SCMA329 Practical Mathematical Financial Modeling

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Contents

Chapter 1

Cashflows, Interest and the Time Value of Money

1.1 Introduction to Financial Modelling

A financial model is a financial representation of a real world financial situation, which is either a mathematical or statistical model that describes the relationship among the variables of the financial problem. Here are some types of financial models.

- **Financial statement model:** The model includes three main components including income statement, cash flow statement and balance sheet. These are accounting reports issued by a company quarterly and annually that are used for decision making and performing financial analysis. (see https://corporatefinanceinstitute.com/resources/knowledge/accounting/three-financial-statements/)
- **Project finance models:** The model incorporates two main elements of the project including loans and debt repayment. It can be used to assess the risk-reward of lending to or investing in a long-term project, i.e. it can be used to tell whether the project has enough cash to cover the debt in the long term. (see https://www.wallstreetprep.com/knowledge/project-finance-model-structure/)
- **Discounted cashflow model:** It is the model to value a company using the net present value of the business's future cashflows, or to estimate the value of an investment based on its future cash flows. (see https://corporatefinanceinstitute.com/resources/templates/excel-modeling/dcf-model-template/)
- **Pricing models:** This models the way prices are set within a market in order to maximise profits.

1.2 Cashflows

Cashflows are amounts of money which are received (or income, positive cashflows) or paid (or outgo, negative cashflows) at particular times. Those payments arise from a financial transaction, e.g

- a bank account,
- a loan,
- an equity,
- a zero-coupon bond: A bond is a fixed income instrument that represents a loan from an investor to a debtor either a government or a corporation. A zero-coupon bond is a bond that pays no interest during its life.

- a fixed interest security: A fixed-income security is a debt instrument such as a bond or debenture that investors use to lend money to a company in exchange for interest payments.
- an index-linked security: An index-linked bonds pay interest that is tied to an underlying index, such as the consumer price index (CPI). Index-linked bonds are issued by governments to mitigate the effects of inflation by paying a real return plus accrued inflation.
- an annuity: An annuity is a contract issued and sold by financial institutions in which funds are invested for the purpose of paying out a fixed income later.
- a capital project etc.

Cash received represents inflows, income or also called **positive cashflows**, while money spent represents outflows, outgo or **negative cashflows**. The net cashflow at a given point in time is the difference between expenses and income.

Example 1.1. *A series of payments into and out of a bank account is given as follows:*

- *payments into the account : ฿1000 on 1 January 2014 and ฿100 on 1 January 2016*
- *payments out of the account : ฿200 on 1 July 2015, ฿300 on 1 July 2016, and ฿400 on 1 January 2018*

In practice, cashflows can be represented by a timeline as can be illustrated in this example.

1.3 Interest and the Time Value of Money

This section introduces the time value of money using the concepts of compound interest and discounting. The effect of interest rates on the present value of future cash flows is discussed. The value of distant cash flows in the present and current cash flows in the future are then considered.

We illustrate the time value of money by considering the following examples.

Example 1.2. *An investor want to make a payment of ฿10000 in 2 years. Suppose that a bank pays compound interest at 4% per annum effective. How much should the initial investment?*

Note The amount we need to invest now (i.e. the initial investment in this example) is called the *present value (PV)* or *discounted value* of the payments. **Solution:** By the end of 2 years an initial payment of ฿X will have accumulated to:

$$X \cdot 1.04^2 = 10000.$$

Hence, $X = 9245.56213$.

Example 1.3. *Consider the following arguments*

- *It is obvious that you would prefer to have ฿1100 now than ฿1000 now.*
- *Is it obvious that your would be better off with ฿1100 in 2 years than ฿1000 now?*
- *If we receive and hold ฿1 now, then it is worth more than receiving and holding ฿1 at some time in the future? Why is this?*

Notes From the above example,

1. One can deposit or invest ฿1 now and will receive ฿1 back and a reward called *interest* at some point in the future. Because of its potential earning power, money in the present is worth more than an equal amount in the future. This is a fundamental financial principle known as **the time value of money**.
2. At a given point of time, cash has a monetary value, but also has a *time value*.
3. The amount deposited or invested is called *capital* or *principal*.

1.3.1 Simple interest

Simple interest is a calculation of interest that does not take into account the effect of compounding. Suppose an amount C is deposited in an account that pays simple interest at the rate of $i\%$ per annum. Then after n years the deposit will have accumulated to

$$C(1 + i \cdot n).$$

Hence, the interest accrued over n years is

$$\text{Simple Interest} = C \cdot i \cdot n.$$

Note Auto loans and short-term personal loans are usually simple interest loans.

Example 1.4. *An investor deposits ฿10000 in a bank account that pays simple interest at a rate of 5% per annum. Calculate*

1. *interest he will earn after the first two years.*
2. *interest he will earn after the first three months.*

Note When n is not an integer, interest is paid on a pro-rate basis (in proportion). **Solution:**

1. At the end of 2 years the interest earned is

$$10000 \cdot 0.05 \cdot 2 = 1000.$$

2. At the end of 3 months the interest earned is

$$10000 \cdot 0.05 \cdot \frac{3}{12} = 125.$$

Alternatively, the interest per month is $5\%/12 = 0.4167\%$ and hence the interest earned can be calculated as

$$10000 \cdot 0.004167 \cdot 3 = 125.$$

1.3.2 Effective rate of interest

The effective rate of interest of i per time unit is the amount of interest received at the end of one time unit per ฿1 invested at the start of that time unit.

Example 1.5. *An investor invests ฿1 at 7.5% p.a. (per annum) effective. Then* $i = 0.075$*. Calculate the value of investment after one year.*

Solution: The value of investment after one year at this rate is

$$1 \times (1 + 0.075) = 1.075.$$

Example 1.6. *An investor invests ฿1000 at 5.25% per half-year effective. Then* $i = 0.0525$*. Calculate the value of investment after half a year.*

Solution: The value of investment after half year at this rate is

$$1000 \times (1 + 0.0525) = 1052.5.$$

Note The time unit is an **essential part of the definition**.

Example 1.7. *An investor invests ฿1 at effective rate i% per time unit for n time units. Calculate the value of investment after two, three, ..., n time units.*

Note Here we assume that we can take money out and reinvest it as new capital (see the timeline).

Example 1.8. *An investor invests ฿200 at 3% pa effective. What will be the deposit have accumulated to after 5 years.*

Solution: The deposit accumulates to $200 \cdot (1.03)^5 = 231.854815$ after 5 years.

Note We refer to the amount to which the capital accumulates with the addition of interest as *accumulation* or *accumulating value*.

Example 1.9.

