3/5 + 1/4 = 4/9 and 4 + .3 = .7.

Previous research on cross-notation knowledge

Several previous studies have employed tasks that assess cross-notation knowledge (although without using that term) such as conversion between fractions and decimals and cross-notation comparison or ordering (Binzak & Hubbard, 2020; Ganor-Stern, 2013; Hurst & Cordes, 2016, 2018; Mazzocco & Devlin, 2008; McMullen, Laakkonen, Hannula-Sormunen, & Lehtinen, 2015; Van Hoof, Janssen, Verschaffel, & Van Dooren, 2015; Zhang, Fang, Gabriel, & Szűcs, 2016). However, these studies did not analyze relations between cross-notation knowledge and other aspects of rational number knowledge.

Other studies did not assess cross-notation knowledge but hinted at influences of decimal knowledge on fraction knowledge and vice versa that could be mediated by cross-notation knowledge. For example, a longitudinal study found that decimal magnitude knowledge in the winter of fourth grade predicted fraction magnitude knowledge in the spring of fourth grade when controlling for initial fraction magnitude knowledge (Resnick et al., 2019; see also McMullen & Van Hoof, 2020). Similarly, using a priming paradigm, Ren and Gunderson (2019) found that activating children's and adults' fraction knowledge affected subsequent performance on a decimal magnitude comparison task. However, an intervention study found no benefit of integrated fraction/decimal instruction over fractions-only instruction for improving fraction outcomes among at-risk fourth graders (Malone, Fuchs, Sterba, Fuchs, & Foreman-Murray, 2019). In summary, the role of cross-notation knowledge in children's numerical development (if any) is not yet clear.

The current study

The main goal of the current study was to test the prediction that individual differences in cross-notation knowledge would explain unique variance in rational number arithmetic skill when controlling for within-notation fraction and decimal magnitude knowledge. This prediction is consistent with the more general hypothesis that cross-notation knowledge confers benefits beyond those of within-notation knowledge alone. The prediction builds on previous research that has found positive relations between individual differences in magnitude knowledge with fractions or decimals and arithmetic skill with the same notation (Bailey, Hansen, & Jordan, 2017; Rittle-Johnson & Koedinger, 2009; Siegler & Pyke, 2013; Torbeyns et al., 2015), consistent with the Integrated Theory of Numerical Development (Siegler et al., 2011).

Another goal was to investigate whether effects of fraction and decimal magnitude knowledge on arithmetic skill are notation specific. If children's knowledge of different notations is highly compartmentalized, magnitude knowledge with each notation should predict arithmetic with the same notation more than arithmetic with the other notation, and should not predict arithmetic with the other notation at all when controlling for magnitude knowledge with that other notation. However, our hypothesis regarding cross-notation knowledge suggests that knowledge of each notation could help to acquire proficiency with the other notation as well. To the extent that this is the case, within-notation magnitude knowledge of each notation could predict arithmetic with both notations about equally, and could predict arithmetic with the other notation even when controlling for magnitude knowledge with that other notation.

Research approach and analyses

To test these predictions, we reanalyzed data from studies of children that included assessments of rational number arithmetic as well as assessments of fraction magnitude knowledge (e.g., 1/2 > 3/4?), decimal magnitude knowledge (e.g., 0.50 > 0.75?), and cross-notation magnitude knowledge (e.g., 4/5 > 0.89?) using comparison or ordering tasks. A literature search by the first author identified three published studies that met these criteria: two conducted by the second author (McMullen et al., 2020; McMullen & Van Hoof, 2020) and one conducted by the third author (Hurst & Cordes, 2018). All of these studies were included in our analyses.

We fit a "within-notation-only" model and a "cross-and-within-notation" model to each dataset, then compared the models. Both models were linear mixed models with rational number arithmetic accuracy as the dependent variable; grade level, fraction magnitude knowledge, and decimal magnitude knowledge as fixed effects; and participant as a random effect. The cross-and-within-notation model, but not the within-notation-only model, also included cross-notation magnitude knowledge as a fixed effect. We expected that model comparisons would favor the cross-and-within-notation model and that in this model, cross-notation magnitude knowledge would have a positive effect on arithmetic accuracy.

To investigate notation specificity of effects of magnitude knowledge on arithmetic skill, in the studies that included both fraction and decimal arithmetic (Studies 2 and 3), all models included arithmetic notation and interactions of arithmetic notation with each magnitude knowledge measure as fixed effects. We also considered that magnitude knowledge might differentially predict skill with different arithmetic operations. For example, magnitude knowledge might be more related to skill with addition and subtraction than to skill with multiplication and division because sums and differences of rational numbers can be estimated by composing or decomposing magnitudes, whereas estimation of products and quotients of rational numbers requires more advanced transformations such as scaling (CCSSI, 2010, 5.NF.5; Devlin, 2008). Therefore, in the studies where arithmetic tasks involved multiple operations (Studies 1 and 3), all models included operation and interactions of operation with each magnitude knowledge measure as fixed effects. In the one study that included both fraction and decimal arithmetic with multiple operations (Study 3), the models also included the three-way interactions of arithmetic notation and operation with each magnitude knowledge measure as fixed effects.

Models were fit using both frequentist and Bayesian approaches in R Version 3.6.1 (R Core Team, 2018). Frequentist analyses employed *lmer* from the lme4 package (Bates, Maechler, Bolker, & Walker, 2015) and *lmerTest* (Kuznetsova, Brockhoff, & Christensen, 2017). Models were fit using maximum likelihood (ML) and were compared using the Akaike information criterion (AIC) and chi-square tests. Bayesian analyses employed *brm* from the brms package in R (Bürkner, 2017), which is a front end for Stan (Carpenter et al., 2017), a probabilistic programming language. Default priors from *brm* were used (an improper flat prior over the reals for fixed effects and a half Student's *t* prior for the standard deviation of random effects); sampling employed the no-U-turn sampler (NUTS; Hoffman & Gelman, 2014) with eight chains of 10,000 iterations, including 2000 warm-up iterations, for each model. Bayesian model comparisons employed the leave-one-out information criterion (LOOIC), an estimate of predictive accuracy (Gelman, Hwang, & Vehtari, 2014), and Bayes factors (BFs), obtained from *LOO* and *bayes_factor* in brms. Andraszewicz et al. (2015) recommended interpreting BFs exceeding 1, 3, 10, 30, and 100 as indicating anecdotal, moderate, strong, very strong, and extremely strong evidence, respectively.

Bayesian analyses were included because they could, in principle, provide evidence favoring the null hypothesis—that is, the within-notation-only models. However, to preview the results, all model comparisons favored the cross-and-within-notation models. Therefore, we report details regarding the cross-and-within-notation models in the main text and report details regarding the within-notation-only models in the online supplementary material. For each effect in the cross-and-with-notation models, we report a point estimate (point estimates from Bayesian and frequentist analyses were identical except in one case, noted below), a 95% credible interval (95% CrI) from the Bayesian analysis, and significance test results from the frequentist analysis.

