SCMA329 Practical Mathematical Financial Modeling

Pairote Satiracoo

2022-08-14

Chapter 1

Cashflows, Interest and the Time Value of Money

1.1 Introduction to Financial Modelling

A financial model is a financial representation of a real world financial situation, which is either a mathematical or statistical model that describes the relationship among the variables of the financial problem. Here are some types of financial models.

- **Financial statement model:** The model includes three main components including income statement, cash flow statement and balance sheet. These are accounting reports issued by a company quarterly and annually that are used for decision making and performing financial analysis. (see https://corporatefinanceinstitute.com/resources/knowledge/accounting/three-financial-statements/)
- **Project finance models:** The model incorporates two main elements of the project including loans and debt repayment. It can be used to assess the risk-reward of lending to or investing in a long-term project, i.e. it can be used to tell whether the project has enough cash to cover the debt in the long term. (see https://www.wallstreetprep.com/knowledge/

project-finance-model-structure/)

- **Discounted cashflow model:** It is the model to value a company using the net present value of the business's future cashflows, or to estimate the value of an investment based on its future cash flows. (see https://corporatefinanceinstitute.com/resources/templates/excel-modeling/dcf-model-template/)

- **Pricing models:** This models the way prices are set within a market in order to maximise profits.

This chapter covers the basic concepts of calculating interest, including simple and compound interest, the frequency of compounding, the effective interest rate and the discount rate, and the present and future values of a single payment.

1.2 Cashflows

Cashflows are amounts of money which are received (or income, positive cashflows) or paid (or outgo, negative cashflows) at particular times. Those payments arise from a financial transaction, e.g

- a bank account,
- a loan,
- an equity,
- a zero-coupon bond: A bond is a fixed income instrument that represents a loan from an investor to a debtor either a government or a corporation. A zero-coupon bond is a bond that pays no interest during its life.
- a fixed interest security: A fixed-income security is a debt instrument such as a bond or debenture that investors use to lend money to a company in exchange for interest payments.
- an index-linked security: An index-linked bonds pay interest that is tied

to an underlying index, such as the consumer price index (CPI). Index-linked bonds are issued by governments to mitigate the effects of inflation by paying a real return plus accrued inflation.

- an annuity: An annuity is a series of payments made at regular intervals, such as equal monthly payments on a mortgage.
- a capital project etc.

Cash received represents inflows, income or also called **positive cashflows**, while money spent represents outflows, outgo or **negative cashflows**. The net cashflow at a given point in time is the difference between expenses and income.

Example 1.1. *A series of payments into and out of a bank account is given as follows:*

- *payments into the account : ฿1000 on 1 January 2014 and ฿100 on 1 January 2016*
- *payments out of the account : ฿200 on 1 July 2015, ฿300 on 1 July 2016, and ฿400 on 1 January 2018*

In practice, cashflows can be represented by a timeline as can be illustrated in this example.

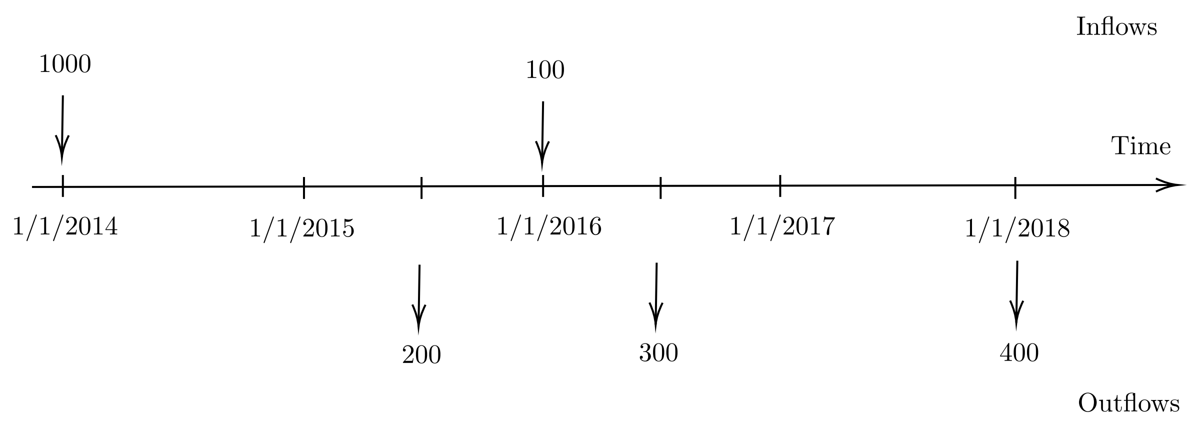


Figure 1.1: an example of a timeline

1.3 Interest and the Time Value of Money

This section introduces the time value of money using the concepts of compound interest and discounting. The effect of interest rates on the present value of future cash flows is discussed. The value of distant cash flows in the present and current cash flows in the future are then considered.

We illustrate the time value of money by considering the following examples.

Example 1.2. *An investor want to make a payment of ฿10000 in 2 years. Suppose that a bank pays compound interest at 4% per annum effective. How much should the initial investment?*

Note The amount we need to invest now (i.e. the initial investment in this example) is called the *present value (PV)* or *discounted value* of the payments.

Solution: The interest for year 1 is

$$X \cdot 0.04.$$

For year 2 the principal is

$$X + X \cdot 0.04 = X \cdot (1 + 0.04)$$

so that the interest for the year is

$$X \cdot (1 + 0.04) \cdot 0.04.$$

By the end of 2 years an initial payment of ฿X will have accumulated to:

$$X \cdot (1+0.04) + X \cdot (1+0.04) \cdot 0.04 = X \cdot 1.04^2 = 10000.$$

Hence,

$$X = \frac{10000}{1.04^2} = 9245.56213,$$

Note We refer to the amount to which the capital accumulates with the addition of interest as *accumulation* or *accumulated value*.

Example 1.3. *Consider the following arguments*

- *It is obvious that you would prefer to have ฿1100 now than ฿1000 now.*
- *If we receive and hold ฿1 now, then it is worth more than receiving and holding ฿1 at some time in the future? Why is this?*
- *Is it obvious that your would be better off with ฿1100 in 2 years than ฿1000 now?*

Solution:

For the second argument, this one baht will grow to $1 + r$ in the first year, $(1 + r)^2$ in two years, and so on. These amounts are clearly worth more than receiving and holding ฿1 at the same time in the future.

For the last argument, we need to compare the values of the amounts **received at different times. To do this, we can look at the today's values of ฿1100 received in 2 years assuming that we can invest at an annual interest rate of r percent.

The present value of this amount X in year 2 is

$$1100/(1+r)^2.$$

Assuming $r = 5\%$, the present value of X is 997.7324263.

Comparing the values in today's baths, it is better to have ฿1000 now than to have ฿1100 in 2 years.

Notes From the above example,

1. One can deposit or invest ฿1 now and will receive ฿1 back and a reward called *interest* at some point in the future. Because of its potential earning power, money in the present is worth more than an equal amount in the future. This is a fundamental financial principle known as **the time value of money**.

2. At a given point of time, cash has a monetary value, but also has a *time value.*

3. The amount deposited or invested is called *capital* or *principal.*

1.3.1 Simple interest

Simple interest is a calculation of interest that does not take into account the effect of compounding. Under simple interest, the amount of interest that accrues over time is proportional to the length of the period.

Suppose an amount C is deposited in an account that pays simple interest at the rate of $i\%$ per annum. Then after n years the deposit will have accumulated to

$$C(1 + i \cdot n).$$

Hence, the interest accrued over n years is

$$\text{Simple Interest} = C \cdot i \cdot n.$$

Note Auto loans and short-term personal loans are usually simple interest loans.

