SCMA329 Practical Mathematical Financial Modeling

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1.6 Principle of Equivalence, Yields and Equation of Value

The principle of equivalence is used to compare two different cashflows whether one is worth more than the other.

Consider two sequences of cashflows

- $C_1, C_2, ...$ with payments at times $t_1, t_2, ...$ and
- $D_1, D_2, ...$ with payments at times $s_1, s_2, ...$.

Assume that the interest rates are given and apply to both of them. The two sequences of cashflows are said to be **equivalent** (or equal in value) if their values at any time t are the same, i.e. there exists $t \in \mathbb{R}$ such that

$$PV^C(t) = PV^D(t).$$

Notes

1. If two sequences of cashflows have the same value at time s, then they have the same value at any time t since

- for $t \leq s$,

  (Value at time t) = (Value at time s) $\times V(t, s)$,

- for $t \geq s$

  (Value at time t) = (Value at time s) $\times A(s, t)$.

2. The two sequences of cashflows are **indifferent** if their present values are the same.

3. The principle of equivalent can be applied for **pricing a financial security**, for example, a price P which will be paid by the investor in return for a series of future cashflows.

Example 1.37. *Calculate the maximum price an investor wish to pay in return for an investment that will pay ฿500 at the end of each of the next 15 months given that the interest rate is 0.2% per month.*

The present value of these payments of 500 at the end of the next 15 months is

$$PV(0) = 500a_{\overline{15}|}^{0.002} = 500 \cdot \left(\frac{1 - (1.002)^{-15}}{0.002} \right) = 7381.35.$$

Therefore, the investor would be willing to pay a maximum of 7381.35.

Example 1.38. *Determine whether the following series of cashflows are equivalent given that an interest rate is 6% per annum effective.*

1. *One single payment of amount 6,691.127888 at year 5.*
2. *a level annuity of 300 payable yearly in arrears for the next 5 years plus a lump sum of 5,000.*
3. *a level annuity of 1,186.982002 payable yearly in arrears for the next 5 years.*

Solution:

1. The present value is $6,691.127888 \times (1.06)^{-5} = 5000$.
2. The present value is
$$300 a_{\overline{5}|}^{0.06} + 5000 \times (1.06)^{-5} = 5000.$$
3. The present value is
$$1186.982002 a_{\overline{5}|}^{0.06} = 5000.$$

Therefore, the three series of cashflows are **indifferent**.

1.6.1 Equation of value and yields

Consider a transaction from an investment that offers

- to pay an investor of amounts (i.e. money received) $B_1, B_2, \ldots, B_n$ at time $t_1, t_2, \ldots, t_n$
- in return for outlays (i.e. money paid out) of amounts $A_1, A_2, \ldots, A_n$ at these times, respectively.

Only one of A_i and B_i will be non-zero in general.

An equation of value equates the present value of money received to the present value of money paid out, which can be written as

$$\sum_{i=1}^{n} A_i v^i = \sum_{i=1}^{n} B_i v^i.$$

The equation of value can also be written in terms of the **net cashflow** at time t_i, i.e. $C_t = B_t - A_t$,

$$PV_i(0) = \sum_{i=1}^{n} C_i v^i = 0.$$

Equations of value are used throughout actuarial work. Some examples are as follows:

- The **fair price** to pay for an investment such as a fixed interest security or an equity (ie, PV outgo) equals the present value of the proceeds from the investment, discounted at the rate of interest required by the investor.
- The **premium** for an insurance policy is calculated by equating the present value of the expected amounts received in premiums to the present value of the expected benefits and other outgo.

We shall be concerned mainly with the question:

At what rate of interest does the series of amounts paid out have the same value as the series of amounts received? The corresponding rate of interest is called the **yield of the cashflows** (or **internal rate of return, money-weighted rate of return**).

Notes

1. Equations of values may have no roots, a unique root or multiple roots.

2. In most practice situations, there is a unique positive real root.

Example 1.39. *An investor pays ฿1,000 in order to receive ฿600 back in 2 years and ฿800 back in 4 years. Calculate the annual effective rate of interest earned on this investment (or the yield on the investment).*

Solution: The yield of the investment $i\%$ satisfies the equation of value

$$PV_i(0) = -1000 + 600(1+i)^{-2} + 800(1+i)^{-4} = 0.$$

To solve the equation for i, we define $z = (1+i)^{-2}$, resulting in

$$8z^2 + 6z - 10 = 0.$$

Therefore $z = 0.804248$ and $i = 0.115078$.