Study 1

Our first dataset came from a single time point of a longitudinal study of Finnish primary school students' rational number development (McMullen & Van Hoof, 2020). Participants were fourth through sixth graders from two schools in southwest Finland. In Finland, fraction and decimal magnitudes and addition and subtraction of fractions with like denominators are typically covered by the end of fourth grade, and decimal arithmetic and more advanced fraction arithmetic are covered in fifth and sixth grades. Within-notation magnitude knowledge was assessed using magnitude comparison and ordering tasks, and cross-notation magnitude knowledge was assessed using a magnitude comparison task. The measure of arithmetic skill was a fraction arithmetic task that included all four arithmetic operations.

Method

Participants

Our analysis included 277 fourth graders (n = 94; M_{age} = 10 years 11 months), fifth graders (n = 83; M_{age} = 11 years 11 months), and sixth graders (n = 100; M_{age} = 12 years 10 months; 138 boys and 139 girls). The sample represented a Finnish urban population, including students from lower-middle-class to middle-class backgrounds and from diverse ethnic backgrounds. The ethics board of University of Turku, the district, and the school administrations approved the study.

Tasks, stimuli, and procedure

Participants completed a paper-and-pencil test of their rational number knowledge in a whole class setting in their regular math classrooms. The test included assessments of magnitude knowledge, understanding of rational number density, and fraction arithmetic administered in that order. The density assessment does not relate to the predictions tested in the current study and so is not discussed further (see McMullen & Van Hoof, 2020 for details on this task). Students had 45 min to complete the test.

Magnitude knowledge. The magnitude knowledge assessment included three tasks: within-notation comparison, within-notation ordering, and cross-notation comparison. The within-notation comparison task required participants to circle the larger of two fractions (e.g., 5/8, 4/3) or two decimals (e.g., 0.36, 0.5) or to circle both numbers if they were equal. The within-notation ordering task required participants to put three fractions (e.g., 6/8, 2/2, 1/3) or three decimals (e.g., 6.79, 6.786, 6.4) in order from smallest to largest. The cross-notation comparison task was identical to the within-notation comparison task except that each trial involved comparing one fraction and one decimal (e.g., 1/7, 0.7).

Each within-notation task included three fraction items and three decimal items, and the cross-notation comparison task included four items. Each correct answer was given 1 point. Measures of within-notation magnitude knowledge were calculated separately for fractions and decimals by adding scores from the comparison and ordering tasks (maximum = 6 for each notation). Scores on the cross-notation comparison task (maximum = 4) served as our measure of cross-notation magnitude knowledge. Cronbach's alpha was .86 for fraction magnitude knowledge, .81 for decimal magnitude knowledge, and .78 for cross-notation magnitude knowledge. All measures of magnitude knowledge were converted to proportions correct and mean-centered for analysis.

Fraction arithmetic. Participants completed 12 fraction arithmetic problems: 7 fraction addition and subtraction items (3/5 + 1/5; 2/9 + 5/9; 2/5 + 3/10; 2 3/5 + 1/5; 4 − 1/2; 5 4/5 − 2 2/5; 3 3/4 + 4) and 5 fraction multiplication and division items (1/2 × 1/2; 2/3 × 4/5; 1 3/8 × 1/8; 6/7 × 3/2; 1/4 ÷ 1/2) with like and unlike denominators. Each correct answer was given 1 point, with a maximum score of 7 for the addition and subtraction items (Cronbach's alpha = .75) and 5 for multiplication and division items (Cronbach's alpha = .71). These scores were converted to proportions correct separately for addition/subtraction and multiplication/division.

Analysis

The within-notation-only model was a linear mixed model with accuracy on the fraction arithmetic tasks as the dependent variable; grade, within-notation fraction magnitude knowledge, within-notation decimal magnitude knowledge, arithmetic operation, and the interactions of fraction and decimal magnitude knowledge with arithmetic operation as fixed effects; and participant as a random effect. The cross-and-within-notation model included all effects that were included in the within-notation-only model and also included cross-notation magnitude knowledge and its interaction with arithmetic operation as fixed effects. Arithmetic operations were classified as either addition/subtraction or multiplication/division, which were dummy-coded as −0.5 and 0.5, respectively.[1]

[1] Assigning numeric codes with a difference of 1.0 for arithmetic operation ensures that the estimated effect of arithmetic operation equals the difference in accuracies between arithmetic operations. Assigning numeric codes centered around 0.0 ensures that the estimated effect of each other predictor in the model equals the average of the predictor's estimated effects for all arithmetic operations, as would be the case in analysis of variance.

Results and discussion

Descriptive statistics and zero-order correlations for the measures of magnitude knowledge and arithmetic are shown in the Appendix (Table A1). As predicted, model comparisons favored the cross-and-within-notation model over the within-notation-only model, as indicated by lower AIC (−6.6 vs. −0.3), a significant chi-square test [$\chi^2(2) = 10.3$, $p = .006$], lower LOOIC (−21.5 vs. −17.3), and BF of 2.9.

Details regarding the cross-and-within-notation model are shown in Table 1. As predicted, higher cross-notation knowledge predicted higher arithmetic accuracy. This result is consistent with the hypothesis that cross-notation knowledge benefits arithmetic learning, at least for fraction arithmetic. The absence of a cross-notation knowledge by arithmetic operation interaction indicates that effects of cross-notation knowledge did not differ between addition/subtraction and multiplication/division.

A significant main effect was also found for decimal magnitude knowledge, but not for fraction magnitude knowledge. Both fraction and decimal magnitude knowledge interacted with arithmetic operation. To investigate these interactions, we used the samples generated in the Bayesian analysis to calculate 95% CrIs for effects of fraction and decimal magnitude knowledge on addition/subtraction and multiplication/division accuracy.[2] Effects of fraction magnitude knowledge were larger for addition/subtraction than for multiplication/division, but the 95% CrIs of both effects included zero (addition/subtraction: estimate = 0.10, 95% CrI = [−0.01, 0.21]; multiplication/division: estimate = −0.08, 95% CrI = [−0.19, 0.03]). Effects of decimal magnitude knowledge were also larger for addition/subtraction than for multiplication/division, and 95% CrIs of the effects excluded zero for addition/subtraction (estimate = 0.19, 95% CrI = [0.07, 0.30]) but not multiplication/division (estimate = 0.01, 95% CrI = [−0.11, 0.13]).

Effects of grade and arithmetic operation were also found. Children in higher grades were more accurate (fourth grade = 28%, fifth grade = 39%, sixth grade = 51%), and accuracy was higher on addition/subtraction problems than on multiplication/division problems (60% vs. 20%).

