Divide-and-Conquer: Polynomial Multiplication

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Data Structures and Algorithms Algorithmic Toolbox

Outline

- 1 Problem Overview
- 2 Naïve Algorithm
- 3 Naïve Divide and Conquer Algorithm
- 4 Faster Divide and Conquer

Uses of multiplying polynomials

- Error-correcting codes
- Large-integer multiplication
- Generating functions
- Convolution in signal processing

$$A(x) = 3x^2 + 2x + 5$$

$$A(x) = 3x^2 + 2x + 5$$
$$B(x) = 5x^2 + x + 2$$

$$A(x) = 3x^{2} + 2x + 5$$

$$B(x) = 5x^{2} + x + 2$$

$$A(x)B(x) = 15x^{4} + 13x^{3} + 33x^{2} + 9x + 10$$

Input: Two n-1 degree polynomials: $a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \cdots + a_1x + a_0$ $b_{n-1}x^{n-1} + b_{n-2}x^{n-2} + \cdots + b_1x + b_0$

Output:

Input: Two
$$n-1$$
 degree polynomials: $a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \cdots + a_1x + a_0$
 $b_{n-1}x^{n-1} + b_{n-2}x^{n-2} + \cdots + b_1x + b_0$

Output: The product polynomial:
$$c_{2n-2}x^{2n-2} + c_{2n-3}x^{2n-3} + \cdots + c_1x + c_0$$

Input: Two
$$n-1$$
 degree polynomials: $a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \cdots + a_1x + a_0$
 $b_{n-1}x^{n-1} + b_{n-2}x^{n-2} + \cdots + b_1x + b_0$

Output: The product polynomial:
$$c_{2n-2}x^{2n-2} + c_{2n-3}x^{2n-3} + \cdots + c_1x + c_0$$
 where:

$$c_{2n-2}=a_{n-1}b_{n-1}$$

Input: Two
$$n-1$$
 degree polynomials: $a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \cdots + a_1x + a_0$
 $b_{n-1}x^{n-1} + b_{n-2}x^{n-2} + \cdots + b_1x + b_0$

Output: The product polynomial: $c_{2n-2}x^{2n-2} + c_{2n-3}x^{2n-3} + \cdots + c_1x + c_0$ where:

$$c_{2n-2} = a_{n-1}b_{n-1}$$

 $c_{2n-3} = a_{n-1}b_{n-2} + a_{n-2}b_{n-1}$

Input: Two
$$n-1$$
 degree polynomials: $a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \cdots + a_1x + a_0$ $b_{n-1}x^{n-1} + b_{n-2}x^{n-2} + \cdots + b_1x + b_0$

Output: The product polynomial: $c_{2n-2}x^{2n-2} + c_{2n-3}x^{2n-3} + \cdots + c_1x + c_0$ where: $c_{2n-2} = a_{n-1}b_{n-1}$ $c_{2n-3} = a_{n-1}b_{n-2} + a_{n-2}b_{n-1}$... $c_2 = a_2b_0 + a_1b_1 + a_0b_2$

Input: Two
$$n-1$$
 degree polynomials: $a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \cdots + a_1x + a_0$
 $b_{n-1}x^{n-1} + b_{n-2}x^{n-2} + \cdots + b_1x + b_0$

Output: The product polynomial: $c_{2n-2}x^{2n-2} + c_{2n-3}x^{2n-3} + \cdots + c_1x + c_0$ where:

$$c_{2n-2} = a_{n-1}b_{n-1}$$

 $c_{2n-3} = a_{n-1}b_{n-2} + a_{n-2}b_{n-1}$

 $c_2 = a_2b_0 + a_1b_1 + a_0b_2$ $c_1 = a_1b_0 + a_0b_1$

Input: Two n-1 degree polynomials: $a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \cdots + a_1x + a_0$ $b_{n-1}x^{n-1} + b_{n-2}x^{n-2} + \cdots + b_1x + b_0$ Output: The product polynomial: $c_{2n-2}x^{2n-2}+c_{2n-3}x^{2n-3}+\cdots+c_1x+c_0$ where: $c_{2n-2} = a_{n-1}b_{n-1}$ $c_{2n-3} = a_{n-1}b_{n-2} + a_{n-2}b_{n-1}$

$$c_{2n-3}$$
 $a_{n-1}b_{n-2}+a_{n-2}b_{n-2}$... $c_2=a_2b_0+a_1b_1+a_0b_2$ $c_1=a_1b_0+a_0b_1$ $c_0=a_0b_0$

Example

Input: n = 3, A = (3, 2, 5), B = (5, 1, 2)

Input:
$$n = 3, A = (3, 2, 5), B = (5, 1, 2)$$

$$A(x) = 3x^2 + 2x + 5$$

Input:
$$n = 3, A = (3, 2, 5), B = (5, 1, 2)$$

$$A(x) = 3x^2 + 2x + 5$$
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Input:
$$n = 3, A = (3, 2, 5), B = (5, 1, 2)$$

$$A(x) = 3x^{2} + 2x + 5$$

$$B(x) = 5x^{2} + x + 2$$

$$A(x)B(x) = 15x^{4} + 13x^{3} + 33x^{2} + 9x + 10$$

Example

Input:
$$n = 3, A = (3, 2, 5), B = (5, 1, 2)$$

$$A(x) = 3x^{2} + 2x + 5$$

$$B(x) = 5x^{2} + x + 2$$

$$A(x)B(x) = 15x^{4} + 13x^{3} + 33x^{2} + 9x + 10$$

Output: C = (15, 13, 33, 9, 10)

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MultPoly(A, B, n)

```
\begin{array}{l} \textit{pair} \leftarrow \textit{Array}[n][n] \\ \textit{for } \textit{i} \;\; \textit{from 0 to } n-1: \\ \textit{for } \textit{j} \;\; \textit{from 0 to } n-1: \\ \textit{pair}[\textit{i}][\textit{j}] \leftarrow \textit{A}[\textit{i}] * \textit{B}[\textit{j}] \end{array}
```

MultPoly(A, B, n)

```
pair \leftarrow Array[n][n]
for i from 0 to n-1:
for j from 0 to n-1:
pair[i][j] \leftarrow A[i] * B[j]
product \leftarrow Array[2n-1]
for i from 0 to 2n-1:
product[i] \leftarrow 0
```

```
MultPoly(A, B, n)
pair \leftarrow Array[n][n]
for i from 0 to n-1:
```

for i from 0 to n-1: $pair[i][j] \leftarrow A[i] * B[j]$ $product \leftarrow Array[2n-1]$ for i from 0 to 2n-1:

