Greedy Algorithms: Main Ideas

Michael Levin

Higher School of Economics

Algorithmic Toolbox
Data Structures and Algorithms

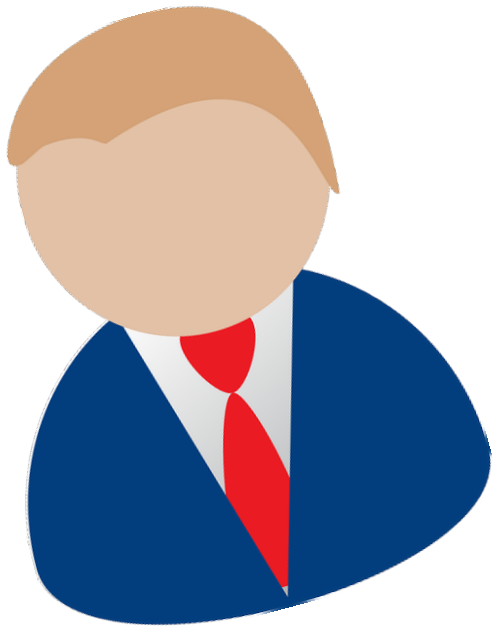
Outline

Learning objectives

- Come up with a greedy algorithm yourself

Job Interview

Job Interview



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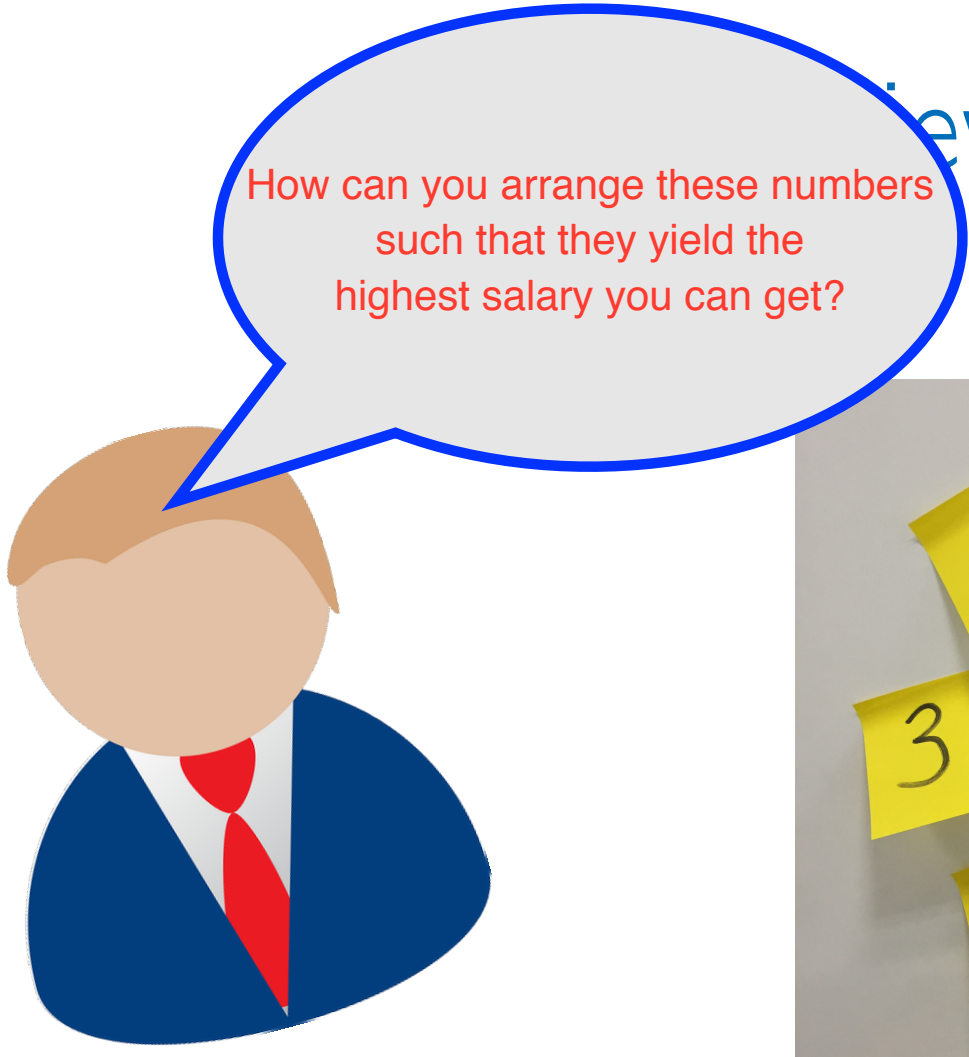
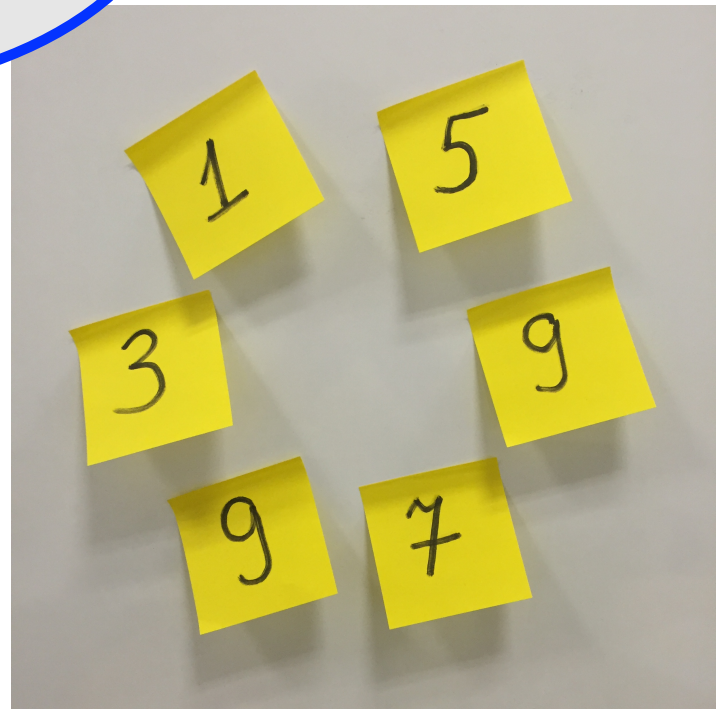
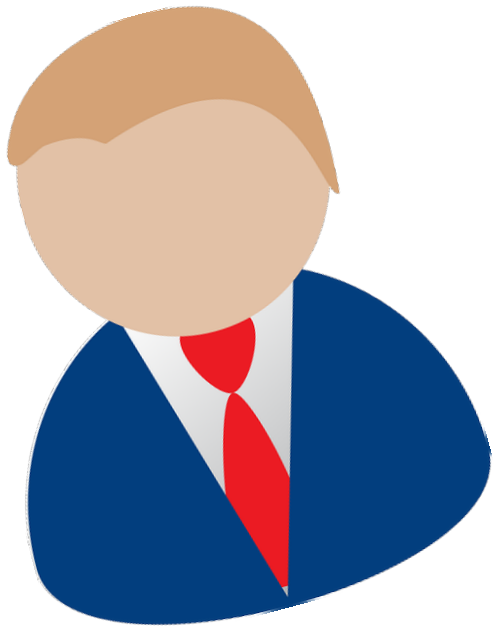
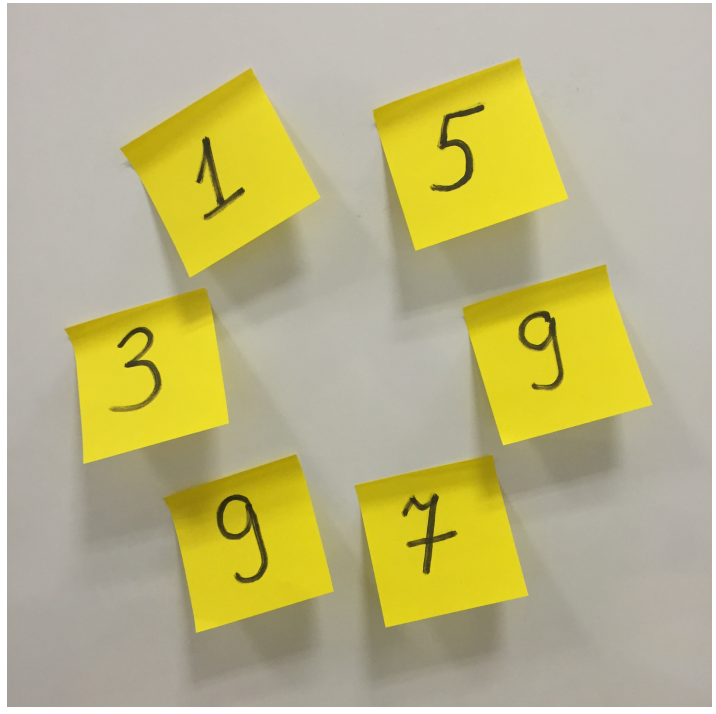
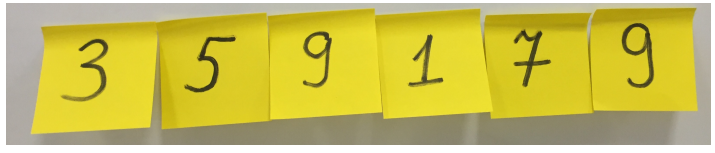

How can you arrange these numbers
such that they yield the
highest salary you can get?


1
5
3
9
9
7

Job Interview







Largest Number

Toy problem

What is the largest number that consists of digits 3, 9, 5, 9, 7, 1? Use all the digits.

Largest Number

Toy problem

What is the largest number that consists of digits 3, 9, 5, 9, 7, 1? Use all the digits.