Example 1.4. *An investor deposits ฿10000 in a bank account that pays simple interest at a rate of 5% per annum. Calculate*

1. *interest he will earn after the first two years.*

2. *interest he will earn after the first three months.*

Note When n is not an integer, interest is paid on a pro-rate basis (in proportion).

Solution:

1. At the end of 2 years the interest earned is

$$10000 \cdot 0.05 \cdot 2 = 1000.$$

2. At the end of 3 months the interest earned is

$$10000 \cdot 0.05 \cdot \frac{3}{12} = 125.$$

Alternatively, the interest per month is $5\%/12 = 0.4167\%$ and hence the interest earned can be calculated as

$$10000 \cdot 0.004167 \cdot 3 = 125.$$

1.3.2 Compound interest

In compound interest, the accumulated amount over a period of time is the capital of the following period. Therefore, a capital of 1 unit at the end of the year increases to $1 + i$ units, which becomes the capital for the following year.

For year 2, the principal is $1 + i$ and the interest for the year is $(1 + i) \cdot i$. By the end of 2 years, an initial payment of 1 will have accumuulated to

$$(1 + i) + (1 + i) \cdot i = (1 + i)^2.$$

As this progression continues, the accumulated amount of X units at the end of year n becomes

$$X \cdot (1 + i)^n.$$

Note In this case, we can take money out and reinvest it as new capital illustrated in the timeline.

Exercise 1.1. (Excel) Use Excel to create a table showing the accumulated amounts at the end of each year for 15 years for a principal of ฿100 under the

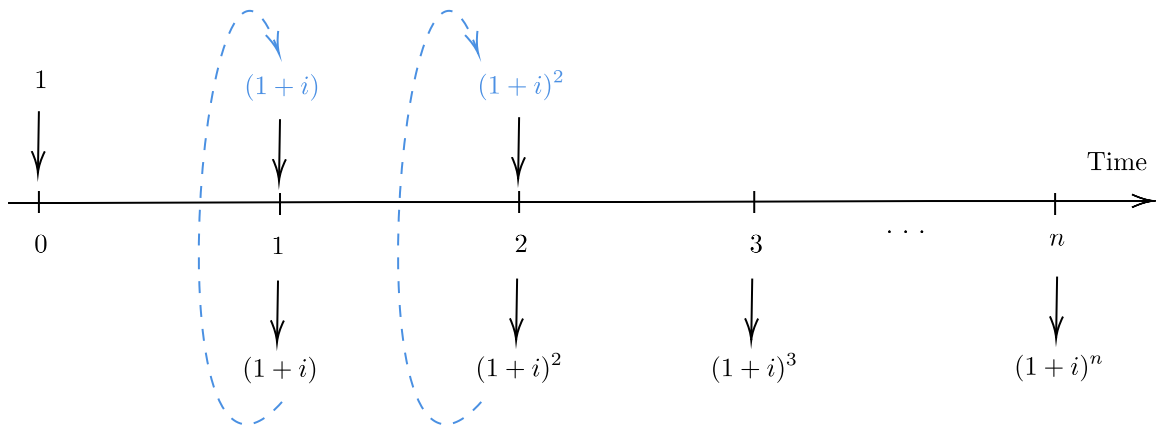


Figure 1.2: a timeline of compounding interest

simple interest approach and the compound interest approach with $r = 6\%$ for both cases. Discuss the results obtained (How long does it take to double the investment? How much will the principal grow over a 15-year period?)

The effect of compounding is to increase the total amount of accumulation. The effect is greater when the interest rate is high. This example shows two examples of the accumulated amount of ฿100 under the simple interest approach and the compound interest approach. As can be seen, the compound interest method makes the principle increase much faster than the simple interest method when the interest rate is high.

1.4 Frequency of Compounding

Even though the interest rate is typically expressed in annual terms, an investment's interest is frequently paid more frequently than once per year. For example, a savings account may offer an interest rate of 4% per year, credited quarterly. This interest rate is usually referred to as **nominal rate of interest**, i.e., 4% due four times per year.

We will see that the frequency of interest payments, also known as the frequency of compounding, has a significant impact on the total amount accrued and the interest collected. Consequently, it is crucial to precisely specify the rate of interest.

We use $i^{(m)}$ to represent the nominal rate of interest payable m times a year in order to underline the significance of the frequency of compounding. Therefore, m is the frequency of compounding per year and $1/m$ year is the **compounding period** or **conversion period**.

Note The nominal rate of interest payable m times per period is also known as the rate of interest convertible $m$thly or compounded $m$thly.

Example 1.5. Calculate the accumulated value in 1 year of a deposit of ฿100 in a saving account that earns interest at 10% payable quarterly.

Solution: In this example, the nominal rate of interest of $i^{(4)} = 10\%$ p.a. convertible quarterly means an interest rate of $10\%/4 = 2.5\%$ per quarter. In this case, the interest rate of 2.5% is called *effective interest*. The effective interest rate of i per unit of time (which may be month, quarter, etc.) is the amount of interest received at the end of a unit of time per ฿1 invested at the beginning of that unit.

Therefore, the nominal interest rate $i^{(4)} = 10\%$ is equivalent to an *effective interest rate* of 2.5% per quarter. The accumulated value in 1 year is $100(1+\frac{10\%}{4})^4 = 100(1+2.5\%)^4 = 110.3813$.

Note that after compound interest is taken into account, the interest income of an investor at the quarterly convertible nominal interest rate of 10% p.a. is 10.3813 (or 10.3813%.p.a. effective)

Example 1.6. *At a rate of 12% p.a. effective, draw a timeline to show cashflows if ฿100 is invested at the start of the year.*

Solution: The accumulated value of ฿100 at the end of the year is $100(1 + 12\%) = 112$.

Example 1.7. *At a rate of 12% p.a. compounding quarterly, draw a time line to show cashflows if ฿100 is invested at the start of the year.*

Solution: The nominal interest rate $i^{(4)} = 12\%$ is equivalent to an effective interest rate of 3% per quarter. The accumulated value in 1 year is $100(1 + 3\%)^4 = 112.55$. After compound interest is taken into account, the interest income of an investor at the quarterly convertible nominal interest rate of 10% p.a. is 12.55 (or 12.55%. p.a. effective)

Note $i^{(m)}$ is a nominal rate of interest which is equivalent to $i^{(m)}/m$ applied for each mth of a period. The interest is paid m times per measurement period.

The value at time n can be considered as the **annuity** with a cashflow of $i^{(m)}/m$ per period for n years together with the capital at time n as shown in the following figure. Therefore, the accumulated value in 1 year can also be calculated as $100(1 + 0.03 s_{\overline{4}|}^{3\%})$. The concept of annuity will be discussed in the subsequent section.