Example 1.40. *An investor pays ฿1,000 in order to receive ฿300 back at the end of the first 2 years and ฿400 back at the end of the third, forth and fifth year. Calculate the annual effective rate of interest earned on this investment (or the yield on the investment).*

Solution: The yield of the investment $i\%$ p.a. satisfies the equation of value

$$PV_i(0) = -1000 + \frac{300}{(1+i)} + \frac{300}{(1+i)^2} + \frac{400}{(1+i)^3} + \frac{400}{(1+i)^4} + \frac{400}{(1+i)^5}.$$

In our next section, we will learn how to approximate the yield of the above equation.

1.6.2 The method to estimate the yield

By using linear interpolation, the yield can be estimated as follows. Let P_1 and P_2 be the present values calculated at interest rates i_1 and i_2, respectively. Then the interest rate corresponding to a present value of P can be approximated by

$$i \approx i_1 + (i_2 - i_1)\frac{P - P_1}{P_2 - P_1}.$$

In order to apply this method to calculate the yield i, we simply set $P = 0$, and hence

$$i \approx i_1 + (i_2 - i_1)\frac{-P_1}{P_2 - P_1}.$$

From the figure above, the yield i can be approximated by i^*, which is the x-intercept of the straight line joining the points (i_1, P_1) and (i_2, P_2). From

$$\frac{i^* - i_1}{i_2 - i_1} = \frac{P_{i^*} - P_1}{P_2 - P_1},$$

we have $P_{i^*} = 0$ and

$$i \approx i^* = i_1 + (i_2 - i_1)\frac{-P_1}{P_2 - P_1}.$$

Note that one can get a good approximation by taking values that are either side of the true value and about 1% apart.

Example 1.41. *Approximate the yield of the transaction in Example* 1.40.

Solution: Here, When $i_1 = 0.21$, $P_1 = PV_{0.21}(0) = 19.448$ and when $i_1 = 0.22$, $P_2 = PV_{0.22}(0) = -3.698$. The yield is approximately equal to

$$\begin{aligned} i &\approx 0.21 - (0.22 - 0.21)\left(\frac{19.448}{-3.698 - 19.448}\right) \\ &= 0.218402 \text{ p.a. effective.} \end{aligned}$$

1.7 Loan schedules

In this section, we describe how a loan may be repaid. A schedule of repayment together with the interest and capital components of an annuity payment will be discussed.

Suppose that a lender lends an individual of amount L for n years with an effective rate of interest i per annum. We say that the **term** of the loan is n years with the loan **amount** of L. How could we repay the loan?

Repay as late as possible:

After n year, the borrower repays the entire loan and all interest that accrued over the period. The total amount to be repaid is equal to

Repay interest only during the term and repay the capital at the end of the term:

These types of loan where the borrower is a government or a company are **bonds** or **fixed interest securities**.

Repay loan by regular instalments of interest and capital throughout term of loan:

Each repayment must pay first for interest due and the remainder is used to repay some of the capital outstanding.

Example 1.42. *You borrow ฿5,000 for a term of 3 years at a fixed interest rate of 10% pa. The loan is to be repaid by 3 level annual repayments of ฿2,010.57 at the end of each year. Calculate the interest content, capital content from each repayment and capital outstanding after such repayment.*

Note The loan payments can be expressed in the form of a **Loan Schedule** as follows:

Time	Repayment	Intest content	Capital content	Capital outstanding
0				5000
1	2010.57	500	1510.57	3489.43
2	2010.57	348,943	1661.627	1827.80
3	2010.57	182.780	1827.79	0.01

1.7.1 The loan schedule

A more general form of loan payments can be expressed as follows: Let

- L_t be the amount of the loan outstanding at time t.
- X_t be the instalment at time t (all instalments may not be the same amount).
- i be the effective rate of interest per time unit charged on the loan.

ime Rep	ayment Int	est content Ca	pital content	Capital outstanding
0				L_0
1 $	X_1$	iL_0 $	(X_1 - iL_0)$	$L_1 = L_0 - (X_1 - iL_0)$
2 $	X_2$	iL_1 $	(X_2 - iL_1)$	$L_2 = L_1 - (X_2 - iL_1)$
⋮				
t $	X_t$ $	iL_{t-1}$ $(X	t - iL{t-1})$ $L_	t = L_{t-1} - (X_t - iL_{t-1})$
⋮				

ime	ayment	est content		
Rep	Int	Ca	pital content	Capital outstanding
n$ $	X_n$ $	iL_{n-1}$ $(X	n - iL{n-1})$	0

Note The capital outstanding after the kth payment is $Xa_{\overline{n-k}|}$, which is the present value of future repayments. This holds even when the repayments and interest rates are not constant.

