Category theory

Def. $\mathrm{Grpd} := \sum_{T:\mathcal{U}}$ is of h-level 3 T.

We have a univalence principle

Thm. $(G =_{\mathrm{Grpd}} H) \simeq (G \simeq H)$.

But what about categories?

How do we define categories in UF?

- A category is normally defined as a bunch of sets.
- We could do this, but
  - we stay in the set level (everything in mathematics is a bunch of sets)
  - the SIP for structures on sets tells us that

$$(\mathcal{C} =_{\mathrm{Cat}} \mathcal{D}) \simeq (\mathcal{C} \cong \mathcal{D})$$

    and isomorphism is not the right kind of sameness for categories.

Instead, we take a groupoid and put extra structure on it.

- Every category has a 'core groupoid'
  - the objects and all invertible morphisms.

Def. A *category* $\mathscr{C}$ consists of

- a ~~set~~ groupoid $\mathrm{ob}\,\mathscr{C}$
- a set $\hom(X,Y)$ for every pair $X,Y \in \mathrm{ob}\,\mathscr{C}$

  $X,Y : \mathrm{ob}\,\mathscr{C} \vdash \hom(X,Y) : \mathrm{Set}$
- an ~~element~~ term $1_X \in \hom(X,X)$ for every $X \in \mathrm{ob}\,\mathscr{C}$

  $X : \mathrm{ob}\,\mathscr{C} \vdash 1_X : \hom(X,X)$
- a function $\circ : \hom(X,Y) \times \hom(Y,Z) \to \hom(X,Z)$ for every $X,Y,Z \in \mathrm{ob}\,\mathscr{C}$.

  $X,Y,Z : \mathrm{ob}\,\mathscr{C} \vdash \circ : \hom(X,Y) \to \hom(Y,Z) \to \hom(X,Z)$
- such that
- the morphism idtoiso : $(X = Y) \to \mathrm{iso}(X,Y)$ is an equivalence

  $\left(\mathrm{iso}(X,Y) := \sum_{f : \hom(X,Y)} \sum_{g : \hom(X,Y)} g \circ f = 1_X \times f \circ g = 1_Y\right)$.

. . .

i.e., the type $\mathscr{C}at := \sum_{\mathrm{ob}\,\mathscr{C} : \mathrm{Gpd}} \; \sum_{\hom : \mathrm{ob}\,\mathscr{C} \to \mathrm{ob}\,\mathscr{C} \to \mathrm{Set}} \; \sum_{1 : \prod_{X : \mathrm{ob}\,\mathscr{C}} \hom X} \cdots$

<u>NB</u>. This is often called a <u>univalent category</u> and the requirement that idtoiso is an equivalence is called (internal) univalence.

<u>Thm (univalence for categories)</u> $(\mathscr{C} =_{\mathscr{C}at} \mathscr{D}) \simeq (\mathscr{C} \simeq \mathscr{D})$.

<u>Cor</u> $\mathscr{C}at$ is a 2-groupoid (h-level 4).

Higher inductive types.

Homotopy type theory = MLTT + UA + higher inductive types

Recall: inductive types are generated by their constructors (terms) / canonical terms

Since we now consider types as having

- terms
- equalities
- equalities between equalities

We can consider higher inductive types, whose constructors can be terms, equalities, equalities between equalities, etc.

Ex. $S^0 :=$ bool has constructors

- true : bool
- false : bool

Def. D^1 (the interval) has constructors

- true : D^1
- false : D^1
- $p : \text{true} =_{D^1} \text{false}$

We could define D' with four rules:

$$\frac{}{D' \text{ type}} \quad D'\text{-form}$$

$$\frac{}{\text{true} : D'} \qquad \frac{}{\text{false} : D'} \qquad \frac{}{p : \text{true} =_{D'} \text{false}} \qquad D'\text{-intro}$$

$$\frac{\begin{array}{l} d : D' \vdash E(d) \text{ Type} \\ \quad \vdash t : E(\text{true}) \\ \quad \vdash f : E(\text{false}) \\ \quad \vdash \pi : p_* t = f : E(\text{false}) \end{array}}{d : D' \vdash \text{ind}_{D',t,f,\pi}(d) : E(d)} \quad D'\text{-elim}$$

$$\left.\begin{array}{l} \vdash \text{ind}_{D',t,f,\pi}(\text{true}) \doteq t : E(\text{true}) \\ \vdash \text{ind}_{D',t,f,\pi}(\text{false}) \doteq f : E(\text{false}) \\ \vdash \text{ind}_{D',t,f,\pi}(p) \doteq \pi : p_* t = f : E(\text{false}) \end{array}\right\} \quad D'\text{-comp}$$

__Def__. S^1 (the circle) has constructors

- $\text{base} : S^1$
- $\text{loop} : \text{base} = \text{base}$

__Thm__. $\pi_1(S^1) = \mathbb{Z}$ where

$\pi_1 : T \to \text{Set}$ is defined as

$S^1 \to T \ldots ?$

- We want to make this a set.
- We also want to make thing into propositions.

Ex. Given $P, Q : \mathrm{Prop}$

$P + Q$ is not a proposition in general

Thm. • If $P, Q : \mathrm{Prop}$, $P \times Q : \mathrm{Prop}$ (write $P \wedge Q$)

• If $P : \mathrm{Prop}$, $E : P \to \mathrm{Prop}$,

$\sum_{p:P} e(p)$ $\prod_{p:P} e(p)$ $\left(\text{write } \exists_{p:P} e(p) \quad \forall_{p:P} e(p) \right)$

are in Prop.

Def. Given a type T, the <u>propositional truncation $\|T\|_1$ of T</u> is the higher inductive type with constructors

- $|-|_1 : T \to \|T\|_1$
- $\prod_{x,y:T} |x|_1 =_{\|T\|_1} |y|_1$

Ex. Show that for any type T, $\|T\|_1$ is a proposition.

Def. Define $P \vee Q := \|P + Q\|_1$ for $P, Q : \mathrm{Prop}$.

Ex. Functions $(T \to P) \simeq (\|T\|_1 \to P)$ for any type T, Proposition P.

Def. Given a type T, the set truncation $\|T\|_2$ of T is the higher inductive type with constructor

- $|-|_2 : T \to \|T\|_2$
- $\prod_{\substack{x,y:T \\ p,q:x=y}} |p|_2 = |q|_2$.

Ex. Show that for any type T, $\|T\|_2$ is a set.

Def. $\pi_1 : U \to \mathrm{Set}$
$:= \lambda T.\ \|S^1 \to T\|_2$.

Thm. $\pi_1(S^1) = \mathbb{Z}$.

More higher inductive types:

Resk completion

Quotient