SCMA329 Practical Mathematical Financial Modeling

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Chapter 1

Cashflows, Interest and the Time Value of Money

1.1 Introduction to Financial Modelling

A financial model is a financial representation of a real world financial situation, which is either a mathematical or statistical model that describes the relationship among the variables of the financial problem. Here are some types of financial models.

- **Financial statement model:** The model includes three main components including income statement, cash flow statement and balance sheet. These are accounting reports issued by a company quarterly and annually that are used for decision making and performing financial analysis. (see https://corporatefinanceinstitute.com/resources/knowledge/accounting/three-financial-statements/)

- **Project finance models:** The model incorporates two main elements of the project including loans and debt repayment. It can be used to assess the risk-reward of lending to or investing in a long-term project, i.e. it can be used to tell whether the project has enough cash to cover the debt in the long term. (see https://www.wallstreetprep.com/knowledge/

project-finance-model-structure/)

- **Discounted cashflow model:** It is the model to value a company using the net present value of the business's future cashflows, or to estimate the value of an investment based on its future cash flows. (see https://corporatefinanceinstitute.com/resources/templates/excel-modeling/dcf-model-template/)

- **Pricing models:** This models the way prices are set within a market in order to maximise profits.

This chapter covers the basic concepts of calculating interest, including simple and compound interest, the frequency of compounding, the effective interest rate and the discount rate, and the present and future values of a single payment.

1.2 Cashflows

Cashflows are amounts of money which are received (or income, positive cashflows) or paid (or outgo, negative cashflows) at particular times. Those payments arise from a financial transaction, e.g

- a bank account,
- a loan,
- an equity,
- a zero-coupon bond: A bond is a fixed income instrument that represents a loan from an investor to a debtor either a government or a corporation. A zero-coupon bond is a bond that pays no interest during its life.
- a fixed interest security: A fixed-income security is a debt instrument such as a bond or debenture that investors use to lend money to a company in exchange for interest payments.
- an index-linked security: An index-linked bonds pay interest that is tied

to an underlying index, such as the consumer price index (CPI). Index-linked bonds are issued by governments to mitigate the effects of inflation by paying a real return plus accrued inflation.

- an annuity: An annuity is a series of payments made at regular intervals, such as equal monthly payments on a mortgage.

- a capital project etc.

Cash received represents inflows, income or also called **positive cashflows**, while money spent represents outflows, outgo or **negative cashflows**. The net cashflow at a given point in time is the difference between expenses and income.

Example 1.1. *A series of payments into and out of a bank account is given as follows:*

- *payments into the account : ฿1000 on 1 January 2014 and ฿100 on 1 January 2016*
- *payments out of the account : ฿200 on 1 July 2015, ฿300 on 1 July 2016, and ฿400 on 1 January 2018*

In practice, cashflows can be represented by a timeline as can be illustrated in this example.

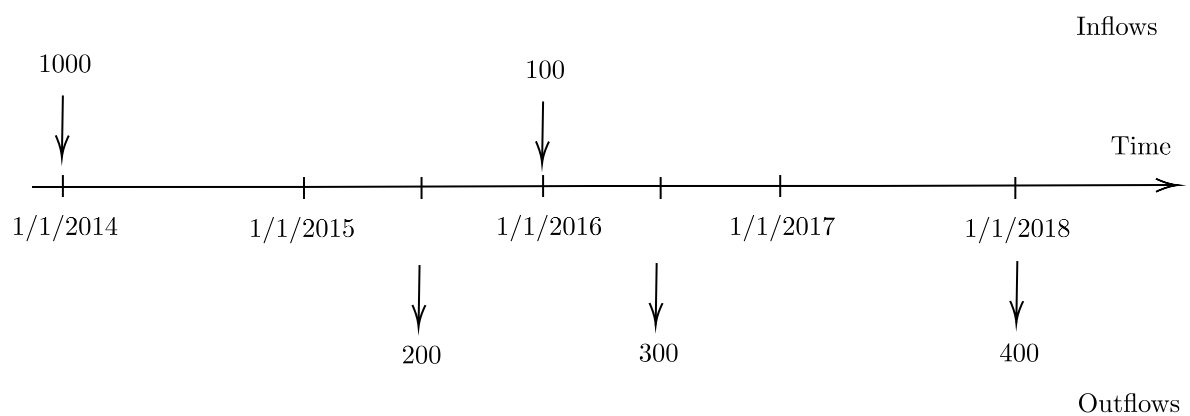


Figure 1.1: an example of a timeline

1.3 Interest and the Time Value of Money

This section introduces the time value of money using the concepts of compound interest and discounting. The effect of interest rates on the present value of future cash flows is discussed. The value of distant cash flows in the present and current cash flows in the future are then considered.