Example 1.43. *A loan of ฿20,000 is repayable by equal monthly payments for 4 years, with interest rate payable at 10% pa effective.*

1. *Calculate the amount of each monthly payment.*

2. *Calculate the interest and capital contents of the 25th repayment.*

Solution:

1. The loan is repaid by level instalments of amount X payable monthly. Working in months, we define $j\%$ per month effective equivalent to 10% pa effective. We have

$$j = (1.1)^{(1/12)} - 1 = 0.007974.$$

The loan equation followed the equation of value is given by

$$PV_j(0) = 20000 - Xa_{\overline{48}|}^{j} = 0$$

Solving for X gives $X = 503.12$.

2. The capital outstanding after 24th repayment $= L_{24} = Xa_{\overline{24}|}^{j} = 10950.23$. Hence, the interest content of the 25th repayment $=$

$j \cdot L_{24} = 87.32$. The capital content of the 25th repayment $= X = 503.12 - 87.32 = 415.8$.

1.7.2 Changing the term of a loan

The term of the loan can be changed in the following circumstances:

- extend or shorten the term,
- miss a number of payments,
- repay part of the loan early.

The repayment amount will then need to be calculated according to the condition(s) as given in the change.

Example 1.44. *A person takes out a loan of ฿100,000 to be repaid by level monthly instalments in arrears over 7 years where the bank charges an effective annual rate of interest of 6%*

1. *Calculate the monthly repayment* ***Solution:*** *Working in months, we define* $j\%$ per month effective equivalent to 6% pa effective.

   $$j = (1.06)^{(1/12)} - 1 = 0.007974.$$

   The loan equation followed the equation of value is given by

   $$PV_j(0) = 100000 - Xa^{j}_{\overline{84|}} = 0$$

   Solving for X gives $X = 1453.25$.

2. *Calculate the new repayment amount if the the term of loan can be extended by 1 year, immediately after the 60th repayment has been made.* ***Solution:*** *The capital outstanding after 60th repayment =* $L_{60} = Xa^{j}_{\overline{24|}} = 32842.48$. Now the remaining term becomes 3 years (or 36 months). The new repayment amount X' satisfies

   $$PV_j(0) = 32842.48 - X'a^{j}_{\overline{36|}} = 0.$$

Solving for X' gives $X' = 996.77$.

3. *Instead of extending the term, the person had requested to miss the 61st and 62nd repayments. Calculate the remaining installments.* ***Solution:*** *After missing the 61st and 62nd repayments, the capital outstanding at time 62* $= L_{60} \cdot (1 + j)^2 = 32842.48(1.004868)^2 = 33162.99$. Hence, the remaining number of payments is 22.

4. *Calculate the new repayment amount if the person repaid ฿10,000 at the time he made the 60th repayment together with the 60th repayment.* ***Solution:*** *The revised capital outstanding after repayment of 10000 (the 60th repayment) is* $32842.48 - 10000 = 22842.48$. The new repayment amount X'' satisfies

$$PV_j(0) = 22842.48 - X'' a_{\overline{24|}}^{j} = 0.$$

Solving for X'' gives $X'' = 1010.76$.

1.7.3 Changing the interest rate

The interest rates for a loan can vary during the term of the loan. The reasons for varying rates of interest could be the following:

1. interest rates have been planned to changed during the term, for example the borrower would repay less during the beginning of the loan, or
2. the lender changes the rates of interest to reflect the market conditions.

Example 1.45. *You borrow ฿20,000 for a term of 20 years to be repaid by level annual instalments. The rate of interest will be 7% pa effective for the first 10 years and 8% pa effective thereafter. Calculate the annual repayment.*

Solution: Let X be the annual repayment. Using an equation of value, we have

$$20000 = Xa_{\overline{10}|}^{7\%} + (1.07)^{-10} Xa_{\overline{10}|}^{8\%}.$$

Then solving for X gives $X = 1916.69$.

Example 1.46. *You borrow ฿20,000 for a term of 15 years to be repaid by level annual instalments where the bank charges an effective annual rate of interest of 6%. After the 10th repayment has been made, the bank raises the interest rate to 6.5% pa effective. Calculate the new repayment amount.*

Solution: The annual repayment X **for a term of 15 years** before the adjustment of interest rate.

$$X = \frac{20000}{a_{\overline{15}|}^{6\%}} = 2059.26.$$

However, after the 10th repayment has been made, the bank raises the interest rate to 6.5% pa effective. Therefore, the capital outstanding after the 10th repayment $= L_{10} = Xa_{\overline{5}|}^{6\%} = 8674.332$. After the adjustment of the interest rate, the new repayment amount X' satisfies satisfies

$$PV_{6.5\%}(0) = 8674.332 - X'a_{\overline{5}|}^{6.5\%} = 0.$$

Solving for X' gives $X' = 2087.34$.