Study 2

Study 2 analyzed data from a study of children's rational number magnitude knowledge, rational number arithmetic, and pre-algebra skills previously reported in Hurst and and Cordes (2018). Participants were fourth to seventh graders in the United States. According to the Common Core State Standards for mathematics, comparison of fractions and decimals, as well as addition of fractions with like denominators and multiplication of fractions by a whole number, should be covered by the end of fourth grade, whereas decimal arithmetic and more advanced fraction arithmetic are to be covered in fifth and sixth grades (CCSSI, 2010). The study included both within-notation and cross-notation magnitude comparison tasks. The measure of arithmetic skill included fraction and decimal addition problems but no other arithmetic operations.

Method

Participants

Our analysis included 39 children (M_{age} = 11.8 years, range = 9.8–14.4; 10 fourth graders, 10 fifth graders, 3 sixth graders, and 16 seventh graders; 24 boys and 15 girls) recruited from the Boston area in the northeastern United States through a variety of recruitment methods and testing locations, including schools, child-care programs, and an on-campus research lab. We do not have additional demographic information about the sample. Two children who were included in Hurst and Cordes's (2018) study were excluded from the current analysis because they had missing arithmetic data, which was the primary purpose of the current analysis. The institutional review board of Boston College approved the study.

[2] Effect on addition/subtraction = (main effect of magnitude knowledge) − 0.5 * (interaction with operation). Effect on multiplication/division = (main effect of magnitude knowledge) + 0.5 * (interaction with operation).

Table 1
Frequentist and Bayesian analysis results from cross-and-within-notation model (Study 1; N = 277).

		Bayesian analysis	Frequentist analysis	
Effect	B	[95% CrI]	t	P
Grade	0.07	[0.03, 0.10]	4.1	<.001
Arithmetic operation	−0.40	[−0.43, −0.36]	−22.4	<.001
Fraction magnitude knowledge	0.01	[−0.08, 0.10]	0.2	.87
Fraction magnitude knowledge * arithmetic operation	−0.18	[−0.32, −0.05]	−2.7	.008
Decimal magnitude knowledge	0.10	[0.01, 0.19]	2.2	.03
Decimal magnitude knowledge * arithmetic operation	−0.18	[−0.32, −0.04]	−2.5	.01
Cross-notation magnitude knowledge	0.14	[0.05, 0.23]	3.2	.001
Cross-notation magnitude knowledge * arithmetic operation	−0.01	[−0.15, 0.12]	−0.2	.86

Note. B indicates estimated effects, which were identical in the frequentist and Bayesian analyses; 95% CrI indicates 95% credible intervals of effects from the Bayesian analyses; and t and p indicate test results from the frequentist analyses.

Tasks, stimuli, and procedure

Children completed assessments of magnitude knowledge, pre-algebra, and rational number arithmetic in that order. The pre-algebra assessment does not relate to the predictions tested in the current study and so is not discussed further (see Hurst & Cordes, 2018, for details about this task). The entire session occurred one on one between a researcher and the child, using a computer for the magnitude knowledge assessment and paper and pencil for all other tasks.

Magnitude knowledge. Magnitude knowledge was measured using a magnitude comparison task in which children were shown two numbers presented on a computer screen and asked to decide which was larger as accurately and quickly as they could by pressing the corresponding key on the keyboard. The task included six types of comparisons: fraction versus fraction, decimal versus decimal, whole number versus whole number, fraction versus decimal, fraction versus whole number, and decimal versus whole number.

Each block consisted of 16 trials made from four unique comparisons from two ratio bins, each shown twice ($4 \times 2 \times 2 = 16$). The ratio bins (larger magnitude/smaller magnitude) were approximately 1.5 (range = 1.35–1.67) and 2.5 (range = 2.20–2.92). The pairs of numbers compared in the fraction versus fraction, decimal versus decimal, and decimal versus fraction blocks had the same magnitudes up to rounding error. For example, the fraction versus fraction trials included the comparison 3/5 versus 2/9; the corresponding comparisons in the decimal versus decimal and fraction versus decimal blocks were 0.60 versus 0.22 and 3/5 versus 0.22. In the fraction versus decimal block, the larger value was presented as a fraction on half the trials and as a decimal on the other half. Magnitude values ranged from 1/5 (0.2) to 7/2 (3.5) to include values above and below 1. For fraction stimuli, all the numerators and denominators were between 1 and 10; for fraction versus fraction comparisons, each component of each fraction differed from both components of the other fraction. Decimals were always presented to the hundredth digit (potentially with a 0 in the hundredth digit, e.g., 0.20) with an integer before the decimal point (sometimes 0).

Each type of comparison was presented in a separate block. The six blocks were presented in a random order for each participant. Although participants were encouraged to answer as quickly as they could, they had unlimited time and all trials were included regardless of their reaction time.

Within-notation magnitude knowledge was calculated as the proportion of trials correct (out of 16) on the fraction versus fraction block (Cronbach's alpha = .77) and decimal versus decimal block (Cronbach's alpha = .62) separately. Proportion correct on the fraction versus decimal block (Cronbach's alpha = .72) served as our measure of cross-notation magnitude knowledge. Data from blocks involving whole numbers were not included in the current analyses. As in Study 1, all measures of magnitude knowledge were mean-centered.

Rational number arithmetic. The rational arithmetic assessment included a fraction addition task and a decimal addition task. The order of these two tasks was counterbalanced between participants.

The fraction addition task included five fraction addition problems (2/3 + 5/6; 2/5 + 3/4; 12 + 4/7; 5/8 + 2/4; 3/9 + 2/3) presented in a booklet with enough space between problems to work out solutions and provide answers. Problems were presented horizontally, with fractions presented in their formal upright notation. All problems involved fractions with different denominators. Children were able to take as long as they needed. Performance was scored as the proportion of problems correct (out of 5; Cronbach's alpha = .95), and any correct answer was accepted (it did not need to be in any particular format or specific simplified fraction).

The decimal addition task included five decimal addition problems (0.5 + 0.38; 0.21 + 0.63; 0.78 + 0.19; 0.45 + 0.8; 0.53 + 0.49) presented in a booklet with enough space between problems to work out solutions and provide answers. Problems were presented horizontally. Three problems involved two addends that each had two decimal digits, and two problems involved one addend with two decimal digits and one addend with one decimal digit. Children were able to take as long as they needed. Performance was scored as the proportion of problems correct (out of 5; Cronbach's alpha = .64), and any correct answer was accepted.

Analysis

The within-notation-only model was a linear mixed model with accuracy on the arithmetic assessment as the dependent variable; grade, within-notation fraction magnitude knowledge, within-notation decimal magnitude knowledge, arithmetic notation, and the interactions of fraction and decimal magnitude knowledge with arithmetic notation as fixed effects; and participant as a random effect. The cross-and-within-notation model included all effects that were included in the within-notation-only model as well as cross-notation magnitude knowledge and its interaction with arithmetic notation as fixed effects. In all analyses, fraction and decimal arithmetic were dummy-coded as -0.5 and 0.5, respectively.[3]

Results and discussion

Descriptive statistics and zero-order correlations for the measures of magnitude knowledge and arithmetic are shown in the Appendix (Table A2). As in Study 1, model comparisons favored the cross-and-within-notation model, as indicated by lower AIC (30.5 vs. 33.4), a significant chi-square test [$\chi^2(2) = 6.8$, $p = .033$], lower LOOIC (30.4 vs. 31.6), and BF of 22.0.