 $product[i] \leftarrow 0$ for i from 0 to n-1: for i from 0 to n-1:

 $product[i + j] \leftarrow product[i + j] + pair[i][j]$

return *product*

```
MultPoly(A, B, n)
pair \leftarrow Array[n][n]
for i from 0 to n-1:
  for i from 0 to n-1:
```

 $pair[i][j] \leftarrow A[i] * B[j]$ Refer next page for the same algo in a more readable form

 $product[i] \leftarrow 0$ for i from 0 to n-1: for i from 0 to n-1:

 $product[i + j] \leftarrow product[i + j] + pair[i][j]$

 $product \leftarrow Array[2n-1]$ for i from 0 to 2n-1:

MultPoly(A, B, n)

```
product \leftarrow Array[2n-1] why 2n-1? for i from 0 to 2n-2: product[i] \leftarrow 0 for i from 0 to n-1: product[i+j] \leftarrow product[i+j] + A[i] * B[j] return product
```

Runtime: $O(n^2)$

Naïve Solution: $O(n^2)$

■ Multiply all $a_i b_j$ pairs (n^2 multiplications)

Naïve Solution: $O(n^2)$

- Multiply all $a_i b_j$ pairs (n^2 multiplications)
- Sum needed pairs $(n^2 \text{ additions})$

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Let
$$A(x) = D_1(x)x^{\frac{n}{2}} + D_0(x)$$
 where $D_1(x) = a_{n-1}x^{\frac{n}{2}-1} + a_{n-2}x^{\frac{n}{2}-2} + ... + a_{n-2}x^{\frac{n}{2}-2}$

$$D_1(x) = a_{n-1}x^{\frac{n}{2}-1} + a_{n-2}x^{\frac{n}{2}-2} + \dots + a_{\frac{n}{2}}$$

$$D_0(x) = a_{\frac{n}{2}-1}x^{\frac{n}{2}-1} + a_{\frac{n}{2}-2}x^{\frac{n}{2}-2} + \dots + a_0$$

Let
$$A(x) = D_1(x)x^{\frac{n}{2}} + D_0(x)$$
 where $D_1(x) = a_{n-1}x^{\frac{n}{2}-1} + a_{n-2}x^{\frac{n}{2}-2} + ... + a_{\frac{n}{2}}$ $D_0(x) = a_{\frac{n}{2}-1}x^{\frac{n}{2}-1} + a_{\frac{n}{2}-2}x^{\frac{n}{2}-2} + ... + a_0$

Let
$$B(x) = E_1(x)x^{\frac{n}{2}} + E_0(x)$$
 where $E_1(x) = b_{n-1}x^{\frac{n}{2}-1} + b_{n-2}x^{\frac{n}{2}-2} + ... + b_{\frac{n}{2}}$ $E_0(x) = b_{\frac{n}{2}-1}x^{\frac{n}{2}-1} + b_{\frac{n}{2}-2}x^{\frac{n}{2}-2} + ... + b_0$

Basically, A(x) and B(x) are divided into a upper half and lower half.

Will become clear in the example given on page 35

Let
$$A(x) = D_1(x)x^{\frac{n}{2}} + D_0(x)$$
 where $D_1(x) = a_{n-1}x^{\frac{n}{2}-1} + a_{n-2}x^{\frac{n}{2}-2} + ... + a_{\frac{n}{2}}$ $D_0(x) = a_{\frac{n}{2}-1}x^{\frac{n}{2}-1} + a_{\frac{n}{2}-2}x^{\frac{n}{2}-2} + ... + a_0$

Let
$$B(x) = E_1(x)x^{\frac{n}{2}} + E_0(x)$$
 where $E_1(x) = b_{n-1}x^{\frac{n}{2}-1} + b_{n-2}x^{\frac{n}{2}-2} + ... + b_{\frac{n}{2}}$ $E_0(x) = b_{\frac{n}{2}-1}x^{\frac{n}{2}-1} + b_{\frac{n}{2}-2}x^{\frac{n}{2}-2} + ... + b_0$

$$AB = (D_1 x^{\frac{n}{2}} + D_0)(E_1 x^{\frac{n}{2}} + E_0)$$

= $(D_1 E_1) x^n + (D_1 E_0 + D_0 E_1) x^{\frac{n}{2}} + D_0 E_0$

Let
$$A(x) = D_1(x)x^{\frac{n}{2}} + D_0(x)$$
 where $D_1(x) = a_{n-1}x^{\frac{n}{2}-1} + a_{n-2}x^{\frac{n}{2}-2} + ... + a_{\frac{n}{2}}$ $D_0(x) = a_{\frac{n}{2}-1}x^{\frac{n}{2}-1} + a_{\frac{n}{2}-2}x^{\frac{n}{2}-2} + ... + a_0$

Let
$$B(x) = E_1(x)x^{\frac{n}{2}} + E_0(x)$$
 where $E_1(x) = b_{n-1}x^{\frac{n}{2}-1} + b_{n-2}x^{\frac{n}{2}-2} + ... + b_{\frac{n}{2}}$ $E_0(x) = b_{\frac{n}{2}-1}x^{\frac{n}{2}-1} + b_{\frac{n}{2}-2}x^{\frac{n}{2}-2} + ... + b_0$

$$AB = (D_1 x^{\frac{n}{2}} + D_0)(E_1 x^{\frac{n}{2}} + E_0)$$

$$= (D_1 E_1) x^n + (D_1 E_0 + D_0 E_1) x^{\frac{n}{2}} + D_0 E_0$$

lacksquare Calculate D_1E_1, D_1E_0, D_0E_1 , and D_0E_0

Let
$$A(x) = D_1(x)x^{\frac{n}{2}} + D_0(x)$$
 where $D_1(x) = a_{n-1}x^{\frac{n}{2}-1} + a_{n-2}x^{\frac{n}{2}-2} + ... + a_{\frac{n}{2}}$ $D_0(x) = a_{\frac{n}{2}-1}x^{\frac{n}{2}-1} + a_{\frac{n}{2}-2}x^{\frac{n}{2}-2} + ... + a_0$

