Greedy Algorithms: Grouping Children

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Algorithmic Toolbox
Data Structures and Algorithms

Outline

1. The Problem
2. Naive Algorithm
3. Efficient Algorithm

Many children came to a celebration. Organize them into the minimum possible number of groups such that the age of any two children in the same group differ by at most one year.

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MinGroups(C)

```
m ← len(C)
for each partition into groups
C = G_1 ∪ G_2 ∪ ··· ∪ G_k:
  good ← true
  for i from 1 to k:
    if max(G_i) − min(G_i) > 1:
      good ← false
  if good:
    m ← min(m, k)
return m
```

Running time

Lemma

The number of operations in `MinGroups`(C) is at least 2^n, where n is the number of children in C.

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- There are 2^n different G_1
- Thus, at least 2^n operations $\square$

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  operations!
- We will improve this significantly

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Before we think about the solution let's try to frame the problem in a mathematical way!

Covering points by segments

Input: A set of n points $x_1, \ldots, x_n \in \mathbb{R}$.

Output: The minimum number of segments of unit length needed to cover all the points.

Here, children are replaced by points on a number line. xi is the coordinate of the i-th point. Since max difference between 2 children in a group is 1, we need to find the min # of segments of length = 1 (ie. unit segments) that can cover all points on the line.

Example

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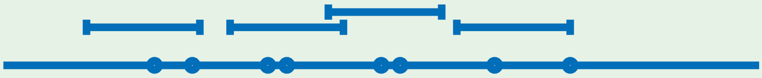

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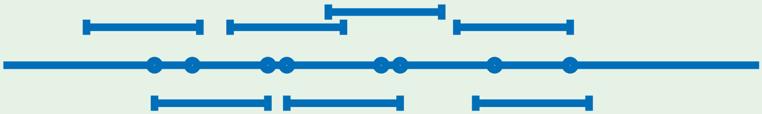
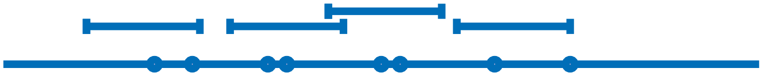


Safe move: cover the leftmost point with a unit segment which starts in this point.



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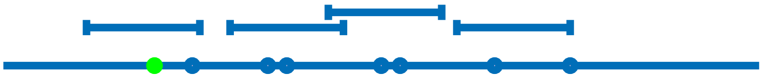

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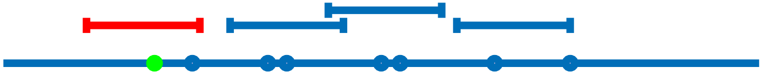

How do we prove this is a “safe move”?

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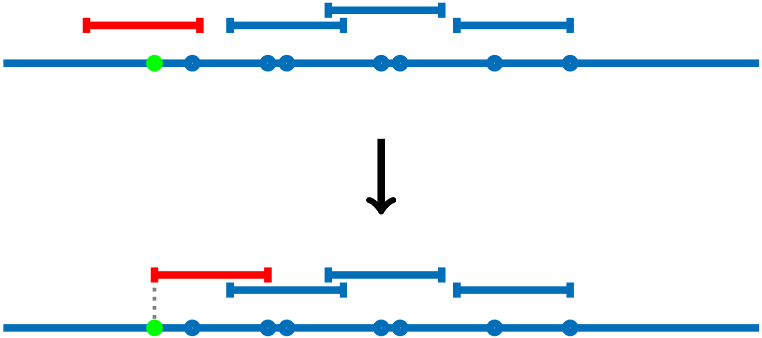

If we slide the segment until the leftmost point, we have made sure that none of the points on the left side of the line are left out.
Also, it makes sure we cover as many points as possible with the unit segment on the right side.

Consider that the points are available in a sorted (increasing) order (just like a number line)

Assume $x_1 \leq x_2 \leq \ldots \leq x_n$

$\texttt{PointsCoverSorted}(x_1, \ldots, x_n)$

$R \leftarrow \{\},\ i \leftarrow 1$

$\texttt{while}\ i \leq n$:

  $[\ell, r] \leftarrow [x_i, x_i + 1]$

  $R \leftarrow R \bigcup \{[\ell, r]\}$

  $i \leftarrow i + 1$

  $\texttt{while}\ i \leq n\ \texttt{and}\ x_i \leq r$:

    $i \leftarrow i + 1$

$\texttt{return}\ R$

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- Overall, running time is $O(n)$ $\square$

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- Sort $\{x_1, x_2, \ldots, x_n\}$, then call `PointsCoverSorted`
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- Sort + `PointsCoverSorted` is $O(n \log n)$

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- Very long for $n = 50$
- Sort + greedy is $O(n \log n)$
- Fast for $n = 10\,000\,000$
- Huge improvement!

Conclusion

- Straightforward solution is exponential
- Important to reformulate the problem in mathematical terms
- Safe move is to cover leftmost point
- Sort in $O(n \log n)$ + greedy in $O(n)$