In general, we have

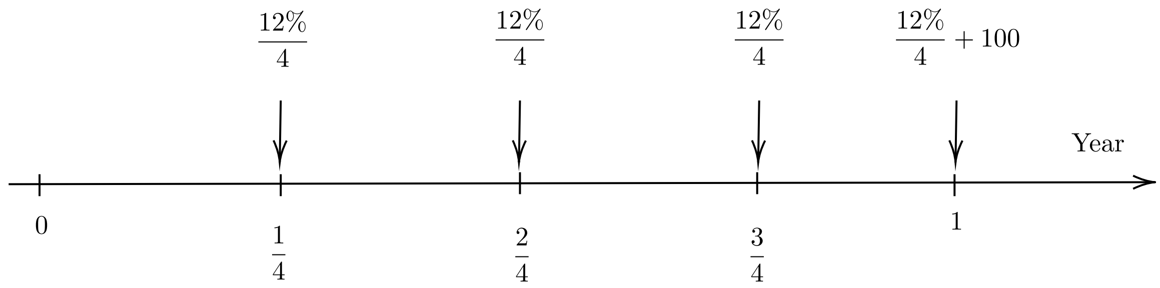


Figure 1.3: Frequency of Compouding vs Annuity

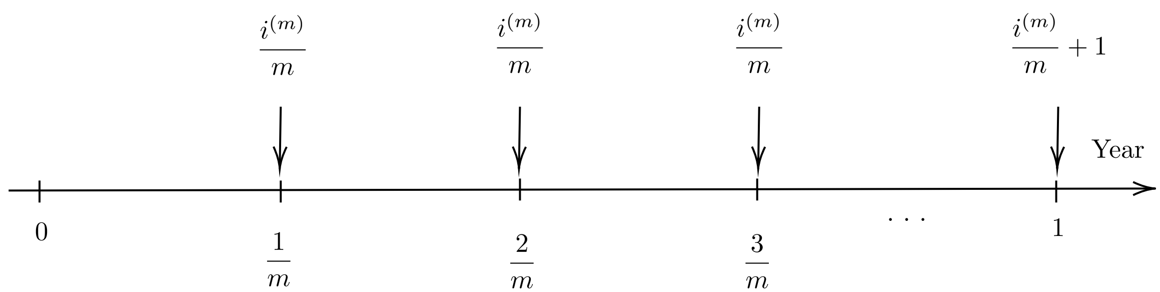


Figure 1.4: Frequency of Compouding vs Annuity

1.4.1 Effective rate of interest

The compounding frequency affects the accumulated amount. As a result, it may be inaccurate to compare two investment strategies only based on their nominal rates of return without also taking into account their frequency of compounding. It is necessary to compare different investment strategies on an equal basis. The measure known as the **effective interest rate** is often used for this purpose.

The effective rate of interest of i per time unit is the amount of interest received at the end of one time unit per ฿1 invested at the start of that time unit.

Example 1.8. *An investor invests ฿1 at 7.5% p.a. (per annum) effective. Then* $i = 0.075$. Calculate the value of investment after one year.

Solution: The value of investment after one year at this rate is

$$1 \times (1 + 0.075) = 1.075.$$

In particular, the amount of interest received at the end of the year per ฿1 invested is 0.075.

Example 1.9. *An investor invests ฿1000 at 5.25% per half-year effective. Then* $i = 0.0525$. Calculate the value of investment after half a year.

Solution: The value of investment after half year at this rate is

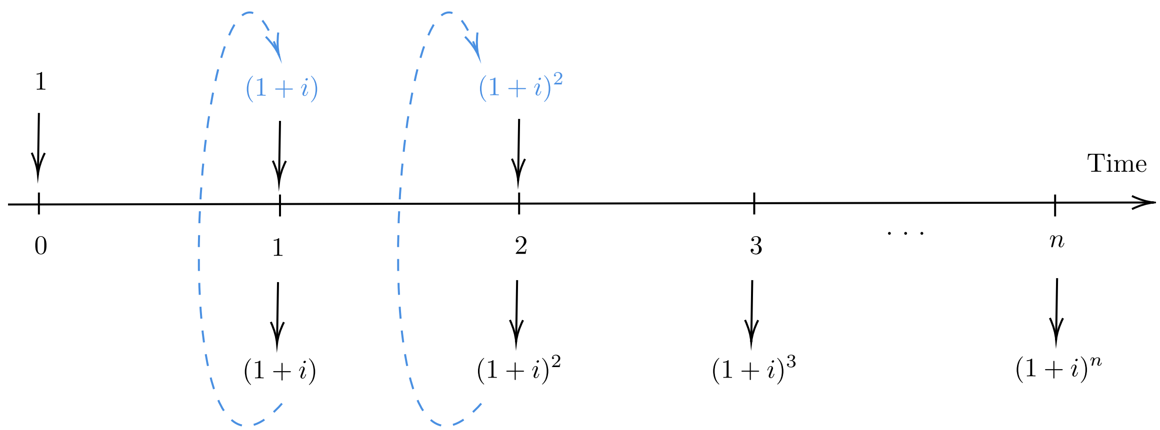
$$1000 \times (1 + 0.0525) = 1052.5.$$

Again, the amount ofinterest received at the end of the quarter of ฿1 invested is 0.0525.

Note The time unit is an **essential part of the definition**.

Example 1.10. *An investor invests ฿1 at effective rate* $i\%$ per time unit for n time units. Calculate the value of investment after two, three, ..., n time units.

Note Here we assume that we can take money out and reinvest it as new capital (see the timeline).



Example 1.11. *An investor invests ฿200 at* 3% pa effective. What will be the deposit have accumulated to after 5 years.

Solution: The deposit accumulates to $200 \cdot (1.03)^5 = 231.854815$ after 5 years.

Example 1.12. Consider the following problems.

1. *An investor invests ฿500 at* 2.75% per quarter effective. What will be the deposit have accumulated to after 9 months.

2. *An investor invests ฿2000 at* 6% per half-year effective. What will be the deposit have accumulated to after 2 years.

Solution:

1. Accumulating the 500 for 9 months at this rate gives

$$500 \cdot (1.0275)^3 = 542.394773.$$

2. After 2 years the accumulation is

$$2000 \cdot (1.06)^4 = 2524.95392.$$

Notes

1. The model under the effective rate of interest condition is a model of *compound interest*, where interest is earned on interest previously earned.

Unless state otherwise, we shall assume that i is the compound interest rate.

2. In practice, it is easier to work with the effective rate of interest which is defined in a suitable time unit.

The following formula can be used to convert between the effective rate i p.a. and the nominal rate $i^{(m)}$ p.a.:

$$(1+i) = \left(1 + \frac{i^{(m)}}{m}\right)^m .$$

Example 1.13. Consider the following problems.

1. *Express a nominal annual interest rate of 9% convertible half-yearly as a monthly effective interest.*

2. *Express a two-monthly effective interest of 3% as a nominal annual interest rate convertible two-monthly.*

Solution:

1. The effective rate $i\%$ p.a. is

$$i = (1 + \frac{0.09}{2})^2 - 1.$$

   Hence the monthly effective rate is $j = (1+i)^{1/12} - 1 = (1+\frac{0.09}{2})^{2/12} - 1 = 0.007363$.

2. A nominal annual interest rate convertible two-monthly is $6 \cdot 3\% = 18\%$.

Example 1.14. *Express each of the following effective rates per annum as a nominal rate, and vice versa.*

Effective Rate	***Nominal Rate***
$i = 0.04$	$i^{(4)} = 0.039412$
$i = 0.10$	$i^{(12)} = 0.095690$
$i = 0.06152$	$i^{(2)} = 0.06$
$i = 0.126825$	$i^{(12)} = 0.12$

1.4.2 Compounding over any number of time units

Suppose an amount ฿1 is invested at the rate of $i\%$ per time unit. At time t the accumulation is $(1 + i)^t$.

Example 1.15. 1. *An investor invests ฿4000 at* 8.5% per quarter effective. What will be the deposit have accumulated to after 1 month.

2. *An investor invests ฿800 at* 6% per half-year effective. What will be the deposit have accumulated to after 2.6 years.

