

Semantics of type theory

Recall the simply typed λ -calculus.

- ~ Like MLTT, except only terms (not types) can depend on contexts
- ~ Have unit, product, function types

Def. The syntactic category of the STLC \mathbb{T} is the category $\mathcal{C}(\mathbb{T})$ whose

- objects are types
- morphisms are terms

$$x:A \vdash f:B$$

- the identity is given by the variable rule

$$x:A \vdash x:A$$

- composition is given by weakening and substitution

$$x:A \vdash f:B \quad y:B \vdash g:C$$

$$\frac{x:A, y:B \vdash g:C}{x:A \vdash g[f/y]:C}$$

$$x:A \vdash g[f/y]:C$$

- unitarity/associativity follow from rules

[https://
www.cl.cam.ac.uk/
teaching/1617/L108/catl-
notes.pdf](https://www.cl.cam.ac.uk/teaching/1617/L108/catl-notes.pdf)

Lem. $\mathcal{C}(\bar{T})$ is cartesian closed.

Pf. The rules for the product type are

Form
 $\frac{A, B \text{ type}}{A \times B \text{ type}}$

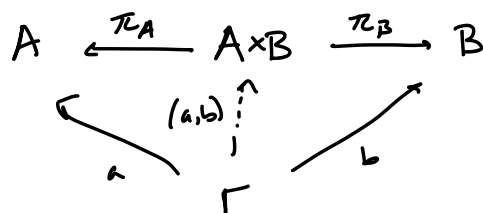
INTRO
 $\frac{\Gamma \vdash a:A \quad \Gamma \vdash b:B}{\Gamma \vdash (a,b):A \times B}$

ELIM
 $\frac{\Gamma \vdash p:A \times B}{\Gamma \vdash \pi_A p:A}$
 $\Gamma \vdash \pi_B p:B$

COMP
 $\frac{\Gamma \vdash \pi_A(a,b) \doteq a:A}{\Gamma \vdash \pi_B(a,b) \doteq b:B}$

UNIQ
 $\Gamma \vdash (\pi_A p, \pi_B p) \doteq p$

Semantically, we get an object $A \times B$ (Form) together with morphisms $\pi_A: A \times B \rightarrow A$ and $\pi_B: A \times B \rightarrow B$ (ELIM). Now given any solid diagram like the following we get the dotted arrow by INTRO.



(similarly for a)

This commutes since $\pi_B(a,b) \doteq b$ by COMP. The dotted arrow is unique since if f were another one, $\pi_A(f) \doteq a$ and $\pi_B(f) \doteq b$, so $f \doteq (a,b)$.

The proof that the unit type presents a terminal object and the arrow type gives an internal hom are left as exercises.

Def. A model for the simply typed λ calculus is a cartesian closed category.

Note: We never explicitly interpreted contexts. This is because

$x_0:T_0, \dots, x_n:T_n \vdash a:A$ is equivalent to $x:T_0 \times \dots \times T_n \vdash a:A$.

We could have defined the syntactic category equivalently as

• objects: contexts

• morphisms: $f: \Gamma \rightarrow \Delta$ given by

$$\Gamma \vdash f_i : \Delta_i$$

for each Δ_i in $\Delta = \langle \Delta_0, \dots, \Delta_n \rangle$.

Note: We can furthermore distinguish extra structure in $\mathcal{C}(\overline{T})$.

A) class of morphisms \mathcal{D} of $\mathcal{C}(\overline{T})$ ^(display maps) that are product projections of the form $\Gamma \times A \xrightarrow{\pi} \Gamma$. This interprets the judgment $\gamma: \Gamma, x:A \vdash \gamma: \Gamma$

so we often write $\Gamma.A$ for $\Gamma \times A$ to indicate that

$\Gamma.A$ is the extension of the context Γ by A .

? Note that every $X \rightarrow * \in \mathcal{D}$, every $X \cong Y \in \mathcal{D}$, and \mathcal{D} is stable under pullback.

B) If we define $\mathcal{C}(\overline{T})$ via contexts, then there is a length function $\ell: \text{ob}(\mathcal{C}(\overline{T})) \rightarrow \mathbb{N}$. Note that

1) there is a unique object of length 0 (empty context)

2) for every object Γ with $\ell(\Gamma) > 0$, there is another object $\text{ft}(\Gamma)$ such that $\ell(\text{ft}(\Gamma)) = \ell(\Gamma) - 1$ and Γ is of

the form $\# \Gamma . A$

3) The pullback

$$\begin{array}{ccc} \cdot & \longrightarrow & \Gamma \\ \downarrow \lrcorner & & \downarrow \\ \Delta & \longrightarrow & \# \Gamma \end{array}$$

?

always exists.