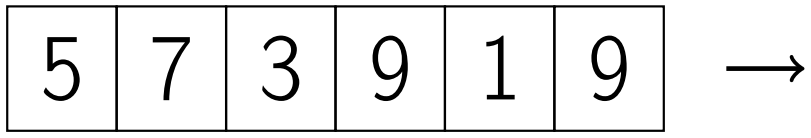
Examples

$$359179, 537991, 913579, \ldots$$

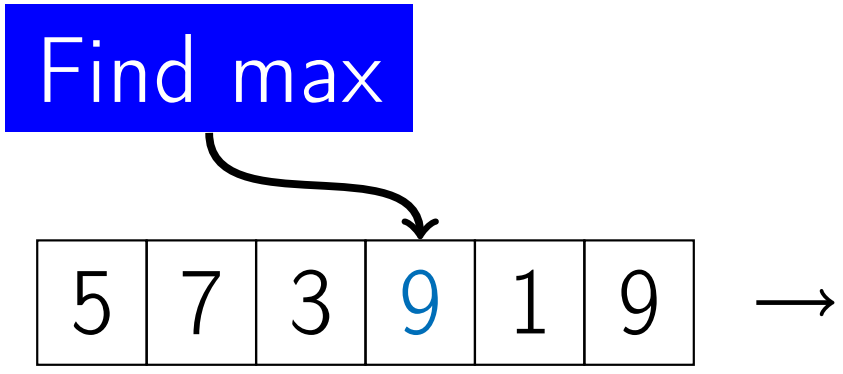
Correct answer

997531

Greedy Strategy

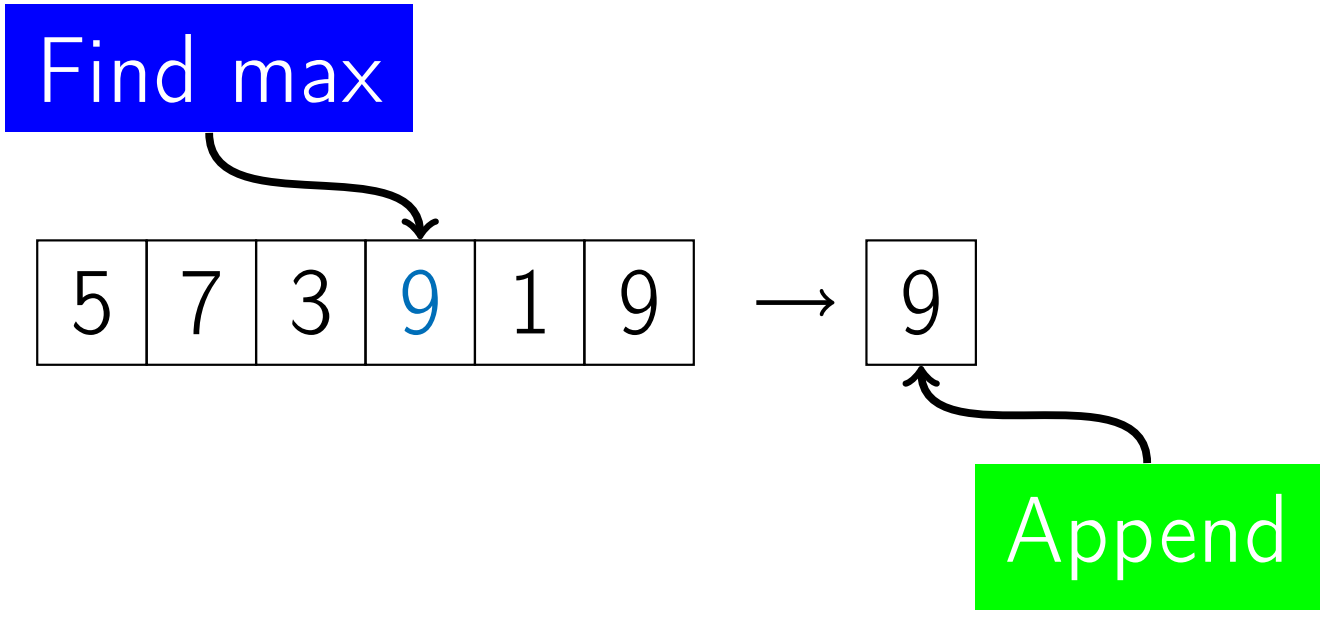


Greedy Strategy



- Find max digit

Greedy Strategy



- Find max digit
- Append it to the number

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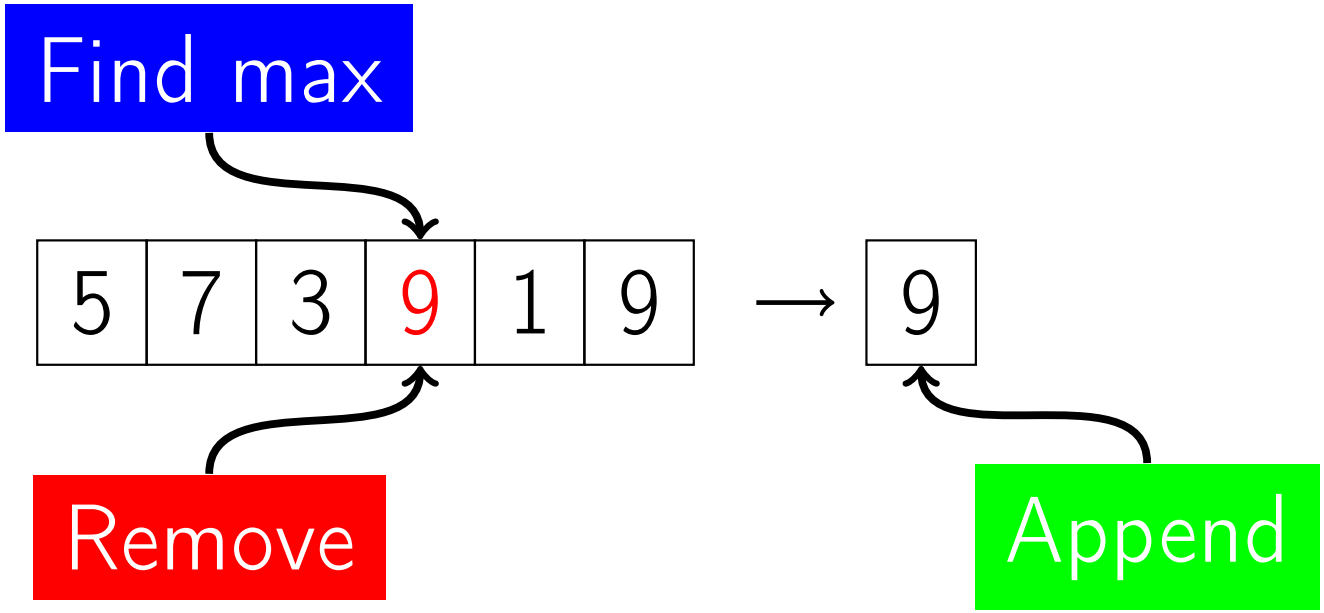

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Greedy Strategy

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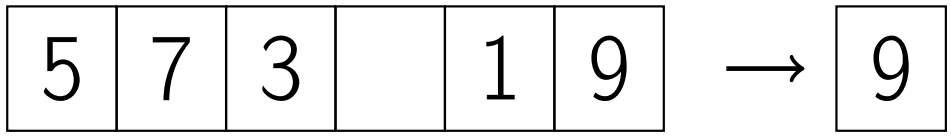

Remove

Append

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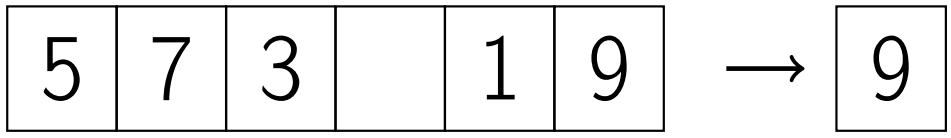
Greedy Strategy

Find max



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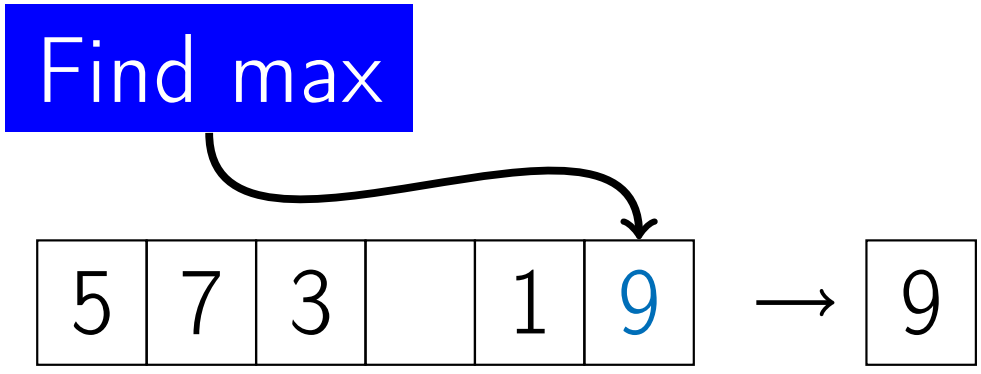

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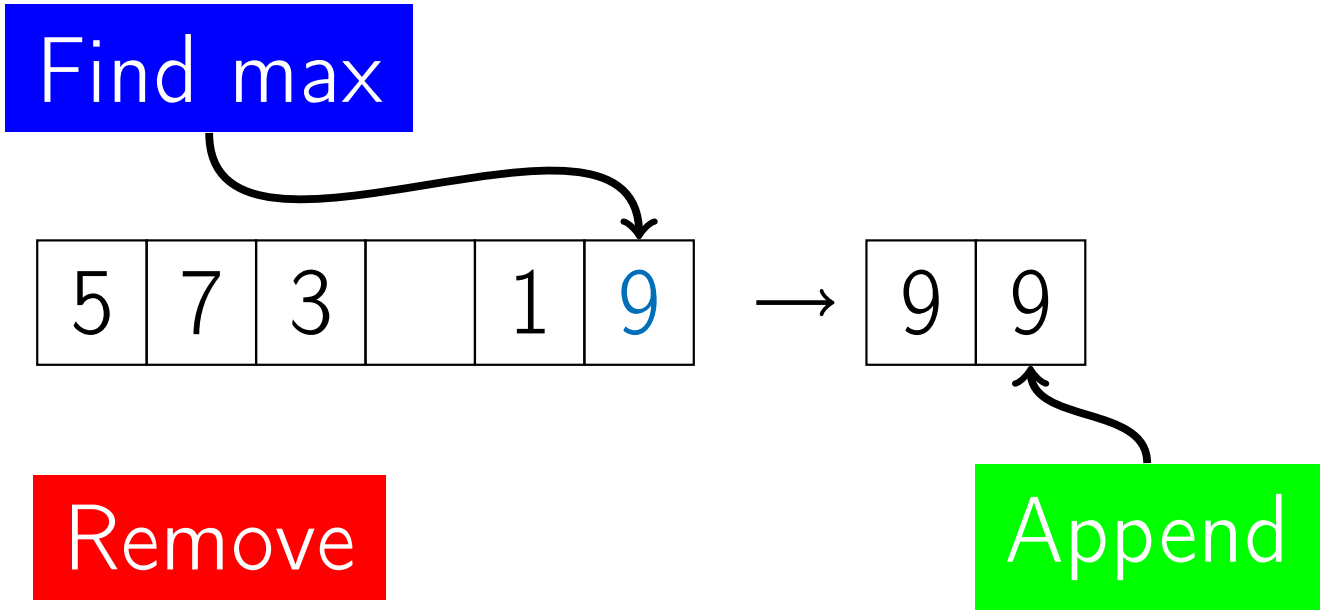

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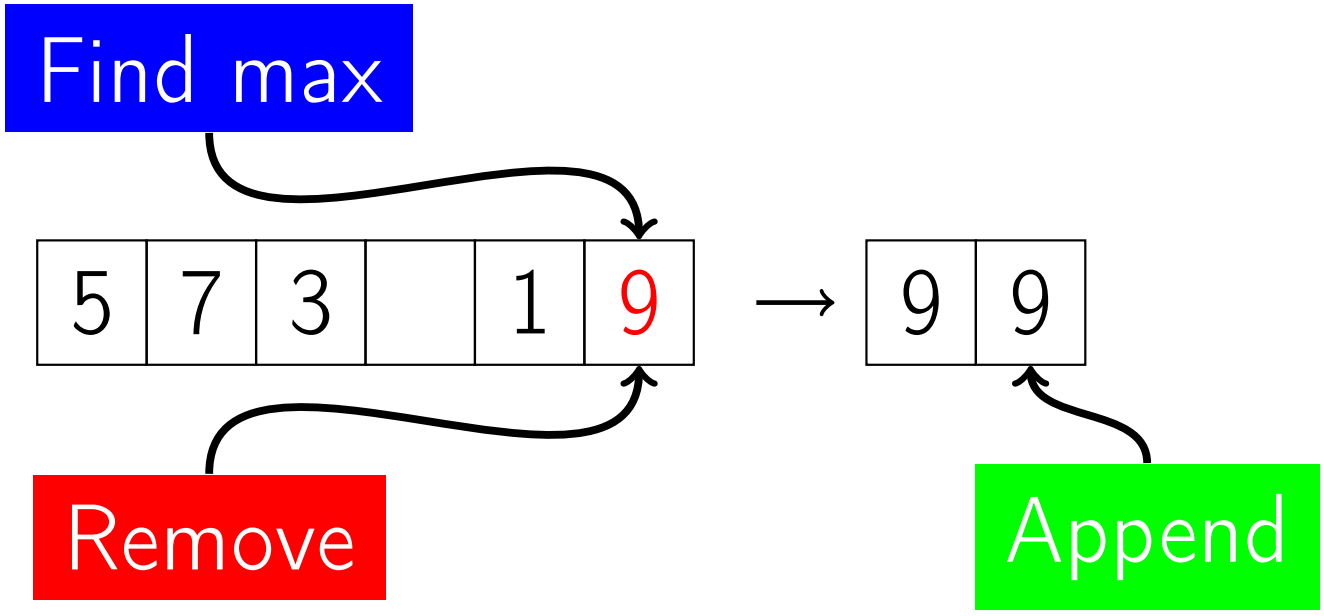

