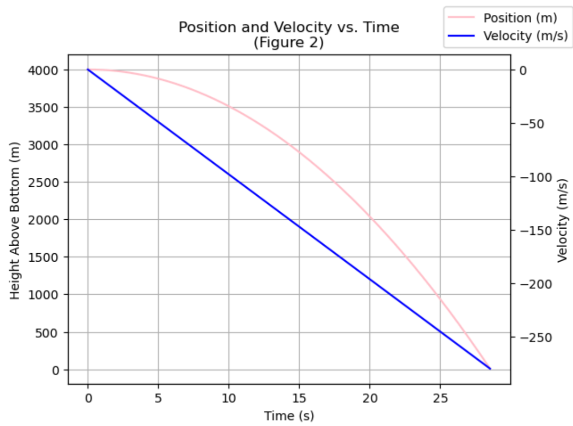
I. Introduction

This report investigates the feasibility of measuring the depth of one of the deepest vertical mine shafts located at the Earth's equator by analyzing the motion of a falling test mass. The proposed investigation method involves dropping a 1-kilogram test mass down the 4-kilometer (4000 meter) mine shaft and using its fall time to estimate its depth. To ensure the accuracy and practicality of this method, I modeled and evaluated the motion of the test mass under increasingly realistic conditions. This includes constant gravity, height-dependent gravity, drag, and the Coriolis effect due to Earth's rotation. Furthermore, I explore the test mass' motion through an idealized infinitely deep shaft through Earth and compare travel times across different internal density distributions. By simulating these scenarios, I aim to determine whether this measurement technique is reliable and to understand the key factors influencing fall time and trajectory.

II. Calculation of Fall Time

To calculate the fall time of the test mass, I begin by using the coupled first-order differential equations (ODEs) $v = \frac{dy}{dt}$ [Equation 1] and $\frac{d^2y}{dt^2} =- g - \alpha v^{\gamma}$ [Equation 2], which were derived from the expression $\frac{d^2y}{dt^2} =- g - \alpha\left|\frac{dy}{dt}\right|^{\gamma}$ [Equation 3], where g is the gravitational acceleration constant (approx. 9.81 m/s^2), α is the drag coefficient, and γ is the speed dependence of the drag. Although this equation considers drag, I will assume that $\alpha = 0$ for the time being, and will also not consider the Coriolis effect when calculating the fall time. The numerically calculated fall time using this free-fall expression was 1,265.8 seconds. This number is a lot larger than 28.6 seconds, which was the number calculated using the basic algebraic free-fall expression $t = \ - v_o + \frac{\sqrt{v_o^2+2gd)}}{g}$ [Equation 4], where v_o is the initial velocity, g is the gravitational acceleration (9.81 m/s^2), and d is the depth of the shaft. Next, I began to consider the effects of a variable g value that is dependent on r (the radius from the center of the Earth). I calculate the test mass' position and velocity using the equation $g(r) = g_o(\frac{r}{R_E})$ [Equation 5], where g_o is the gravity at the surface, and R_E is the radius of the Earth, and use them to calculate the fall time for a variable gravity. The fall time I calculated came out to be approximately 28.6 seconds, which is very different from the time calculated using the ODEs, but about the same as the time calculated using the basic free-fall algebraic expression. Finally, I accounted for both linear gravity and drag using Equations 2 and 5. After determining the drag coefficient (4^{-3}), I was able to calculate the effect that drag had on the velocity and fall time. The calculated fall time considering both linear gravity and drag came out to be 161.4 seconds. Compared to the time calculated without considering drag, it can be concluded that drag has a significant effect on the falling test mass. Through these calculations, it was determined that the fall time calculated without taking linear gravity or drag into account would be massively inaccurate in this situation.