Def. Given a Martin-Löf type theory \bar{T} , its syntactic category $\mathcal{C}(\bar{T})$ is given by:

• objects: contexts

• morphisms: context morphisms:

an $f: \Gamma \rightarrow \Delta$ consists of

$$\Gamma \vdash f_0: \Delta_0$$

$$\Gamma \vdash \Delta_1 [f_0/x_0]$$

\vdots

$$\Gamma \vdash \Delta_n [f_0/x_0][f_1/x_1] \dots [f_{n-1}/x_{n-1}]$$

Note. We could have equivalently defined $\mathcal{C}(\bar{T})$ to have object types and morphisms terms, using Σ -types.

Now we can similarly identify a class \mathcal{D} of display maps in $\mathcal{C}(\bar{T})$: these consist of maps of the form $(\Gamma, A) \xrightarrow{\pi} \Gamma$ where π is repeated applications of the variable rule. We write $\Gamma . A$ for (Γ, A) . They contain all $X \rightarrow *$.

Lem. For every diagram of the form below, there is a pullback.

$$\begin{array}{ccc} & & \Gamma.A \\ & & \downarrow \tau \in \mathcal{D} \\ \Delta & \xrightarrow[\text{mor } \mathcal{C}(\Gamma)]{f} & \Gamma \end{array}$$

Pf. If Γ is empty, then (Δ, Γ, A) is a pullback. It makes the square commute, and maps $Z \rightarrow \Delta$, $Z \rightarrow \Gamma.A$ uniquely factor through (Δ, Γ, A) .

Otherwise, take $(\Delta, A[f])$ where if

$$x_i: \Gamma_0, \dots, x_n: \Gamma_n \vdash A \text{ type}$$

$$\text{then } A[f] \text{ is } A[f_1/x_0][f_2/x_1] \dots [f_n/x_n].$$

This makes the square commute. If there are maps $Z \xrightarrow{\delta} \Delta$, $Z \xrightarrow{\alpha} \Gamma.A$, then there

is a unique map $Z \rightarrow (\Delta, A[f])$ given by

$$Z \vdash \delta_i: \Delta; [\delta_0/y_0] \dots [\delta_{i-1}/y_{i-1}]$$

$$Z \vdash (A[f_1(y_0, \dots, y_n)/x_0] \dots [f_n(y_0, \dots, y_n)/x_n]) [\delta_0/y_0] \dots [\delta_n/y_n]$$

$$\parallel$$

$$A[f_1(\delta_0, \dots, \delta_n)/x_0] \dots [f_n(\delta_0, \dots, \delta_n)/x_n]$$

$$\alpha[f_1(\delta_0, \dots, \delta_n)/\alpha_0] \dots [f_n(\delta_0, \dots, \delta_n)/\alpha_n]$$

$$\text{where } Z \vdash \alpha: A[\alpha_0/x_0][\alpha_1/x_1] \dots [\alpha_n/x_n].$$

□

NB: Substitution is represented semantically by pullback.

Def. A display map category is a category \mathcal{C} with a distinguished class of morphisms \mathcal{D} called display maps such that

- 1) every $X \rightarrow *$ $\in \mathcal{D}$
- 2) every iso $\in \mathcal{D}$
- 3) \mathcal{D} is closed under pullback

This is the weakest notion of model.

Thm. The category of groupoids is a display map category where the display maps are isofibrations.

Thm. The category of Kan complexes is a display map category where the display maps are Kan fibrations.

Def. A C-system is a category \mathcal{C} with

- 1) $\ell: \text{ob } \mathcal{C} \rightarrow \mathbb{N}$
- 2) a terminal object $*$ such that $\ell^{-1}(0) = \{*\}$
- 3) $f! : \text{ob } \mathcal{C} \setminus * \rightarrow \text{ob } \mathcal{C}$ such that $\ell(f!(r)) = \ell(r) - 1$
- 4) for r with $\ell(r) > 0$, $\pi_r : r \rightarrow f! r$
- 5) pullbacks of diagrams of the form

$$\begin{array}{ccc}
 \pi^* \Delta & \longrightarrow & r \\
 \downarrow & \lrcorner & \downarrow \pi \\
 \Delta & \longrightarrow & f! r
 \end{array}$$

where $f_!(\pi^*\Delta) = \Delta$.

More notions of model