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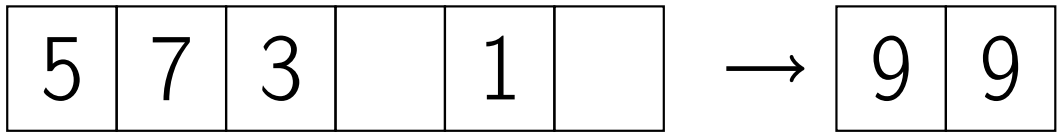

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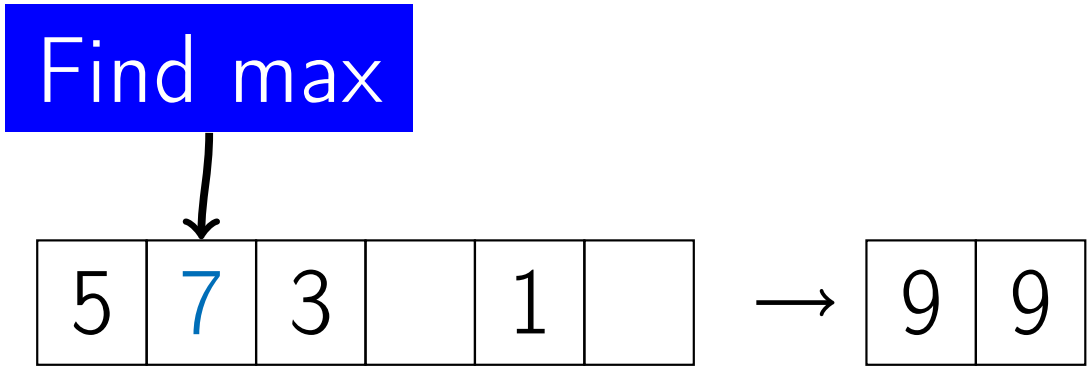

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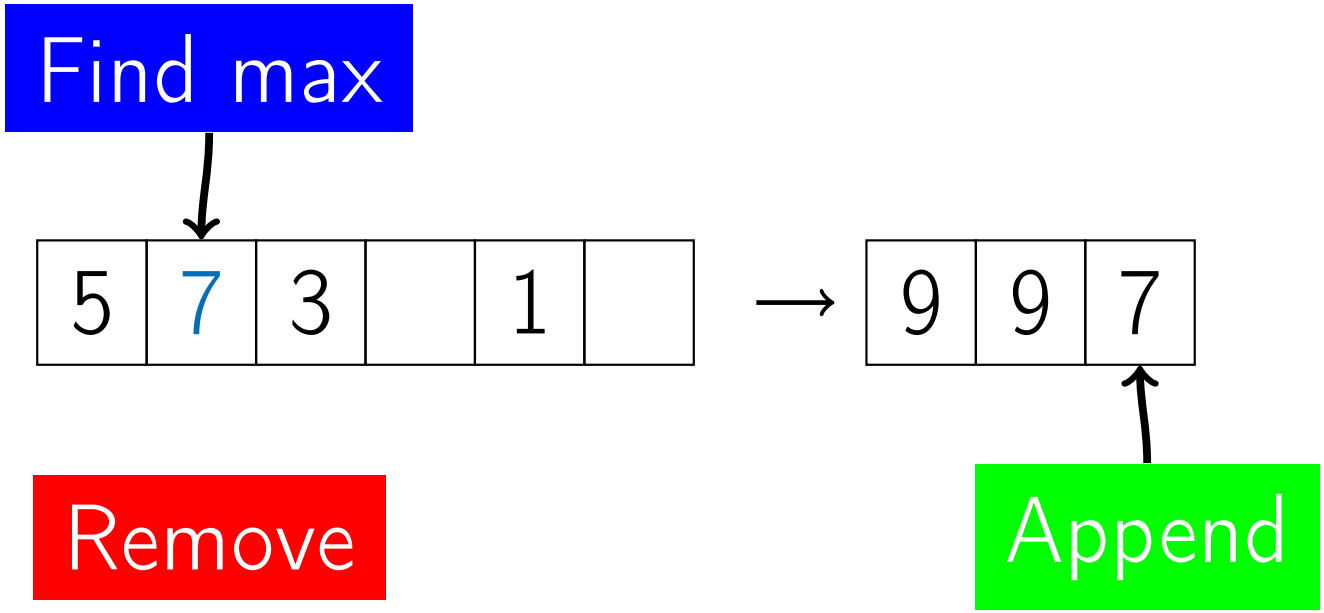

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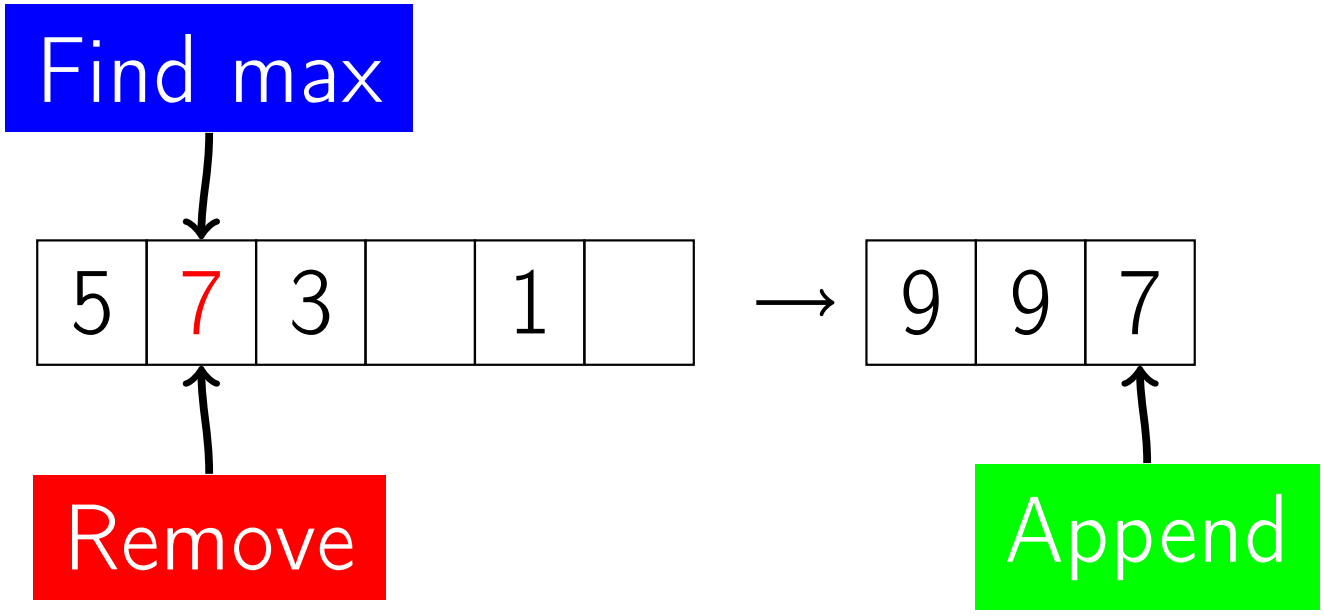

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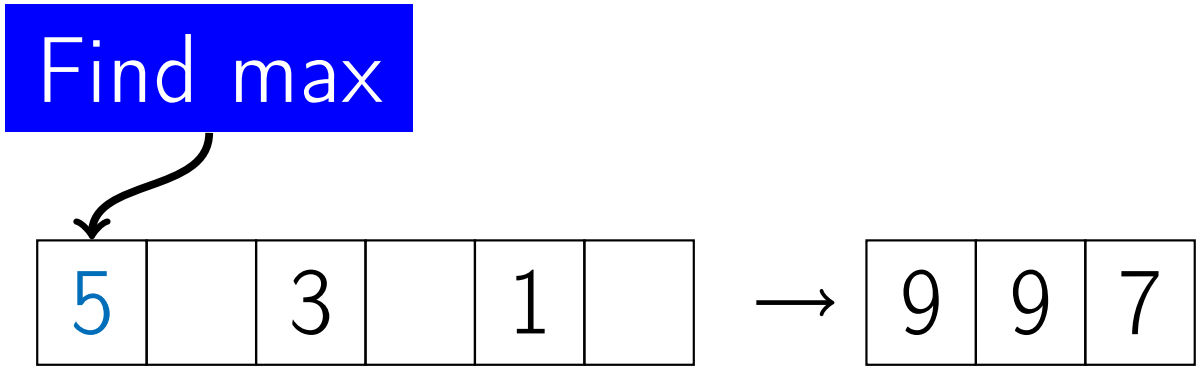

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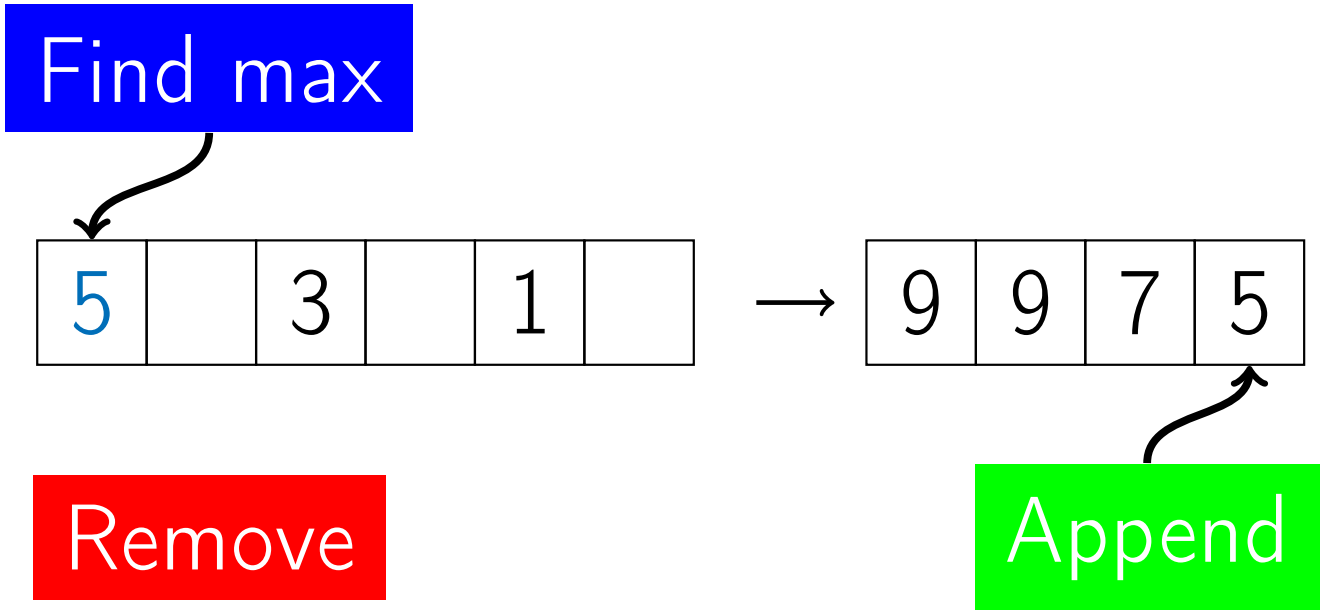

