HW 6 (Mastermath HoTT)

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May 8, 2025

Problem 1. Define when a category with display maps models $\mathbb{N}$-types, and show that the syntactic category $\mathcal{C}(\mathbb{T})$ satisfies the definition, whenever the theory $\mathbb{T}$ has natural numbers.

For the rest of this document, we work within a category $\mathcal{C}$ with all pullbacks. Recall the following definition.

Definition 0.1. A *path object* for an object A is an object Path A equipped with the morphisms r_A, ϵ_A making the following diagram commute.

$$\begin{tikzcd} A \arrow[rr, "\Delta_A"] \arrow[dr, "r_A"'] & & A \times A \\ & \text{Path } A \arrow[ur, "\epsilon_A"'] & \end{tikzcd}$$

Say that a choice of path objects is *nice* when each r_A has the left lifting property against each ϵ_B.

Problem 2. Show the following.

Lemma 0.2. Nice path objects give reflexive and symmetric relations:

If there are classes of morphisms $\mathcal{L}, \mathcal{R}$ such that (1) each $r_A \in \mathcal{L}$, (2) each $\epsilon_B \in \mathcal{R}$, (3) $\mathcal{R}$ is stable under pullback and closed under composition, (4) $\mathcal{L}$ is stable under pullback along $\mathcal{R}$, (5) each morphism to the terminal object is in $\mathcal{R}$, and (6) $\mathcal{L} \boxslash \mathcal{R}$, then the path objects give a reflexive and symmetric (pseudo-)relation (see the nlab).

Problem 3. Show that the relation defined in the previous lemma is transitive, hence an equivalence relation. Hint: compare with the syntactical proof of transitivity.

Problem 4. Show the following.

Lemma 0.3. Consider a weak factorization system $(\mathcal{L}, \mathcal{R})$ on the category $\mathcal{C}$, and write the factorization of $f : X \to Y$ as $X \xrightarrow{\lambda_f} Mf \xrightarrow{\rho_f} Y$. Then, $\mathcal{L}$ is the class of all maps $f : X \to Y$ for which the following lifting problem has a solution.

$$\begin{tikzcd} X \arrow[r, "\lambda_f"] \arrow[d, "f"] & Mf \arrow[d, "\rho_f"] \\ Y \arrow[r, equal] & Y \end{tikzcd} \tag{$*$}$$

Dually, $\mathcal{R}$ is the class of all maps $f : X \to Y$ for which the following lifting problem has a solution.

$$\begin{tikzcd} X \arrow[r, equal] \arrow[d, "\lambda_f"] & X \arrow[d, "f"] \\ Mf \arrow[r, "\rho_f"] & Y \end{tikzcd} \tag{$**$}$$