We illustrate the time value of money by considering the following examples.

Example 1.2. *An investor want to make a payment of ฿10000 in 2 years. Suppose that a bank pays compound interest at 4% per annum effective. How much should the initial investment?*

Note The amount we need to invest now (i.e. the initial investment in this example) is called the *present value (PV)* or *discounted value* of the payments.

Solution: The interest for year 1 is

$$X \cdot 0.04.$$

For year 2 the principal is

$$X + X \cdot 0.04 = X \cdot (1 + 0.04)$$

so that the interest for the year is

$$X \cdot (1 + 0.04) \cdot 0.04.$$

By the end of 2 years an initial payment of ฿X will have accumulated to:

$$X \cdot (1 + 0.04) + X \cdot (1 + 0.04) \cdot 0.04 = X \cdot 1.04^2 = 10000.$$

Hence,

$$X = \frac{10000}{1.04^2} = 9245.56213,$$

Note We refer to the amount to which the capital accumulates with the addition of interest as *accumulation* or *accumulated value*.

Example 1.3. *Consider the following arguments*

- *It is obvious that you would prefer to have ฿1100 now than ฿1000 now.*
- *If we receive and hold ฿1 now, then it is worth more than receiving and holding ฿1 at some time in the future? Why is this?*
- *Is it obvious that your would be better off with ฿1100 in 2 years than ฿1000 now?*

Solution:

For the second argument, this one baht will grow to $1 + r$ in the first year, $(1 + r)^2$ in two years, and so on. These amounts are clearly worth more than receiving and holding ฿1 at the same time in the future.

For the last argument, we need to compare the values of the amounts **received at different times. To do this, we can look at the today's values of ฿1100 received in 2 years assuming that we can invest at an annual interest rate of r percent.

The present value of this amount X in year 2 is

$$1100/(1+r)^2.$$

Assuming $r = 5\%$, the present value of X is 997.7324263.

Comparing the values in today's baths, it is better to have ฿1000 now than to have ฿1100 in 2 years.

Notes From the above example,

1. One can deposit or invest ฿1 now and will receive ฿1 back and a reward called *interest* at some point in the future. Because of its potential earning power, money in the present is worth more than an equal amount in the future. This is a fundamental financial principle known as **the time value of money**.

2. At a given point of time, cash has a monetary value, but also has a *time value.*

3. The amount deposited or invested is called *capital* or *principal.*

1.3.1 Simple interest

Simple interest is a calculation of interest that does not take into account the effect of compounding. Under simple interest, the amount of interest that accrues over time is proportional to the length of the period.

Suppose an amount C is deposited in an account that pays simple interest at the rate of $i\%$ per annum. Then after n years the deposit will have accumulated to

$$C(1 + i \cdot n).$$

Hence, the interest accrued over n years is

$$\text{Simple Interest} = C \cdot i \cdot n.$$

Note Auto loans and short-term personal loans are usually simple interest loans.

Example 1.4. *An investor deposits ฿10000 in a bank account that pays simple interest at a rate of 5% per annum. Calculate*

1. *interest he will earn after the first two years.*
2. *interest he will earn after the first three months.*

Note When n is not an integer, interest is paid on a pro-rate basis (in proportion).

Solution:

1. At the end of 2 years the interest earned is
$$10000 \cdot 0.05 \cdot 2 = 1000.$$
2. At the end of 3 months the interest earned is
$$10000 \cdot 0.05 \cdot \frac{3}{12} = 125.$$
Alternatively, the interest per month is $5\%/12 = 0.4167\%$ and hence the

interest earned can be calculated as

$$10000 \cdot 0.004167 \cdot 3 = 125.$$

1.3.2 Compound interest

In compound interest, the accumulated amount over a period of time is the capital of the following period. Therefore, a capital of 1 unit at the end of the year increases to $1 + i$ units, which becomes the capital for the following year.

