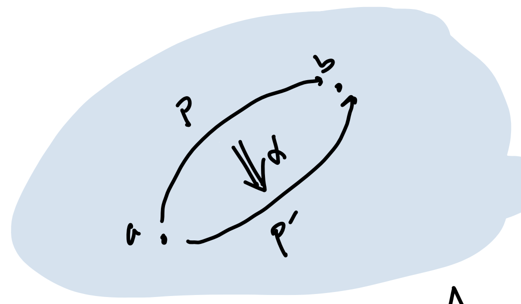
The groupoidal behavior of types

(The first homotopical phenomena)



A

We can now think of types as collections of points (terms) connected by homotopies/paths (equalities).

We can:

- have multiple equalities of the same type (ex: $p, p' : a =_A b$)
- (Ex.) take the inverse of an equality (if $q : b =_A c$, then $q^{-1} : c =_A b$)
- (Ex.) take composition of equalities (if $p : a =_A b$ and $q : b =_A c$, then $p \cdot q : a =_A c$)
- (Ex.) have equalities of equalities ($\alpha : p =_{a =_A b} p'$)

Moreover: (Ex.) functions $A \to B$ respect equality (i.e. map $a =_A a'$ to $fa =_B fa'$)

This is how homotopies in spaces behave.

The space interpretation

Thm. (Voevodsky) There is an interpretation of dependent type theory into Spaces (the category of Kan complexes) in which

types $\rightsquigarrow$ spaces
terms $\rightsquigarrow$ points
equalities $\rightsquigarrow$ paths

Transport

Prop. (Ex) For any dependent type $x : B \vdash E(x)$ type, any terms $b, b' : B$, and any equality $p : b =_B b'$, there is a function $tr_p : E(b) \to E(b')$.

- This ensures that everything respects propositional equality. If we think of E as a predicate on B, then if $E(b)$ is true and $b =_B b'$, so is $E(b')$.

- This is part of a more sophisticated relationship between type theo and homotopy theory (Quillen model category theory). Transport say that $\pi : \sum_{b:B} E(b) \xrightarrow{\delta} B$ behaves like a fibration in a QMC.

Equivalence. For types S, T, there is a notion of equivalence
$$S \simeq T$$
Similar to
$$\sum_{f : S \to T} \sum_{g : T \to S} \left(\prod_{x : S} gfx = x \right) \times \left(\prod_{y : T} fgy = y \right).$$
(To be revisited later.)

Characterizing equality in standard types

bool: We can show false = false, true = true, false $\neq$ true.

$\mathbb{N}$: We have similar: $sn = sm \Rightarrow n = m$, $0 \neq sn$

Σ-types: For $s,t : \sum_{a:A} B(a)$, have $(s=t) \simeq \sum_{p:\pi_1 s = \pi_1 t} tr_p \, \pi_2 s = \pi_2 t$.

Π-types: For $f,g : \prod_{a:A} B(a)$, maybe want $(f=g) \simeq \prod_{x:A} fx = gx$.

Not provable. Called functional extensionality. (funext)

Validated by interpretations in logic, sets, spaces.

$=$-types: For $p,q : a=b$, maybe want

$(p=q) \simeq \mathbb{1}$.

Not provable. Called uniqueness of identity proofs. (UIP)

Validated by interpretations in logic, sets.

U-types: For $S,T : U$, maybe want

$(S=T) \simeq (S \simeq T)$.

Not provable. Called univalence. (UA)

Validated by interpretation in spaces.

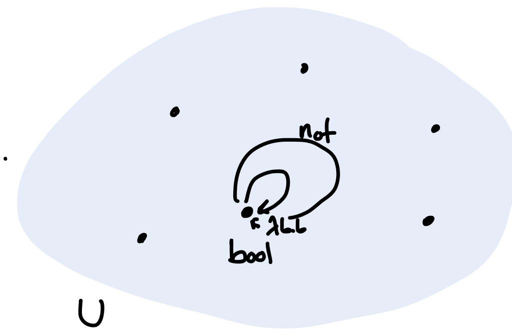
- UA $\Rightarrow$ funext.
- UIP $\wedge$ funext $\not\Rightarrow \perp$
- UA + UIP $\Rightarrow \perp$.

We choose UA.

Homotopy levels

We want to say things like "U is not a set".

- A set is something whose $=$-types don't have structure.



<u>Def.</u> A type T's <u>h-level is 0</u> if

$$\text{hlevel } 0\, T := \sum_{t:T} \prod_{s:T} s =_T t.$$

A type T's <u>h-level</u> is sn if

$$\text{hlevel } sn\, T := \prod_{s,t:T} \text{hlevel } n\ s=t.$$

We've defined a function $\text{hlevel} : \mathbb{N} \to \text{Type} \to \text{Type}$.

<u>h-level 0</u>. AKA contractible, isContr

<u>Ex</u>. $\mathbb{1}$ is contractible.

Most boring.

<u>h-level 1</u>. AKA propositions, isProp

<u>Fact</u>. Equivalent to $\prod_{x,y:P} x=y$.

<u>Ex</u>. $\emptyset, \mathbb{1}$ are propositions

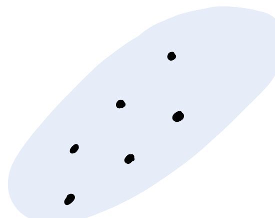
<u>Ex</u>. In fact, any contractible type is a proposition.

Ex. If a proposition is inhabited, it is contractible.

→ So roughly, a proposition is = to $\emptyset$ or $\mathbb{1}$.

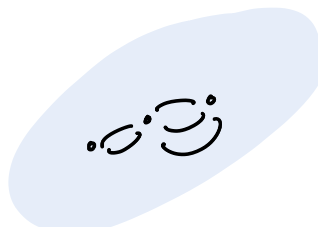
So these behave like logical propositions where Σ behaves like $\exists$, etc.

h-level 2. AKA sets, isSet



Fact. bool, $\mathbb{N}$ are propositions

h-level 3. AKA groupoids



Fact. Type has h-level at least 3.

Ex. If a type T has h-level n, then it has h-level $n+1$.