Figure 1 from Atkey

$$\frac{\Xi \vdash T : \triangleright}{\Xi \vdash El_{\triangleright} T \text{ dtype}}\ \text{SigCGAT} \qquad \frac{\Xi \vdash T : \circ}{\Xi \vdash El_{\circ} T \text{ utype}}\ \text{SigCGAT}$$

Kind-segment of dual-ctx Sys Fω

$KCtx : \triangleright,$

$(\diamond : KCtx,)$

$Kind : \circ,$

$(\langle\rangle, \eta)$

$(\langle\_,\_\rangle, \pi, \xi, \beta_\pi, \beta_\xi, \eta)$

$\_,\_ : KCtx \xrightarrow{+} \langle Disc \mid Kind\rangle \to KCtx,$

$\_ : Hom_{KCtx}(\Theta, \diamond) \cong \top$

$\_ : Hom_{KCtx}(\Theta; (\Theta', \kappa)) \cong Hom_{KCtx}(\Theta, \Theta') \times Tm(\Theta, \kappa)$

$\Delta\ (\Theta, \Theta' : KCtx) \xrightarrow{nat} (\kappa : Kind) \to$

Be more sloppy?

$KTm : (\Theta : KCtx) \xrightarrow{\Theta} (\kappa : Kind) \to \circ$

$\_[\_]$ on KTms

$\_\to\_ : Kind \to Kind \to Kind$

$* : Kind$

(We have ξ for kinds above)

$lam : (\Theta : KCtx) \xrightarrow{nat} (\kappa_1 \kappa_2 : Kind) \to KTm(\Theta, \kappa_1; \kappa_2)$
$\to KTm(\Theta; \kappa_1 \to \kappa_2)$

app: " " $\to KTm(\Theta; \kappa_1 \to \kappa_2)$
$\to KTm(\Theta, \kappa_1) \to KTm(\Theta; \kappa_2)$

$app\beta : \forall \Theta, \kappa_1, \kappa_2, (B : KTm(\Theta, \kappa_1; \kappa_2)) \to$
$(A : KTm(\Theta; \kappa_1)) \to app(lam(B), A) = B[\langle id, A\rangle]$

$app\eta : \forall \Theta, \kappa_1, \kappa_2, (F : KTm(\Theta; \kappa_1 \to \kappa_2)) \to$
$F = lam(app(F[\pi], \xi))$

$\_\to\_ : (\Theta : KCtx) \xrightarrow{nat} (A, B : KTm(\Theta, *)) \to KTm(\Theta, *)$

$\forall : (\Theta : KCtx) \xrightarrow{nat} (\kappa : Kind) \to KTm(\Theta, \kappa; *) \to KTm(\Theta; *)$

Figure 2 from Atkey

Typing segment of dual-ctx SysFω

$\mathrm{TCtx} : \mathrm{KCtx} \xrightarrow{-} \triangleright$

$(\diamond : (\Theta : \mathrm{KCtx}) \xrightarrow{nat} \mathrm{TCtx}(\Theta))$

$-,- : (\Theta : \mathrm{KCtx}) \xrightarrow{nat} \mathrm{TCtx}(\Theta) \oplus \langle \mathrm{Disc}(\mathrm{Ty}(\Theta)) \rangle \to \mathrm{TCtx}(\Theta)$

▷ Difficult semantics

$\mathrm{Ty} : \mathrm{KCtx} \xrightarrow{-} \circ$

$\mathrm{Ty}(\Theta) := \mathrm{KTm}(\Theta, *)$

universal properties of $\diamond$ and $(-,-)$ $\to$ $\mathrm{Hom}_{\mathrm{TCtx}(\Theta)}(\Gamma; \Gamma', T) \cong \mathrm{Hom}_{\mathrm{TCtx}(\Theta)}(\Gamma, \Gamma') \times \mathrm{Tm}_{\Theta}(\Gamma, T)$

$\mathrm{Tm} : \underbrace{(\Theta : \mathrm{KCtx}) \xrightarrow{\times} (\Gamma : \mathrm{TCtx}(\Theta))}_{\text{Curries to Groth. constr.}} \xrightarrow{-} (T : \mathrm{Ty}(\Theta)) \to \circ$