Solution:

1. The accumulation after 1 month is $4000 \cdot 1.085^{1/3} = 4110.265768$.

2. The accumulation after 2.6 years is $800 \cdot 1.06^{5.2} = 1083.129754$.

Exercise 1.2. (Excel) Use Excel to create a table showing the accumulated amounts after 1 year under several different compounding frequencies (yearly, quarterly, monthly, daily) for a principal of ฿100 under with nominal rate of $r = 4\%$ per annum.

Discuss the results obtained. What happens if the compounding is made over infinitely small intervals (i.e. as $m \to \infty$)?

1.4.3 Changing the time period of the effective rates of interest

It is often very useful to change the effective rate of interest per time period to another. For example, if the effective rate of interest is defined per annum but cashflows occur monthly.

Let i be the effective rate of interest per t_i years (which can be any positive number, for e.g. $t_i = 1/2$). Here t_i years can be regarded as one time unit. Let j be the effective rate of interest per t_j years.

Example 1.16. *Find the condition under which the two effective rates of interest* i and j are equivalent.

Solution: Suppose we invest 1 for one year. Then at the end of the year under each rate of interest, we will have

$$(1+i)^{1/t_i} \text{ and } (1+j)^{1/t_j}.$$

Two rates of interest are equivalent if the given amount of principal invested for the same length of time produces the same accumulated value, i.e.

$$(1+i)^{1/t_i} = (1+j)^{1/t_j}.$$

Solving the equation for j yields

$$j = (1+i)^{t_j/t_i} - 1.$$

Example 1.17. 1. *If the effective rate of interest is 6% per annum, what is the effective rate of interest per half-year?*

2. *If the effective rate of interest is 12% per two-years effective, what is the effective rate of interest per quarter-year?*

3. *If the effective rate of interest is 2% per month effective, what is the effective rate of interest per 1.5-years?*

Solution:

1. $i = 6\%$ p.a. Then
$$j = (1.06)^{1/2} - 1 = 0.029563 \text{ per half-year.}$$

2. $i = 12\%$ per two-years. Then
$$j = (1.12)^{1/(2\times 4)} - 1 = 0.0142669 \text{ per quarter-year.}$$

3. $i = 2\%$ per month. Then
$$j = (1.02)^{1.5/(1/12)} - 1 = 0.428246 \text{ per 1.5-years.}$$

1.4.4 Non-constant interest rates

The effective rate may not be the same during every time period. We shall assume that the rates in every future time periods are known in advance.

Example 1.18. *The effective rate of interest per annum was 4% during 2015, 4.5% during 2016 and 5% during 2017. Calculate the accumulation of ฿200 invested on*

1. *01/01/2015 for 3 years*
2. *01/07/2015 for 2 years*
3. *01/04/2016 for 1.5 years*

Solution:

1. Accumulating the ฿200 for the first year at the rate of 4% p.a. gives

   $$200 \cdot 1.04.$$

   The accumulated value was then invested at the rate of 4.5% p.a. for another year, and its value at after 2 years was

   $$200 \cdot 1.04 \cdot 1.045.$$

   At the rate of 5% in the final year, the value after 3 years was

   $$200 \cdot 1.04 \cdot 1.045 \cdot 1.05 = 228.228.$$

2. The accumulation is

$$200 \cdot 1.04^{1/2} \cdot 1.045 \cdot 1.05^{1/2} = 218.4025.$$

3. The accumulation is

$$200 \cdot 1.045^{9/12} \cdot 1.05^{3/4} = 214.416986.$$

1.4.5 Accumulation factors

Let i be the effective rate of interest per one time unit and $s < t$. We define

- the accumulation factor per one time unit

$$A(0,1) = (1+i).$$

- the accumulation factor per t time units

$$A(0,t) = (1+i)^t.$$

- the accumulation factor at time t of 1 unit invested at time s

$$A(s,t).$$

Example 1.19. *The effective rate of interest per annum was 6% during 2015, 8% during 2016 and 10% during 2017. Calculate the following accumulation factors.*

1. $A(01/01/15, 01/01/18)$, i.e. the accumulation at 01/01/18 of an investent of 1 at 01/01/15
2. $A(01/07/15, 01/07/17)$
3. $A(01/04/16, 01/10/17)$

Solution:

1. $A(01/01/15, 01/01/18) = (1.06)(1.08)(1.1) = 1.25928$
2. $A(01/07/15, 01/07/17) = (1.06)^{1/2}(1.08)(1.1)^{1/2} = 1.166200$
3. $A(01/04/16, 01/10/17) = (1.08)^{3/4}(1.1)^{3/4} = 1.137922$

1.4.6 Present values and discount factors

Recall from Example 1.2 that the amount $\dfrac{10000}{1.04^2}$ we need to invest now to obtain ฿10000 in two years is called the *present value (PV)* or *discounted value* of the payments.

We define the discount factor v per annum, at rate i p.a. effective to be the present value of a payment of 1 due in 1 year?s time, i.e.

$$v = \frac{1}{1+i}.$$

Example 1.20. *Calculate the present of ฿25000 due in 3 years at an effective rate of interest of 6% per annum.*

Solution: The present value is

$$25000 \cdot \frac{1}{1.06^3} = 20990.482076.$$

It is the discounted value of 25000 due in 3 years.

Example 1.21. *How much should we invest now to meet a liability of ฿50000 in 5 years at an effective rate of interest of 3% per half-year.*

Solution: The amount we need to invest now to meet the future liability of 50000 in 5 years is the present value

$$50000 \cdot \frac{1}{1.03^{10}} = 37204.695745.$$

Note It follows that the PV of ฿1 in t time units at i effective rate of interest per time unit is

$$PV = \frac{1}{(1+i)^t} = v^t.$$

Example 1.22. *Given the discount factor per year* $v = 0.9$, calculate

1. *the effective rate of interest per year.*

2. *the equivalent discount factor per half-year.*

Solution:

1. From $v = \dfrac{1}{1+i} = 0.9$, solving the equation for i gives

$$i = \frac{1}{v} - 1 = 0.111111 \text{ per year.}$$

2. Let j be the effective rate of interest per half-year. Then

$$j = (1+i)^{1/2} - 1 = 0.054093.$$

Then, the discount factor per half-year is

$$v = \frac{1}{1+j} = \frac{1}{1.054093} = 0.948683.$$

Similarly, we define

- the discount factor per one time unit

$$V(0,1) = 1/(1+i).$$

- the discount factor per t time units

$$V(0,t) = 1/(1+i)^t.$$

- for $s < t$, the discount factor at time s of 1 unit receivable at time t

$$V(s,t) = (1+i)^{s-t}.$$

Notes

1. $V(s,t) = A(s,t)^{-1}$
2. For $r < s < t$, the following holds:
   - $A(r,t) = A(r,s)A(s,t)$
   - $V(r,t) = V(r,s)V(s,t)$

1.5 Cashflows and Annuities

Consider a series of cashflows defined by (see the timeline)

1. the times of payments (cashflows), denoted by $t_1, t_2, \dots$, and
2. the amount of payments, denoted by C_r (or C_{t_r}), which will be paid at time t_r, for $r = 1, 2, \dots$. The amounts can be positive or negative