Let
$$B(x) = E_1(x)x^{\frac{n}{2}} + E_0(x)$$
 where $E_1(x) = b_{n-1}x^{\frac{n}{2}-1} + b_{n-2}x^{\frac{n}{2}-2} + ... + b_{\frac{n}{2}}$ $E_0(x) = b_{\frac{n}{2}-1}x^{\frac{n}{2}-1} + b_{\frac{n}{2}-2}x^{\frac{n}{2}-2} + ... + b_0$

$$AB = (D_1 x^{\frac{n}{2}} + D_0)(E_1 x^{\frac{n}{2}} + E_0)$$

$$= (D_1 E_1) x^n + (D_1 E_0 + D_0 E_1) x^{\frac{n}{2}} + D_0 E_0$$

lacksquare Calculate D_1E_1, D_1E_0, D_0E_1 , and D_0E_0

Let
$$A(x) = D_1(x)x^{\frac{n}{2}} + D_0(x)$$
 where $D_1(x) = a_{n-1}x^{\frac{n}{2}-1} + a_{n-2}x^{\frac{n}{2}-2} + ... + a_{\frac{n}{2}}$ $D_0(x) = a_{\frac{n}{2}-1}x^{\frac{n}{2}-1} + a_{\frac{n}{2}-2}x^{\frac{n}{2}-2} + ... + a_0$

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$$AB = (D_1 x^{\frac{n}{2}} + D_0)(E_1 x^{\frac{n}{2}} + E_0)$$

$$= (D_1 E_1) x^n + (D_1 E_0 + D_0 E_1) x^{\frac{n}{2}} + D_0 E_0$$

 \blacksquare Calculate D_1E_1 , D_1E_0 , D_0E_1 , and D_0E_0

Recurrence: $T(n) = 4T(\frac{n}{2}) + kn$. 4 D*E terms of size n/2 yielding 4T(n/2) and summation time given by kn