Details regarding the cross-and-within-notation model are shown in Table 2. As predicted, higher cross-notation magnitude knowledge predicted higher arithmetic accuracy, replicating the main finding of Study 1. Effects of fraction and decimal magnitude knowledge were not significant, and no interactions reached significance. Thus, there was not statistically significant evidence that fraction magnitude knowledge relates more strongly to fraction arithmetic than to decimal arithmetic or that decimal magnitude knowledge relates more strongly to decimal arithmetic than to fraction arithmetic.

Finally, significant effects of grade and arithmetic notation were also found. Children in higher grades were more accurate (fourth grade = 48%, fifth grade = 82%, sixth grade = 80%, seventh grade = 85%). Accuracy was higher on decimal than fraction arithmetic (84% vs. 65%).

Study 3

The third dataset came from a single time point of a longitudinal study of middle school students' rational number development (Braithwaite et al., 2019, McMullen et al., 2020). Participants were seventh and eighth graders in the United States, where formal instruction in fractions and decimals is typically completed prior to seventh grade (CCSSI, 2010). The magnitude knowledge assessment employed within-notation ordering tasks with fractions and decimals and a cross-notation ordering

[3] Assigning numeric codes with a difference of 1.0 for arithmetic notation implies that the estimated effect of arithmetic operation equals the difference in accuracies between arithmetic notations. Assigning numeric codes centered around 0.0 implies that the estimated effect of each other predictor in the model equals the average of the predictor's estimated effects for both arithmetic notations, as would be the case in analysis of variance.

Table 2
Frequentist and Bayesian analysis results from cross-and-within-notation model (Study 2; N = 39).

Effect	B	Bayesian analysis [95% CrI]	Frequentist analysis t	p
Grade	0.07	[0.01, 0.13]	2.5	.02
Arithmetic notation	0.18	[0.08, 0.29]	3.8	<.001
Fraction magnitude knowledge	0.14	[−0.44, 0.71]	0.5	.62
Fraction magnitude knowledge * arithmetic notation	−0.31[a]	[−1.16, 0.53]	−0.8	.41
Decimal magnitude knowledge	0.01	[−0.91, 0.93]	0.02	.98
Decimal magnitude knowledge * arithmetic notation	−0.86	[−2.21, 0.51]	−1.4	.17
Cross-notation magnitude knowledge	0.68	[0.004, 1.35]	2.2	.04
Cross-notation magnitude knowledge * arithmetic notation	−0.70	[−1.70, 0.29]	−1.6	.13

Note. B indicates estimated effects, which were identical in the frequentist and Bayesian analyses except as stated below; 95% CrI indicates 95% credible intervals of effects from the Bayesian analyses; and t and p indicate test results from the frequentist analyses.

[a] −0.31 denotes the estimated interaction of fraction magnitude knowledge with arithmetic notation in the Bayesian analysis; the corresponding estimate from the frequentist analysis was −0.32.

task. The arithmetic assessment in this study included all four operations with both fractions and decimals.

Method

Participants

Our analysis included 394 children (232 seventh graders and 162 eighth graders; 209 girls and 185 boys). All participants were from a single school in Gainesville, Florida, in the southeastern United States, that was made up of students who were identified as 51% White, 28% African American, 11% Hispanic, 5% Asian, and 5% other in district records, with 43% of students at the school being eligible for free or reduced-price lunch. All participants had parental consent to participate in the study and gave their own assent before participating. The ethics board of University of Turku, the district, and school administrations approved the study.

Tasks, stimuli, and procedure

Participants completed a series of assessments of various aspects of rational number knowledge (detailed in McMullen et al., 2020). Only the magnitude knowledge assessment and rational number arithmetic assessment are examined in the current study. The arithmetic assessment was completed immediately prior to the magnitude knowledge assessment. Students had 50 min to complete all assessments. The assessments were administered in a whole class format during students' regular science class by the second author.

Magnitude knowledge. The magnitude knowledge assessment included fraction ordering and decimal ordering tasks, which served as our measures of within-notation magnitude knowledge, and a cross-notation ordering task, which served as our measure of cross-notation magnitude knowledge. Each task included two items. Each item required participants to put either three or four numbers in order from smallest to largest. The fraction items were {5/8, 11/37, 3/4} and {6/12, 5/7, 2/6, 2/3}, the decimal items were {7.351, 7.8, 7.09, 7.71} and {0.68, 0.29, 0.351, 0.5}, and the mixed items were {0.5, 1/4, 5/100, 0.356} and {13/52, 0.111, 2/3, 0.8}.

Because each task included only two items, but each item required multiple pairwise comparisons, to increase the captured variance across participants, participants received one point for each pair of numbers that they ordered correctly. For example, for the item {5/8, 11/37, 3/4}, participants received 1 point for correctly ordering each of the following pairs of numbers: 5/8 versus 11/37, 5/8 versus 3/4, and 11/37 versus 3/4. The maximum possible scores were 9 for fractions (Cronbach's alpha = .93) and 12 for decimals (Cronbach's alpha = .95). For the cross-notation magnitude knowledge task, only cross-notation pairs (e.g., 0.5 vs. 1/4 but not 1/4 vs. 5/100) were scored, for a maximum of 8 points for cross-notation magnitude knowledge (Cronbach's alpha = .80). As in Studies 1 and 2, magnitude knowledge scores were converted to proportions correct and mean-centered.

Rational number arithmetic. Participants completed 24 rational number arithmetic problems. Items included 12 fraction arithmetic items (2/3 − 1/3; 4/7 ÷ 1/2; 3/4 × 1/5; 8 1/2 ÷ 4 1/8; 5/7 − 1/2; 1/5 + 2/3; 7/8 + 2/8; 2 3/4 + 4 1/8; 2 6/7 + 5 1/2; 5/8 ÷ 3/8; 3 2/3 − 3/4; 3/5 × 1/5) and 12 decimal arithmetic items (1.05 × 0.2; 0.71 − 0.4; 0.11 + 0.7; 5.29 − 4.2; 3.4 + 1.02; 0.38 − 0.14; 0.4 + 0.2; 0.9 ÷ 0.3; 0.4 × 0.52; 0.111 × 0.097; 3.06 × 5.3; 0.84 ÷ 0.4). Each correct answer was given 1 point. Scores on each combination of notation (fraction and decimal) and arithmetic operation (addition/subtraction and multiplication/division) were converted to proportions correct for analysis. Cronbach's alpha was .86 for fraction addition/subtraction, .67 for fraction multiplication/division, .73 for decimal addition/subtraction, and .78 for decimal multiplication/division.