For year 2, the principal is $1 + i$ and the interest for the year is $(1 + i) \cdot i$. By the end of 2 years, an initial payment of 1 will have accumuulated to

$$(1 + i) + (1 + i) \cdot i = (1 + i)^2.$$

As this progression continues, the accumulated amount of X units at the end of year n becomes

$$X \cdot (1 + i)^n.$$

Note In this case, we can take money out and reinvest it as new capital illustrated in the timeline.

Exercise 1.1. (Excel) Use Excel to create a table showing the accumulated amounts at the end of each year for 15 years for a principal of ฿100 under the

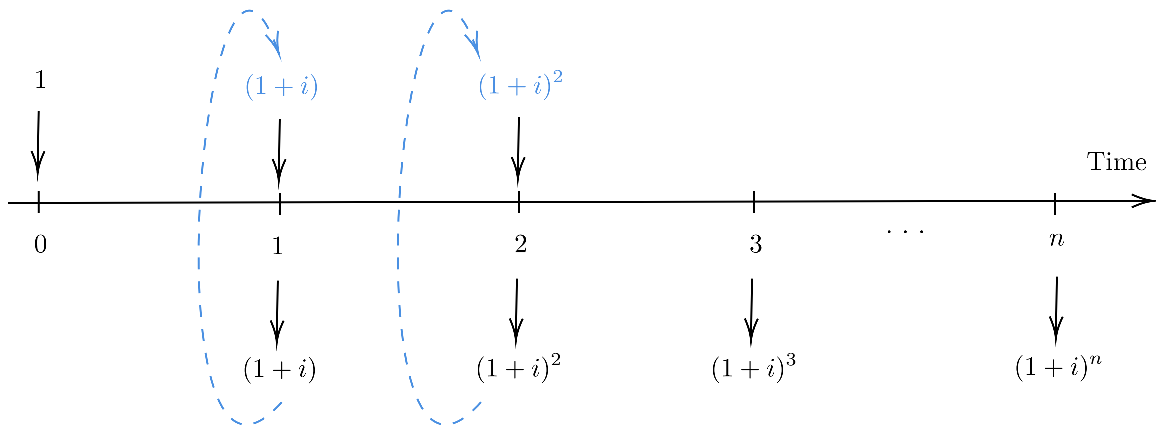


Figure 1.2: a timeline of compounding interest

simple interest approach and the compound interest approach with $r = 6\%$ for both cases. Discuss the results obtained (How long does it take to double the investment? How much will the principal grow over a 15-year period?)

The effect of compounding is to increase the total amount of accumulation. The effect is greater when the interest rate is high. This example shows two examples of the accumulated amount of ฿100 under the simple interest approach and the compound interest approach. As can be seen, the compound interest method makes the principle increase much faster than the simple interest method when the interest rate is high.

1.4 Frequency of Compounding

Even though the interest rate is typically expressed in annual terms, an investment's interest is frequently paid more frequently than once per year. For example, a savings account may offer an interest rate of 4% per year, credited quarterly. This interest rate is usually referred to as **nominal rate of interest**, i.e., 4% due four times per year.

We will see that the frequency of interest payments, also known as the frequency of compounding, has a significant impact on the total amount accrued and the interest collected. Consequently, it is crucial to precisely specify the rate of interest.

We use $i^{(m)}$ to represent the nominal rate of interest payable m times a year in order to underline the significance of the frequency of compounding. Therefore, m is the frequency of compounding per year and $1/m$ year is the **compounding period** or **conversion period**.

Note The nominal rate of interest payable m times per period is also known as the rate of interest convertible $m$thly or compounded $m$thly.

Example 1.5. Calculate the accumulated value in 1 year of a deposit of ฿100 in a saving account that earns interest at 10% payable quarterly.

Solution: In this example, the nominal rate of interest of $i^{(4)} = 10\%$ p.a. convertible quarterly means an interest rate of $10\%/4 = 2.5\%$ per quarter. In this case, the interest rate of 2.5% is called *effective interest*. The effective interest rate of i per unit of time (which may be month, quarter, etc.) is the amount of interest received at the end of a unit of time per ฿1 invested at the beginning of that unit.