$\mathrm{lam} : (\Theta : \mathrm{KCtx}) \xrightarrow{nat} (\Gamma : \mathrm{TCtx}(\Theta)) \xrightarrow{nat} (A\,B : \mathrm{Ty}(\Theta)) \to \mathrm{Tm}_{\Theta}(\Gamma, A; B) \to \mathrm{Tm}_{\Theta}(\Gamma; A \to B)$

$\mathrm{app} :$ " " " $\to \mathrm{Tm}_{\Theta}(\Gamma; A \to B) \to \mathrm{Tm}_{\Theta}(\Gamma; A) \to \mathrm{Tm}_{\Theta}(\Gamma; B)$

Atkey refers to this fig 2 conversion. This is copy of Tm w.r.t T. {+β and η}

$\mathrm{Lam} : (\Theta : \mathrm{KCtx}) \xrightarrow{nat} (\kappa : \mathrm{Kind}) \to (\Gamma : \mathrm{TCtx}(\Theta)) \xrightarrow{nat} (A : \mathrm{Ty}(\Theta, \kappa)) \to \mathrm{Tm}_{\Theta,\kappa}(\Gamma[\pi]; A) \to \mathrm{Tm}_{\Theta}(\Gamma, \forall\kappa.A)$

$\mathrm{App} : (\Theta : \mathrm{KCtx}) \xrightarrow{nat}$ " " " $\to \mathrm{Tm}_{\Theta}(\Gamma; \forall\kappa.A) \to (B : \mathrm{KTm}(\Theta; \kappa)) \to \mathrm{Tm}_{\Theta}(\Gamma; A[\langle id_{\Theta}, B \rangle])$

$\mathrm{App}\beta :$ " " " " $\to (a : \mathrm{Tm}_{\Theta,\kappa}(\Gamma[\pi]; A)) \to (B : \mathrm{KTm}(\Theta; \kappa))$

$\to \underbrace{\mathrm{App}(\mathrm{Lam}(a), B)}_{} = \underbrace{a[\langle id_{\Theta}, B \rangle]}_{}$ ⟶ $\mathrm{Tm}_{\Theta}(\Gamma; A[\langle id_{\Theta}, B \rangle])$

$\mathrm{App}\eta :$ " " " " $\to (f : \mathrm{Tm}_{\Theta}(\Gamma; \forall\kappa.A))$

$\to f = \mathrm{Lam}(\mathrm{App}(f[\pi^{\kappa}_{\Theta}], \xi^{\kappa}_{\Theta}))$

$$\frac{\Gamma, x : A \vdash b : B}{\Gamma \vdash \lambda x. b : (x : A) \to B}$$

$$\frac{\Gamma, \text{🔒}_\mu \vdash a : A}{\Gamma \vdash \mathsf{mod}_\mu\ a : \langle \mu \mid A \rangle}$$

$\vdash$

$\uparrow$ inf rule

$f : (\underbrace{A \Rightarrow A}) \to (A \xrightarrow{\text{🔒}} A)$

SOGAT-style

$\mathsf{mod}_\mu :$

$(\underbrace{R_{\text{🔒}_\mu}}\ A) \to \langle \mu \mid A \rangle$

GAT-style

$C : *$

$T : C \to *$

$\diamond : C$

ex $\vdash : \prod_{\Gamma : C} T(\Gamma) \to C$

$(C \to C) \to ?$

$$\frac{\begin{array}{c}\Gamma \vdash J\ @ d \\ \mu : c \to d\end{array}}{\Gamma, \text{🔒}_\mu \vdash J^{\text{🔒}_\mu}\ @ c}$$

$\Pi(c, d : \mathrm{Mode}). \Pi(\mu : c \Rightarrow d)$

$\mathsf{mod}_- : \langle - \mid - \rangle$ radme

$\prod_{\Gamma : C_d} \prod_{A : T_c(\Gamma, \text{🔒}_\mu)} A$

$\mathrm{Tm}_c(\Gamma.\text{🔒}_\mu, A) \longrightarrow \mathrm{Tm}(\Gamma, \langle \mu \mid A \rangle)$

