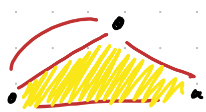
Lecture 9 - The Circle

CW complexes

- A fundamental notion in algebraic topology is CW complexes
  - how you build (nice) spaces (i.e., the ones studied in alg top).

- Start with a disjoint union of '0-cells', i.e., points



$\to$ glue in '1-cells', i.e., line segments along boundary (endpoints)

$\to$ glue in '2-cells', i.e., discs (D^2) along boundary (S^1)

etc...

With a higher inductive type, we can specify

- terms ($\rightsquigarrow$ 0-cells)
- equalities ($\rightsquigarrow$ 1-cells)
- equalities between equalities ($\rightsquigarrow$ 2-cells).

So CW-complexes $\leq$ higher inductive types.

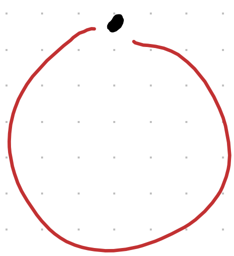
CW-complexes generalize our basic inductive types where we give one constructor for each point.

i.e., $\emptyset$, $\mathbb{1}$, bool, etc.

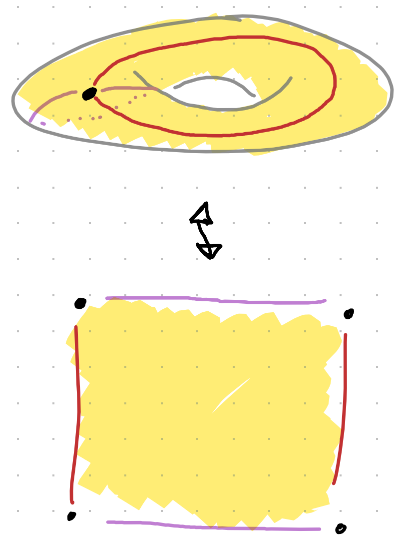
The do not generalize more complicated inductive types like $\mathbb{N}$, Σ, Id, etc.
e.g. they do not include propositional truncation, Rezk completion
(at least without adding hypotheses and proving things)

Examples.

Circle (S^1)



Torus (T^1)



NB: There are different ways to construct 'the circle' as a CW complex.

The circle as a higher inductive type

S^1-form $$\frac{}{S^1 \text{ type}}$$

S^1-intro $$\frac{}{\text{base} : S^1} \qquad \frac{}{\text{loop} : \text{base} =_{S^1} \text{base}}$$

S^1-elim

$$\frac{x : S^1 \vdash D(x) \text{ type} \qquad \vdash b : D(\mathrm{base}) \qquad \vdash p : \mathrm{tr}_{\mathrm{loop}}\, \mathrm{base} =_{D(\mathrm{base})} \mathrm{base}}{x : S^1 \vdash \text{ind-}S^1_{b,p}(x) : D(x)}$$

S^1-comp

$$\frac{\cdots}{\vdash \mathrm{ind}\, S^1_{b,p}(\mathrm{base}) =_{D(\mathrm{base})} b}$$

$$\vdash \mathrm{ind}\, S^1_{b,p}(\mathrm{loop}) = p$$

dependent mapонpaths / functoriality

<u>Thm</u>. S^1 has the following universal properties, natural in X:

$$(S^1 \to X) \xrightarrow{\sim} \left(\sum_{x:X} x = x\right)$$

$$\left(\prod_{s:S^1} X(s)\right) \xrightarrow{\sim} \left(\sum_{x:X(\mathrm{base})} \mathrm{tr}_{\mathrm{loop}}\, x = x\right)$$

Ex. Give a (non-constant) function $S^1 \to \mathcal{U}$.

Take base $\longmapsto$ bool

loop $\longmapsto$ idtoequiv^{-1}(not

Cor. loop $\neq r_{base}$.

Pf. If it were, then $\mathrm{not} = \mathrm{id}_{bool}$ (so false = true). $\square$

Cor. S^1 is not a set.

Lem. There is a nontrivial term $H : \prod_{x:S^1} x = x$

Pf. Define H by sending

base $\mapsto$ loop

$$\text{loop} \mapsto tr_{\text{loop}}\, \text{loop} \underset{\text{base}=\text{base}}{=} \text{loop}^{-1}\,\text{loop}\,\text{loop} \underset{\text{base}=\text{base}}{=} \text{loop}.$$

Cor. $\mathcal{U}$ is not a groupoid.

Pf. We show $S^1 = S^1$ is not a set by showing $\mathrm{id}_{S^1} \underset{S^1 \simeq S^1}{=} \mathrm{id}_{S^1}$ is not a proposition. But this is $\prod_{x:S^1} x = x$. $\square$

Multiplication on the circle

Recall that the circle (in normal math) can be obtained as the set of complex numbers z such that $|z|=1$, and that multiplication on $\mathbb{C}$ restricts to this circle.

- Defining $\mathbb{C}$ in HoTT or any constructive foundation is difficult, but we can still try to recover this multiplication.

Def. A (coherent) h-space is a pointed type (T,e) equipped together with

$$\mu : \prod_{x:A} \sum_{f:A\to A} f(e)=x$$

such that $\mu(e) = (\mathrm{id}, \mathrm{refl})$.