Analysis

The within-notation-only model was a linear mixed model with accuracy on the arithmetic assessment as the dependent variable; grade, within-notation fraction magnitude knowledge, within-notation decimal magnitude knowledge, arithmetic operation, arithmetic notation, the interaction of operation and notation, the interactions of fraction and decimal magnitude knowledge with arithmetic operation and arithmetic notation, and the three-way interactions of fraction and decimal magnitude knowledge with both operation and notation as fixed effects; and participant as a random effect. The cross-and-within-notation model included all effects that were included in the within-notation-only model as well as cross-notation magnitude knowledge, its interactions with operation and notation, and the three-way interaction of cross-notation magnitude knowledge with both operation and notation. As in Study 1, arithmetic operation was dummy-coded as −0.5 for addition or subtraction and 0.5 for multiplication or division. As in Study 2, arithmetic notation was dummy-coded as −0.5 for fraction arithmetic and 0.5 for decimal arithmetic.

Results and discussion

Descriptive statistics and zero-order correlations for the measures of magnitude knowledge and arithmetic are shown in the Appendix (Table A3). As in Studies 1 and 2, all model comparisons favored the cross-and-within-notation model, as indicated by lower AIC (185.5 vs. 201.2), a significant chi-square test [$\chi^2(4) = 23.7$, $p < .001$), lower LOOIC (46.7 vs. 50.4), and BF of 63.9.

Details regarding the cross-and-within-notation model are shown in Table 3. As predicted, cross-notation magnitude knowledge positively predicted arithmetic accuracy. No interactions involving cross-notation magnitude knowledge were found. Thus, cross-notation magnitude knowledge had about equally strong relations to fraction and decimal arithmetic and to accuracy with different arithmetic operations.

The main effect of fraction magnitude knowledge did not reach significance, but a three-way interaction of fraction magnitude knowledge, arithmetic operation, and arithmetic notation was found. To investigate this interaction, we used the samples generated in the Bayesian analysis to calculate estimated effects of fraction magnitude knowledge, and 95% CrIs thereof, for each combination of arithmetic operation and notation.[4] The 95% CrI of the effect on fraction addition/subtraction excluded zero (estimate = 0.16, 95% CrI = [0.03, 0.29]), whereas the 95% CrIs of the effects on other problem types included zero (fraction multiplication/division: estimate = −0.05, 95% CrI = [−0.18, 0.08]; decimal addi-

[4] Effect on fraction addition/subtraction = (main effect of magnitude knowledge)) − 0.5 * (interaction with operation) − 0.5 * (interaction with notation) + 0.25 * (three-way interaction). Effect on fraction multiplication/division = (main effect of magnitude knowledge) + 0.5 * (interaction with operation)) − 0.5 * (interaction with notation)) − 0.25 * (three-way interaction). Effect on decimal addition/subtraction = (main effect of magnitude knowledge)) − 0.5 * (interaction with operation) + 0.5 * (interaction with notation)) − 0.25 * (three-way interaction). Effect on decimal multiplication/division = (main effect of magnitude knowledge) + 0.5 * (interaction with operation) + 0.5 * (interaction with notation) + 0.25 * (three-way interaction).

Table 3
Frequentist and Bayesian analysis results from cross-and-within-notation model (Study 3; N = 394).

Effect	B	Bayesian analysis [95% CrI]	Frequentist analysis t	p
Grade	0.03	[−0.01, 0.06]	1.3	.19
Arithmetic operation	−0.24	[−0.26, −0.22]	−21.3	<.001
Arithmetic notation	0.07	[0.04, 0.09]	6.0	<.001
Arithmetic operation * arithmetic notation	−0.37	[−0.41, −0.33]	−16.6	<.001
Fraction magnitude knowledge	0.06	[−0.03, 0.15]	1.2	.21
Fraction magnitude knowledge * arithmetic operation	−0.07	[−0.18, 0.04]	−1.3	.19
Fraction magnitude knowledge * arithmetic notation	0.00	[−0.10, 0.11]	0.0	.97
Fraction magnitude knowledge * arithmetic operation * arithmetic notation	0.28	[0.07, 0.50]	2.6	.01
Decimal magnitude knowledge	0.26	[0.18, 0.35]	6.1	<.001
Decimal magnitude knowledge * arithmetic operation	−0.15	[−0.25, −0.05]	−3.0	.002
Decimal magnitude knowledge * arithmetic notation	−0.02	[−0.12, 0.08]	−0.4	.66
Decimal magnitude knowledge * arithmetic operation * arithmetic notation	0.08	[−0.12, 0.28]	0.8	.42
Cross-notation magnitude knowledge	0.20	[0.11, 0.29]	4.5	.001
Cross-notation magnitude knowledge * arithmetic operation	−0.04	[−0.14, 0.06]	−0.7	.47
Cross-notation magnitude knowledge * arithmetic notation	0.00	[−0.10, 0.10]	0.0	.99
Cross-notation magnitude knowledge * arithmetic operation * arithmetic notation	0.18	[−0.02, 0.39]	1.8	.08

Note. B indicates estimated effects, which were identical in the frequentist and Bayesian analyses; 95% CrI indicates 95% credible intervals of effects from the Bayesian analyses; and t and p indicate test results from the frequentist analyses.

tion/subtraction: estimate = 0.02, 95% CrI = [−0.11, 0.15]; decimal multiplication/division: estimate = = 0.09, 95% CrI = [−0.04, 0.22]).

An effect of decimal magnitude knowledge was also found, qualified by an interaction with arithmetic operation. We investigated the interaction by calculating means and 95% CrIs of effects of decimal magnitude knowledge on addition/subtraction accuracy and multiplication/division accuracy, as in Study 1. Effects of decimal magnitude knowledge were larger for addition/subtraction (estimate = 0.34, 95% CrI = [0.24, 0.44]) than for multiplication/division (estimate = 0.18, 95% CrI = [0.09, 0.29]), but in contrast to Study 1, 95% CrIs for both effects excluded zero. Effects of decimal magnitude knowledge on fraction arithmetic did not differ from effects of decimal magnitude knowledge on decimal arithmetic.

Finally, effects of arithmetic operation and arithmetic notation were found, indicating that accuracy was higher on addition/subtraction than on multiplication/division, as in Study 1, and was higher on decimal arithmetic than on fraction arithmetic, as in Study 2. These effects were qualified by an operation by notation interaction, indicating that for addition/subtraction, accuracy was higher with decimals than with fractions (fractions: 41%; decimals: 66%), whereas the opposite was true for multiplication/division (fractions: 36%; decimals: 24%).

In contrast to Studies 1 and 2, there was no effect of grade. This null result may reflect the fact that participants in Study 3 were in seventh and eighth grades, during which rational number arithmetic is not a focus of instruction. Studies 1 and 2 included children in fourth, fifth, and sixth grades, during which rational number arithmetic is a major focus of instruction.