$B : T_d(\Gamma)$

$C : \mathrm{Mode} \to D*$

$\mathrm{Ty}_d : \mathrm{Ctx}_d \to \mathrm{Set}$

$$\begin{tikzcd}
c \arrow[dd, "\mu"'] & \mathrm{Ctx}_c \arrow[rd, bend left, "\mathrm{Ty}_c"] & \\
 & & \mathrm{Set} \\
d & \mathrm{Ctx}_d \arrow[uu, "\text{🔒}_\mu"] \arrow[ru, bend right, "\mathrm{Ty}_d"'] \arrow[uu, phantom, shift right=4, "\langle \mu \mid - \rangle \Downarrow"] &
\end{tikzcd}$$

Judl forms:

$DCtx : \triangleright$

$Ctx : \triangleright$

$DTy : C \to \triangleright$

Ty

DTm

Tm

$\triangleright$

$\circ$

$DCtx \sim Cat$, Functor

$Ctx \sim \sum_{C : Cat} Psh(C)$, ?

$((C|\Gamma)/D)) : (C : DSort, \Gamma : C) \to (D : DSort)$

$$\frac{}{\vdash \star\ type}\ \overset{CAT}{SOGAT}$$

$$\frac{\Gamma \vdash T : \star}{\Gamma \vdash El\, T\ type}$$

$$\frac{}{\vdash \circ\ ctype}\ CGAT \qquad \frac{}{\vdash \triangleright\ dtype}$$

$$\frac{\Gamma \vdash T : \circ}{\Gamma \vdash T\ ctype} \qquad \frac{\Gamma \vdash T : \triangleright}{\Gamma \vdash T\ dtype}$$

Sys F: $\vdash KCtx : \triangleright$

$\Gamma : KCtx \vdash TCtx : \triangleright$ } functorial

$\Gamma : KCtx \vdash \Delta : TCtx$ } nat transf.

MTT: $d : Mode \vdash Ctx_d : \triangleright$

$\Gamma : Ctx_d \vdash (\Gamma, 🔒_\mu) : \underline{Ctx_c}$ } functor

$$\underbrace{\Gamma \mid \Delta}\ \underbrace{\vdash\ t : T}$$

Atkey – Relational Parametricity for higher
Pierre – Types and programming

$$\mathcal{C} : \triangleright, \quad \Gamma, \overset{\mathcal{C}}{\cancel{\Delta}}, \quad \sigma : \mathrm{Hom}(\Gamma, \Delta) \overset{\text{functor}}{\vdash} (\Gamma, \sigma) : \underline{\mathcal{C}/\Delta}$$

$\Delta : \mathrm{Obj}(\mathcal{C})$

↳ na-functin Δ

$$\frac{f(\Gamma) \vdash M}{\Gamma \vdash M'} \qquad \text{if } f : \underline{\mathcal{C}/\Gamma} \to \mathcal{D}$$

$\mathcal{C}, \mathcal{D} : \triangleright$

$M(f(\Gamma)) \qquad \to M'(\Gamma)$

$$\mathrm{Ctx}/\Gamma \xrightarrow{\ \Sigma_\Omega\ } \quad -/- : \mathrm{Ctx}/\mathbb{I} \to \mathrm{Ctx}$$

$$\frac{\begin{array}{c}\boxed{\Gamma/\Omega} \vdash b : (i : \mathbb{I}) \multimap B : \\ \Gamma \vdash \Omega : \mathbb{I}\end{array}}{\boxed{\Gamma} \vdash b_\Omega : B_\Omega}$$

$$\frac{\begin{array}{c}\Gamma \vdash a : A \\ \Gamma/a \vdash b : B\end{array}}{\Gamma \vdash (a, b) : A \otimes B}$$

$\Gamma = \Delta, \cancel{(i : \mathbb{I}, \ominus)}$

$(\Delta * \mathbb{I}) : \underset{L}{\ominus}$

- SysFω (TCtx as fib or as functor)

$$\begin{tikzcd} TCtx \arrow[d] \\ KCtx \end{tikzcd}$$

- MTT
- BCM/CH, TraSCwoD
- something truly substruct? } infer resource distribution
- QTT?