1. *An investor invests ฿500 at 2.75% per quarter effective. What will be the deposit have accumulated to after 9 months.*
2. *An investor invests ฿2000 at 6% per half-year effective. What will be the deposit have accumulated to after 2 years.*

Solution:

1. Accumulating the 500 for 9 months at this rate gives
$$500 \cdot (1.0275)^3 = 542.394773.$$
2. After 2 years the accumulation is
$$2000 \cdot (1.06)^4 = 2524.95392.$$

Note The model under the effective rate of interest condition is a model of *compound interest*, where interest is earned on interest previously earned. Unless state otherwise, we shall assume that i is the compound interest rate.

1.3.3 Compounding over any number of time units

Suppose an amount ฿1 is invested at the rate of i% per time unit. At time t the accumulation is $(1+i)^t$.

Example 1.10.

1. *An investor invests ฿4000 at 8.5% per quarter effective. What will be the deposit have accumulated to after 1 month.*
2. *An investor invests ฿800 at 6% per half-year effective. What will be the deposit have accumulated to after 2.6 years.*

Solution:

1. The accumulation after 1 month is $4000 \cdot 1.085^{1/3} = 4110.265768$.
2. The accumulation after 2.6 years is $800 \cdot 1.06^{5.2} = 1083.129754$.

1.3.4 Changing the time period of the effective rates of interest

It is often very useful to change the effective rate of interest per time period to another. For example, if the effective rate of interest is defined per annum but cashflows occur monthly.

Let i be the effective rate of interest per t_i years (which can be any positive number, for e.g. $t_i = 1/2$). Here t_i years can be regarded as one time unit. Let j be the effective rate of interest per t_j years.

Example 1.11. *Find the condition under which the two effective rates of interest i and j are equivalent.*

Solution: Suppose we invest 1 for one year. Then at the end of the year under each rate of interest, we will have

$$(1+i)^{1/t_i} \text{ and } (1+j)^{1/t_j}.$$

Two rates of interest are equivalent if the given amount of principal invested for the same length of time produces the same accumulated value, i.e.

$$(1+i)^{1/t_i} = (1+j)^{1/t_j}.$$

Solving the equation for j yields

$$j = (1+i)^{t_j/t_i} - 1.$$

Example 1.12.

1. *If the effective rate of interest is 6% per annum, what is the effective rate of interest per half-year?*
2. *If the effective rate of interest is 12% per two-years effective, what is the effective rate of interest per quarter-year?*
3. *If the effective rate of interest is 2% per month effective, what is the effective rate of interest per 1.5-years?*

Solution:

1. $i = 6\%$ p.a. Then
$$j = (1.06)^{1/2} - 1 = 0.029563 \text{ per half-year.}$$
2. $i = 12\%$ per two-years. Then
$$j = (1.12)^{1/(2\times 4)} - 1 = 0.0142669 \text{ per quarter-year.}$$
3. $i = 2\%$ per month. Then
$$j = (1.02)^{1.5/(1/12)} - 1 = 0.428246 \text{ per 1.5-years.}$$

1.3.5 Non-constant interest rates

The effective rate may not be the same during every time period. We shall assume that the rates in every future time periods are known in advance.

Example 1.13. *The effective rate of interest per annum was 4% during 2015, 4.5% during 2016 and 5% during 2017. Calculate the accumulation of ฿200 invested on*

1. *01/01/2015 for 3 years*
2. *01/07/2015 for 2 years*
3. *01/04/2016 for 1.5 years*

Solution:

1. Accumulating the ฿200 for the first year at the rate of 4% p.a. gives
$$200 \cdot 1.04.$$
The accumulated value was then invested at the rate of 4.5% p.a. for another year, and its value at after 2 years was
$$200 \cdot 1.04 \cdot 1.045.$$
At the rate of 5% in the final year, the value after 3 years was
$$200 \cdot 1.04 \cdot 1.045 \cdot 1.05 = 228.228.$$

2. The accumulation is
$$200 \cdot 1.04^{1/2} \cdot 1.045 \cdot 1.05^{1/2} = 218.4025.$$

3. The accumulation is
$$200 \cdot 1.045^{9/12} \cdot 1.05^{3/4} = 214.416986.$$

1.3.6 Accumulation factors

Let i be the effective rate of interest per one time unit and $s < t$. We define

- the accumulation factor per one time unit
$$A(0,1) = (1+i).$$
- the accumulation factor per t time units
$$A(0,t) = (1+i)^t.$$
- the accumulation factor at time t of 1 unit invested at time s
$$A(s,t).$$

Example 1.14. *The effective rate of interest per annum was 6% during 2015, 8% during 2016 and 10% during 2017. Calculate the following accumulation factors.*

1. $A(01/01/15, 01/01/18)$, *i.e. the accumulation at 01/01/18 of an investent of 1 at 01/01/15*
2. $A(01/07/15, 01/07/17)$
3. $A(01/04/16, 01/10/17)$

Solution:

1. $A(01/01/15, 01/01/18) = (1.06)(1.08)(1.1) = 1.25928$
2. $A(01/07/15, 01/07/17) = (1.06)^{1/2}(1.08)(1.1)^{1/2} = 1.166200$
3. $A(01/04/16, 01/10/17) = (1.08)^{3/4}(1.1)^{3/4} = 1.137922$

1.3.7 Present values and discount factors

Recall from Example 1.2 that the amount $\dfrac{10000}{1.04^2}$ we need to invest now to obtain B10000 in two years is called the *present value (PV)* or *discounted value* of the payments.

We define the discount factor v per annum, at rate i p.a. effective to be the present value of a payment of 1 due in 1 year?s time, i.e.
$$v = \frac{1}{1+i}.$$

Example 1.15. *Calculate the present of B25000 due in 3 years at an effective rate of interest of 6% per annum.*

Solution: The present value is
$$25000 \cdot \frac{1}{1.06^3} = 20990.482076.$$
It is the discounted value of 25000 due in 3 years.

Example 1.16. *How much should we invest now to meet a liability of ฿50000 in 5 years at an effective rate of interest of 3% per half-year.*

Solution: The amount we need to invest now to meet the future liability of 50000 in 5 years is the present value

$$50000 \cdot \frac{1}{1.03^{10}} = 37204.695745.$$

Note It follows that the PV of ฿1 in t time units at i effective rate of interest per time unit is

$$PV = \frac{1}{(1+i)^t} = v^t.$$

Example 1.17. *Given the discount factor per year* $v = 0.9$, *calculate*

1. *the effective rate of interest per year.*
2. *the equivalent discount factor per half-year.*

Solution:

1. From $v = \dfrac{1}{1+i} = 0.9$, solving the equation for i gives

$$i = \frac{1}{v} - 1 = 0.111111 \text{ per year.}$$

2. Let j be the effective rate of interest per half-year. Then

$$j = (1+i)^{1/2} - 1 = 0.054093.$$

Then, the discount factor per half-year is

$$v = \frac{1}{1+j} = \frac{1}{1.054093} = 0.948683.$$

Similarly, we define

- the discount factor per one time unit

$$V(0,1) = 1/(1+i).$$

- the discount factor per t time units

$$V(0,t) = 1/(1+i)^t.$$

- for $s < t$, the discount factor at time s of 1 unit receivable at time t

$$V(s,t) = (1+i)^{s-t}.$$

Notes

1. $V(s,t) = A(s,t)^{-1}$
2. For $r < s < t$, the following holds:
   - $A(r,t) = A(r,s)A(s,t)$
   - $V(r,t) = V(r,s)V(s,t)$

1.4 Cashflows and Annuities

Consider a series of cashflows defined by (see the timeline)

1. the times of payments (cashflows), denoted by $t_1, t_2, \dots$, and
2. the amount of payments, denoted by C_r (or C_{t_r}), which will be paid at time t_r, for $r = 1, 2, \dots$. The amounts can be positive or negative

The present value at any time t of this series of cashflow is

$$PV(t) = \sum_{r=1}^{\infty} C_r(1+i)^{t-t_r} = \sum_{r=1}^{\infty} C_r v^{t_r - t}$$

where i is the effective rate of interest.

The above formula can be obtained by summing these two components:

- for all $t_r < t$, adding up the accumulations of these individual cashflows up to time t, and
- for all $t_r > t$, adding up the discounted values of these individual cashflows back to time t.

Notes

1. At a fixed effective rate of interest, the original series of cashflows is equivalent to a single payment of amount $PV(t)$ at time t.
2. If two different series of cashflows have the same PV at one time at a given effective rate of interest, then they have the same PV at any time at that effective rate of interest.

Example 1.18. *Let $i = 4\%$ effective per time unit. Cashflows are given as follows:*

- $C_1 = 200$ *at time* $t_1 = 1$.
- $C_2 = 300$ *at time* $t_2 = 3$.
- $C_3 = -100$ *at time* $t_3 = 5$.
- $C_4 = -50$ *at time* $t_4 = 6$.

Calculate

1. *the accumulation at time* $t = 7$.
2. *the present value at time* $t = 0$.
3. *the present value at time* $t = 4$.

Solution:

1. The series of cashflows is shown in the following timeline. The accumulation at time $t = 7$ is

$$\begin{aligned}\sum_{r=1}^{4} A(t_r, 7) &= 200 \cdot A(1,7) + 300 \cdot A(3,7) - 100 \cdot A(5,7) - 50 \cdot A(6,7) \\ &= 200 \cdot 1.04^6 + 300 \cdot 1.04^4 - 100 \cdot 1.04^2 - 50 \cdot 1.04 \\ &= 443.861372\end{aligned}$$

2. The present value at time $t = 0$ can be obtained by discounting the accumulation at time $t = 7$ back to time $t = 0$, which is

$$443.861372 \cdot V(0,7) = 443.861372 \cdot \frac{1}{1.04^7} = 337.298163.$$

3. The present value at time $t = 4$ is

$$443.861372 \cdot V(4,7) = 443.861372 \cdot \frac{1}{1.04^3} = 394.591143.$$

1.4.1 Level Annuities certain

Annuities are financial products that provide a guaranteed income stream and are primarily used for retirement savings. They are regular series of payments (cashflows). When they are specific payments to be made for a specific period of time, they are called *annuity certain.*

- If the payments are made at the end of each time period, they are paid *in arrears.*
- Otherwise, payments are made at the beginning of each time period, they are pain *in advance.*
- An annuity paid in advance is also known as an *annuity due*
- If each payment is for the same amount, this is a *level* annuity.

Example 1.19. *Let i be the constant effective rate of interest per time unit. Show that the accumulated value of a level annuity certain having cashflow of 1 unit at the end of each of the next n time units is*

$$\frac{(1+i)^n - 1}{i}.$$

Such accumulated value of the annuity is denoted by $s_{\overline{n}|}$ (pronounced "S.N.")

Solution: Based on the first principles,

$$s_{\overline{n}|} = \sum_{r=1}^{n} C_r \cdot A(t_r, n) \tag{1.1}$$

$$= (1+i)^{n-1} + (1+i)^{n-2} + ... + (1+i) + 1. \tag{1.2}$$

Multiplying Eq.(1.2) through by (1+i) gives

$$(1+i) \cdot s_{\overline{n}|} = (1+i)^n + (1+i)^{n-1} + ... + (1+i)^2 + (1+i). \tag{1.3}$$

Subtracting the two equations results in

$$\begin{aligned} i \cdot s_{\overline{n}|} &= (1+i)^n - 1 \\ s_{\overline{n}|} &= \frac{(1+i)^n - 1}{i}. \end{aligned}$$

Example 1.20. *Let i be the constant effective rate of interest per time unit. Show that the present value at time 0 of a level annuity certain, denoted by $a_{\overline{n}|}$ (pronounced "A.N.") , having cashflow of 1 unit at the end of each of the next n time units is*

$$a_{\overline{n}|} = \frac{1 - v^n}{i}.$$

Solution: Taking the accumulated value at time n and discounting back to time 0 gives

$$\begin{aligned} a_{\overline{n}|} &= s_{\overline{n}|} \cdot v^n \\ &= \frac{(1+i)^n - 1}{i} \cdot v^n \\ &= \frac{1 - v^n}{i}. \end{aligned}$$

Example 1.21. *Given the effective rate of interest of* 8% *p.a., calculate*

1. *the accumulation at 12 years of* ฿*500 payable yearly in arrears for the next 12 years.*
2. *the present value now of* ฿*2,000 payable yearly in arrears for the next 6 years.*
3. *the present value now of* ฿*1,000 payable half-yearly in arrears for the next 12.5 years.*

Solution:

1. The timeline of this transaction is shown in the figure below.

   The accumulation of the payments is

$$500 \cdot s_{\overline{12}|} = 500 \cdot \frac{1.08^{12} - 1}{0.08} = 9488.563230.$$

2. The present value of the payments is

$$2000 \cdot a_{\overline{6}|} = 2000 \cdot \frac{1 - 1.08^{-6}}{0.08} = 9245.759328.$$

3. An interest rate of 8% p.a. is equivalent to an effective half-yearly interest rate, denoted by j, of

$$j = 1.08^{1/2} - 1 = 0.039230.$$

   There are 25 payments of 1000 each, starting in six months' time.

   Working in terms of half year, the present value of the payment is

$$1000 \cdot a^{j}_{\overline{25}|} = 1000 \cdot \frac{1 - 1.039230^{-25}}{0.039230} = 15750.003911.$$

1.4.2 Level Annuities Due

An *annuity-due* is an annuity where the payments made at the start of each time period (instead of a the end), i.e. the payments are paid *in advance*.