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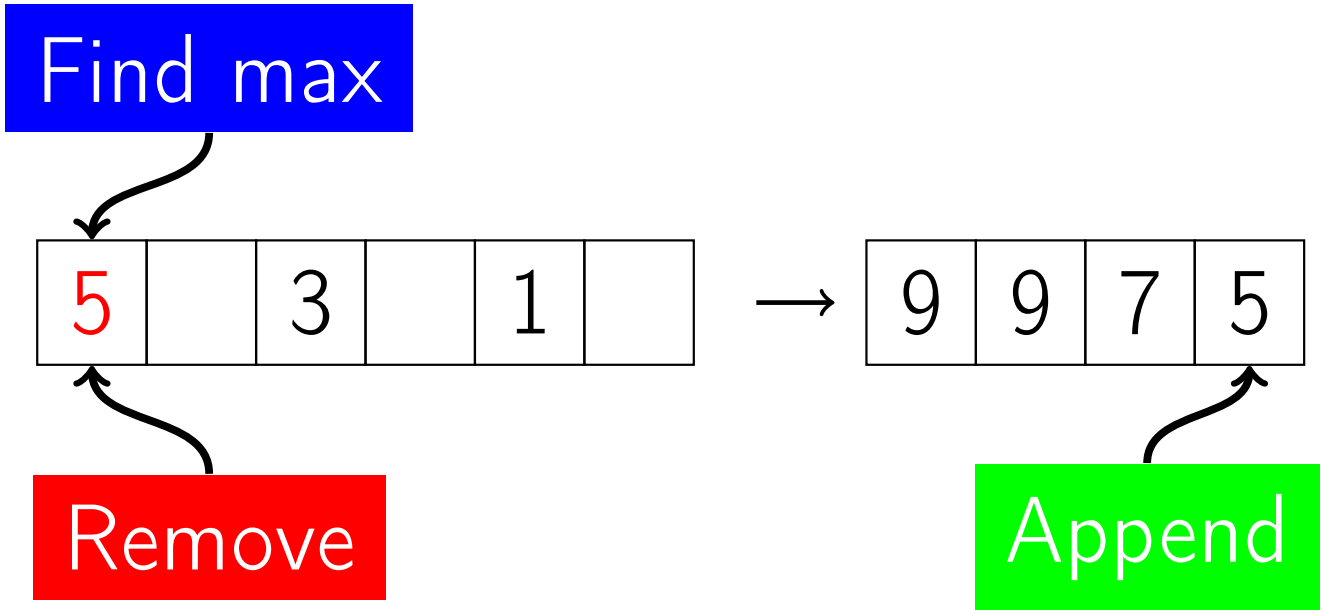

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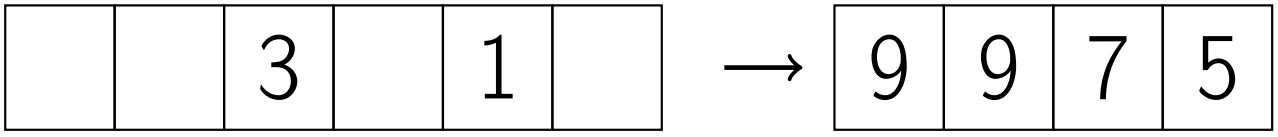

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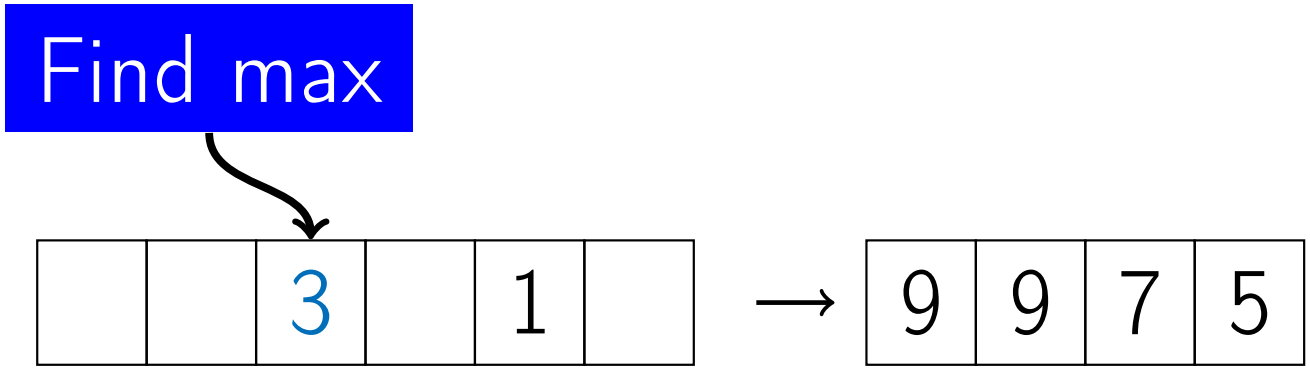

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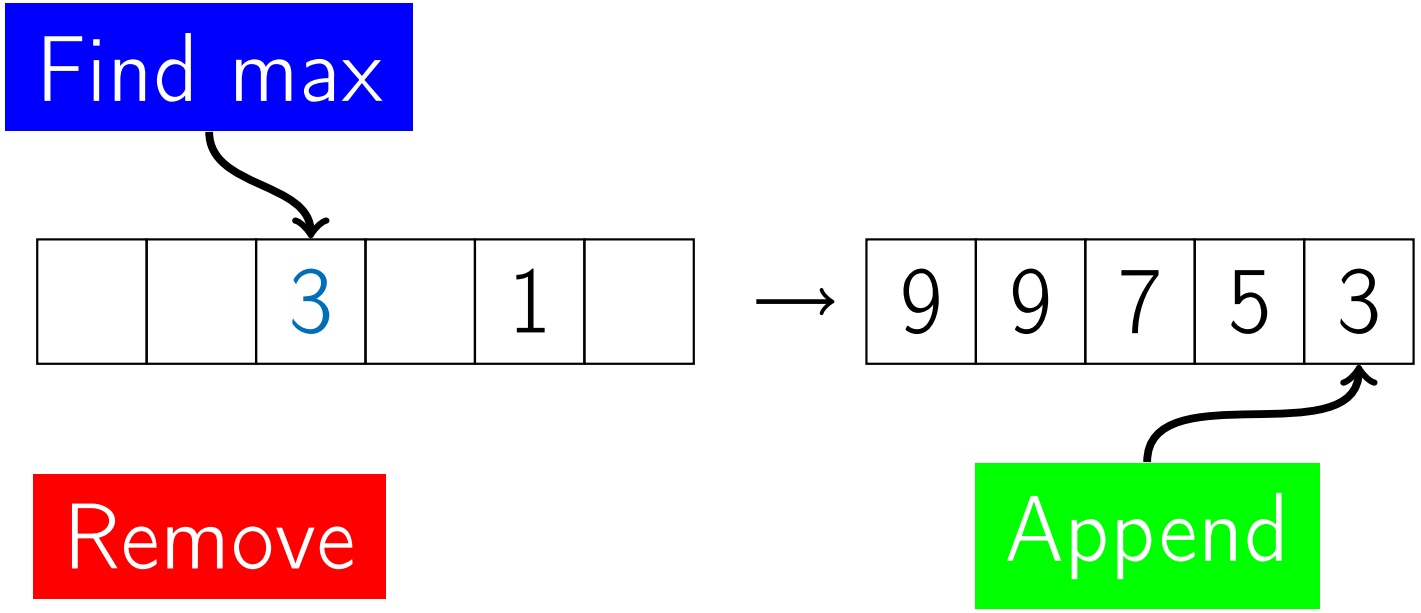

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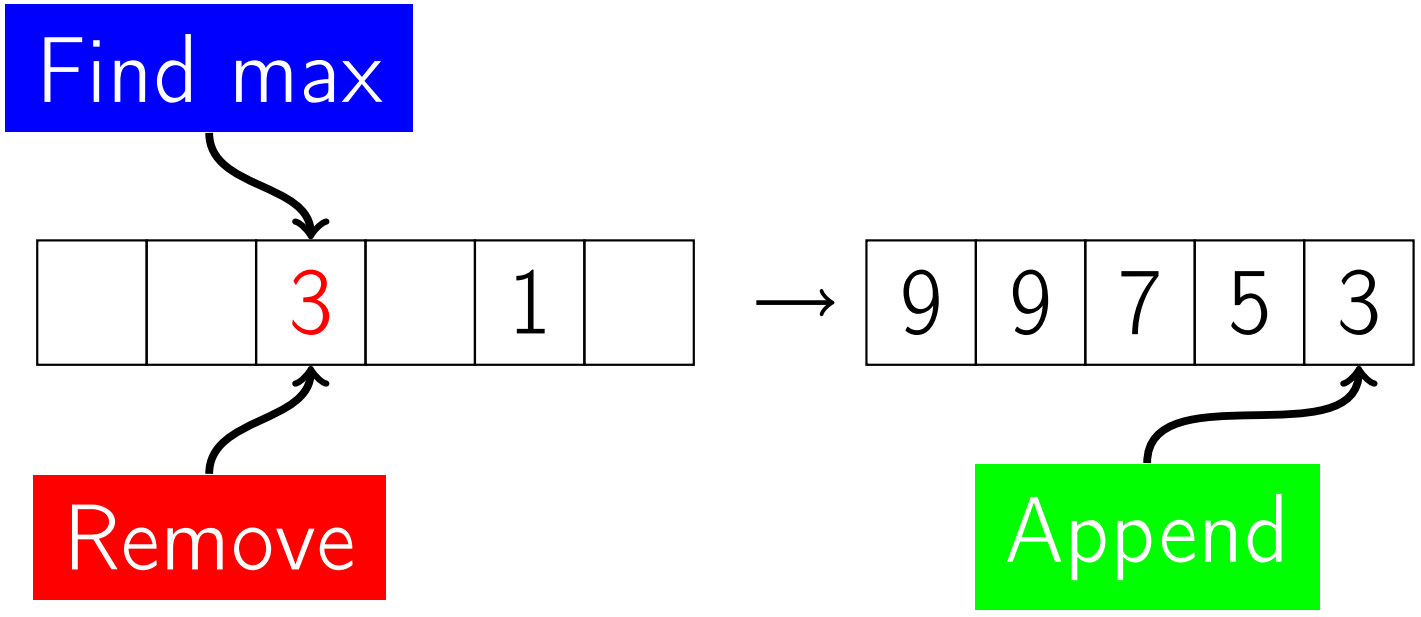