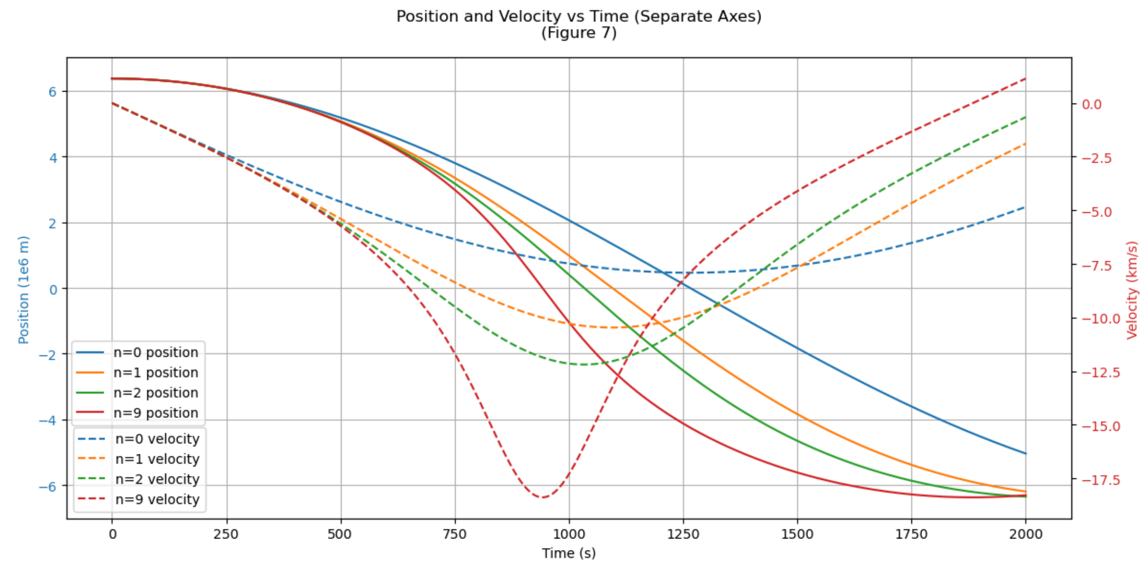


III. Feasibility of Depth Measurement Approach

Another limitation that must be considered is Coriolis forces, which make the test mass appear to curve and possibly bump into the wall of the shaft. The Coriolis force is $F_c = -2m(\Omega \times v)$, where Ω is the Earth's rotation rate for a vector along z and m is the mass of the object. When considering this without the effect of drag, the mass ended up hitting the wall at t = 22.02 seconds at a depth of d = 2,378.9 meters. When taking both the Coriolis effect and drag into consideration, the mass hits the bottom of the mine shaft before hitting the wall. When comparing these two outcomes, it can be determined that drag has a major impact on fall time and should be considered when calculating fall times.

IV. Calculation of Crossing Times for Homogeneous and Non-Homogeneous Earth

To determine the times it takes to get from one side of the Earth to the other by going through the center of the Earth (crossing times) for a homogeneous and a non-homogeneous Earth, it is important to use the r-dependent density function $\rho(r) = \rho_n(1 - \frac{r^2}{R_E^2})^n$ where ρ_n is a normalizing constant and n is some exponent. The case of n = 0 is the constant density of Earth, while n = 2 is closer to the real value. Using n = 0, 1, 2, and 9, I was able to compare the positions and velocities of each n exponent. For a homogeneous Earth where n = 0, the crossing time came out to be around 1,265.2 seconds. For non-homogeneous Earth where n = 9, the crossing time came out to be 942.2 seconds. As shown by these results, the density of the Earth has a significant impact on the crossing time of the mass. It appears that the less homogeneous the density is for Earth, the lower the crossing time is.



If we are considering the Moon for a trans-lunar tunnel in terms of density, we notice that the time it would take to fall to the *center* of the Moon would be approximately 1625.1 seconds. Since the Earth is significantly denser than the Moon, the travel time to the Moon's center would be longer than to the Earth's center.

V. Discussion and Future Work

This study shows that while measuring mine shaft depth by fall time is theoretically possible, it strongly depends on factors such as drag, variable gravity, and the Coriolis Force. Without accounting for these factors, the fall times can be extremely inaccurate, and the test mass may even hit the wall before reaching the bottom. The Earth-Crossing simulations also helped to determine the strong effect that internal density has on fall time — denser cores lead to faster crossings. This highlights the relationship between gravity and mass distribution. Further improvements could include using more realistic models of Earth, altitude-dependent air density, and considering angled shafts to better reflect mining scenarios.

Signed:
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