Polynomial Mult: Divide & Conquer $A(x) = 4x^3 + 3x^2 + 2x + 1$

$$A(x) = 4x^{3} + 3x^{2} + 2x + B(x) = x^{3} + 2x^{2} + 3x + 4$$

Polynomial Mult: Divide & Conquer $A(x) = 4x^3 + 3x^2 + 2x + 1$

$$A(x) = \frac{4x^3 + 3x^2 + 2x + 1}{B(x)}$$

$$B(x) = x^3 + 2x^2 + 3x + 4$$

 $D_1(x) = 4x + 3$

$$A(x) = 4x^3 + 3x^2 + 2x + 1$$

$$B(x) = x^3 + 2x^2 + 3x + 4$$

 $D_1(x) = 4x + 3$

$$A(x) = 4x^3 + 3x^2 + 2x + 1$$

$$B(x) = x^3 + 2x^2 + 3x + 4$$

$$A(x) = 4x^{3} + 3x^{2} + 2x + 1$$

$$B(x) = x^{3} + 2x^{2} + 3x + 4$$

Since $A(x) = D1(x).x^{3/2} + D0(x)$

 $D_0(x) = 2x + 1$

$$A(x) = 4x^3 + 3x^2 + 2x + 1$$

$$B(x) = x^3 + 2x^2 + 3x + 4$$

 $E_1(x) = x + 2$

$$A(x) = 4x^{3} + 3x^{2} + 2x + 1$$

$$B(x) = x^{3} + 2x^{2} + 3x + 4$$

$$B(x) = 4x + 3x + 2x + 1$$

$$B(x) = x^3 + 2x^2 + 3x + 4$$

$$D_1(x) = 4x + 3$$

$$D_0(x) = 2x + 1$$

$$A(x) = 4x^3 + 3x^2 + 2x + 1$$

$$B(x) = x^3 + 2x^2 + 3x + 4$$

 $D_0(x) = 2x + 1$

 $E_0(x) = 3x + 4$

 $D_1(x) = 4x + 3$

 $E_1(x) = x + 2$

$$A(x) = 4x^3 + 3x^2 + 2x + 1$$

$$B(x) = x^3 + 2x^2 + 3x + 4$$

$$B(x) = x^3 + 2x^2 + 3x + 4$$

$$D_1(x) = 4x + 3$$

$$E_1(x) = x + 2$$

 $D_1 E_1 = 4x^2 + 11x + 6$

$$= 4x + 3$$

$$D_0(x)=2x$$

$$D_0(x) = 2x + 1$$

$$E_0(x) = 3x + 4$$

$$= 2x + 1$$

$$2x + 1$$

$$B(x) = x^3 + 2x^2 + 3x + 4$$

$$B(x) = x^3 + 2x^2 + 3x + D_1(x) = 4x + 3$$

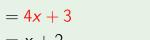
$$D_1(x) = 4x + 3$$

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$$D_1(x) = 4x + 3$$

 $E_1(x) = x + 2$

 $D_1 E_1 = 4x^2 + 11x + 6$



$$D_0(x) = 2x + 1$$

 $E_2(x) = 3x + 4$

$$E_0(x) = 3x + 4$$

$$E_0(x) = 3x + 4$$

 $D_1 E_0 = 12x^2 + 25x + 12$

$$B(x) = x^3 + 2x^2 + 3x + 4$$

$$D_1(x) = x^2 + 2x^2 + 3x + 2x^2$$
$$D_1(x) = 4x + 3$$

$$D_1(x) = 4x + 3$$

 $D_0 E_1 = 2x^2 + 5x + 2$

$$D_1(x) = 4x + 3$$

$$E_1(x) = x + 2$$

$$x + 2$$

$$E_1(x) = x + 2$$

 $D_1 E_1 = 4x^2 + 11x + 6$

$$E_0(x) = 3x + 4$$

 $D_1E_0 = 12x^2 + 4$

$$\nu_1 \nu_0 = 1$$

 $D_0(x) = 2x + 1$

$$D_1 E_0 = 12x^2 + 25x + 12$$

$$.2x^{2} + .25$$

$$B(x) = 4x^{2} + 3x^{2} + 2x + 1$$

$$B(x) = x^{3} + 2x^{2} + 3x + 4$$

$$D_1(x) = 4x + 3$$

$$D_1(x) = 4x + 3$$

 $E_1(x) = x + 2$

$$(x) = x + 2$$

$$E_1(x) = x + 2$$

 $D_1E_1 = 4x^2 + 11x + 6$

$$x^2 + 11x + 6$$

$$-11x + 6$$

$$D_1 E_1 = 4x^2 + 11x + 6$$
$$D_0 E_1 = 2x^2 + 5x + 2$$

$$D_1 E_0 = 12x^2 + 25x +$$

$$D_0 E_0 = 6x^2 + 11x + 4$$

$$= 12$$

 $D_0(x) = 2x + 1$

$$D_1 E_0 = 12x^2 + 25x + 12$$

$$E_0(x) = \frac{3x + 4}{D_1 E_0} = 12x^2 + 25$$

$$B(x) = x^3 + 2x^2 + 3x + 4$$

$$D_1(x) = 4x + 3$$

$$D_1(x) = 4x + 3$$

$$D_1(x) = 4x + 3$$

AB =

$$E_1(x) = x + 2$$

$$D_1 E_1 = 4x^2 + 11x + 6$$

$$D_1E_1 = 4x^2 + 11x + 0$$
$$D_0E_1 = 2x^2 + 5x + 2$$

$$= 6$$

 $D_0(x) = 2x + 1$

 $E_0(x) = 3x + 4$

$$D_1 E_0 = 12x + 23x + 4$$
$$D_0 E_0 = 6x^2 + 11x + 4$$

$$D_1 E_0 = 12x^2 + 25x + 12$$

$$D_2 E_2 = 6x^2 + 11x + 4$$

$$B(x) = x^3 + 2x^2 + 3x + 4$$

$$D_1(x) = 4x + 3$$

$$4x + 3$$
$$x + 2$$

$$E_1(x) = x + 2$$

$$D_1 E_1 = 4x^2 + 11x + 6$$

$$D_1 E_1 = 4x^2 + 11x + 6$$
$$D_0 E_1 = 2x^2 + 5x + 2$$

$$AB = (4x^2 + 11x + 6)x^4 + 6$$

$$D_0 E_0 =$$

 $D_0(x) = 2x + 1$

 $E_0(x) = 3x + 4$

$$D_0 E_0 = 6x^2 + 11x + 4$$

$$D_1 E_0 = 12x^2 + 25x + 12$$
$$D_0 E_0 = 6x^2 + 11x + 4$$

$$A(x) = 4x^3 + 3x^2 + 2x + 1$$

$$B(x) = x^3 + 2x^2 + 3x + 4$$

$$D_1(x) = 4x + 3$$

$$D_0(x) = 2x + 1$$

 $E_0(x) = 3x + 4$

$$D_1(x) = 4x + 3$$

 $E_1(x) = x + 2$

 $D_0E_1 = 2x^2 + 5x + 2$

$$E_1(x) = x + 2$$

 $D_1 E_1 = 4x^2 + 11x + 6$

$$D_1E_0 = 12x^2 + 25x + 12$$

$$D_0 E_1 = 2x^2 + 5x + 2$$

$$AB = (4x^2 + 11x + 6)x^4 + 6x^4 + 6x^$$

$$E_0 =$$

$$D_0 E_0 = 6x^2 + 11x + 4$$

 $)x^{2} +$

$$A(x) = 4x^3 + 3x^2 + 2x + 1$$

$$B(x) = x^3 + 2x^2 + 3x + 4$$

$$B(x) = x^3 + 2x^2 + 3x + 4$$
$$D_1(x) = 4x + 3$$

$$D_1(x) = 4x + 3$$

$$E_1(x) = x + 2$$

$$E_1(x) = x + 2$$

 $D_1E_1 = 4x^2 + 11x + 6$

 $(12x^2 + 25x + 12)$

$$x^2 + 11x + 6$$

$$D_0E_1 = 2x^2 + 5x + 2$$

 $AB = (4x^2 + 11x + 6)x^4 + 6x^4$

 $D_0(x) = 2x + 1$

 $E_0(x) = 3x + 4$

$$D_0 E_0 = 6x^2 + 11x + 4$$

 $)x^{2} +$

 $D_1 E_0 = 12x^2 + 25x + 12$

$$+11x$$

$$A(x) = 4x^3 + 3x^2 + 2x + 1$$

$$B(x) = x^3 + 2x^2 + 3x + 4$$

$$D_1(x) = 4x + 3$$

$$D_1(x) = 4x + 3$$

$$E_1(x) = x + 2$$

$$E_1(x) = x + 2$$

$$D_1 E_1 = 4x^2 + 11x + 6$$

$$D_1 E_1 = 4x^2 + 11x + 6$$

$$D_0 E_1 = 2x^2 + 5x + 2$$

$$D_1E_1 = 4x^2 + 11x + 6$$
 $D_1E_0 = 12x^2 + 25x + 12$
 $D_0E_1 = 2x^2 + 5x + 2$ $D_0E_0 = 6x^2 + 11x + 4$
 $AB = (4x^2 + 11x + 6)x^4 +$

 $(12x^2 + 25x + 12 + 2x^2 + 5x + 2)x^2 +$

$$D_0 E_0 =$$

 $D_0(x) = 2x + 1$

 $E_0(x) = 3x + 4$

$$D_0 E_0 = 6x^2 + 11x + 4$$

$$A(x) = 4x^3 + 3x^2 + 2x + 1$$

$$B(x) = x^3 + 2x^2 + 3x + 4$$

$$D_1(x) = 4x + 3$$

$$D_1(x) = 4x + 3$$

$$E_1(x) = x + 2$$

 $D_0E_1 = 2x^2 + 5x + 2$

 $6x^2 + 11x + 4$

$$D_1 E_1 = 4x^2 + 11x + 6$$

$$AB = (4x^2 + 11x + 6)x^4 + B$$

 $(12x^2 + 25x + 12 + 2x^2 + 5x + 2)x^2 +$

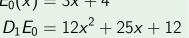
 $D_0(x) = 2x + 1$

 $E_0(x) = 3x + 4$

$$= 0x + 11x$$

$$D_0 E_0 = 6x^2 + 11x + 4$$

$$+25x + 1$$



$$A(x) = 4x^3 + 3x^2 + 2x + 1$$

$$B(x) = x^3 + 2x^2 + 3x + 4$$

$$A(x) = 4x^{3} + 3x^{2} + 2x + 1$$

$$B(x) = x^{3} + 2x^{2} + 3x + 4$$

$$D_{1}(x) = 4x + 3$$

$$D_{0}(x) = 2x + 1$$

$$D_1(x) = 4x + 3$$
 $D_0(x) = 2x + 4$
 $E_1(x) = x + 2$ $E_0(x) = 3x + 4$

$$E_1(x) = x + 2$$
 $E_0(x) = 3x + 2$
 $D_1E_1 = 4x^2 + 11x + 6$ $D_1E_0 = 12x^2$