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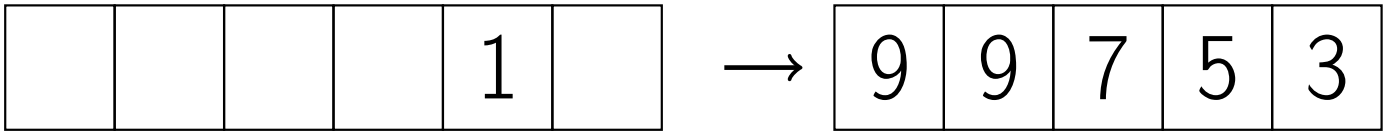

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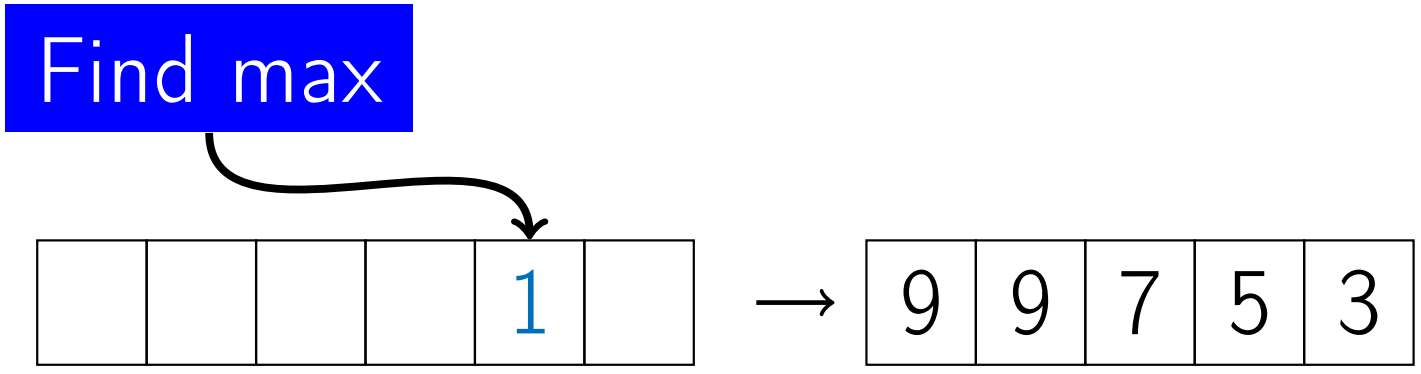

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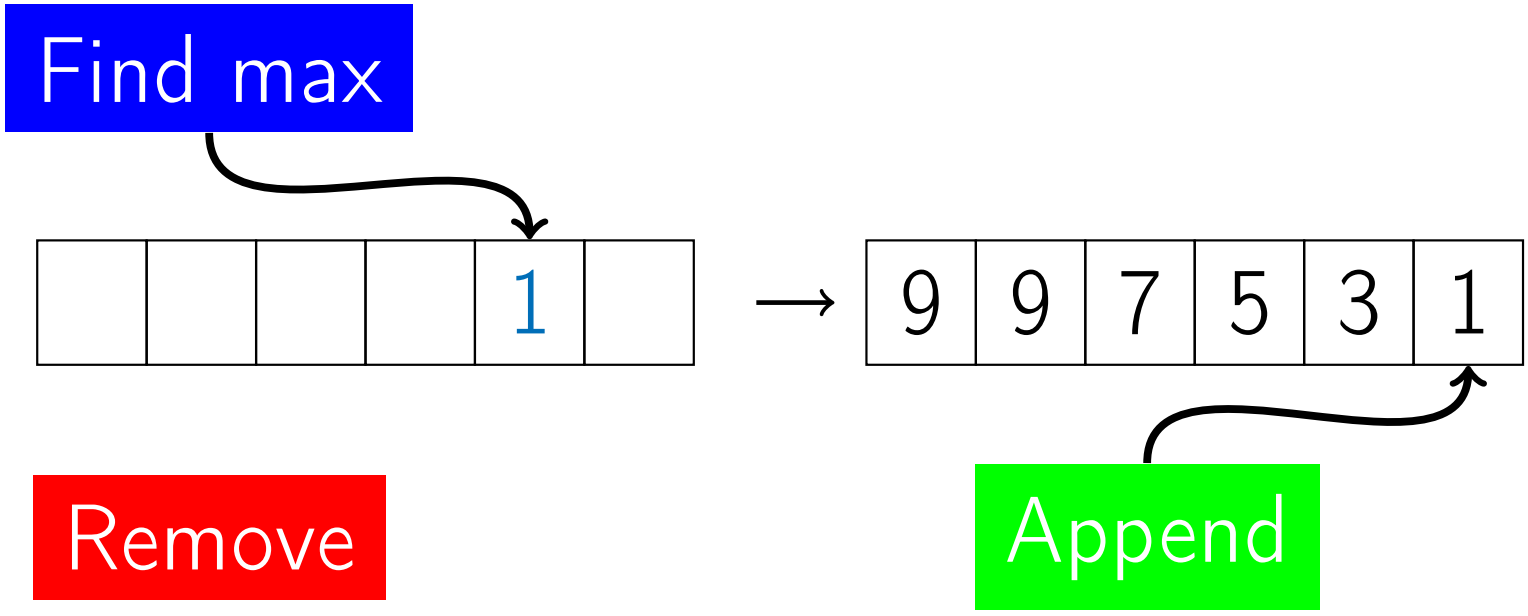

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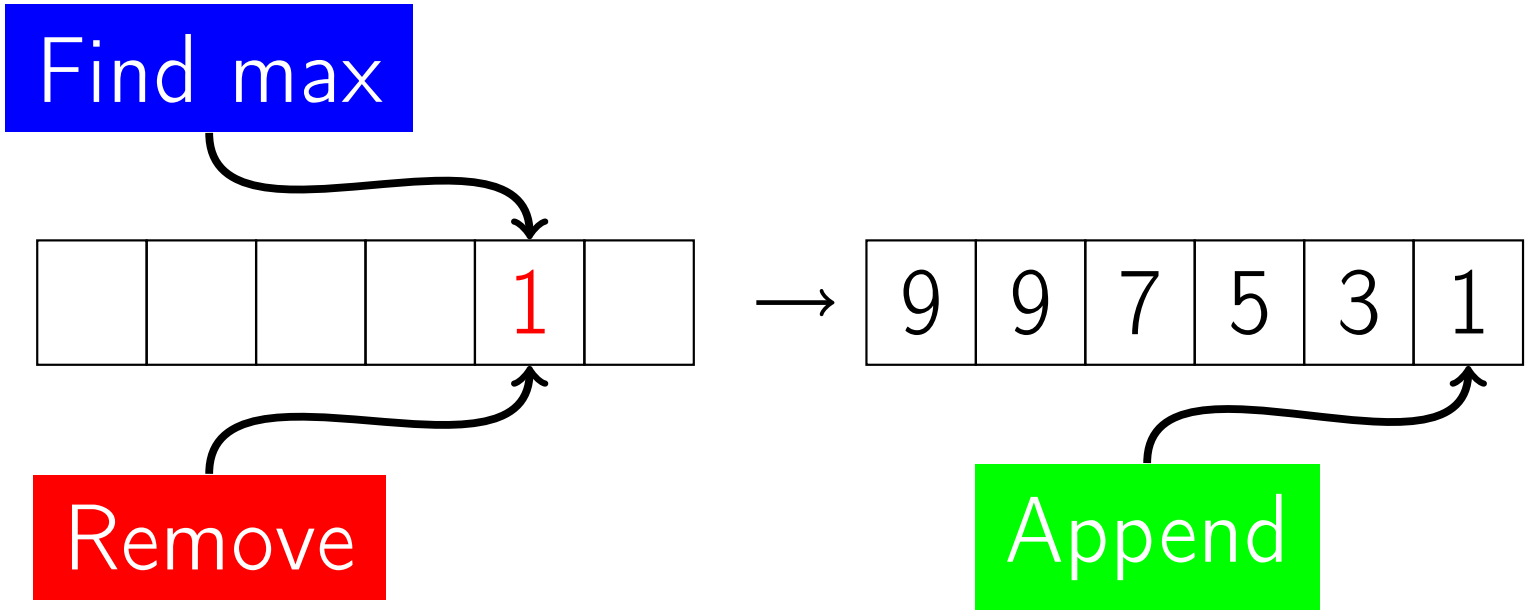

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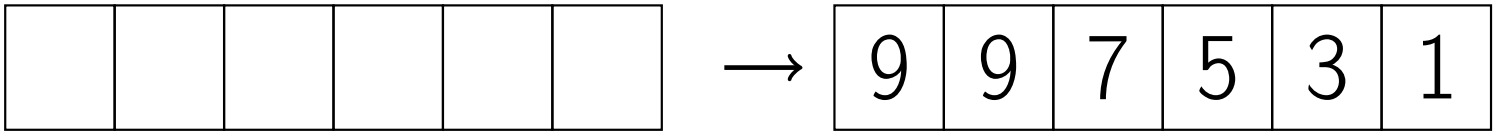

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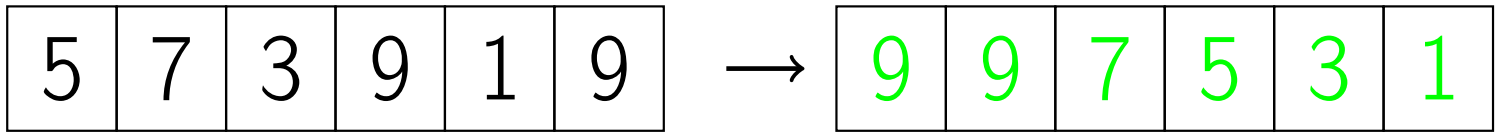

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Outline

Car Fueling

Distance with full tank = 400km

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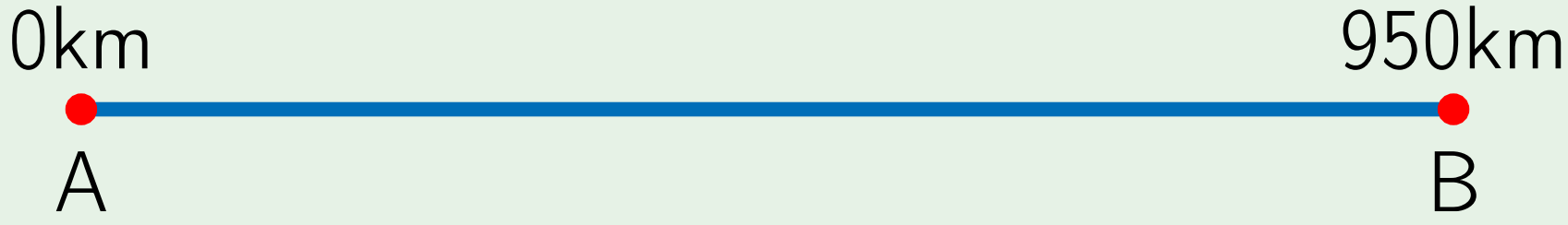
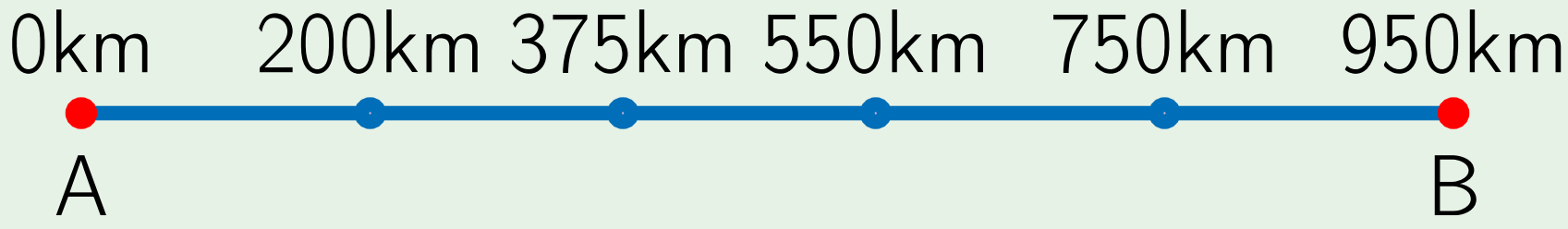


You wanna drive from A to B (=950 km).
But you only have a full tank that can last 400km max!