Therefore, the nominal interest rate $i^{(4)} = 10\%$ is equivalent to an *effective interest rate* of 2.5% per quarter. The accumulated value in 1 year is $100(1 + \frac{10\%}{4})^4 = 100(1 + 2.5\%)^4 = 110.3813$.

Note that after compound interest is taken into account, the interest income of an investor at the quarterly convertible nominal interest rate of 10% p.a. is 10.3813 (or 10.3813%.p.a. effective)

Example 1.6. *At a rate of 12% p.a. effective, draw a timeline to show cashflows if ฿100 is invested at the start of the year.*

Solution: The accumulated value of ฿100 at the end of the year is $100(1 + 12\%) = 112$.

Example 1.7. *At a rate of 12% p.a. compounding quarterly, draw a time line to show cashflows if ฿100 is invested at the start of the year.*

Solution: The nominal interest rate $i^{(4)} = 12\%$ is equivalent to an effective interest rate of 3% per quarter. The accumulated value in 1 year is $100(1 + 3\%)^4 = 112.55$. After compound interest is taken into account, the interest income of an investor at the quarterly convertible nominal interest rate of 10% p.a. is 12.55 (or 12.55%. p.a. effective)

Note $i^{(m)}$ is a nominal rate of interest which is equivalent to $i^{(m)}/m$ applied for each mth of a period. The interest is paid m times per measurement period.

The value at time n can be considered as the **annuity** with a cashflow of $i^{(m)}/m$ per period for n years together with the capital at time n as shown in the following figure. Therefore, the accumulated value in 1 year can also be calculated as $100(1 + 0.03 s_{\overline{4|}}^{3\%})$. The concept of annuity will be discussed in the subsequent section.

In general, we have

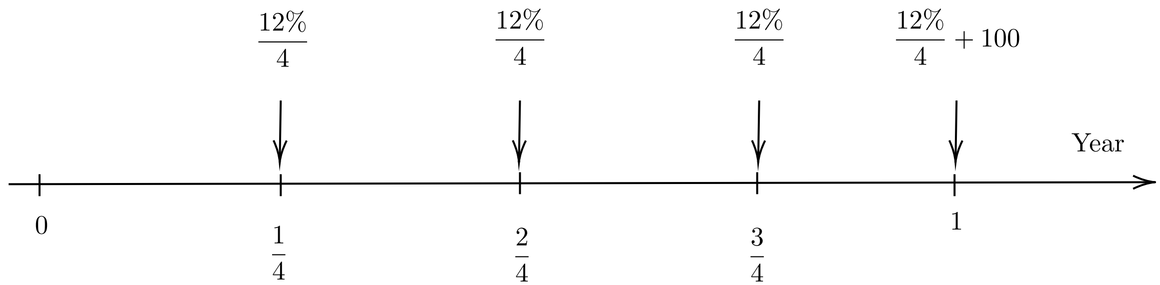


Figure 1.3: Frequency of Compouding vs Annuity

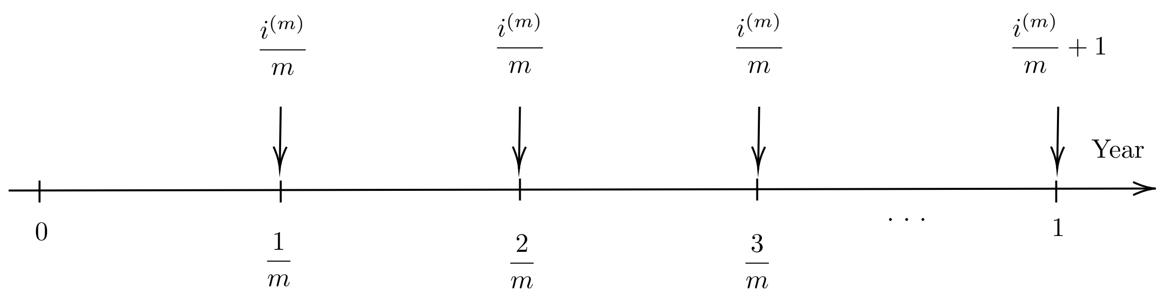


Figure 1.4: Frequency of Compouding vs Annuity

1.4.1 Effective rate of interest

The compounding frequency affects the accumulated amount. As a result, it may be inaccurate to compare two investment strategies only based on their nominal rates of return without also taking into account their frequency of compounding. It is necessary to compare different investment strategies on an equal basis. The measure known as the **effective interest rate** is often used for this purpose.