In order to calculate the present value or accumulation of an annuity due, we first introduce the concept of the rate of discount.

The rate of discount

As opposed to the interest rate where the accumulation of initial investment can be obtained by multiplying it by the accumulation factor $(1 + i)^n$, we can obtain the discounted value of payment by using discount rates.

Suppose an amount of ฿1 is due after 1 year with an effective rate of i% p.a. (see the timeline below). What is the amount of money required to invested now to accumulate to 1?

The amount of money required now to accumulate to ฿1 in one year is

$$v = \frac{1}{1+i}.$$

Note that

$$\frac{1}{1+i} = 1 - \frac{i}{1+i}.$$

We define the effective rate of discount d per annum as

$$d = \frac{i}{1+i}.$$

It follows that

$$v = \frac{1}{1+i} = 1 - \frac{i}{1+i} = 1 - d$$

represents the discount of ฿1 for 1 year using the effective rate of interest of $i\%$ p.a.

Similarly, suppose an amount of ฿1 is due after n year with an effective rate of $i\%$ p.a. The amount of money required to invested now to accumulate to 1 in n year is

$$\frac{1}{(1+i)^n} = (1-d)^n.$$

See the timeline below for illustration.

Example 1.22. *Discount ฿2,000 for 3 years using the effective rate of discount of 5% per annum.*

Solution: After 1 year the discount will be $0.05 \cdot 2000 = 100$, and the discounted value of the payment will be

$$2000 \cdot (1-d) = 2000 \cdot (1-0.05) = 1900.$$

Similarly, after 2 years, the discounted value will be

$$2000 \cdot (1-d)^2 = 2000 \cdot (1-0.05)^2 = 1805.$$

After 3 years, the discounted value of the payment will be

$$2000 \cdot (1-d)^3 = 2000 \cdot (1-0.05)^3 = 1714.75.$$

Example 1.23. *The effective rate of discount d per time unit can be regarded as the interest paid in advance at time 0, which is equivalent to the effective rate of interest i payable in arrears.*

Solution: To show this, suppose that the bank added interest of x to an account of an amount of 1 unit at the start of the period. Assume that the interest amount of x can be withdrawn and invested in another bank that earn the rate of interest $i\%$ effective per time unit. The principle of 1 unit is still in the first bank.

At the end of the year, we have

- the principle of 1 unit in the first bank, and
- the interest paid in advance which accumulates to $x(1+i)$ in the second bank.

For this to be equivalent to the interest paid in arrears, we can find x which solves

$$\begin{aligned} 1 + x(1+i) &= 1+i, \\ x &= \frac{i}{1+i} = \frac{1+i}{1+i} - \frac{1}{1+i} = 1 - v = d. \end{aligned}$$

Therefore, the effective rate of discount d per time unit can be regarded as the interest paid in advance at time 0, which is equivalent to the effective rate of interest i payable in arrears.

Example 1.24. *Let i be the constant effective rate of interest per time unit. Show that the accumulated value of a level annuity due, denoted by $\ddot{s}_{\overline{n}|}$ (pronounced "S-due N", having cashflow of 1 unit at the start of each of the next n time units is*

$$\ddot{s}_{\overline{n}|} = \frac{(1+i)^n - 1}{d}.$$

Solution: Using the previous results, it follows that

$$\begin{aligned}
\ddot{s}_{\overline{n}|} &= (1+i)^n + (1+i)^{n-1} + \cdots + (1+i)^2 + (1+i) \\
&= (1+i) \cdot \left[(1+i)^{n-1} + \cdots + (1+i)^1 + 1\right] \\
&= (1+i) \cdot s_{\overline{n}|} \\
&= (1+i) \cdot \frac{(1+i)^n - 1}{i} \\
&= \frac{(1+i)^n - 1}{i/(1+i)} \\
&= \frac{(1+i)^n - 1}{d}.
\end{aligned}$$

Example 1.25. *Let i be the constant effective rate of interest per time unit. Show that the present value at time 0 of a level annuity due having cashflow of 1 unit at the start of each of the next n time units is*

$$\ddot{a}_{\overline{n}|} = \frac{1 - v^n}{d}.$$

Solution: The present values of the payments can be obtained by discounting $\ddot{a}_{\overline{n}|}$ back to time 0, i.e.

$$\begin{aligned}
\ddot{a}_{\overline{n}|} &= v^n \cdot \ddot{s}_{\overline{n}|} \\
&= v^n \frac{(1+i)^n - 1}{d} \\
&= \frac{1 - v^n}{d}.
\end{aligned}$$

Example 1.26. *Given the effective rate of interest of 8% p.a., calculate*

1. *the accumulation at 12 years of ฿500 payable yearly in advance for the next 12 years.*
2. *the present value now of ฿2,000 payable yearly in advance for the next 6 years.*
3. *the present value now of ฿1,000 payable half-yearly in advance for the next 12.5 years.*

Solution:

1. The accumulation of the annuity-due of 12 years is

$$500 \cdot \ddot{s}_{\overline{12}|} = 500 \cdot \frac{1.08^{12} - 1}{0.08/1.08} = 10247.648289.$$

2. The present value of the annuity-due of 6 years is

$$2000 \cdot \ddot{a}_{\overline{6}|} = 2000 \cdot \frac{1 - 1.08^{-6}}{0.08/1.08} = 9985.420074.$$

3. An interest rate of 8% p.a. is equivalent to an effective half-yearly interest rate, denoted by j, of

$$j = 1.08^{1/2} - 1 = 0.039230.$$

   There are 25 payments of 1000 each, starting in six months' time.

   Working in terms of half year, the present value of the payment is

$$1000 \cdot \ddot{a}^{j}_{\overline{25}|} = 1000 \cdot \frac{1 - 1.039230^{-25}}{0.039230/1.039230} = 16367.876564.$$

1.4.3 Deferred annuities

An annuity whose first payment is made during the first time period (either in arrears or in advance) is called *immediate annuity*. Otherwise, the annuity is known as *deferred* annuity, i.e. the first payment starts some time in the future.