Car Fueling

Distance with full tank = 400km

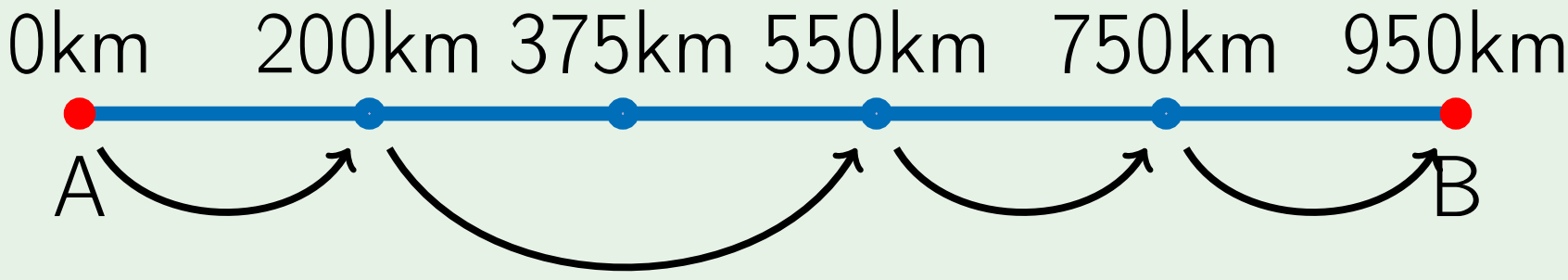


The blue dots represent gas stations

Car Fueling

Would this be an optimal solution?

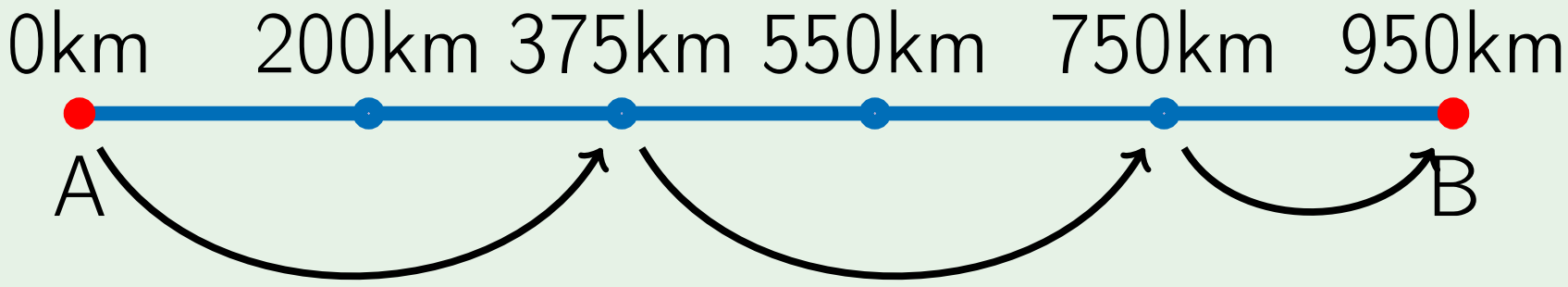
Distance with full tank = 400km



Now, you can choose to first go 200kms.
Then refill and drive another 350kms.
Then maybe drive another 200km and
then the final 200kms, refilling at each stop.
But are we refilling the least number of times possible??

Car Fueling

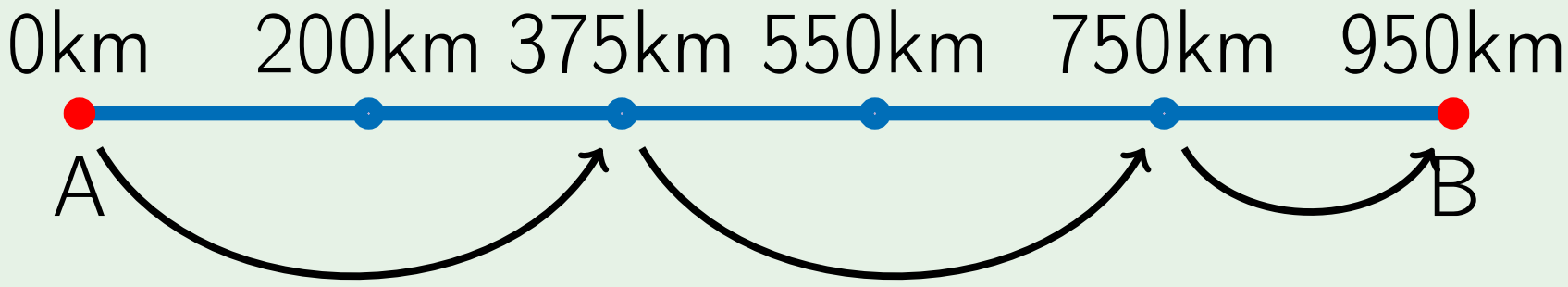
Distance with full tank = 400km



The optimal solution yields only 2 refills

Car Fueling

Distance with full tank = 400km



Minimum number of refills = 2

How can we prove this?

Car Fueling

Input: A car which can travel at most L kilometers with full tank, a source point A, a destination point B and n gas stations at distances $x_1 \leq x_2 \leq x_3 \leq \cdots \leq x_n$ in kilometers from A along the path from A to B.

Output: The minimum number of refills to get from A to B, besides refill at A.

Greedy Strategy

- Make some greedy choice
- Reduce to a smaller problem
- Iterate

Greedy Choice

Let's try to make a greedy choice

We have the following options:

- Refill at the the closest gas station
- Refill at the farthest reachable gas station
- Go until there is no fuel

Greedy Choice

- Refill at the the closest gas station
- Refill at the farthest reachable gas station

Think why it makes sense to choose this option

- Go until there is no fuel

Greedy Algorithm

- Start at A

Greedy Algorithm

- Start at A
- Refill at the farthest reachable gas station G

Greedy Algorithm

- Start at A
- Refill at the farthest reachable gas station G
- Make G the new A

Greedy Algorithm

- Start at A
- Refill at the farthest reachable gas station G
- Make G the new A
- Get from new A to B with minimum number of refills

Definition

Subproblem is a similar problem of smaller size.

Subproblem

Examples

- $\texttt{LargestNumber}(3, 9, 5, 9, 7, 1) =$

Subproblem

Examples

- $\texttt{LargestNumber}(3, 9, 5, 9, 7, 1) =$ “9” $+$

Subproblem

Examples

- $\texttt{LargestNumber}(3, 9, 5, 9, 7, 1) =$
  $\text{“9”} + \texttt{LargestNumber}(3, 5, 9, 7, 1)$

Subproblem

Examples

- $\texttt{LargestNumber}(3, 9, 5, 9, 7, 1) =$ “9” $+ \texttt{LargestNumber}(3, 5, 9, 7, 1)$
- Min number of refills from A to $B =$

Subproblem

Examples

- $\texttt{LargestNumber}(3, 9, 5, 9, 7, 1) =$ “9” $+ \texttt{LargestNumber}(3, 5, 9, 7, 1)$
- Min number of refills from A to $B =$ first refill at G $+$

Subproblem

Examples

- $\texttt{LargestNumber}(3, 9, 5, 9, 7, 1) =$ “9” $+ \texttt{LargestNumber}(3, 5, 9, 7, 1)$
- Min number of refills from A to $B =$ first refill at G + min number of refills from G to B

Safe Move

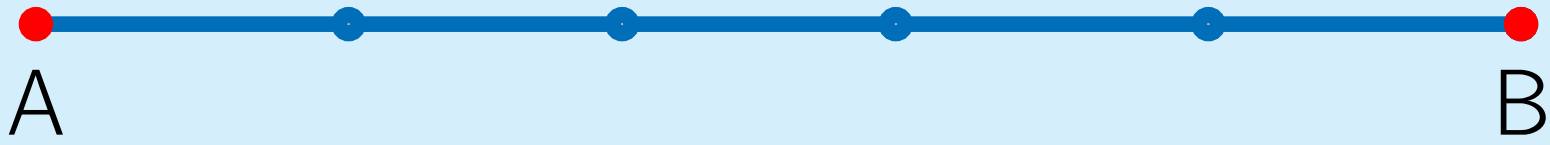
Definition

A greedy choice is called safe move if there is an optimal solution consistent with this first move.

Lemma

To refill at the farthest reachable gas station is a safe move.

Proof



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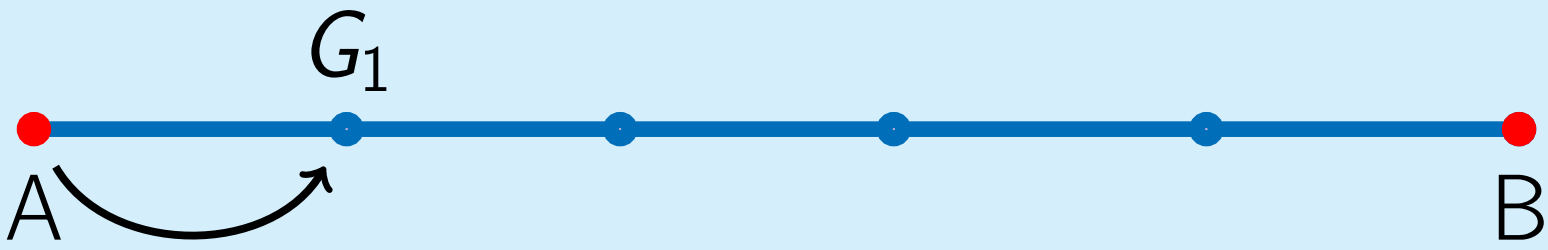