The present value at any time t of this series of cashflow is

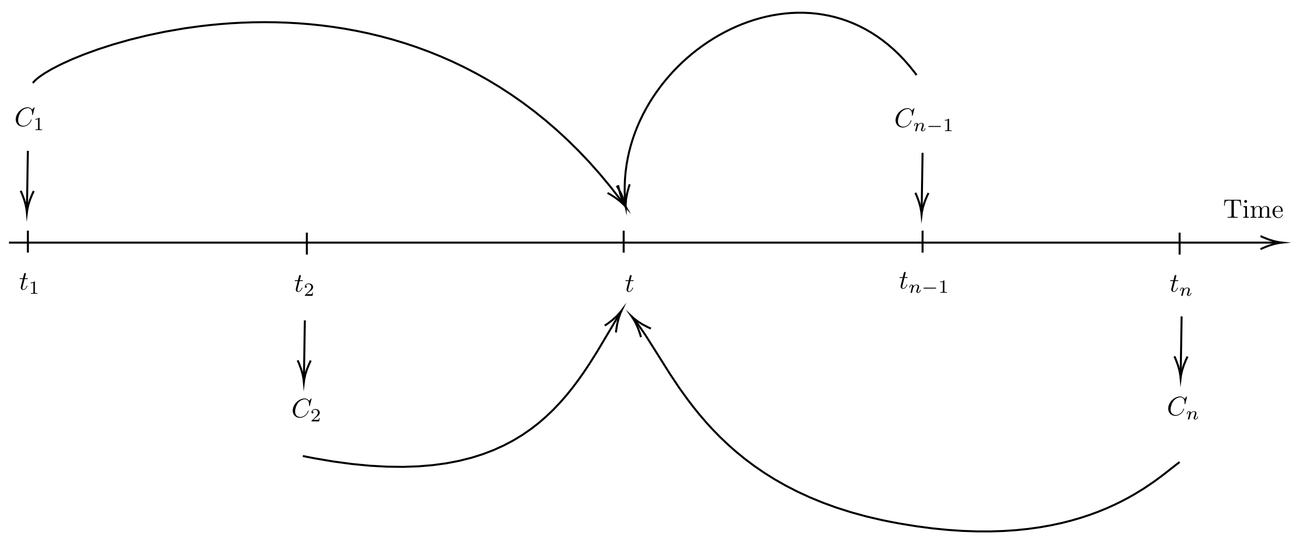
$$PV(t) = \sum_{r=1}^{\infty} C_r(1+i)^{t-t_r} = \sum_{r=1}^{\infty} C_r v^{t_r - t}$$

where i is the effective rate of interest.

The above formula can be obtained by summing these two components:

- for all $t_r < t$, adding up the accumulations of these individual cashflows up to time t, and
- for all $t_r > t$, adding up the discounted values of these individual cash-

flows back to time t.



Notes

1. At a fixed effective rate of interest, the original series of cashflows is equivalent to a single payment of amount $PV(t)$ at time t.

2. If two different series of cashflows have the same PV at one time at a given effective rate of interest, then they have the same PV at any time at that effective rate of interest.

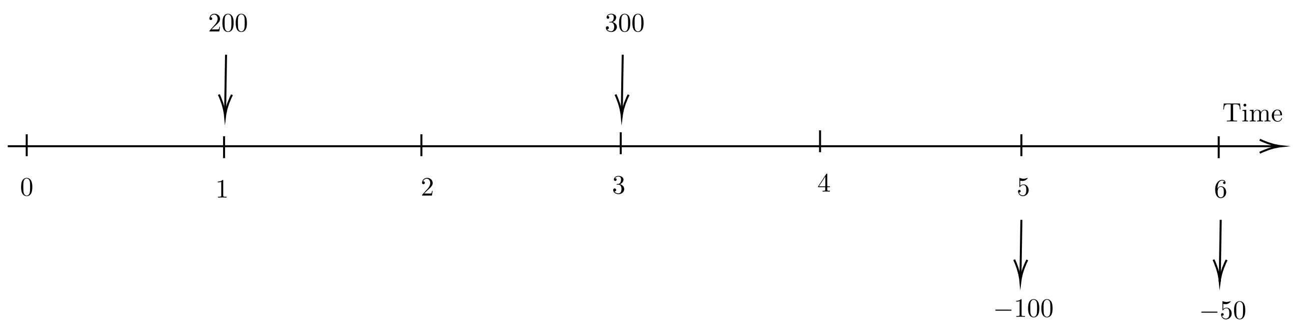
Example 1.23. *Let* $i = 4\%$ effective per time unit. Cashflows are given as follows:

- $C_1 = 200$ at time $t_1 = 1$.
- $C_2 = 300$ at time $t_2 = 3$.
- $C_3 = -100$ at time $t_3 = 5$.
- $C_4 = -50$ at time $t_4 = 6$.

Calculate

1. *the accumulation at time* $t = 7$.
2. *the present value at time* $t = 0$.
3. *the present value at time* $t = 4$.

Solution:



1. The series of cashflows is shown in the following timeline. The accumulation at time $t = 7$ is

$$\begin{aligned}\sum_{r=1}^{4} A(t_r, 7) &= 200 \cdot A(1,7) + 300 \cdot A(3,7) - 100 \cdot A(5,7) - 50 \cdot A(6,7) \\ &= 200 \cdot 1.04^6 + 300 \cdot 1.04^4 - 100 \cdot 1.04^2 - 50 \cdot 1.04 \\ &= 443.861372\end{aligned}$$

2. The present value at time $t = 0$ can be obtained by discounting the accumulation at time $t = 7$ back to time $t = 0$, which is

$$443.861372 \cdot V(0,7) = 443.861372 \cdot \frac{1}{1.04^7} = 337.298163.$$

3. The present value at time $t = 4$ is

$$443.861372 \cdot V(4,7) = 443.861372 \cdot \frac{1}{1.04^3} = 394.591143.$$

1.5.1 Level Annuities certain

An **annuity** is a series of payments made at equal intervals. There are many practical examples of financial transactions involving annuities, such as.

- a car loan that is repaid in equal monthly instalments
- a pensioner who purchases an annuity from an insurance company upon retirement
- a life insurance policy that is taken out with monthly premiums

When certain payments are to be made for a certain period of time, they are called *annuity certain*.

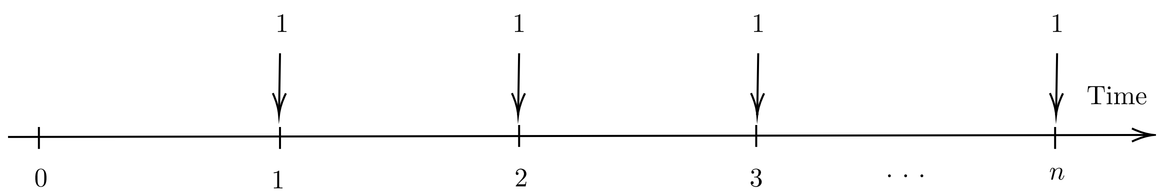
- If the payments are made at the end of each time period, they are paid *in arrears.*

- Otherwise, payments are made at the beginning of each time period, they are pain *in advance.*
- An annuity paid in advance is also known as an *annuity due*
- If each payment is for the same amount, this is a *level* annuity.

Example 1.24. *Let i be* the constant effective rate of interest per time unit. Show that the accumulated value of a level annuity certain having cashflow of 1 unit at the end of each of the next n time units is

$$\frac{(1+i)^n - 1}{i}.$$

Such accumulated value of the annuity is denoted by $s_{\overline{n|}}$ (pronounced "S.N.")