To calculate the present value of the annuity of a series of n payments deferred for m time units (the first payment is due at time $m+1$), denoted by ${}_{m|}a_{\overline{n}|}$, we first calculate the present value at the end of the deferred period and then discount back to the start of the period.

$$\begin{aligned} {}_{m|}a_{\overline{n}|} &= v^{m+1} + v^{m+2} + \cdots + v^{m+n} \\ &= v^m \left(v + v^2 + \cdots + v^n\right) \\ &= v^m \cdot a_{\overline{n}|}. \end{aligned}$$

Example 1.27. *Calculate the present value at time 0 of an annuity of 1 p.a. in arrears for 6 years and deferred for 10 at 6% effective rate p.a.*

This is an annuity with 6 unit payments for which the first payment is at time 11. Hence the present values of such payments is

$$\begin{aligned} {}_{10|}a_{\overline{6}|} &= v^{11} + v^{12} + \cdots + v^{16} \\ &= v^{10} \left(v + v^2 + \cdots + v^6\right) \\ &= v^{10} \cdot a_{\overline{6}|} \\ &= \left(\frac{1}{1.06}\right)^{10} \cdot \left(\frac{1 - 1.06^{-6}}{0.06}\right) \quad = 2.745808. \end{aligned}$$

Example 1.28. *Give the reason or show that the present value of a series of $(n+m)$ payments of one unit payable at the end of each time period is equal to the sum of*

1. *present value of m payments of one units payable at the end of each time period (denoted by $a_{\overline{m}|}$) and*
2. *present value of n payments of one units payable at the end of each time period deferred for m years (denoted by ${}_{m|}a_{\overline{n}|}$).*

Solution: The present value of a series of (m+n) payments is

$$\begin{aligned} a_{\overline{m+n}|} &= \left(v + v^2 + \cdots v^m\right) + \left(v^{m+1} + v^{m+2} + \cdots + v^{m+n}\right) \\ &= a_{\overline{m}|} +_{m|} a_{\overline{n}|}. \end{aligned}$$

It follows that ${}_{m|}a_{\overline{n}|} = a_{\overline{m+n}|} - a_{\overline{m}|}$.

1.4.4 Increasing annuities

An annuity in which the ith payment of the amount i is made at time $t_i = i$ is called an *(simple) increasing* annuity. The present and accumulated value of this annuity can be obtained from the first principles. For example, the present value of the increasing annuity can be evaluated by

$$\sum_{i=1}^{n} X_i v^{t_i} = \sum_{i=1}^{n} i v^{t_i},$$

where the ith payment of amount $X_i = i$ at time $t_i = i$.

Example 1.29. *Derive the formula for the present value of a simple increasing annuity payable yearly in arrears with the effective rate i% p.a. for n years.*

Solution: The cashflows of the simple increasing annuity payable yearly in arrears is illustrated below. The present value of payments of 1 at time 1, 2 at time 2, ..., n at time n denoted by $(Ia)_{\overline{n}|}$ is given by

$$(Ia)^{i}_{\overline{n}|} = \frac{\ddot{a}^{i}_{\overline{n}|} - nv^n}{i}.$$

Notes

1. An increasing annuity but with payments in advance is given by

$$(I\ddot{a})^{i}_{\overline{n}|} = \frac{\ddot{a}^{i}_{\overline{n}|} - nv^n}{d}.$$

2. The formulas for the accumulated values are

$$(Is)^{i}_{\overline{n}|} = \frac{\ddot{s}^{i}_{\overline{n}|} - n}{i} \quad \text{(in arrears)}$$

$$(I\ddot{s})^{i}_{\overline{n}|} = \frac{\ddot{s}^{i}_{\overline{n}|} - n}{d} \quad \text{(in advance)}$$

1.4.5 Compound increasing annuities

The following example considers the value of compound increasing annuities where the payments increase by a constant factor each time.

Example 1.30. *Assume that the effective rate of interest is 6% p.a. Calculate the present value as at 1 January 2010 of an annuity payable annually in arrears for 8 years. The first payment is ฿10 and subsequent payments increase by 2% per annum compound.*

Solution:

At 1/1/2010, the present value of the payment is given by

$$\begin{aligned} PV &= 10 \cdot \frac{1}{1.06} + 10 \cdot \frac{1.02}{(1.06)^2} + \dots + 10 \cdot \frac{(1.02)^7}{(1.06)^8} \\ &= \frac{10}{1.02}\left(\frac{1.02}{1.06} + \left(\frac{1.02}{1.06}\right)^2 + \dots + \left(\frac{1.02}{1.06}\right)^8\right) \end{aligned}$$

The above equation can be arranged so that the annuity formula can be applied. We can define j such that $1 + j = 1.06/1.02$, and hence,

$$\begin{aligned} PV &= \frac{10}{1.02}\left(\frac{1}{1+j} + \left(\frac{1}{1+j}\right)^2 + \dots + \left(\frac{1}{1+j}\right)^8\right) \\ &= \frac{10}{1.02} a_{\overline{8}|} \quad \text{at } j\% \\ &= \frac{10}{1.02}\left(\frac{1 - \left(\frac{1.02}{1.06}\right)^8}{\left(\frac{1.06}{1.02} - 1\right)}\right) \\ &= 66.2216 \end{aligned}$$

1.4.6 Annuities payable more than once per time unit

Consider the value of an annuity payable in arrears m times per time unit at an effective rate of interest i per time unit. The annuity is still payable for n time units and a total amount of 1 unit per time unit. The present and accumulated values of the corresponding annuity are denoted by $a_{\overline{n}|}^{(m)i}$ and $s_{\overline{n}|}^{(m)i}$, respectively.

To calculate either the present or accumulation value of this annuity, we can simply apply the first principles by using the effective rate of interest per $1/m$ time unit. In particular, we have

$$a_{\overline{n}|}^{(m)i} = \frac{1}{m} a_{\overline{n \cdot m}|}^{j},$$

and

$$s_{\overline{n}|}^{(m)i} = \frac{1}{m} s_{\overline{n \cdot m}|}^{j},$$

where j is the effective rate per $1/m$ time unit.

Example 1.31. *Calculate the accumulation at 1 January 2020 of an annuity of ฿100 per month, payable in arrears from 1 January 2010 at an effective rate of interest of 4% p.a.*

Solution: The annual payment is 1200 and the effective rate per month equivalent to 4% p.a. is $j = (1.04)^{1/12} - 1 = 0.003274$ per month. Hence,

$$1200 s_{\overline{10}|}^{(12)4\%} = 100 s_{\overline{12 \cdot 10}|}^{j} = 14669.59.$$

1.5 Nominal Rates of Interest

Nominal interest rates are the interest rates before taking inflation into account. They may also refer to the advertised (in bank accounts) or stated rates of interest on a loan, without regard to fees or compound interest. Throughout this section, the time unit used is assumed to be **one year**.

- **Effective rate of interest** is the interest i paid at the end of the year on an amount ฿1 at the start of the year.
- **Nominal interest rate payable p times per period**, denoted by $i^{(p)}$ is an effective rate of interest of $i^{(p)}/p$ applied for each pth of a period. The interest is paid more frequently than once per measurement period.