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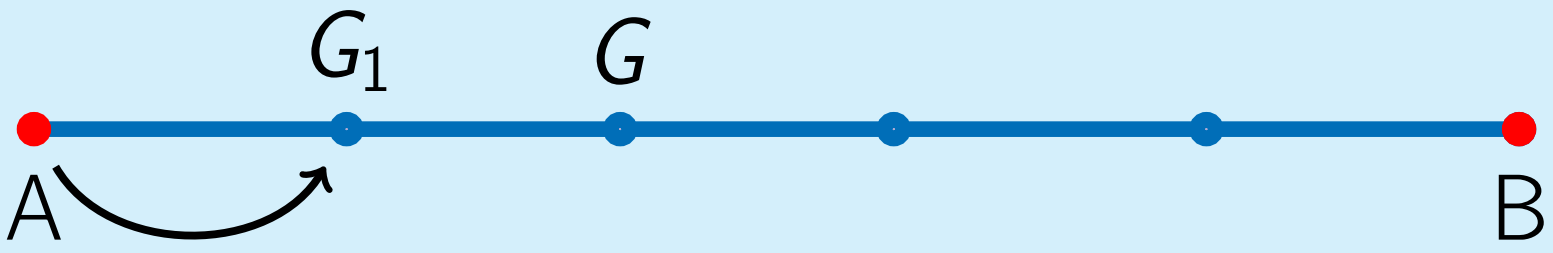
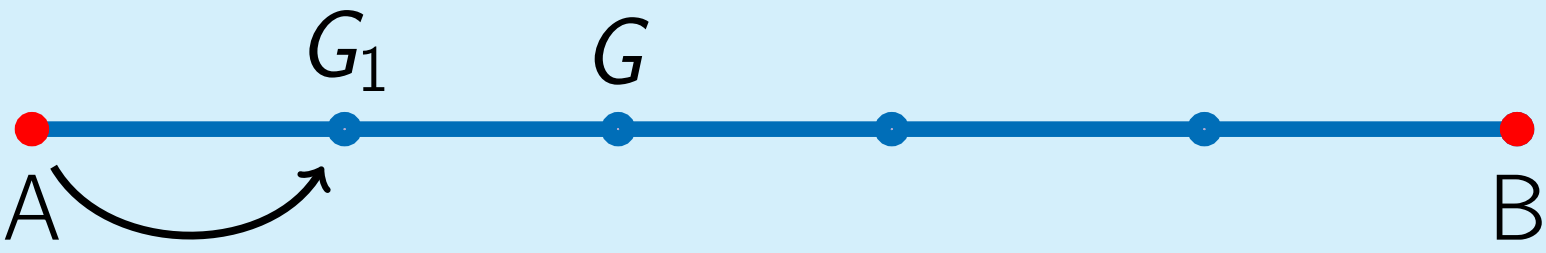


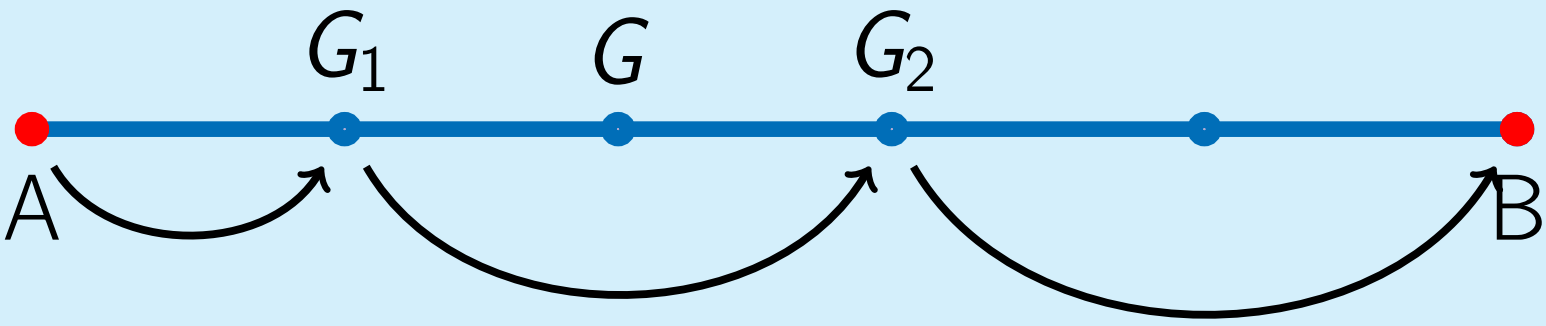
Let G1 be the optimal refill stop

Proof



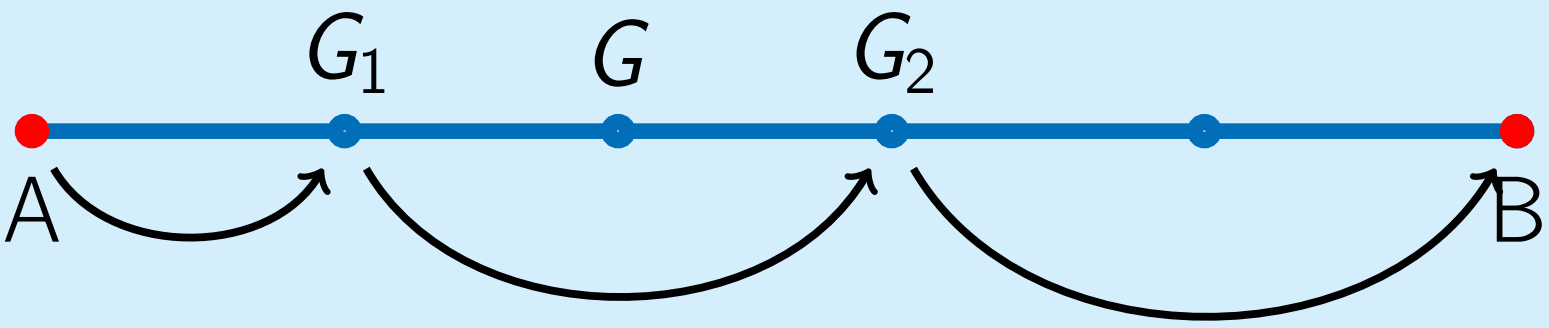
First case: G is closer than G_2

Proof



First case: G is closer to A than G_2

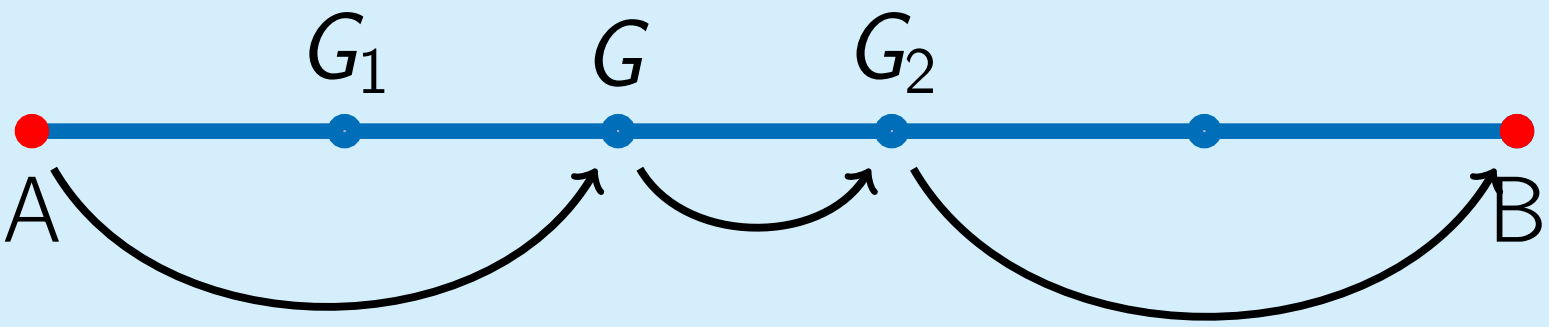
Proof



First case: G is closer than G_2

Refill at G instead of G_1

Proof



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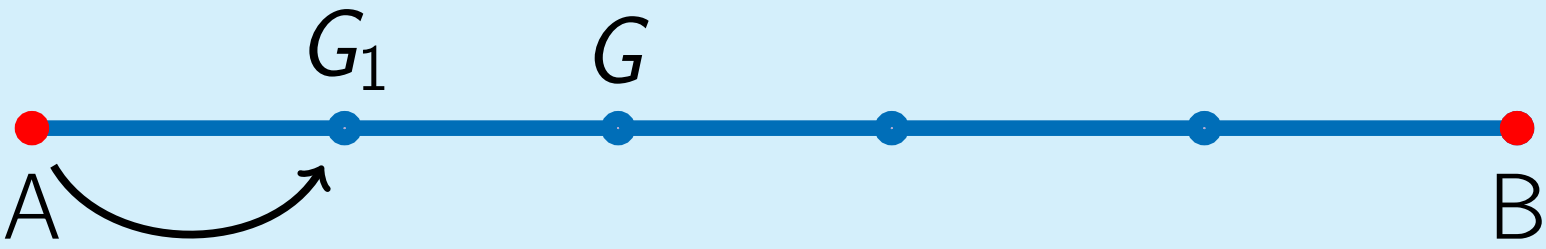
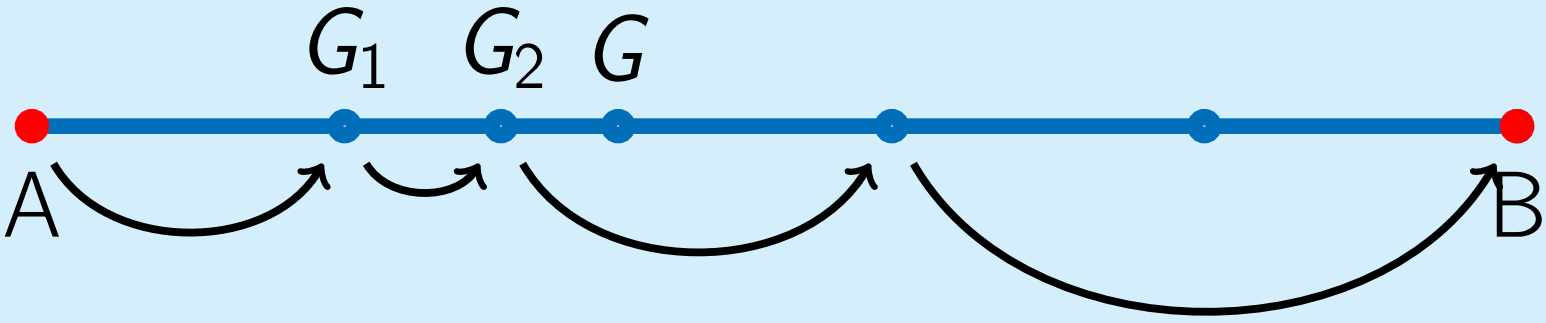
Refill at G instead of G_1

Proof



Second case: G_2 is closer than G

Proof



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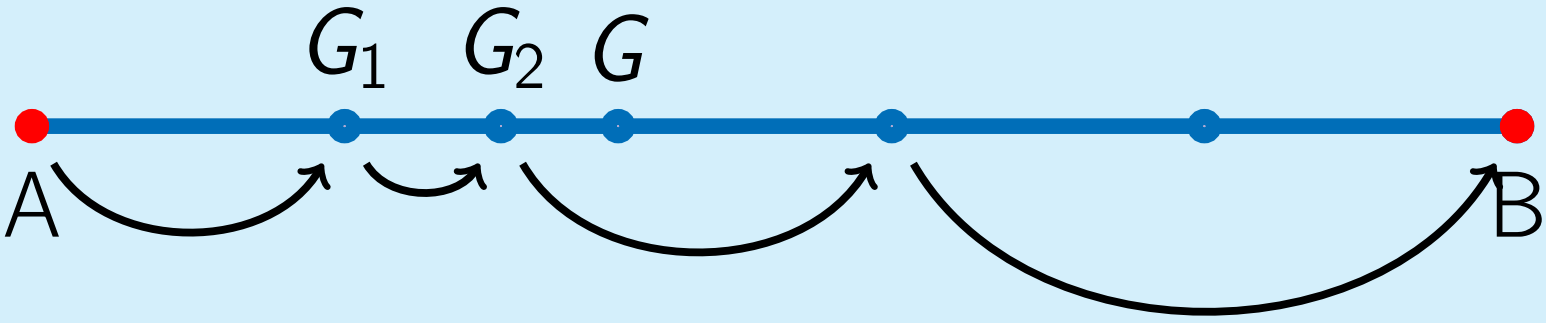