 $6x^2 + 11x + 4$

$$E_1(x) = x + 2$$
 $E_0(x) = 3x + 4$
 $D_1E_1 = 4x^2 + 11x + 6$ $D_1E_0 = 12x^2 + 11x + 12$

$$D_1E_1 = 4x^2 + 11x + 6$$
 $D_1E_0 = 12x^2 + 2$
 $D_0E_1 = 2x^2 + 5x + 2$ $D_0E_0 = 6x^2 + 11$

$$D_1E_1 = 4x^2 + 11x + 6$$
 $D_1E_0 = 12x^2 + 25x + 12$
 $D_0E_1 = 2x^2 + 5x + 2$ $D_0E_0 = 6x^2 + 11x + 4$

$$D_1E_1 = 4x^2 + 11x + 6$$
 $D_1E_0 = 12x^2 + 25x + 6$
 $D_0E_1 = 2x^2 + 5x + 2$ $D_0E_0 = 6x^2 + 11x + 6$
 $AB = (4x^2 + 11x + 6)x^4 + 6$

$$D_0E_1 = 2x^2 + 5x + 2$$
 $D_0E_0 = 6x^2 + 11x + 4$
 $AB = (4x^2 + 11x + 6)x^4 + 26x^4$

 $(12x^2 + 25x + 12 + 2x^2 + 5x + 2)x^2 +$

 $=4x^{6} + 11x^{5} + 20x^{4} + 30x^{3} + 20x^{2} + 11x + 4$

$$D_1E_1 = 4x^2 + 11x + 6$$
 $D_1E_0 = 12x^2 + 25x$
 $D_0E_1 = 2x^2 + 5x + 2$ $D_0E_0 = 6x^2 + 11x$

R = array[0..2n - 2] al = low index of array A bl = low index of array B

Function Mult2(A, B, n, a_l, b_l) R = array[0..2n - 2]

if n = 1: $R[0] = A[a_I] * B[b_I]$; return R

$$R = \text{array}[0..2n - 2]$$

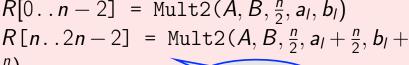
if $n = 1$:
 $R[0] = A[a_I] * B[b_I]$; return R
 $R[0..n - 2] = \text{Mult2}(A, B, \frac{n}{2}, a_I, b_I)$

This is D0*E0 (the low coeffs)

$$R = array[0..2n - 2]$$

if $n = 1$:

$$R[0] = A[a_I] * B[b_I]$$
; return R
 $R[0..n-2] = Mult2(A, B, \frac{n}{2}, a_I, b_I)$



$$\frac{n}{2}$$
This is D1*E1 (the high coeffs)

Function Mult2(A, B, n, a_l, b_l) R = array[0..2n - 2]

if
$$n = 1$$
:
 $R[0] = A[a_i] * B[b_i]$

$$R[0] = A[a_I] * B[b_I]$$
; return R
 $R[0..n-2] = Mult2(A, B, \frac{n}{2}, a_I, b_I)$

$$R[0..n-2] = Mult2(A, B, \frac{n}{2}, a_l, b_l)$$

 $R[n..2n-2] = Mult2(A, B, \frac{n}{2}, a_l + \frac{n}{2}, b_l + \frac{n}{2})$

$$R[n..2n-2] = Mult2(A, B, \frac{n}{2}, a)$$

 $D_0 E_1 = \text{Mult2}(A, B, \frac{n}{2}, a_l, b_l + \frac{n}{2})$

$$R = array[0..2n - 2]$$
 if $n = 1$:

$$R[0] = A[a_I] * B[b_I]$$
; return R
 $R[0, n-2] = Mult2(A, B, \frac{n}{n}, a_I, b_I)$

$$R[0..n-2] = Mult2(A, B, \frac{n}{2}, a_I, b_I)$$

 $R[n..2n-2] = Mult2(A, B, \frac{n}{2}, a_I + \frac{n}{2}, b_I + \frac{n}{2})$

$$D_0 E_1 = \text{Mult2}(A, B, \frac{n}{2}, a_l, b_l + \frac{n}{2})$$

 $D_1 E_0 = \text{Mult2}(A, B, \frac{n}{2}, a_l + \frac{n}{2}, b_l)$

$$R\left[\frac{n}{2}\dots n + \frac{n}{2} - 2\right] += D_1E0 + D_0E_1$$
The middle coeffs

Function Mult2(A, B, n, a_l, b_l) R = array[0..2n - 2]

if
$$n = 1$$
:
 $R[0] = A[a_I] * B[b_I]$; return R
 $R[0..n-2] = Mult2(A, B, \frac{n}{2}, a_I, b_I)$

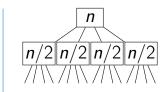
 $R[n..2n-2] = Mult2(A, B, \frac{n}{2}, a_l + \frac{n}{2}, b_l +$ $D_0 E_1 = \text{Mult2}(A, B, \frac{n}{2}, a_I, b_I + \frac{n}{2})$

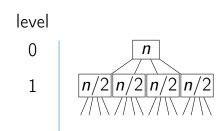
$$\begin{array}{l} \frac{n}{2} \\ D_0 E_1 &= \text{Mult2}(A, B, \frac{n}{2}, a_l, b_l + \frac{n}{2}) \\ D_1 E_0 &= \text{Mult2}(A, B, \frac{n}{2}, a_l + \frac{n}{2}, b_l) \\ R[\frac{n}{2} \dots n + \frac{n}{2} - 2] &+= D_1 E_0 + D_0 E_1 \end{array}$$