The nominal rate of interest payable p times per period is also known as **the rate of interest convertible $p$thly or compounded $p$thly.**

Example 1.32. *A nominal rate of interest of $i^{(4)} = 10\%$ p.a. convertible quarterly means an interest rate of 10/4 = 2.5% per quarter effective. Calculate the accumulated value in 1 year of a payment of ฿100 at the given nominal rate.*

Solution: When working with the nominal interest rate, the nominal interest rate is often converted to an effective interest rate. In this example, the nominal interest rate $i^{(4)} = 10\%$ is equivalent to an effective interest rate of 2.5% per quarter. The accumulated value in 1 year is $100(1 + 2.5\%)^4 = 110.3813$. After compound interest is taken into account, the interest income of an investor at the quarterly convertible nominal interest rate of 10% p.a. is 10.3813 (or 10.3813%. p.a. effective)

Nominal is used where interest is paid more frequently than once per unit year.

Example 1.33. *At a rate of 12% p.a. effective, draw a timeline to show cashflows if ฿100 is invested at the start of the year.*

Solution: The accumulated value of ฿100 at the end of the year is $100(1 + 12\%) = 112$.

Example 1.34. *At a rate of 12% p.a. compounding quarterly, draw a time line to show cashflows if ฿100 is invested at the start of the year.*

Solution: The nominal interest rate $i^{(4)} = 12\%$ is equivalent to an effective interest rate of 3% per quarter. The accumulated value in 1 year is $100(1 + 3\%)^4 = 112.55$. After compound interest is taken into account, the interest income of an investor at the quarterly convertible nominal interest rate of 10% p.a. is 12.55 (or 12.55%. p.a. effective)

$i^{(p)}$ is an effective rate of interest of $i^{(p)}/p$ applied for each pth of a period. The interest is paid more p times per measurement period (i.e. per year). The value at time can be regarded as the annuity having cashflow of $i^{(p)}/p$ per each period as shown in the figure below. Therefore, the accumulated value in 1 year can also be calculated as $100(1 + 0.03s_{\overline{4}|}^{3\%})$.

Note In practice, it is easier to work with the effective rate of interest which is defined in a suitable time unit.

The following formula can be used to convert between the effective rate i p.a. and the nominal rate $i^{(m)}$ p.a.:

$$(1+i) = \left(1 + \frac{i^{(m)}}{m}\right)^m .$$

Example 1.35.
1. *Express a nominal annual interest rate of 9% convertible half-yearly as a monthly effective interest.*
2. *Express a two-monthly effective interest of 3% as a nominal annual interest rate convertible two-monthly.*

Solution:

1. The effective rate $i\%$ p.a. is
$$i = (1 + \frac{0.09}{2})^2 - 1.$$
Hence the monthly effective rate is $j = (1+i)^{1/12} - 1 = (1 + \frac{0.09}{2})^{2/12} - 1 = 0.007363$.
2. A nominal annual interest rate convertible two-monthly is $6 \cdot 3\% = 18\%$.

Example 1.36. *Express each of the following effective rates per annum as a nominal rate, and vice versa.*

Effective Rate	*Nominal Rate*
$i = 0.04$	$i^{(4)} = 0.039412$
$i = 0.10$	$i^{(12)} = 0.095690$
$i = 0.06152$	$i^{(2)} = 0.06$
$i = 0.126825$	$i^{(12)} = 0.12$

1.5.1 Nominal Rates of Discount

The effective rate of discount per annum is $d = 1 - v$. It is the amount of interest payable at the start of the time unit which is equivalent to i payable at the end of the time unit.

The nominal rate of discount payable p times per period $d^{(m)}$ (or convertible $p$thly or compounded $p$thly is interest of $d^{(m)}/m$ paid at the start of each $1/m$ of a year.

The relationship between the effective discount rate d p.a. and the nominal rate of discount payable m times a year is

$$1 - d = \left(1 - \frac{d^{(m)}}{m}\right)^m .$$

Example 1.37. 1. *Express a nominal annual discount rate of 6% convertible half-yearly as an annual effective discount.*

2. *Express an effective discount of 10% per half year as a nominal annual discount rate convertible quarterly.*

Solution:

1. The annual effective discount d is

$$d = 1 - \left(1 - \frac{d^{(m)}}{m}\right)^m = 1 - \left(1 - \frac{6\%}{2}\right)^2 = 0.0591 = 5.91\% \text{ per annum.}$$

2. We know that the discount factor v and the rate of discount d satisfy the following equation.

$$v = 1 - d.$$

Hence the discount factor over half-year is $1 - 0.1 = 0.9$. The discount factor v for one year (or 2 half-year) is

$$(1 - 0.1)^2 = 0.9^2 = 0.81.$$

It follows that the discount rate per annum is $d = 1 - 0.81 = 0.19$, and the nominal annual discount rate convertible quarterly $d^{(4)}$ is given by

$$d^{(m)} = m \cdot \left(1 - (1 - d)^{1/m}\right)$$
$$d^{(4)} = 4 \cdot \left(1 - (1 - 0.19)^{1/4}\right) = 0.205267.$$

1.6 Principle of Equivalence, Yields and Equation of Value

The principle of equivalence is used to compare two different cashflows whether one is worth more than the other.

Consider two sequences of cashflows

- $C_1, C_2, \ldots$ with payments at times $t_1, t_2, \ldots$ and
- $D_1, D_2, \ldots$ with payments at times $s_1, s_2, \ldots$.

Assume that the interest rates are given and apply to both of them. The two sequences of cashflows are said to be **equivalent** (or equal in value) if their values at any time t are the same, i.e. there exists $t \in \mathbb{R}$ such that

$$PV^C(t) = PV^D(t).$$

Notes

1. If two sequences of cashflows have the same value at time s, then they have the same value at any time t since

   - for $t \leq s$,

     (Value at time t) = (Value at time s) $\times V(t, s)$,

   - for $t \geq s$

     (Value at time t) = (Value at time s) $\times A(s, t)$.

2. The two sequences of cashflows are **indifferent** if their present values are the same.

3. The principle of equivalent can be applied for **pricing a financial security**, for example, a price P which will be paid by the investor in return for a series of future cashflows.

Example 1.38. *Calculate the maximum price an investor wish to pay in return for an investment that will pay ฿500 at the end of each of the next 15 months given that the interest rate is 0.2% per month.*

The present value of these payments of 500 at the end of the next 15 months is

$$PV(0) = 500a_{\overline{15}|}^{0.002} = 500 \cdot \left(\frac{1 - (1.002)^{-15}}{0.002} \right) = 7381.35.$$

Therefore, the investor would be willing to pay a maximum of 7381.35.

Example 1.39. *Determine whether the following series of cashflows are equivalent given that an interest rate is 6% per annum effective.*

1. *One single payment of amount 6,691.127888 at year 5.*
2. *a level annuity of 300 payable yearly in arrears for the next 5 years plus a lump sum of 5,000.*
3. *a level annuity of 1,186.982002 payable yearly in arrears for the next 5 years.*

Solution:

1. The present value is $6,691.127888 \times (1.06)^{-5} = 5000$.
2. The present value is
$$300a_{\overline{5}|}^{0.06} + 5000 \times (1.06)^{-5} = 5000.$$
3. The present value is
$$1186.982002a_{\overline{5}|}^{0.06} = 5000.$$

Therefore, the three series of cashflows are **indifferent**.