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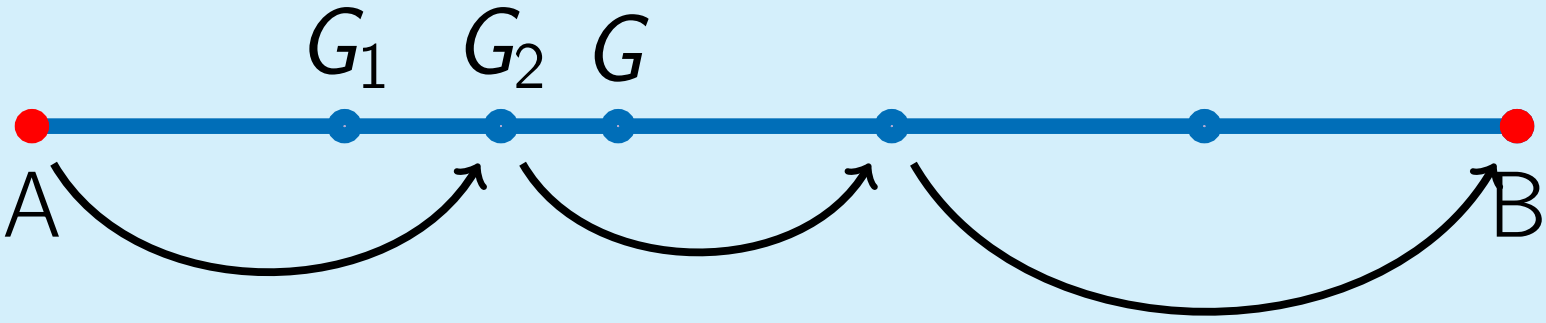

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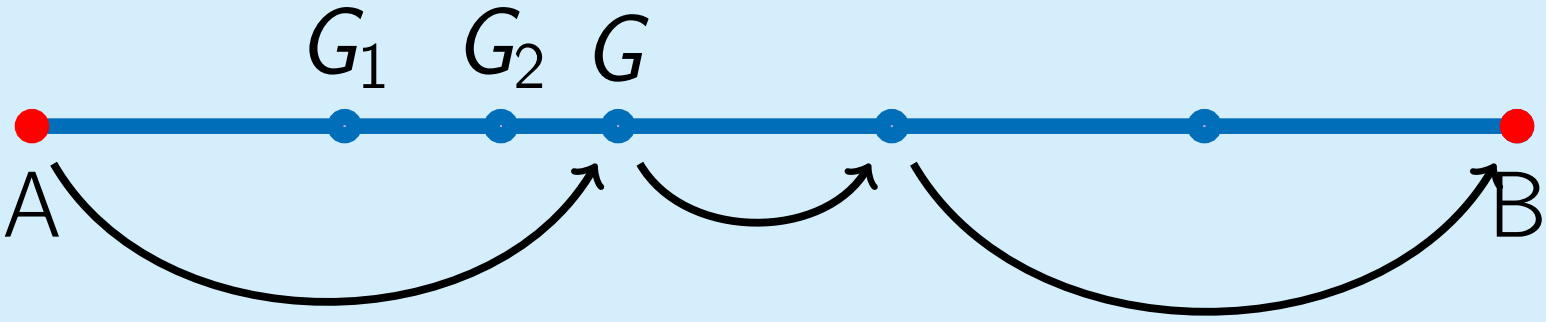
Proof



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- Route R with the minimum number of refills

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Proof

- Route R with the minimum number of refills
- G_1 — position of first refill in R
- G_2 — next stop in R (refill or B)
- G — farthest refill reachable from A
- If G is closer than G_2, refill at G instead of G_1
- Otherwise, avoid refill at G_1 □

Outline

$A = x_0 \le x_1 \le x_2 \le \cdots \le x_n \le x_{n+1} = B$

MinRefills(x, n, L)

```
numRefills ← 0,  currentRefill ← 0
while currentRefill ≤ n:
  lastRefill ← currentRefill
  while (currentRefill ≤ n and
         x[currentRefill + 1] − x[lastRefill] ≤ L):
    currentRefill ← currentRefill + 1
  if currentRefill == lastRefill:
    return IMPOSSIBLE
  if currentRefill ≤ n:
    numRefills ← numRefills + 1
return numRefills
```

Lemma

The running time of `MinRefills`(x, n, L) is $O(n)$.

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The running time of `MinRefills`(x, n, L) is $O(n)$.

Proof

- *currentRefill* changes from 0 to $n + 1$, one-by-one
- *numRefills* changes from 0 to at most n, one-by-one
- Thus, $O(n)$ iterations □

Outline

Reduction to Subproblem

- Make a first move
- Then solve a problem of the same kind
- Smaller: fewer digits, fewer fuel stations
- This is called a "subproblem"

Safe move

- A move is called safe if there is an optimal solution consistent with this first move

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- Not all first moves are safe
- Often greedy moves are not safe

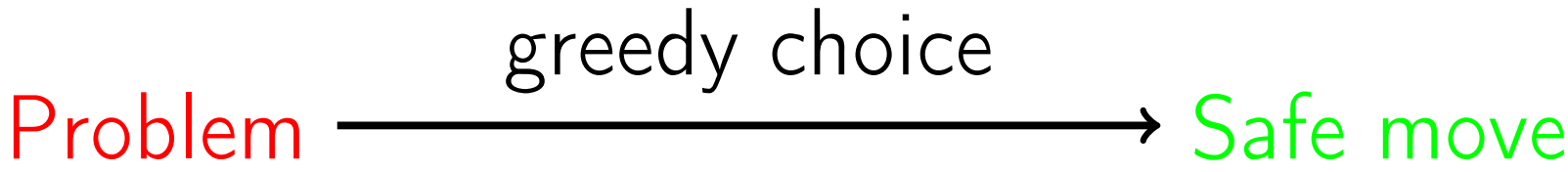
General Strategy

Problem

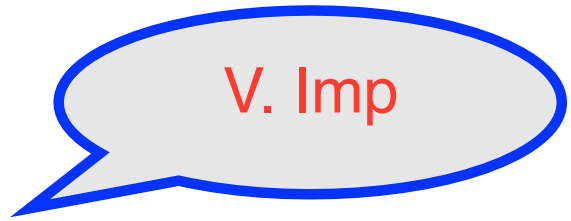
General Strategy

- Make a greedy choice

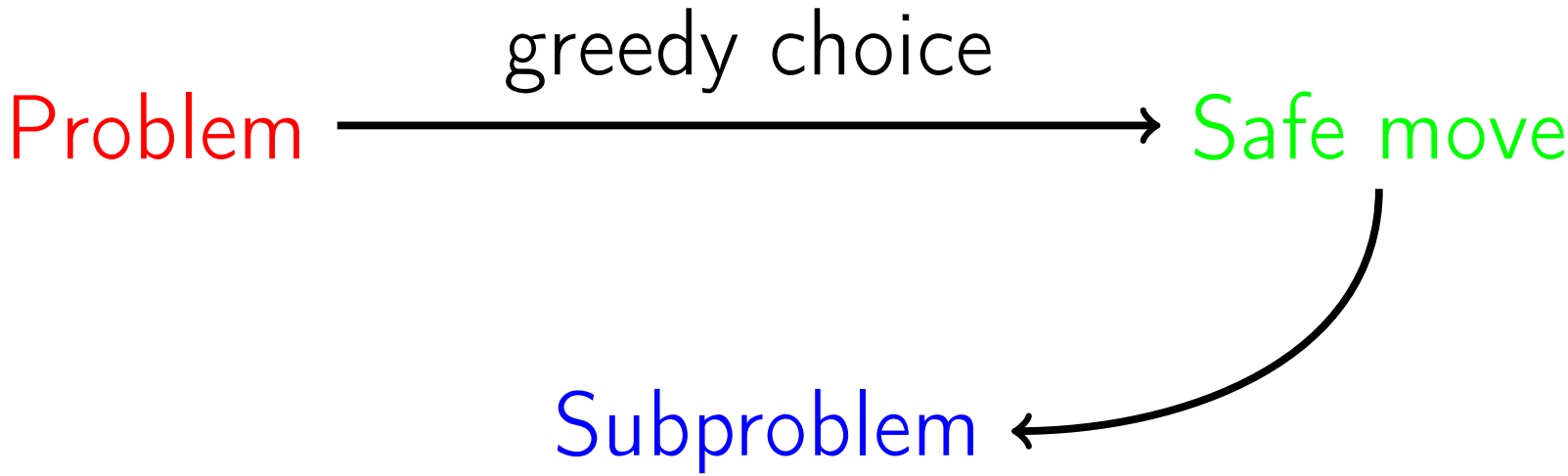
General Strategy



- Make a greedy choice
- Prove that it is a safe move



General Strategy



- Make a greedy choice
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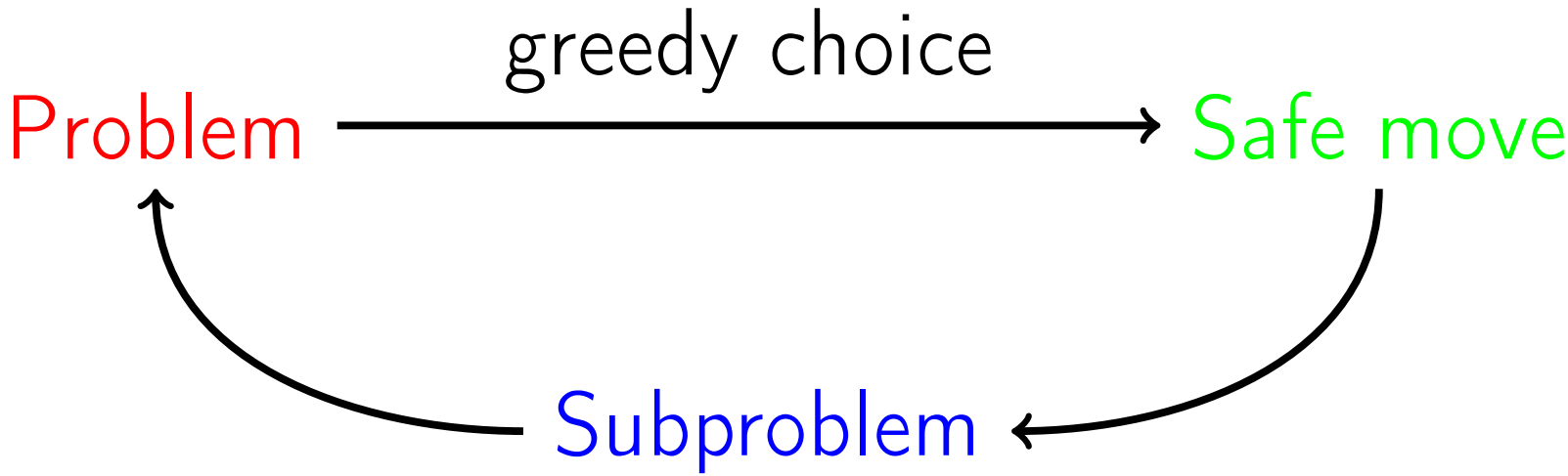
- Reduce to a subproblem

General Strategy



- Make a greedy choice
- Prove that it is a safe move
- Reduce to a subproblem
- Solve the subproblem