The effective rate of interest of i per time unit is the amount of interest received at the end of one time unit per ฿1 invested at the start of that time unit.

Example 1.8. *An investor invests ฿1 at 7.5% p.a. (per annum) effective. Then* $i = 0.075$. Calculate the value of investment after one year.

Solution: The value of investment after one year at this rate is

$$1 \times (1 + 0.075) = 1.075.$$

In particular, the amount of interest received at the end of the year per ฿1 invested is 0.075.

Example 1.9. *An investor invests ฿1000 at 5.25% per half-year effective. Then* $i = 0.0525$. Calculate the value of investment after half a year.

Solution: The value of investment after half year at this rate is

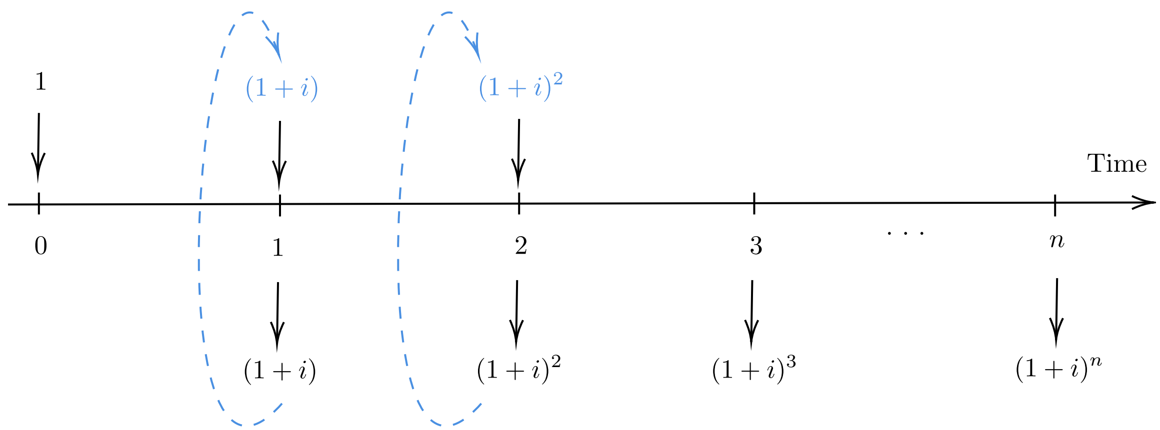
$$1000 \times (1 + 0.0525) = 1052.5.$$

Again, the amount ofinterest received at the end of the quarter of ฿1 invested is 0.0525.

Note The time unit is an **essential part of the definition**.

Example 1.10. *An investor invests ฿1 at effective rate* $i\%$ per time unit for n time units. Calculate the value of investment after two, three, ..., n time units.

Note Here we assume that we can take money out and reinvest it as new capital (see the timeline).



Example 1.11. *An investor invests ฿200 at* 3% pa effective. What will be the deposit have accumulated to after 5 years.

Solution: The deposit accumulates to $200 \cdot (1.03)^5 = 231.854815$ after 5 years.

Example 1.12. Consider the following problems.

1. *An investor invests ฿500 at* 2.75% per quarter effective. What will be the deposit have accumulated to after 9 months.

2. *An investor invests ฿2000 at* 6% per half-year effective. What will be the deposit have accumulated to after 2 years.

Solution:

1. Accumulating the 500 for 9 months at this rate gives

$$500 \cdot (1.0275)^3 = 542.394773.$$

2. After 2 years the accumulation is

$$2000 \cdot (1.06)^4 = 2524.95392.$$

Notes

1. The model under the effective rate of interest condition is a model of *compound interest*, where interest is earned on interest previously earned.

Unless state otherwise, we shall assume that i is the compound interest rate.

2. In practice, it is easier to work with the effective rate of interest which is defined in a suitable time unit.

The following formula can be used to convert between the effective rate i p.a. and the nominal rate $i^{(m)}$ p.a.:

$$(1+i) = \left(1 + \frac{i^{(m)}}{m}\right)^m .$$

Example 1.13. Consider the following problems.

