HW 6 (Mastermath HoTT)

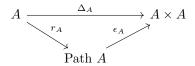
Paige Randall North

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Problem 1. Define when a category with display maps models \mathbb{N} -types, and show that the syntactic category $\mathcal{C}(\mathbb{T})$ satisfies the definition, whenever the theory \mathbb{T} has natural numbers.

For the rest of this document, we work within a category $\mathcal C$ with all pullbacks. Recall the following definition.

Definition 0.1. A path object for an object A is an object Path A equipped with the morphisms r_A , ϵ_A making the following diagram commute.



Say that a choice of path objects is *nice* when each r_A has the left lifting property against each ϵ_B .

Problem 2. Show the following.

Lemma 0.2. Nice path objects give reflexive and symmetric relations:

If there are classes of morphisms \mathcal{L}, \mathcal{R} such that (1) each $r_A \in \mathcal{L}$, (2) each $\epsilon_B \in \mathcal{R}$, (3) \mathcal{R} is stable under pullback and closed under composition, (4) \mathcal{L} is stable under pullback along \mathcal{R} , (5) each morphism to the terminal object is in \mathcal{R} , and (6) $\mathcal{L} \boxtimes \mathcal{R}$, then the path objects give a reflexive and symmetric (pseudo-)relation (see the nlab).

Problem 3. Show that the relation defined in the previous lemma is transitive, hence an equivalence relation. Hint: compare with the syntactical proof of transitivity.

Problem 4. Show the following.

Lemma 0.3. Consider a weak factorization system $(\mathcal{L}, \mathcal{R})$ on the category \mathcal{C} , and write the factorization of $f: X \to Y$ as $X \xrightarrow{\lambda_f} Mf \xrightarrow{\rho_f} Y$. Then, \mathcal{L} is

the class of all maps $f:X\to Y$ for which the following lifting problem has a solution.

$$\begin{array}{ccc}
X & \xrightarrow{\lambda_f} & Mf \\
\downarrow^f & & \downarrow^{\rho_f} \\
Y & = & Y
\end{array}$$
(*)

Dually, $\mathcal R$ is the class of all maps $f:X\to Y$ for which the following lifting problem has a solution.

$$X = X$$

$$\downarrow \lambda_f \qquad \downarrow f$$

$$Mf \xrightarrow{\rho_f} Y$$

$$(**)$$