1.6.1 Equation of value and yields

Consider a transaction from an investment that offers

- to pay an investor of amounts (i.e. money received) $B_1, B_2, \dots, B_n$ at time $t_1, t_2, \dots, t_n$
- in return for outlays (i.e. money paid out) of amounts $A_1, A_2, \dots, A_n$ at these times, respectively.

Only one of A_i and B_i will be non-zero in general.

An equation of value equates the present value of money received to the present value of money paid out, which can be written as

$$\sum_{i=1}^{n} A_i v^i = \sum_{i=1}^{n} B_i v^i.$$

The equation of value can also be written in terms of the **net cashflow** at time t_i, i.e. $C_t = B_t - A_t$,

$$PV_i(0) = \sum_{i=1}^{n} C_i v^i = 0.$$

Equations of value are used throughout actuarial work. Some examples are as follows:

- The **fair price** to pay for an investment such as a fixed interest security or an equity (ie, PV outgo) equals the present value of the proceeds from the investment, discounted at the rate of interest required by the investor.
- The **premium** for an insurance policy is calculated by equating the present value of the expected amounts received in premiums to the present value of the expected benefits and other outgo.

We shall be concerned mainly with the question:

At what rate of interest does the series of amounts paid out have the same value as the series of amounts received? The corresponding rate of interest is called the **yield of the cashflows** (or **internal rate of return, money-weighted rate of return**).

Notes

1. Equations of values may have no roots, a unique root or multiple roots.
2. In most practice situations, there is a unique positive real root.

Example 1.40. *An investor pays ฿1,000 in order to receive ฿600 back in 2 years and ฿800 back in 4 years. Calculate the annual effective rate of interest earned on this investment (or the yield on the investment).*

Solution: The yield of the investment $i\%$ satisfies the equation of value

$$PV_i(0) = -1000 + 600(1+i)^{-2} + 800(1+i)^{-4} = 0.$$

To solve the equation for i, we define $z = (1+i)^{-2}$, resulting in

$$8z^2 + 6z - 10 = 0.$$

Therefore $z = 0.804248$ and $i = 0.115078$.

Example 1.41. *An investor pays ฿1,000 in order to receive ฿300 back at the end of the first 2 years and ฿400 back at the end of the third, forth and fifth year. Calculate the annual effective rate of interest earned on this investment (or the yield on the investment).*

Solution: The yield of the investment $i\%$ p.a. satisfies the equation of value

$$PV_i(0) = -1000 + \frac{300}{(1+i)} + \frac{300}{(1+i)^2} + \frac{400}{(1+i)^3} + \frac{400}{(1+i)^4} + \frac{400}{(1+i)^5}.$$

In our next section, we will learn how to approximate the yield of the above equation.

1.6.2 The method to estimate the yield

By using linear interpolation, the yield can be estimated as follows. Let P_1 and P_2 be the present values calculated at interest rates i_1 and i_2, respectively. Then the interest rate corresponding to a present value of P can be approximated by

$$i \approx i_1 + (i_2 - i_1)\frac{P - P_1}{P_2 - P_1}.$$

In order to apply this method to calculate the yield i, we simply set $P = 0$, and hence

$$i \approx i_1 + (i_2 - i_1)\frac{-P_1}{P_2 - P_1}.$$

From the figure above, the yield i can be approximated by i^*, which is the x-intercept of the straight line joining the points (i_1, P_1) and (i_2, P_2). From

$$\frac{i^* - i_1}{i_2 - i_1} = \frac{P_{i^*} - P_1}{P_2 - P_1},$$

we have $P_{i^*} = 0$ and

$$i \approx i^* = i_1 + (i_2 - i_1)\frac{-P_1}{P_2 - P_1}.$$

Note that one can get a good approximation by taking values that are either side of the true value and about 1% apart.

Example 1.42. *Approximate the yield of the transaction in Example 1.41.*

Solution: Here, When $i_1 = 0.21$, $P_1 = PV_{0.21}(0) = 19.448$ and when $i_1 = 0.22$, $P_2 = PV_{0.22}(0) = -3.698$. The yield is approximately equal to

$$\begin{aligned} i &\approx 0.21 - (0.22 - 0.21)\left(\frac{19.448}{-3.698 - 19.448}\right) \\ &= 0.218402 \text{ p.a. effective.} \end{aligned}$$

1.7 Loan schedules

In this section, we describe how a loan may be repaid. A schedule of repayment together with the interest and capital components of an annuity payment will be discussed.

Suppose that a lender lends an individual of amount L for n years with an effective rate of interest i per annum. We say that the **term** of the loan is n years with the loan **amount** of L. How could we repay the loan?

Repay as late as possible:

After n year, the borrower repays the entire loan and all interest that accrued over the period. The total amount to be repaid is equal to

Repay interest only during the term and repay the capital at the end of the term:

These types of loan where the borrower is a government or a company are **bonds** or **fixed interest securities**.

Repay loan by regular instalments of interest and capital throughout term of loan:

Each repayment must pay first for interest due and the remainder is used to repay some of the capital outstanding.

Example 1.43. *You borrow ฿5,000 for a term of 3 years at a fixed interest rate of 10% pa. The loan is to be repaid by 3 level annual repayments of ฿2,010.57 at the end of each year. Calculate the interest content, capital content from each repayment and capital outstanding after such repayment.*

Note The loan payments can be expressed in the form of a **Loan Schedule** as follows:

Time	Repayment	Intest content	Capital content	Capital outstanding
0				5000
1	2010.57	500	1510.57	3489.43
2	2010.57	348,943	1661.627	1827.80
3	2010.57	182.780	1827.79	0.01

1.7.1 The loan schedule

A more general form of loan payments can be expressed as follows: Let

- L_t be the amount of the loan outstanding at time t.
- X_t be the instalment at time t (all instalments may not be the same amount).
- i be the effective rate of interest per time unit charged on the loan.

ime Rep	ayment Int	est content Ca	pital content	Capital outstanding
0				L_0
1 \$	X_1\$	iL_0 \$	(X_1 - iL_0)\$	$L_1 = L_0 - (X_1 - iL_0)$
2 \$	X_2\$	iL_1 \$	(X_2 - iL_1)\$	$L_2 = L_1 - (X_2 - iL_1)$
⋮				
t \$	X_t\$ \$	iL_{t-1}\$ \$(X	t - iL{t-1})\$ \$L_	t = L_{t-1} - (X_t - iL_{t-1})\$
⋮				
n\$ \$	X_n\$ \$	iL_{n-1}\$ \$(X	n - iL{n-1})\$	0

Note The capital outstanding after the kth payment is $Xa_{\overline{n-k|}}$, which is the present value of future repayments. This holds even when the repayments and interest rates are not constant.