1. *Express a nominal annual interest rate of 9% convertible half-yearly as a monthly effective interest.*
2. *Express a two-monthly effective interest of 3% as a nominal annual interest rate convertible two-monthly.*

Solution:

1. The effective rate $i\%$ p.a. is
$$i = (1 + \frac{0.09}{2})^2 - 1.$$
Hence the monthly effective rate is $j = (1+i)^{1/12} - 1 = (1+\frac{0.09}{2})^{2/12} - 1 = 0.007363$.
2. A nominal annual interest rate convertible two-monthly is $6 \cdot 3\% = 18\%$.

Example 1.14. *Express each of the following effective rates per annum as a nominal rate, and vice versa.*

Effective Rate	***Nominal Rate***
$i = 0.04$	$i^{(4)} = 0.039412$
$i = 0.10$	$i^{(12)} = 0.095690$
$i = 0.06152$	$i^{(2)} = 0.06$
$i = 0.126825$	$i^{(12)} = 0.12$

1.4.2 Compounding over any number of time units

Suppose an amount $\mathbb{B}1$ is invested at the rate of $i\%$ per time unit. At time t the accumulation is $(1+i)^t$.

Example 1.15. 1. *An investor invests ฿4000 at* 8.5% per quarter effective. What will be the deposit have accumulated to after 1 month.

2. *An investor invests ฿800 at* 6% per half-year effective. What will be the deposit have accumulated to after 2.6 years.

Solution:

1. The accumulation after 1 month is $4000 \cdot 1.085^{1/3} = 4110.265768$.

2. The accumulation after 2.6 years is $800 \cdot 1.06^{5.2} = 1083.129754$.

Exercise 1.2. (Excel) Use Excel to create a table showing the accumulated amounts after 1 year under several different compounding frequencies (yearly, quarterly, monthly, daily) for a principal of ฿100 under with nominal rate of $r = 4\%$ per annum.

Discuss the results obtained. What happens if the compounding is made over infinitely small intervals (i.e. as $m \to \infty$)?

1.4.3 Changing the time period of the effective rates of interest

It is often very useful to change the effective rate of interest per time period to another. For example, if the effective rate of interest is defined per annum but cashflows occur monthly.

Let i be the effective rate of interest per t_i years (which can be any positive number, for e.g. $t_i = 1/2$). Here t_i years can be regarded as one time unit. Let j be the effective rate of interest per t_j years.

Example 1.16. *Find the condition under which the two effective rates of interest* i and j are equivalent.

Solution: Suppose we invest 1 for one year. Then at the end of the year under each rate of interest, we will have

$$(1+i)^{1/t_i} \text{ and } (1+j)^{1/t_j}.$$

Two rates of interest are equivalent if the given amount of principal invested for the same length of time produces the same accumulated value, i.e.

$$(1+i)^{1/t_i} = (1+j)^{1/t_j}.$$

Solving the equation for j yields

$$j = (1+i)^{t_j/t_i} - 1.$$

Example 1.17. 1. *If the effective rate of interest is 6% per annum, what is the effective rate of interest per half-year?*

2. *If the effective rate of interest is 12% per two-years effective, what is the effective rate of interest per quarter-year?*

3. *If the effective rate of interest is 2% per month effective, what is the effective rate of interest per 1.5-years?*

Solution:

1. $i = 6\%$ p.a. Then

$$j = (1.06)^{1/2} - 1 = 0.029563 \text{ per half-year.}$$

2. $i = 12\%$ per two-years. Then

$$j = (1.12)^{1/(2\times 4)} - 1 = 0.0142669 \text{ per quarter-year.}$$

3. $i = 2\%$ per month. Then

$$j = (1.02)^{1.5/(1/12)} - 1 = 0.428246 \text{ per 1.5-years.}$$

1.4.4 Non-constant interest rates

The effective rate may not be the same during every time period. We shall assume that the rates in every future time periods are known in advance.