Solution: Based on the first principles,

$$s_{\overline{n|}} = \sum_{r=1}^{n} C_r \cdot A(t_r, n) \tag{1.1}$$

$$= (1+i)^{n-1} + (1+i)^{n-2} + ... + (1+i) + 1. \tag{1.2}$$

Multiplying Eq.(1.2) through by (1+i) gives

$$(1+i) \cdot s_{\overline{n}|} = (1+i)^n + (1+i)^{n-1} + ... + (1+i)^2 + (1+i). \tag{1.3}$$

Subtracting the two equations results in

$$\begin{aligned} i \cdot s_{\overline{n}|} &= (1+i)^n - 1 \\ s_{\overline{n}|} &= \frac{(1+i)^n - 1}{i}. \end{aligned}$$

Example 1.25. *Let* i be the constant effective rate of interest per time unit. Show that the present value at time 0 of a level annuity certain, denoted by $a_{\overline{n}|}$ (pronounced "A.N.") , having cashflow of 1 unit at the end of each of the next n time units is

$$a_{\overline{n}|} = \frac{1 - v^n}{i}.$$

Solution: Taking the accumulated value at time n and discounting back to time 0 gives

$$\begin{aligned} a_{\overline{n}|} &= s_{\overline{n}|} \cdot v^n \\ &= \frac{(1+i)^n - 1}{i} \cdot v^n \\ &= \frac{1 - v^n}{i}. \end{aligned}$$

Example 1.26. *Given the effective rate of interest of* 8% p.a., calculate

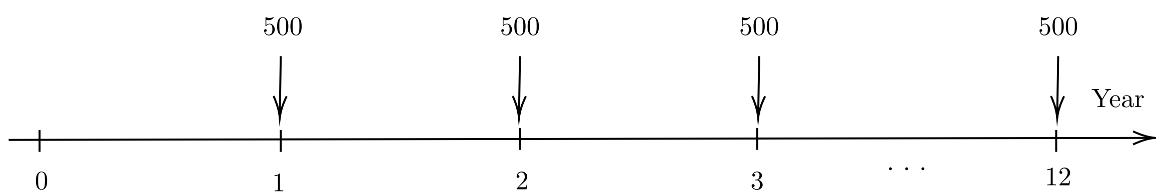
1. *the accumulation at 12 years of ฿500 payable yearly in arrears for the next 12 years.*

2. *the present value now of ฿2,000 payable yearly in arrears for the next 6 years.*

3. *the present value now of ฿1,000 payable half-yearly in arrears for the next 12.5 years.*

Solution:

1. The timeline of this transaction is shown in the figure below.

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The accumulation of the payments is
$$500 \cdot s_\angl{12}  = 500 \cdot \frac{1.08^{12} - 1}{0.08} = 94
```

2. The present value of the payments is

$$2000 \cdot a_{\overline{6}|} = 2000 \cdot \frac{1 - 1.08^{-6}}{0.08} = 9245.759328.$$

3. An interest rate of 8% p.a. is equivalent to an effective half-yearly interest rate, denoted by j, of

$$j = 1.08^{1/2} - 1 = 0.039230.$$

There are 25 payments of 1000 each, starting in six months' time.

Working in terms of half year, the present value of the payment is

$$1000 \cdot a^{j}_{\overline{25}|} = 1000 \cdot \frac{1 - 1.039230^{-25}}{0.039230} = 15750.003911.$$

1.5.2 Level Annuities Due

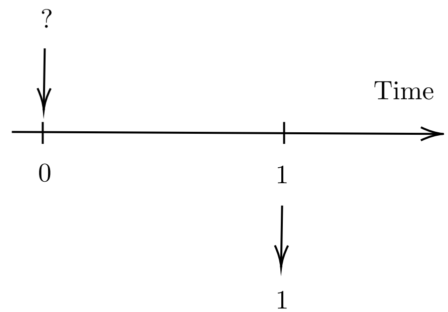
An *annuity-due* is an annuity where the payments made at the start of each time period (instead of a the end), i.e. the payments are paid *in advance*.

In order to calculate the present value or accumulation of an annuity due, we first introduce the concept of the rate of discount.

The rate of discount

As opposed to the interest rate where the accumulation of initial investment can be obtained by multiplying it by the accumulation factor $(1+i)^n$, we can obtain the discounted value of payment by using discount rates.

Suppose an amount of ฿1 is due after 1 year with an effective rate of $i\%$ p.a. (see the timeline below). What is the amount of money required to invested now to accumulate to 1?



The amount of money required now to accumulate to ฿1 in one year is

$$v = \frac{1}{1+i}.$$

Note that

$$\frac{1}{1+i} = 1 - \frac{i}{1+i}.$$

We define the effective rate of discount d per annum as

$$d = \frac{i}{1+i}.$$

It follows that

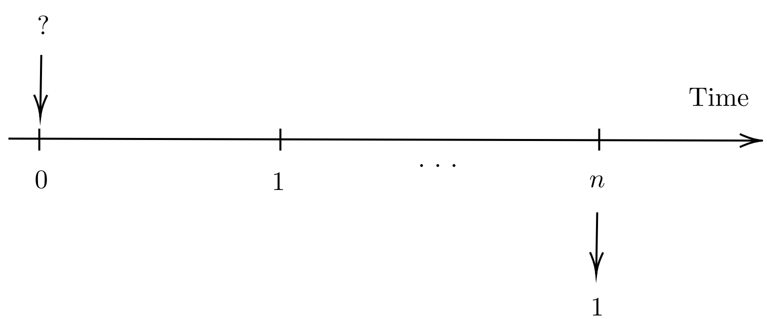
$$v = \frac{1}{1+i} = 1 - \frac{i}{1+i} = 1 - d$$

represents the discount of ฿1 for 1 year using the effective rate of interest of $i\%$ p.a.

Similarly, suppose an amount of ฿1 is due after n year with an effective rate of $i\%$ p.a. The amount of money required to invested now to accumulate to 1 in n year is

$$\frac{1}{(1+i)^n} = (1-d)^n.$$

See the timeline below for illustration.



Example 1.27. *Discount ฿2,000 for 3 years using the effective rate of discount of 5% per annum.*

Solution: After 1 year the discount will be $0.05 \cdot 2000 = 100$, and the discounted value of the payment will be

$$2000 \cdot (1 - d) = 2000 \cdot (1 - 0.05) = 1900.$$

Similarly, after 2 years, the discounted value will be

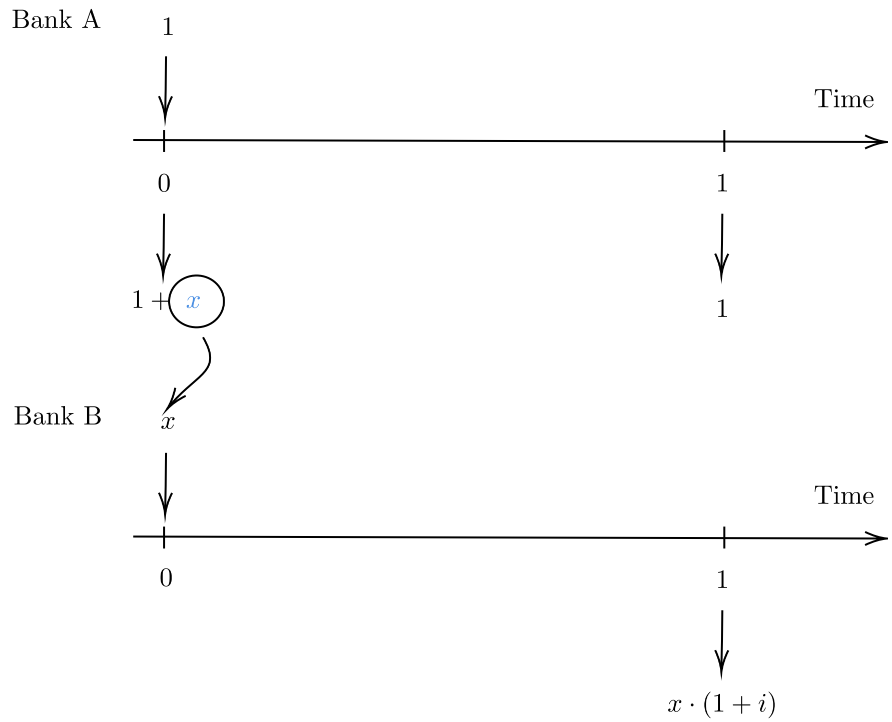
$$2000 \cdot (1 - d)^2 = 2000 \cdot (1 - 0.05)^2 = 1805.$$

After 3 years, the discounted value of the payment will be

$$2000 \cdot (1 - d)^3 = 2000 \cdot (1 - 0.05)^3 = 1714.75.$$

Example 1.28. *The effective rate of discount d* per time unit can be regarded as the interest paid in advance at time 0, which is equivalent to the effective rate of interest i payable in arrears.

Solution: To show this, suppose that the bank added interest of x to an account of an amount of 1 unit at the start of the period. Assume that the interest amount of x can be withdrawn and invested in another bank that earn the rate of interest $i\%$ effective per time unit. The principle of 1 unit is still in the first bank.



At the end of the year, we have

- the principle of 1 unit in the first bank, and
- the interest paid in advance which accumulates to $x(1+i)$ in the second bank.

For this to be equivalent to the interest paid in arrears, we can find x which solves

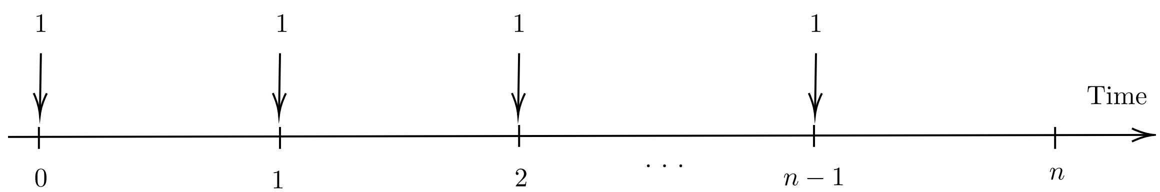
$$1 + x(1 + i) = 1 + i,$$

$$x = \frac{i}{1+i} = \frac{1+i}{1+i} - \frac{1}{1+i} = 1 - v = d.$$

Therefore, the effective rate of discount d per time unit can be regarded as the interest paid in advance at time 0, which is equivalent to the effective rate of interest i payable in arrears.

Example 1.29. *Let* i be the constant effective rate of interest per time unit. Show that the accumulated value of a level annuity due, denoted by $\ddot{s}_{\overline{n}|}$ (pronounced "S-due N", having cashflow of 1 unit at the start of each of the next n time units is

$$\ddot{s}_{\overline{n}|} = \frac{(1+i)^n - 1}{d}.$$



Solution: Using the previous results, it follows that

$$\begin{aligned}
\ddot{s}_{\overline{n}|} &= (1+i)^n + (1+i)^{n-1} + \cdots + (1+i)^2 + (1+i) \\
&= (1+i) \cdot \left[(1+i)^{n-1} + \cdots + (1+i)^1 + 1\right] \\
&= (1+i) \cdot s_{\overline{n}|} \\
&= (1+i) \cdot \frac{(1+i)^n - 1}{i} \\
&= \frac{(1+i)^n - 1}{i/(1+i)} \\
&= \frac{(1+i)^n - 1}{d}.
\end{aligned}$$

Example 1.30. *Let* i be the constant effective rate of interest per time unit. Show that the present value at time 0 of a level annuity due having cashflow of 1 unit at the start of each of the next n time units is

$$\ddot{a}_{\overline{n}|} = \frac{1 - v^n}{d}.$$

Solution: The present values of the payments can be obtained by discounting $\ddot{a}_{\overline{n}|}$ back to time 0, i.e.

$$\begin{aligned}
\ddot{a}_{\overline{n}|} &= v^n \cdot \ddot{s}_{\overline{n}|} \\
&= v^n \frac{(1+i)^n - 1}{d} \\
&= \frac{1 - v^n}{d}.
\end{aligned}$$

Example 1.31. *Given the effective rate of interest of* 8% p.a., calculate

1. *the accumulation at 12 years of ฿500 payable yearly in advance for the next 12 years.*
2. *the present value now of ฿2,000 payable yearly in advance for the next 6 years.*
3. *the present value now of ฿1,000 payable half-yearly in advance for the next 12.5 years.*

Solution:

1. The accumulation of the annuity-due of 12 years is

$$500 \cdot \ddot{s}_{\overline{12}|} = 500 \cdot \frac{1.08^{12} - 1}{0.08/1.08} = 10247.648289.$$

2. The present value of the annuity-due of 6 years is

$$2000 \cdot \ddot{a}_{\overline{6}|} = 2000 \cdot \frac{1 - 1.08^{-6}}{0.08/1.08} = 9985.420074.$$

3. An interest rate of 8% p.a. is equivalent to an effective half-yearly interest rate, denoted by j, of

$$j = 1.08^{1/2} - 1 = 0.039230.$$

There are 25 payments of 1000 each, starting in six months' time.

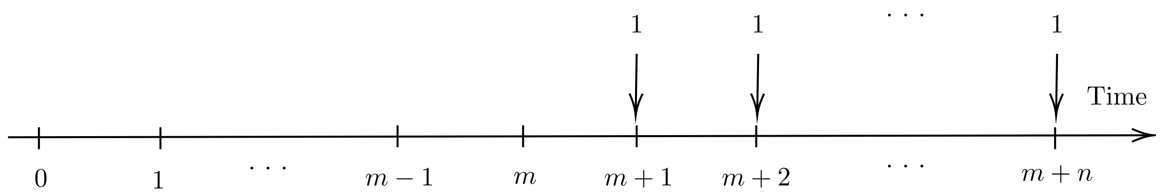
Working in terms of half year, the present value of the payment is