Combined analyses

Results of Studies 1, 2, and 3 were broadly consistent but differed in some details. For example, not all model comparisons favored the cross-and-within-notation model equally strongly (BFs = 2.9, 22.0, and 63.9 in Studies 1, 2, and 3, respectively), and some effects were found in only some studies. To address these issues, pooled data from all three studies were submitted to a within-notation-only model and a cross-and-within-notation model. Predictors were as in Study 3 except that experiment was included as a random effect, with participants nested within experiments. Model comparisons favored the cross-and-within-notation model, as indicated by lower AIC (426.8 vs. 456.0), a significant

Table 4
Frequentist and Bayesian analysis results from cross-and-within-notation model (combined data from Studies 1–3).

Effect	*B*	Bayesian analysis [95% CrI]	Frequentist analysis *t*	*p*
Grade	0.04	[0.02, 0.06]	3.4	<.001
Arithmetic operation	−0.31	[−0.33, −0.29]	−29.6	<.001
Arithmetic notation	0.07	[0.04, 0.09]	5.8	<.001
Arithmetic operation * arithmetic notation	−0.23	[−0.27, −0.19]	−11.1	<.001
Fraction magnitude knowledge	0.04	[−0.03, 0.11]	1.2	.24
Fraction magnitude knowledge * arithmetic operation	−0.07	[−0.16, 0.03]	−1.3	.18
Fraction magnitude knowledge * arithmetic notation	0.02	[−0.08, 0.12]	0.3	.74
Fraction magnitude knowledge * arithmetic operation * arithmetic notation	0.27	[0.08, 0.46]	2.7	.006
Decimal magnitude knowledge	0.20	[0.13, 0.27]	5.9	<.001
Decimal magnitude knowledge * arithmetic operation	−0.14	[−0.23, −0.05]	−3.1	.002
Decimal magnitude knowledge * arithmetic notation	0.02	[−0.07, 0.12]	0.5	.63
Decimal magnitude knowledge * arithmetic operation * arithmetic notation	0.07	[−0.11, 0.25]	0.8	.45
Cross-notation magnitude knowledge	0.19	[0.12, 0.26]	5.6	<.001
Cross-notation magnitude knowledge * arithmetic operation	−0.01	[−0.10, 0.08]	−0.3	.79
Cross-notation magnitude knowledge * arithmetic notation	0.00	[−0.10, 0.10]	0.1	.96
Cross-notation magnitude knowledge * arithmetic operation * arithmetic notation	0.14	[−0.04, 0.33]	1.5	.12

Note. B indicates estimated effects, which were identical in the frequentist and Bayesian analyses; 95% CrI indicates 95% credible intervals of effects from the Bayesian analyses; and *t* and *p* indicate test results from the frequentist analyses.

chi-square test [$\chi^2(4) = 37.2$, $p < .001$], lower LOOIC (293.9 vs. 310.0), and BF of 34,376.7. This BF indicates extremely strong evidence for the cross-and-within-notation model (Andraszewicz et al., 2015).

Details regarding the cross-and-within-notation model are shown in Table 4. Significant effects were the same as in Study 3, with the exception that an effect of grade appeared in this analysis but not in Study 3. Cross-notation magnitude knowledge predicted arithmetic accuracy and did not interact with any other predictor. Interactions involving within-notation magnitude knowledge did appear and were investigated as in Study 3, yielding very similar results. The effect of fraction magnitude knowledge was credibly greater than zero for fraction addition/subtraction (estimate = 0.13, 95% CrI = [0.04, 0.22]) but not for any other problem category (fraction multiplication/division: estimate = −0.07, 95% CrI = [−0.16, 0.02]; decimal addition/subtraction: estimate = 0.02, 95% CrI = [−0.11, 0.14]; decimal multiplication/division: estimate = 0.08, 95% CrI = [−0.04, 0.21]). The effect of decimal magnitude knowledge was larger for addition/subtraction (estimate = 0.27, 95% CrI = [0.19, 0.35]) than for multiplication/division (estimate = 0.13, 95% CrI = [0.05, 0.21]) but was credibly greater than zero in both cases.

General discussion

The current study investigated relations between individual differences in magnitude knowledge and individual differences in rational number arithmetic skill. In contrast to previous studies, which investigated such relations primarily using within-notation measures of magnitude knowledge, the current study estimated unique effects of both cross-notation and within-notation magnitude knowledge. Below, we review the key findings and discuss their implications for the role of cross-notation knowledge and within-notation knowledge in learning about rational numbers, instruction in rational numbers, and math learning in general.

The role of cross-notation knowledge in learning about rational numbers

Individual differences in cross-notation magnitude knowledge predicted arithmetic skill with rational numbers while controlling for within-notation magnitude knowledge. To our knowledge, the current study is the first to show this effect. Attesting to the robustness of the effect, it appeared in three datasets that involved children of different ages and nationalities, different measures of magnitude knowledge, and different measures of rational number arithmetic. The effect did not vary as a function of arithmetic notation or arithmetic operation.

Many measures of children's knowledge of rational number magnitudes have relied exclusively on within-notation tasks such as fraction magnitude comparison (Fazio, DeWolf, & Siegler, 2016; Gabriel, Szucs, & Content, 2013; Meert, Grégoire, & Noël, 2010), fraction number line estimation (Booth et al., 2014; Resnick et al., 2016), decimal magnitude comparison (DeWolf, Grounds, Bassok, & Holyoak, 2014; Roell, Viarouge, Houdé, & Borst, 2017), and decimal number line estimation (Durkin & Rittle-Johnson, 2015; Rittle-Johnson et al., 2001). The current findings demonstrate that purely within-notation magnitude tasks may fail to capture important aspects of individual differences and therefore suggest that it could be useful to include cross-notation tasks in assessments of rational number magnitude knowledge. Relatedly, it could also be useful to investigate children's uses of cross-notation knowledge, such as translating between notations, when performing within-notation tasks (e.g., Siegler et al., 2011).

Which aspects of individual differences are captured by cross-notation magnitude tasks but are not captured, or are captured less well, by differences in accuracy on within-notation magnitude tasks? One possibility is that cross-notation tasks tap into children's conceptual understanding of fraction–decimal equivalence and their procedural skill at converting fractions into decimals and converting decimals into fractions. For example, when asked to compare 0.5 and 1/4, a child might reason that $0.5 = 1/2$ and $1/2 > 1/4$, so $0.5 > 1/4$. Alternatively, the child might reason that $1/4 = 0.25$ and $0.25 < 0.5$, so $1/4 < 0.5$.

Another possibility is that cross-notation tasks tap into analog magnitude representations that ground children's understanding of both fraction and decimal magnitudes (Matthews, Lewis, & Hubbard, 2016; Zhang et al., 2016). Binzak and Hubbard (2020) found that adults perform cross-notation comparisons faster and more accurately as the distance between the to-be-compared numbers increases, consistent with reliance on analog magnitude representations during such comparisons. However, distance effects also appeared for within-notation comparisons and were generally as large for fraction versus fraction comparisons as for cross-notation comparisons. Thus, within-notation and cross-notation comparisons likely both elicit similar levels of reliance on analog magnitude representations. It remains to be seen whether such representations are involved in the aspects of individual differences that are uniquely captured by cross-notation tasks.