Example 1.18. *The effective rate of interest per annum was 4% during 2015, 4.5% during 2016 and 5% during 2017. Calculate the accumulation of ฿200 invested on*

1. *01/01/2015 for 3 years*
2. *01/07/2015 for 2 years*
3. *01/04/2016 for 1.5 years*

Solution:

1. Accumulating the ฿200 for the first year at the rate of 4% p.a. gives

$$200 \cdot 1.04.$$

The accumulated value was then invested at the rate of 4.5% p.a. for another year, and its value at after 2 years was

$$200 \cdot 1.04 \cdot 1.045.$$

At the rate of 5% in the final year, the value after 3 years was

$$200 \cdot 1.04 \cdot 1.045 \cdot 1.05 = 228.228.$$

2. The accumulation is

$$200 \cdot 1.04^{1/2} \cdot 1.045 \cdot 1.05^{1/2} = 218.4025.$$

3. The accumulation is

$$200 \cdot 1.045^{9/12} \cdot 1.05^{3/4} = 214.416986.$$

1.4.5 Accumulation factors

Let i be the effective rate of interest per one time unit and $s < t$. We define

- the accumulation factor per one time unit

$$A(0,1) = (1+i).$$

- the accumulation factor per t time units

$$A(0,t) = (1+i)^t.$$

- the accumulation factor at time t of 1 unit invested at time s

$$A(s,t).$$

Example 1.19. *The effective rate of interest per annum was 6% during 2015, 8% during 2016 and 10% during 2017. Calculate the following accumulation factors.*

1. $A(01/01/15, 01/01/18)$, i.e. the accumulation at 01/01/18 of an investent of 1 at 01/01/15
2. $A(01/07/15, 01/07/17)$
3. $A(01/04/16, 01/10/17)$

Solution:

1. $A(01/01/15, 01/01/18) = (1.06)(1.08)(1.1) = 1.25928$
2. $A(01/07/15, 01/07/17) = (1.06)^{1/2}(1.08)(1.1)^{1/2} = 1.166200$
3. $A(01/04/16, 01/10/17) = (1.08)^{3/4}(1.1)^{3/4} = 1.137922$

1.4.6 Present values and discount factors

Recall from Example 1.2 that the amount $\dfrac{10000}{1.04^2}$ we need to invest now to obtain ฿10000 in two years is called the *present value (PV)* or *discounted value* of the payments.

We define the discount factor v per annum, at rate i p.a. effective to be the present value of a payment of 1 due in 1 year?s time, i.e.

$$v = \frac{1}{1+i}.$$

Example 1.20. *Calculate the present of ฿25000 due in 3 years at an effective rate of interest of 6% per annum.*

Solution: The present value is

$$25000 \cdot \frac{1}{1.06^3} = 20990.482076.$$

It is the discounted value of 25000 due in 3 years.

Example 1.21. *How much should we invest now to meet a liability of ฿50000 in 5 years at an effective rate of interest of 3% per half-year.*

Solution: The amount we need to invest now to meet the future liability of 50000 in 5 years is the present value

$$50000 \cdot \frac{1}{1.03^{10}} = 37204.695745.$$

Note It follows that the PV of ฿1 in t time units at i effective rate of interest per time unit is

$$PV = \frac{1}{(1+i)^t} = v^t.$$

Example 1.22. *Given the discount factor per year $v = 0.9$*, calculate

1. *the effective rate of interest per year.*
2. *the equivalent discount factor per half-year.*

Solution:

1. From $v = \dfrac{1}{1+i} = 0.9$, solving the equation for i gives
$$i = \frac{1}{v} - 1 = 0.111111 \text{ per year.}$$

2. Let j be the effective rate of interest per half-year. Then
$$j = (1+i)^{1/2} - 1 = 0.054093.$$
Then, the discount factor per half-year is
$$v = \frac{1}{1+j} = \frac{1}{1.054093} = 0.948683.$$

Similarly, we define

- the discount factor per one time unit

$$V(0,1) = 1/(1+i).$$

- the discount factor per t time units

$$V(0,t) = 1/(1+i)^t.$$

- for $s < t$, the discount factor at time s of 1 unit receivable at time t

$$V(s,t) = (1+i)^{s-t}.$$

Notes

1. $V(s,t) = A(s,t)^{-1}$
2. For $r < s < t$, the following holds:
   - $A(r,t) = A(r,s)A(s,t)$
   - $V(r,t) = V(r,s)V(s,t)$