→ Have multiplication $x,y \mapsto \pi_1(\mu x)\, y$

unit e

Thm. S^1 has an h-space structure. The function $\prod_{x:S^1} \sum_{f:S^1\to S^1} f(\mathrm{base})=x$ is defined by

- the requirement $\mu(\mathrm{base}) = (\mathrm{id}, \mathrm{refl})$

and an equality

$$\mathrm{tr}_{\mathrm{loop}}(\mathrm{id},\mathrm{refl}) = (\mathrm{id}, \mathrm{loop}) = (\mathrm{id}, \mathrm{refl})$$

$$\sum_{f:S^1\to S^1} f(\mathrm{base}) = \mathrm{base}$$

given by (H, h) where $H : \mathrm{id}_{S^1} \sim \mathrm{id}_{S^1}$ is from the above corollary.

Observe that $\mathrm{tr}_{H^{-1}}\,\mathrm{loop} = \mathrm{loop}\,\mathrm{loop}^{-1} = r_{\mathrm{base}}$ $\square$

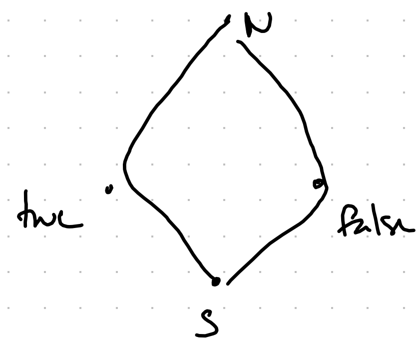
Suspensions.

Def. Given a type T, the suspension ΣT of T is the higher inductive type with constructors

- $N : \Sigma T$
- $S : \Sigma T$
- for all $t : T$, an equality $\mathrm{merid}_t : N = S$.

Lem. $\Sigma \mathrm{bool} \simeq S^1$

(Lem 6.5.1 of HoTT book)



Loops

Def Given a type T and term t, define

$$\Omega(T,t) := (t =_T t)$$

$$\pi_1(T,t) := \| \Omega(T,t) \|_0 \leftarrow \text{set truncation}$$

Thm. $\Omega(S^1, \mathrm{base}) \simeq \mathbb{Z}$, so $\pi_1(T,t) \simeq \mathbb{Z}$.

<u>Def.</u> $\mathbb{Z}$ is the higher inductive type given by constructors

- $- : \mathbb{N} \times \mathbb{N} \longrightarrow \mathbb{Z}$
- $e_{a,b,c,d} : (a,b) = (c,d)$ if $(a+d = b+c)$

and then taking set truncation $\|-\|_0$.

(As a set, it is the usual quotient of $\mathbb{N} \times \mathbb{N}$.)

<u>Lem.</u> The successor function $s : z \mapsto z+1$ is an equivalence $\mathbb{Z} \simeq \mathbb{Z}$.

Thus, we get a function $S^1 \xrightarrow{\text{code}} \mathcal{U}$ by sending base $\mapsto \mathbb{Z}$, loop $\mapsto$ idtoequiv^{-1}s.

This is, equivalently a dependent type $x : S^1 \vdash \mathbb{Z}$ type.

We think of this as a covering space

<u>Prob.</u> $\text{loop}^- : \mathbb{Z} \longrightarrow \Omega(S^1, \text{base})$.

<u>Lm.</u> Take $\text{loop}^z := \text{loop}^z$

<u>Prob.</u> deloop $: \Omega(S^1, \text{base}) \longrightarrow \mathbb{Z}$

<u>Lm.</u> Take $\text{deloop}(p) := \text{tr}_p\, 0$, where we can transport $0 : \mathbb{Z}$ along $p : \text{base} =_{S^1} \text{base}$ because we have a dependent type code $: S^1 \longrightarrow \mathcal{U}$ that sends base $\mapsto \mathbb{Z}$.

<u>Lem</u>. $\mathrm{deloop}(\mathrm{loop}) \sim \mathrm{id}_{\mathbb{Z}}$.

<u>Pf</u>. Given $z : \mathbb{Z}$, we have

$$\begin{aligned}\mathrm{deloop}(\mathrm{loop}^z) &:\equiv \mathrm{tr}_{\mathrm{loop}^z} 0 \\ &= s^z 0 \\ &= z.\end{aligned}$$

<u>Lem</u>. $\mathrm{Loop}^{\mathrm{deloop}} \simeq \mathrm{id}_{\Omega(S^1, \mathrm{base})}$

<u>Pf</u>. We generalize:

define

$$\mathrm{encode} : \prod_{x : S^1} (\mathrm{base} = x) \to \mathrm{code}(x)$$
$$p \longmapsto \mathrm{tr}_p 0$$

$$\mathrm{decode} : \prod_{x : S^1} \mathrm{code}(x) \to (\mathrm{base} = x)$$
$$\mathrm{base}, z \longmapsto \mathrm{loop}^z$$

Now we show $\mathrm{decode}_x\, \mathrm{encode}_x\, p = p$ for all x, p.
By path induction, it suffices to show this for $x :\equiv \mathrm{base}$, $p :\equiv r_{\mathrm{base}}$.
But $\mathrm{decode}_{\mathrm{base}}\, \mathrm{encode}_{\mathrm{base}}\, r_{\mathrm{base}} :\doteq \mathrm{decode}_{\mathrm{base}}\, \mathrm{tr}_{r_{\mathrm{base}}} 0$

$$\begin{aligned} &:\doteq \mathrm{decode}_{\mathrm{base}}\, 0 \\ &:\equiv \mathrm{loop}^0 \\ &= r_{\mathrm{base}}. \end{aligned}$$

$\square$

<u>Pf of thm</u>. Thus, we have shown a quasi-equivalence

$$\mathbb{Z} \xrightarrow{\simeq} \Omega(S^1, \mathrm{bas})$$

and hence an equivalence. $\square$