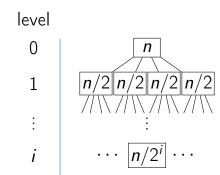
return R

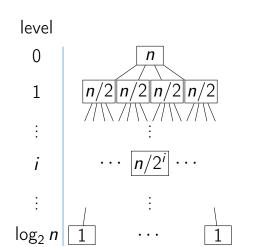


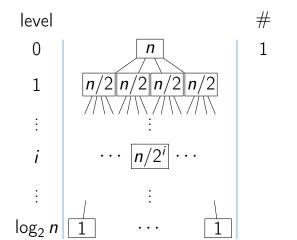
level

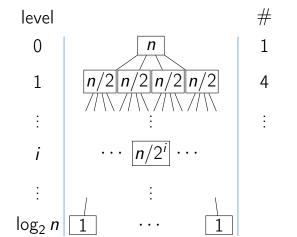


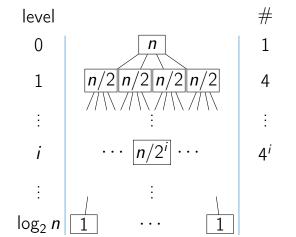


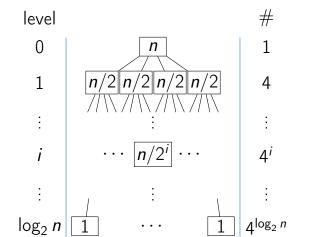


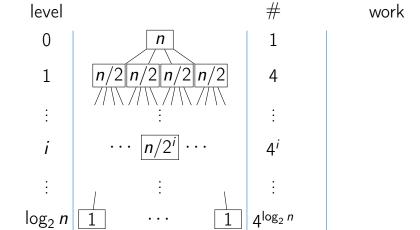


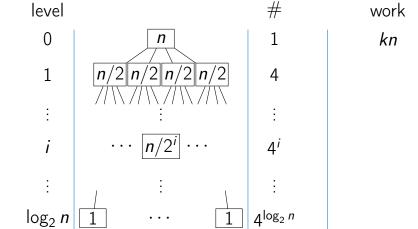


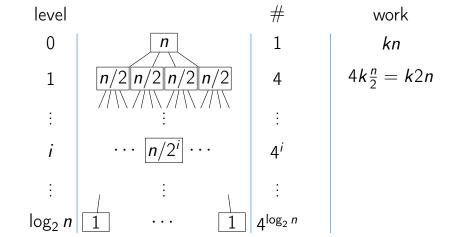


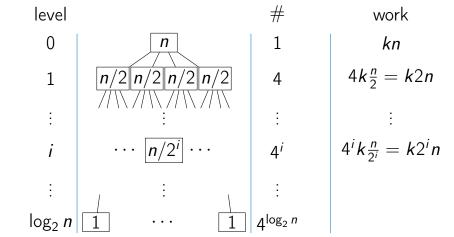


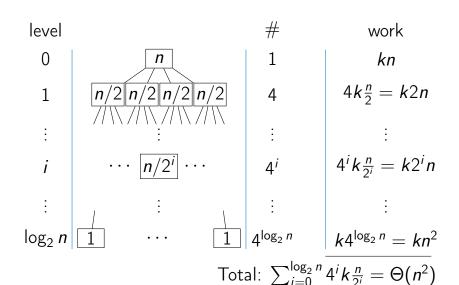






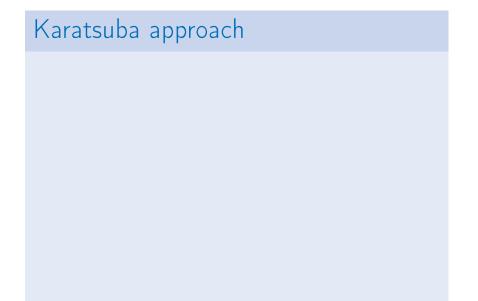






Outline

- 1 Problem Overview
- 2 Naïve Algorithm
- 3 Naïve Divide and Conquer Algorithm
- 4 Faster Divide and Conquer



$$A(x) = a_1x + a_0$$

$$A(x) = a_1x + a_0$$
$$B(x) = b_1x + b_0$$

$$A(x) = a_1x + a_0$$

$$B(x) = b_1 x + b_0$$

$$B(x) = b_1 x + b_0$$

 $C(x) = a_1 b_1 x^2 + (a_1 b_0 + a_0 b_1) x + a_0 b_0$

$$A(x) = a_1x + a_0$$

$$B(x) = b_1 x + b_0$$

$$x + t$$
 $b_1 x^2$

$$C(x) = \frac{a_1b_1x^2}{a_1b_0} + \frac{a_0b_1}{a_0b_1} + \frac{a_0b_0}{a_0b_1}$$

Needs 4 multiplications

$$A(x) = a_1x + a_0$$

$$B(x) = b_1 x + b_0$$

Rewrite as:

Needs 4 multiplications

$$C(x) = a_1b_1x^2 + (a_1b_0 + a_0b_1)x + a_0b_0$$

$$A(x) = a_1 x + a_0$$

$$B(x) = b_1 x + b_0$$

 $C(x) = a_1 b_1 x^2 + b$

$$C(x) = a_1b_1x^2 + (a_1b_0 + a_0b_1)x + a_0b_0$$

Needs 4 multiplications

Rewrite as:
$$C(x) = a_1b_1x^2 + ((a_1 + a_0)(b_1 + b_0) - a_1b_1 - a_0b_0)x + a_0b_0$$

$$A(x) = a_1 x + a_0$$

Rewrite as:

 $C(x) = \frac{a_1b_1x^2}{a_1b_1x^2} + \frac{a_1b_1x$

 a_0b_0

Needs 3 multiplications

$$B(x) = b_1 x + b_0$$

Needs 4 multiplications

 $C(x) = a_1b_1x^2 + (a_1b_0 + a_0b_1)x + a_0b_0$

 $((a_1 + a_0)(b_1 + b_0) - a_1b_1 - a_0b_0)x +$

$$A(x) = a_1 x + a_0$$

$$B(x) = b_1 x + b_0$$

 $C(x) = a_1 b_1 x^2 + (a_1 b_0 + a_0 b_1) x + a_0 b_0$

$$_{1}b_{1}x^{2}$$

Needs 4 multiplications

 $C(x) = a_1b_1x^2 +$

 a_0b_0

Needs 3 multiplications

Rewrite as:

 $((a_1 + a_0)(b_1 + b_0) - a_1b_1 - a_0b_0)x +$

$$A(x) = a_1 x + a_0$$

$$B(x) = b_1 x + b_0$$

 $C(x) = a_1 b_1 x^2 + (a_1 b_0 + a_0 b_1) x + a_0 b_0$

$$_1b_1x^2$$

Needs 4 multiplications

 $C(x) = a_1b_1x^2 +$

 a_0b_0

Needs 3 multiplications

Rewrite as:

 $((a_1 + a_0)(b_1 + b_0) - a_1b_1 - a_0b_0)x +$

$$A(x) = 4x^3 + 3x^2 + 2x + B(x) = x^3 + 2x^2 + 3x + 4$$

$$A(x) = 4x^{2} + 3x^{2} + 2x + 4$$

$$B(x) = x^{3} + 2x^{2} + 3x + 4$$

$$D_{1}(x) = 4x + 3$$

$$B(x) = x^3 + 2x^2 + 3x + 4$$

$$B(x) = x^3 + 2x^2 + 3x + 4$$
$$D_1(x) = 4x + 3$$

$$D_1(x)=4x+3$$

$$D_0$$

$$D_0(x)=2x+1$$

$$B(x) = x^3 + 2x^2 + 3x + 4$$

$$D_1(x) = 4x + 3$$

$$D_1(x) = 4x + 3$$
 $D_0(x) = 2x + 1$

$$D_1(x) = 4x + 3$$

$$E_1(x) = x + 2$$

$$B(x) = x^3 + 2x^2 + 3x + 4$$

$$D_1(x) = 4x + 3$$

$$D_1(x) = 4x + 3$$

$$E_1(x) = x + 2$$

$$D_1(x) = 4x + 3$$

$$E_1(x) = x + 2$$

$$D_0(x) = 2x + 1 E_0(x) = 3x + 4$$

$$E_0(x) = 3x + 4$$

$$B(x) = x^3 + 2x^2 + 3x + 4$$

$$D_1(x) = 4x + 3$$

$$E_1(x) = x + 2$$

 $D_0(x) = 2x + 1$

 $E_0(x) = 3x + 4$

$$E_1(x) = 4x + 3$$

$$E_1(x) = x + 2$$

$$D_1E_1 = 4x^2 + 11x + 6$$

$$B(x) = x^3 + 2x^2 + 3x + 4$$

$$D_1(x) = 4x + 3$$

 $E_1(x) = x + 2$

$$D_1(x) = 4x + 3$$

 $E_1(x) = x + 2$
 $D_1E_1 = 4x^2 + 11x + 6$

$$E_0(x) = 3x + 4$$

 $D_0 E_0 = 6x^2 + 3$

$$5x^2$$

$$D_0 E_0 = 6x^2 + 11x + 4$$

$$B(x) = x^3 + 2x^2 + 3x + 4$$

$$D_1(x) = 4x + 3$$

$$E_1(x) = x + 2$$

$$E_1(x) = x + 2$$

 $D_1E_1 = 4x^2 + 11x + 6$

 $(D_1 + D_0)(E_1 + E_0) =$

$$1x + 6$$

$$E_0(x) = 3x + 4$$
$$D_0 E_0 = 6x^2 + 11x + 4$$

$$6x^2$$

$$x^2 + 11x$$

$$B(x) = x^3 + 2x^2 + 3x + 4$$

$$D_1(x) = 4x + 3$$

$$D_1(x) = 4x + 3$$

$$E_1(x) = x + 2$$

$$E_1(x) = \frac{4x + 3}{1}$$

 $(D_1 + D_0)(E_1 + E_0) = (6x + 4)$

$$E_1(x) = x + 2$$

 $D_1E_1 = 4x^2 + 11x + 6$

$$E_0(x) = 3x + 4$$
$$D_0 E_0 = 6x^2 + 11x + 4$$

$$= 6x^2$$

$$x^2 + 11x -$$

$$B(x) = x^3 + 2x^2 + 3x + 4$$

$$D_1(x) = 4x + 3$$

$$D_1(x) = 4x + 3$$

$$E_1(x) = x + 2$$

$$D_1(x) = 4x + 3$$

$$E_1(x) = x + 2$$

 $D_1E_1 = 4x^2 + 11x + 6$

$$D_0(x) = 2x + 1$$

$$E_0(x) = 3x + 4$$

$$E_0(x) = 3x + 4$$
$$D_0 E_0 = 6x^2 + 1$$

$$=6x^{2}$$

$$D_0 E_0 = 6x^2 + 11x + 4$$

$$D_1E_1 = 4x^2 + 11x + 6$$
 $D_0E_0 = 6x^2$
 $(D_1 + D_0)(E_1 + E_0) = (6x + 4)(4x + 6)$

$$B(x) = x^3 + 2x^2 + 3x + 4$$

$$D_1(x) = 4x + 3$$

$$D_1(x) = 4x + 3$$

$$E_1(x) = x + 2$$

$$E_1(x) = x + 2$$

 $D_1 E_1 = 4x^2 + 11x + 6$

$$E_0(x) = 3x + 4$$

$$D_1E_1 = 4x^2 + 11x + 6$$
 $D_0E_0 = 6x^2 + 11x + 4$
 $(D_1 + D_0)(E_1 + E_0) = (6x + 4)(4x + 6)$

 $= 24x^2 + 52x + 24$

$$x_0 = 6x^2 + 6$$

$$x^2 + 11x + 11$$

$$x^2 + 11x -$$

$$B(x) = x^3 + 2x^2 + 3x + 4$$

$$D_1(x) = 4x + 3$$

$$E_1(x) = x + 2$$

$$E_1(x) = x + 2$$

$$E_2(x) = x + 2$$

 $AB = (4x^2 + 11x + 6)x^4 +$

$$E_1(x) = x + 2$$

 $D_1 E_1 = 4x^2 + 11x + 6$

$$11x + 6$$

 $= 24x^2 + 52x + 24$

$$D_1E_1 = 4x^2 + 11x + 6$$
 $D_0E_0 = 6x^2 + 11x + 4$
 $(D_1 + D_0)(E_1 + E_0) = (6x + 4)(4x + 6)$

 $D_0(x) = 2x + 1$

 $E_0(x) = 3x + 4$

$$B(x) = x^3 + 2x^2 + 3x + 4$$

$$D_1(x) = 4x + 3$$

$$E_1(x) = x + 2$$

$$E_1(x) = x + 2$$

 $D_1E_1 = 4x^2 + 11x + 6$

 $AB = (4x^2 + 11x + 6)x^4 +$

$$11x + 6$$

$$E_0(x) = 3x + 4$$
$$D_0 E_0 = 6x^2 + 11x + 4$$

$$(D_1 + D_0)(E_1 + E_0) = (6x + 4)(4x + 6)$$

 $D_0(x) = 2x + 1$

$$= (6x + 4)(4x + 6)$$
$$= 24x^2 + 52x + 24$$

 $)x^{2} +$

$$B(x) = x^3 + 2x^2 + 3x + 4$$

$$D_1(x) = 4x + 3$$

$$E_1(x) = x + 2$$

$$E_1(x) = 4x + 3$$

$$E_1(x) = x + 2$$

$$D_1 E_1 = 4x^2 + 11x + 6$$

 $AB = (4x^2 + 11x + 6)x^4 +$

 $(24x^2 + 52x + 24)$

$$D_1E_1 = 4x^2 + 11x + 6$$
 $D_0E_0 = 6x^2 + 11x + 4$
 $(D_1 + D_0)(E_1 + E_0) = (6x + 4)(4x + 6)$

 $D_0(x) = 2x + 1$

 $E_0(x) = 3x + 4$

$$= (6x + 4)(4x + 6)$$
$$= 24x^2 + 52x + 24$$

 $)x^{2} +$

$$B(x) = x^3 + 2x^2 + 3x + 4$$

$$D_1(x) = 4x + 3$$

 $E_1(x) = x + 2$

 $AB = (4x^2 + 11x + 6)x^4 +$

$$E_1(x) = x + 2$$

$$D_1 E_1 = 4x^2 + 11x + 6$$

$$(D_1 + D_0)(E_1 + E_0) = (6x + 4)(4x + 6)$$

$$E_0$$
) = (6)