Both accounts of cross-notation magnitude knowledge are consistent with the possibility that this knowledge has a causal effect on rational number arithmetic. First, conversion between fractions and decimals could enable children to use knowledge of each notation to reason about arithmetic problems involving the other notation. For example, a child who has this ability, when presented with $.4 \times .2$, could reason that $.4 \times .2 = 4/10 \times 2/10 = 8/100 = .08$, thereby avoiding the common error $.4 \times .2 = .8$. Second, analog magnitude representations could enable children to evaluate answers to arithmetic problems based on plausibility. For example, a child who can form an analog magnitude representation of $3/5 + 1/4$ might recognize that the common incorrect response 4/9 is implausibly small. As these examples illustrate, magnitude knowledge could help children to avoid common errors in rational arithmetic, thereby increasing the likelihood of learning correct procedures and concepts.

In principle, variables not included in our analyses could account for some of or all the variance shared by cross-notation magnitude knowledge and rational number arithmetic. However, many variables that might play such a role are also correlated with fraction and decimal magnitude knowledge, for example general math achievement, nonverbal reasoning, working memory, and vocabulary (Bailey et al., 2017; Malone et al., 2017; Resnick et al., 2019; Siegler et al., 2012). The fact that our analyses controlled for fraction and decimal magnitude knowledge reduces the likelihood that such variables completely explain the relations that were found between cross-notation magnitude knowledge and arithmetic skill. Future research should further assess this possibility by controlling for a more extensive set of covariates than were included in the current study and directly testing the possibility of a causal relation.

The role of within-notation knowledge in learning about rational numbers

In principle, relations between within-notation magnitude knowledge and arithmetic could be notation specific, with fraction magnitude knowledge related specifically to fraction arithmetic and decimal magnitude knowledge related specifically to decimal arithmetic. This intuitively plausible pattern of relations is what would be expected if children's knowledge of rational numbers is compartmentalized by notation, with knowledge of magnitudes and knowledge of arithmetic being connected within each compartment. However, the current findings did not display such a pattern. Fraction magnitude knowledge did not predict fraction arithmetic more strongly than decimal arithmetic, and decimal magnitude knowledge did not predict decimal arithmetic more strongly than fraction arithmetic.

A possible explanation for these results is that within-notation measures of magnitude knowledge tap into a general understanding of rational number magnitudes that is not specific to either notation and that is related to arithmetic proficiency with both notations. This general understanding could include analog representations of numerical magnitudes, as mentioned above (Binzak & Hubbard, 2020; Matthews et al., 2016; Zhang et al., 2016), as well as concepts and principles that apply to both fractions and decimals such as the concept of representing numerical magnitudes as positions on a number line or the principle that a sum of positive numbers is greater than either addend. The fact that, in the current study, performance on decimal magnitude tasks predicted arithmetic skill more consistently than performance on fraction magnitude tasks predicted arithmetic skill could be because decimal magnitude tasks are more sensitive measures of general magnitude understanding than fraction magnitude tasks, a possibility worth exploring in the future.

In our combined analysis, fraction magnitude knowledge uniquely predicted accuracy with fraction addition and subtraction but not with fraction multiplication and division. Similarly, decimal magnitude knowledge predicted addition and subtraction accuracy more strongly than multiplication and division accuracy. These results may reflect students being more likely to think about numerical magnitudes in the context of addition and subtraction than in the context of multiplication and division, perhaps because addition and subtraction transparently involve composition and decomposition of magnitudes. Consistent with this explanation, most middle school children know that adding positive fractions or decimals "makes bigger" and subtracting them "makes smaller," whereas far fewer of them correctly understand how multiplication or division by fractions or decimals affects numerical magnitudes (Lortie-Forgues & Siegler, 2017; Siegler & Lortie-Forgues, 2015).

Relations between within- and cross-notation knowledge

Distinguishing between within- and cross-notation knowledge of rational numbers naturally leads to the question of how these types of knowledge relate to each other. We propose an iterative model in which each form of knowledge contributes to development of the other. Initially, children likely primarily acquire within-notation knowledge, including notation-specific concepts such as numerator and denominator (for fractions) and place value (for decimals). Such knowledge may create a foundation for subsequent development of cross-notation knowledge; for example, understanding the concepts of numerator and denominator provides a basis for converting fractions into decimals via division of numerator by denominator. Cross-notation knowledge may lead to further improvements in within-notation knowledge by one or both of the mechanisms proposed in the Introduction: (1) learners using one notation to help understand the other and (2) similarities between fractions and decimals drawing attention to general properties of rational numbers. Our iterative model implies that individual differences in each type of knowledge should predict improvements in the other type of knowledge both over time and after intervention. Future research should test these predictions.

The above proposal is compatible with and extends the Integrated Theory of Numerical Development (Siegler & Braithwaite, 2017). According to the theory, numerical development during middle childhood involves children integrating their knowledge of different types of numbers into a unified framework represented by the number line. Previous formulations of the theory have emphasized one aspect of this process of integration, namely integrating rational number knowledge with whole number knowledge. The iterative model described above draws attention to a complementary aspect of the process, namely integrating knowledge of fractions and knowledge of decimals, and implies that this aspect of integration may lead to refinements in knowledge of fractions, decimals, and rational numbers in general. Thus, consideration of cross-notation knowledge permits more detailed description of the developmental processes proposed by the Integrated Theory.

Implications for instruction in rational numbers

If, as proposed, connections between fractions and decimals can facilitate learning rational arithmetic, then students might benefit from instruction that explicitly relies on such connections during instruction. For example, a student who incorrectly claims that $3/5 + 1/4 = 4/9$ might benefit from a teacher pointing out that $3/5 + 1/4 = 0.6 + 0.25 = 0.85$ and that 4/9 is much smaller than 0.85. Similarly, a student who incorrectly claims that $.4 \times .2 = .8$ might benefit from a teacher showing that $.4 \times .2 = 4/10 \times 2/10 = 8/100$ and that $8/100 = 0.08$, not 0.8. As these examples illustrate, making connections between fractions and decimals could help to remediate misconceptions and encourage integration of previously encountered information.

The examples also illustrate how instruction that relies on connections between fractions and decimals may require prior knowledge on the part of students to be effective. For example, explaining $.4 \times .2$ by referring to $4/10 \times 2/10$ assumes that students know how to multiply a fraction by a fraction, a fifth-grade topic in U.S. schools (CCSSI, 2010). Thus, this explanation is most likely to be useful in fifth grade after fraction multiplication has been covered or in sixth grade. Similarly, explaining $3/5 + 1/4$ by referring to $0.6 + 0.25$ is most likely to be useful in fifth grade after covering how to add decimals with unequal numbers of decimal digits (CCSSI, 2010) or in sixth grade. Educators who use such approaches should ensure that children have sufficient prior knowledge to benefit from the instruction, and studies testing interventions that use such approaches should also test whether prior knowledge moderates the effects of the interventions.