$$1000 \cdot \ddot{a}^{j}_{\overline{25}|} = 1000 \cdot \frac{1 - 1.039230^{-25}}{0.039230/1.039230} = 16367.876564.$$

1.5.3 Deferred annuities

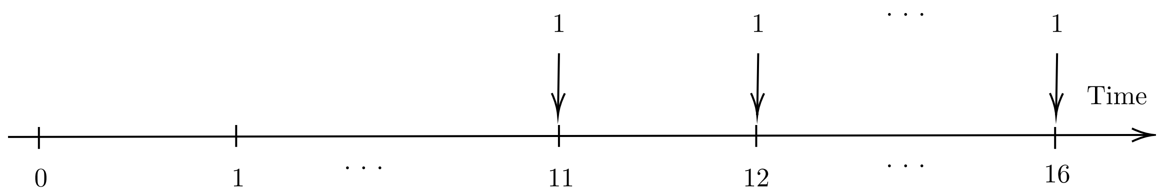
An annuity whose first payment is made during the first time period (either in arrears or in advance) is called *immediate annuity*. Otherwise, the annuity is known as *deferred* annuity, i.e. the first payment starts some time in the future.

To calculate the present value of the annuity of a series of n payments deferred for m time units (the first payment is due at time $m+1$), denoted by ${}_{m|}a_{\overline{n}|}$, we first calculate the present value at the end of the deferred period and then discount back to the start of the period.



$$
\begin{aligned}
{}_{m|}a_{\overline{n}|} &= v^{m+1} + v^{m+2} + \cdots + v^{m+n} \\
&= v^m \left(v + v^2 + \cdots + v^n \right) \\
&= v^m \cdot a_{\overline{n}|}.
\end{aligned}
$$

Example 1.32. *Calculate the present value at time 0 of an annuity of 1 p.a. in arrears for 6 years and deferred for 10 at 6% effective rate p.a.*



This is an annuity with 6 unit payments for which the first payment is at time 11. Hence the present values of such payments is

$$\begin{aligned}
{}_{10|}a_{\overline{6}|} &= v^{11} + v^{12} + \cdots + v^{16} \\
&= v^{10}\left(v + v^2 + \cdots + v^6\right) \\
&= v^{10} \cdot a_{\overline{6}|} \\
&= \left(\frac{1}{1.06}\right)^{10} \cdot \left(\frac{1 - 1.06^{-6}}{0.06}\right) = 2.745808.
\end{aligned}$$

Example 1.33. *Give the reason or show that the present value of a series of* $(n+m)$ payments of one unit payable at the end of each time period is equal to the sum of

1. *present value of* m payments of one units payable at the end of each time period (denoted by $a_{\overline{m}|}$) and
2. *present value of* n payments of one units payable at the end of each time period deferred for m years (denoted by ${}_{m|}a_{\overline{n}|}$).

Solution: The present value of a series of (m+n) payments is

$$\begin{aligned} a_{\overline{m+n}|} &= \left(v + v^2 + \cdots v^m\right) + \left(v^{m+1} + v^{m+2} + \cdots + v^{m+n}\right) \\ &= a_{\overline{m}|} +_{m|} a_{\overline{n}|}. \end{aligned}$$

It follows that ${}_{m|}a_{\overline{n}|} = a_{\overline{m+n}|} - a_{\overline{m}|}$.

1.5.4 Increasing annuities

An annuity in which the ith payment of the amount i is made at time $t_i = i$ is called an *(simple) increasing* annuity. The present and accumulated value of this annuity can be obtained from the first principles. For example, the present value of the increasing annuity can be evaluated by

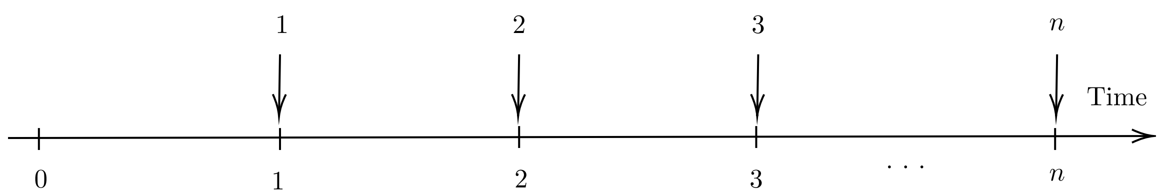
$$\sum_{i=1}^{n} X_i v^{t_i} = \sum_{i=1}^{n} i v^{t_i},$$

where the ith payment of amount $X_i = i$ at time $t_i = i$.

Example 1.34. *Derive the formula for the present value of a simple increasing annuity payable yearly in arrears with the effective rate* $i\%$ p.a. for n years.

Solution: The cashflows of the simple increasing annuity payable yearly in arrears is illustrated below. The present value of payments of 1 at time 1, 2 at time 2, ..., n at time n denoted by $(Ia)_{\overline{n}|}$ is given by

$$(Ia)^{i}_{\overline{n}|} = \frac{\ddot{a}^{i}_{\overline{n}|} - nv^n}{i}.$$



Notes

1. An increasing annuity but with payments in advance is given by

$$(I\ddot{a})^{i}_{\overline{n}|} = \frac{\ddot{a}^{i}_{\overline{n}|} - nv^n}{d}.$$

2. The formulas for the accumulated values are

$$(Is)_{\overline{n}|}^{i} = \frac{\ddot{s}_{\overline{n}|}^{i} - n}{i} \quad \text{(in arrears)}$$

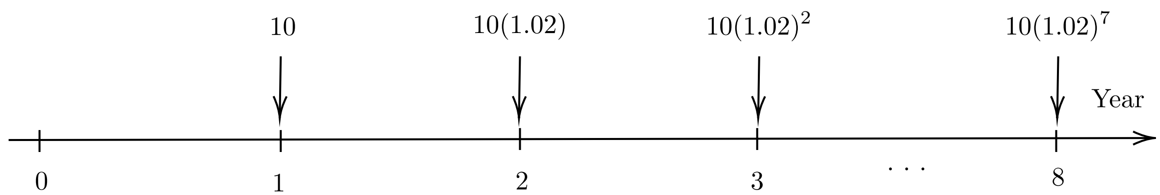
$$(I\ddot{s})_{\overline{n}|}^{i} = \frac{\ddot{s}_{\overline{n}|}^{i} - n}{d} \quad \text{(in advance)}$$

1.5.5 Compound increasing annuities

The following example considers the value of compound increasing annuities where the payments increase by a constant factor each time.

Example 1.35. *Assume that the effective rate of interest is 6% p.a. Calculate the present value as at 1 January 2010 of an annuity payable annually in arrears for 8 years. The first payment is ฿10 and subsequent payments increase by 2% per annum compound.*

Solution:



At 1/1/2010, the present value of the payment is given by

$$PV = 10 \cdot \frac{1}{1.06} + 10 \cdot \frac{1.02}{(1.06)^2} + \cdots + 10 \cdot \frac{(1.02)^7}{(1.06)^8}$$
$$= \frac{10}{1.02}\left(\frac{1.02}{1.06} + \left(\frac{1.02}{1.06}\right)^2 + \cdots + \left(\frac{1.02}{1.06}\right)^8\right)$$

The above equation can be arranged so that the annuity formula can be applied.

We can define j such that $1+j = 1.06/1.02$, and hence,

$$\begin{aligned}
PV &= \frac{10}{1.02}\left(\frac{1}{1+j} + \left(\frac{1}{1+j}\right)^2 + \cdots + \left(\frac{1}{1+j}\right)^8\right) \\
&= \frac{10}{1.02} a_{\overline{8}|} \quad \text{at } j\% \\
&= \frac{10}{1.02}\left(\frac{1-\left(\frac{1.02}{1.06}\right)^8}{\left(\frac{1.06}{1.02}-1\right)}\right) \\
&= 66.2216
\end{aligned}$$

1.5.6 Annuities payable more than once per time unit

Consider the value of an annuity payable in arrears m times per time unit at an effective rate of interest i per time unit. The annuity is still payable for n time units and a total amount of 1 unit per time unit. The present and accumulated values of the corresponding annuity are denoted by $a_{\overline{n}|}^{(m)i}$ and $s_{\overline{n}|}^{(m)i}$, respectively.

To calculate either the present or accumulation value of this annuity, we can simply apply the first principles by using the effective rate of interest per $1/m$ time unit. In particular, we have

$$a_{\overline{n}|}^{(m)i} = \frac{1}{m} a_{\overline{n\cdot m}|}^{j},$$

and

$$s_{\overline{n}|}^{(m)i} = \frac{1}{m} s_{\overline{n\cdot m}|}^{j},$$

where j is the effective rate per $1/m$ time unit.

Example 1.36. *Calculate the accumulation at 1 January 2020 of an annuity of ฿100 per month, payable in arrears from 1 January 2010 at an effective rate of interest of 4% p.a.*

Solution: The annual payment is 1200 and the effective rate per month equivalent to 4% p.a. is $j = (1.04)^{1/12} - 1 = 0.003274$ per month. Hence,

$$1200 s_{\overline{10}|}^{(12)4\%} = 100 s_{\overline{12 \cdot 10}|}^{j} = 14669.59.$$