 $(24x^2 + 52x + 24 - (4x^2 + 11x + 6))$

$$= (6x + 4)(4x + 6)$$
$$= 24x^2 + 52x + 24$$

 $D_0(x) = 2x + 1$

 $E_0(x) = 3x + 4$

 $D_0 E_0 = 6x^2 + 11x + 4$

 $)x^{2} +$

$$B(x) = x^3 + 2x^2 + 3x + 4$$

$$D_1(x) = 4x + 3$$

 $AB = (4x^2 + 11x + 6)x^4 +$

$$E_1(x) = x + 2$$

$$E_2(x) = x + 2$$

$$E_1(x) = x + 2$$

$$D_1 E_1 = 4x^2 + 11x + 6$$

$$11x + 6$$

 $(24x^2 + 52x + 24 - (4x^2 + 11x + 6))$

$$(D_1 + D_0)(E_1 + E_0) = (6x + 4)(4x + 6)$$

$$= (0x + 4)(4x + 0)$$
$$= 24x^2 + 52x + 24$$

 $-(6x^2+11x+4))x^2+$

 $D_0(x) = 2x + 1$

 $E_0(x) = 3x + 4$

 $D_0 E_0 = 6x^2 + 11x + 4$

$$B(x) = x^3 + 2x^2 + 3x + 4$$

$$D_1(x) = 4x + 3$$

$$D_1(x) = 4x + 3$$

$$E_1(x) = x + 2$$

$$E_1(x) = 4x + 3$$
$$E_1(x) = x + 2$$

 $AB = (4x^2 + 11x + 6)x^4 +$

 $6x^2 + 11x + 4$

$$E_1(x) = x + 2$$

 $D_1E_1 = 4x^2 + 11x + 6$

$$+11x + 6 + E_0) = (6x + 6)$$

$$(D_1 + D_0)(E_1 + E_0) = (6x + 4)(4x + 6)$$
$$= 24x^2 + 52x + 24$$

 $(24x^2 + 52x + 24 - (4x^2 + 11x + 6))$

$$(5x + 4)(4x - 4)$$

$$2x + 24$$

 $-(6x^2+11x+4))x^2+$

 $D_0(x) = 2x + 1$

 $E_0(x) = 3x + 4$

 $D_0 E_0 = 6x^2 + 11x + 4$

$$B(x) = x^3 + 2x^2 + 3x + 4$$

$$D_1(x) = 4x + 3$$

$$D_1(x) = 4x + 3$$

$$E_1(x) = x + 2$$

$$E_1(x) = x + 2$$

$$D_1 E_1 = 4x^2 + 11x + 6$$

 $6x^2 + 11x + 4$

$$(D_1 + D_0)(E_1 + E_0) = (6x + 4)(4x + 6)$$

$$(6x + E_0) = (6x + E_0)$$

= $24x^2 + E_0$

$$= 24x^2 +$$

$$= 24x^2 + 52x + 24$$

$$AB = (4x^2 + 11x + 6)x^4 +$$

 $=4x^{6} + 11x^{5} + 20x^{4} + 30x^{3} + 20x^{2} + 11x + 4$

 $D_0 E_0 = 6x^2 + 11x + 4$

 $D_0(x) = 2x + 1$

 $E_0(x) = 3x + 4$

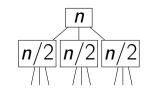
 $-(6x^2+11x+4))x^2+$

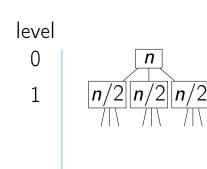
$$(4x^2 + 11x + 6)x^4 +$$

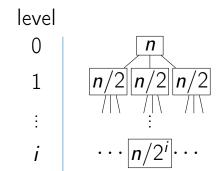
 $(24x^2 + 52x + 24 - (4x^2 + 11x + 6)$

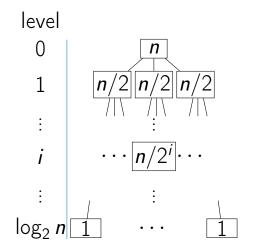


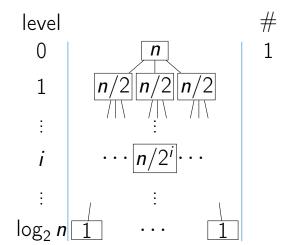
level

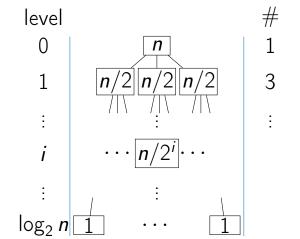


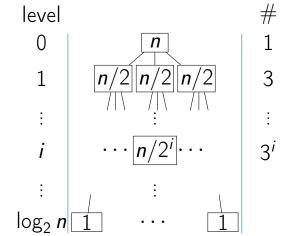


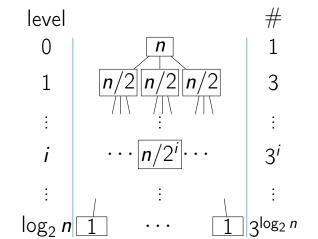


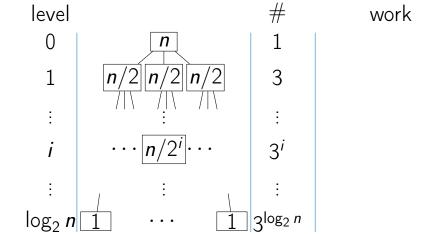


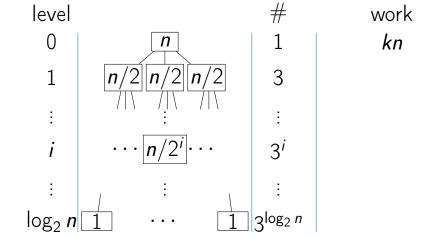












0
1
$$n/2$$
 $n/2$ $n/2$
 3
 $3k\frac{n}{2} = k\frac{3}{2}n$
 $3i$
 $3ik\frac{n}{2} = k(\frac{3}{2})^{i}n$

level

 $\log_2 n$ 1

#

work

 $3^{\log_2 n} k 3^{\log_2 n} = k n^{\log_2 3}$

 $=\Theta(n^{1.58})$