This perspective may help to interpret results of a recent study that investigated effects of emphasizing cross-notation knowledge when teaching about fractions and decimals (Malone et al., 2019). At-risk fourth graders were randomly assigned to a business-as-usual control condition, a fractions-only intervention, or an integrated intervention that covered fractions, decimals, and connections between them. Fraction magnitude knowledge and fraction arithmetic improved more in both experimental conditions than in the control condition, but the integrated intervention did not lead to greater improvement on these measures than the fractions-only intervention. A possible explanation is that participants' knowledge of decimal magnitudes was insufficient to help participants benefit from the cross-notation instruction. Consistent with this possibility, accuracy on a decimal magnitude task was rather low at pretest (11%) and still low at posttest (36%), although it was higher than at pretest. Students with greater decimal magnitude knowledge might be more able to leverage that knowledge to help understand fractions in the context of an integrated intervention.

Implications for math learning in general

The current study's emphasis on cross-notation knowledge aligns well with research that emphasizes the importance of understanding and flexibly using multiple external representations (MERs) when learning and doing math (Ainsworth, 2006). Previous research on MERs has largely emphasized connections between symbolic and graphical representations or between multiple graphical representations (e.g., Acevedo Nistal, Van Dooren, & Verschaffel, 2014; Ainsworth, 2006; Braithwaite & Goldstone, 2013) such as between fractions and number lines, between fractions and pie charts, and between number lines and pie charts (Rau & Matthews, 2017). The current findings suggest that connections between multiple symbolic representations may also play an important role in math education.

Although the current study focused on cross-notation knowledge involving fractions and decimals, cross-notation knowledge is a more general concept. It also includes, to name a few, knowledge of relations of fractions and decimals to percentages and ratios; relations between different units of measure; relations among different notations for large whole numbers such as 4,100,000, 4.1E6, and 4.1×10^6; and relations between degree and radian notations for the sizes of angles. In each example, understanding and flexibly using different notations requires relatively deep understanding of the concepts that the notations represent, but the effort to achieve cross-notation knowledge may also help learners to achieve such deep understanding and improve performance in related tasks. Future research should test this possibility in a wider range of mathematical domains.

Limitations

One limitation of the current study is that our literature search was not exhaustive. Other studies may exist that met our criteria for inclusion but were not included in the current study. Relatedly, we did not include data on within-notation magnitude knowledge obtained using number line estimation tasks. This exclusion may have negatively biased our estimates of effects of within-notation magnitude knowledge on arithmetic because arithmetic accuracy is more strongly correlated with number line estimation than with comparison (Schneider et al., 2018). Our analyses may also have underestimated effects of cross-notation knowledge because we did not include measures of cross-notation knowledge involving percentages. Percentages offer distinct affordances from both fractions and decimals (Tian, Braithwaite, & Siegler, 2020), so knowledge of relations between fractions and percentages, and between decimals and percentages, may make unique contributions to arithmetic development (Moss & Case, 1999). Future research should investigate the unique contributions of within- and cross-notation magnitude knowledge to rational number arithmetic using a wider range of datasets with more varied measures of both types of magnitude knowledge.

Conclusion

Cross-notation knowledge is an important aspect of mathematical knowledge in general and rational number knowledge in particular. The current study shows that cross-notation knowledge of rational number magnitudes captures variation among individuals that is not fully captured by within-notation measures of magnitude knowledge and that predicts individual differences in rational arithmetic proficiency. Thus, it could be useful to include cross-notation tasks in assessments of rational number knowledge. Furthermore, future research should explore the possibility of using connections between fractions and decimals to improve children's understanding of both notations individually and of rational number knowledge more generally.

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Appendix A

See Tables A1–A3.

Table A1
Descriptive statistics and zero-order correlations between measures (Study 1)

	Mean	*SD*	Correlations					
			Within-notation fraction magnitude	Within-notation decimal magnitude	Cross-notation magnitude	Fraction arithmetic	Fraction addition and subtraction	Fraction multiplication and division
Within-notation fraction magnitude	.55	.37	–					
Within-notation decimal magnitude	.67	.33	.60***	–				
Cross-notation magnitude	.55	.38	.67***	.59***	–			
Fraction arithmetic	.43	.23	.41***	.44***	.48***	–		
Fraction addition and subtraction	.60	.26	.48***	.50***	.50***	.89***	–	
Fraction multiplication and division	.20	.26	.11†	.15*	.22	.72***	.32***	–

Note. Mean denotes mean proportion correct.
† $.05 < p < .10$.
* $.01 < p < .05$.
*** $p < .001$.

Table A2
Descriptive statistics and zero-order correlations between measures (Study 2).

	Mean	*SD*	Correlations					
			Within-notation fraction magnitude	Within-notation decimal magnitude	Cross-notation magnitude	Fraction and decimal addition	Fraction addition	Decimal addition
Within-notation fraction magnitude	.75	.19	–					
Within-notation decimal magnitude	.94	.09	.17	–				
Cross-notation magnitude	.78	.18	.72***	.46**	–			
Fraction and decimal addition	.74	.29	.49***	.24	.58***	–		
Fraction addition	.65	.43	.52***	.33*	.62***	.94***	–	
Decimal addition	.84	.22	.27†	−.03	.28†	.75***	.48**	–

Note. Mean denotes mean proportion correct.
† $.05 < p < .10$.
* $.01 < p < .05$.
** $.001 < p < .01$.
*** $p < .001$.

Table A3
Descriptive statistics and zero-order correlations between measures (Study 3).

	Mean	*SD*	Correlations							
			Within-notation fraction magnitude	Within-notation decimal magnitude	Cross-notation magnitude	Rational number arithmetic	Fraction addition and subtraction	Fraction multiplication and division	Decimal addition and subtraction	Decimal multiplication and division
Within-notation fraction magnitude	.66	.28	–							
Within-notation decimal magnitude	.80	.27	.51***	–						
Cross-notation magnitude	.65	.31	.66***	.56***	–					
Rational number arithmetic	.42	.23	.42***	.51***	.50***	–				
Fraction addition and subtraction	.41	.34	.44***	.50***	.50***	.86***	–			
Fraction multiplication and division	.36	.28	.14**	.23***	.21***	.66***	.43***	–		
Decimal addition and subtraction	.66	.30	.29***	.40***	.36***	.68***	.45***	.23***	–	
Decimal multiplication and division	.24	.22	.34***	.36***	.40***	.82***	.62***	.50***	.37***	–

Note. Mean denotes mean proportion correct.
** $.001 < p < .01$.
*** $p < .001$.

Appendix B. Supplementary material

Supplementary data to this article can be found online at https://doi.org/10.1016/j.jecp.2021.105210.

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