- dual context modal TT
- Fitch style modal TT

$$\frac{\Theta \vdash P : Uni_1 \qquad \Theta, x : P \vdash Q : Uni_0}{\Theta \vdash \Pi(x : P) . Q : Uni_0} \qquad \frac{\Theta \vdash \kappa : Kind \qquad \Theta, \alpha : \kappa \vdash Q : \star}{\Theta \vdash \forall(\alpha : \kappa) . Q : \star}$$

$\Gamma \vdash T$ type

$[\![\Gamma]\!] : Cat$

$[\![T]\!] : [\![\Gamma]\!] \to Cat$

$$[\![\Gamma, t \overset{+}{:} T]\!] = \int^{Groth}_{[\![\Gamma]\!]} [\![T]\!]$$

$$[\![\Gamma, t \overset{-}{:} T]\!] = \int^{Groth}_{[\![\Gamma]\!]} \overset{Cat \xrightarrow{\pm} Cat}{Op} \circ [\![T]\!]$$

$$[\![\Gamma, t \overset{nat}{:} T]\!] = \int^{Groth}_{[\![\Gamma]\!]} \cancel{Loc \circ [\![T]\!]}$$

$$\Gamma \vdash \_ : (t \overset{nat}{:} T) \to St$$

$t : T \xrightarrow{s} St$

$(t \rightsquigarrow t') \longmapsto st : St \qquad st' : St'$

$\Diamond : (\Theta : KCtx) \longrightarrow TCtx(\Theta)$

$$\begin{tikzcd}
\Diamond = \tilde{\Diamond} & \Diamond \arrow[l, mapsto, "TCtx(F)"'] \arrow[d, dotted, no head] \\
\Theta \arrow[r, "F"] & \Theta'
\end{tikzcd}$$

~~TCtx~~ $:$

$$\begin{tikzcd}
KCtx \arrow[r, bend left=60, "\mathbb{1}"{name=U}] \arrow[r, "TCtx"'{name=D}] & D \arrow[from=U, to=D, Rightarrow, "\Diamond"']
\end{tikzcd}$$

$\Diamond : (\Theta : KCtx) \to (\mathbb{1} \xrightarrow{nat} TCtx(\Theta))$ / $\Diamond : (\Theta : KCtx) \xrightarrow{+} (\mathbb{1} \to TCtx(\Theta))$

$\_,\_ : (\Theta : KCtx) \to (TCtx(\Theta) \times (Disc\ Ty(\Theta)) \xrightarrow{nat} TCtx(\Theta))$

Wanted: a category C and prof. $\vdash : C^{op} \times Cat \to Set$

$\Gamma \vdash$

Sig CGATs	Denot Semantics
Γ ctx	[Γ is a cat with a class P "precartesian arrows"]
$\Gamma \vdash T$ dtype	$T : \Gamma \to \mathrm{Cat}^{op}$
$\Gamma \vdash t : T$	$\begin{tikzcd} \Gamma \arrow[r, bend left, "t"] \arrow[r, phantom, "\mathrm{id}"] & \Gamma . T \arrow[l, bend left] \end{tikzcd}$ but t sends precart to cart arrows
$\Gamma, \overset{cart}{x} : A \vdash$	$\begin{tikzcd} (\gamma, a) \arrow[d, "{(\gamma,\alpha)}"] \\ (\gamma', a') \end{tikzcd}$ (γ,α) is precart if γ is precart
$\Gamma, \overset{\dagger}{x} : A \vdash$ __	(χ, α) is precart if χ is precart and α is cart
$\overset{nat}{\Theta} : \mathrm{KCtx} \vdash \Diamond : \mathrm{TCtx}(\Theta)$	
$\Gamma \vdash T$ utype	$T : \Gamma \to \mathrm{Set}^{op}$ $\pi : \Gamma . T \to \Gamma$ has only cart arrows.