Example 1.44. *A loan of ฿20,000 is repayable by equal monthly payments for 4 years, with interest rate payable at 10% pa effective.*

1. *Calculate the amount of each monthly payment.*
2. *Calculate the interest and capital contents of the 25th repayment.*

Solution:

1. The loan is repaid by level instalments of amount X payable monthly. Working in months, we define j% per month effective equivalent to 10% pa effective. We have

$$j = (1.1)^{(1/12)} - 1 = 0.007974.$$

   The loan equation followed the equation of value is given by

$$PV_j(0) = 20000 - Xa^{j}_{\overline{48|}} = 0$$

   Solving for X gives $X = 503.12$.

2. The capital outstanding after 24th repayment $= L_{24} = Xa^{j}_{\overline{24|}} = 10950.23$. Hence, the interest content of the 25th repayment $= j \cdot L_{24} = 87.32$. The capital content of the 25th repayment $= X = 503.12 - 87.32 = 415.8$.

1.7.2 Changing the term of a loan

The term of the loan can be changed in the following circumstances:

- extend or shorten the term,
- miss a number of payments,
- repay part of the loan early.

The repayment amount will then need to be calculated according to the condition(s) as given in the change.

Example 1.45. *A person takes out a loan of ฿100,000 to be repaid by level monthly instalments in arrears over 7 years where the bank charges an effective annual rate of interest of 6%*

1. *Calculate the monthly repayment* ***Solution:*** *Working in months, we define j% per month effective equivalent to 6% pa effective.*
$$j = (1.06)^{(1/12)} - 1 = 0.007974.$$
The loan equation followed the equation of value is given by
$$PV_j(0) = 100000 - Xa^j_{\overline{84}|} = 0$$
Solving for X gives $X = 1453.25$.

2. *Calculate the new repayment amount if the the term of loan can be extended by 1 year, immediately after the 60th repayment has been made.* ***Solution:*** *The capital outstanding after 60th repayment =* $L_{60} = Xa^j_{\overline{24}|} = 32842.48$. *Now the remaining term becomes 3 years (or 36 months). The new repayment amount X' satisfies*
$$PV_j(0) = 32842.48 - X'a^j_{\overline{36}|} = 0.$$
Solving for X' gives $X' = 996.77$.

3. *Instead of extending the term, the person had requested to miss the 61st and 62nd repayments. Calculate the remaining installments.* ***Solution:*** *After missing the 61st and 62nd repayments, the capital outstanding at time 62 =* $L_{60} \cdot (1 + j)^2 = 32842.48(1.004868)^2 = 33162.99$. *Hence, the remaining number of payments is 22.*

4. *Calculate the new repayment amount if the person repaid ฿10,000 at the time he made the 60th repayment together with the 60th repayment.* ***Solution:*** *The revised capital outstanding after repayment of 10000 (the 60th repayment) is* $32842.48 - 10000 = 22842.48$. *The new repayment amount X'' satisfies*
$$PV_j(0) = 22842.48 - X''a^j_{\overline{24}|} = 0.$$
Solving for X'' gives $X'' = 1010.76$.

1.7.3 Changing the interest rate

The interest rates for a loan can vary during the term of the loan. The reasons for varying rates of interest could be the following:

1. interest rates have been planned to changed during the term, for example the borrower would repay less during the beginning of the loan, or
2. the lender changes the rates of interest to reflect the market conditions.

Example 1.46. *You borrow ฿20,000 for a term of 20 years to be repaid by level annual instalments. The rate of interest will be 7% pa effective for the first 10 years and 8% pa effective thereafter. Calculate the annual repayment.*

Solution: Let X be the annual repayment. Using an equation of value, we have
$$20000 = Xa^{7\%}_{\overline{10}|} + (1.07)^{-10}Xa^{8\%}_{\overline{10}|}.$$
Then solving for X gives $X = 1916.69$.

Example 1.47. *You borrow ฿20,000 for a term of 15 years to be repaid by level annual instalments where the bank charges an effective annual rate of interest of 6%. After the 10th repayment has been made, the bank raises the interest rate to 6.5% pa effective. Calculate the new repayment amount.*

Solution: The annual repayment X **for a term of 15 years** before the adjustment of interest rate.

$$X = \frac{20000}{a_{\overline{15}|}^{6\%}} = 2059.26.$$

However, after the 10th repayment has been made, the bank raises the interest rate to 6.5% pa effective. Therefore, the capital outstanding after the 10th repayment $= L_{10} = Xa_{\overline{5}|}^{6\%} = 8674.332$. After the adjustment of the interest rate, the new repayment amount X' satisfies satisfies

$$PV_{6.5\%}(0) = 8674.332 - X'a_{\overline{5}|}^{6.5\%} = 0.$$

Solving for X' gives $X' = 2087.34$.

Chapter 2

Bonds and Inflation

2.1 Bonds

A government or corporation can raise money in the capital markets by issuing *fixed interest securities (FIS)*, also called *bonds*. Bonds are a form of medium and long-term securities.

This means that investors will lend money to the issuer (for e.g. the government or coporation) and in return will receive fixed interest payments known as *coupons* at fixed dates plus repayment of the loan at the end of the term.

Note The loan is usually split into smaller units that can be traded on a stock exchange. For example, a company raises $\mathbb{B}$1,000,000,000 by issuing 10,000,000 bonds, each one a loan of face value $\mathbb{B}$100. These can be bought and sold on a stock exchange.

2.1.1 Characteristics of Bonds

1. The *nominal amount or face value* of a bond is the amount of the loan it represents. The nominal amount is usually $\mathbb{B}$1,000 (without further specific, we will set the nominal amount to be $\mathbb{B}$100.)
2. The interest payments are called *coupons*, usually expressed as a percentage per year of the nominal amount. The rate of interest denoted by D is also known as *coupon rate*. They are always **in arrears**.
3. Coupons are usually expressed as the amount of interest payable in a year, but are paid half-yearly (twice per year) or quarterly (4 times per year)
4. *Coupon dates* are the dates on which the bond issuer will make interest payments.
5. Bonds have *maturity dates* at which point the principal amount must be paid back in full.
6. The loan is repaid or redeemed at the end of the term. The redemption amount per 100 nominal is the *redemption rate*, often expressed as a percentage.

   A loan is redeemed

   - at a premium if redemption rate $>$ 100%
   - at par if redemption rate $=$ 100%
   - at a discount if redemption rate $<$ 100%
7. Many corporate and government bonds are publicly traded; others are traded only over-the-counter (OTC) or privately between the borrower and lender.

Example 2.1. *Each bond of ฿100 nominal value carries coupons of 6% pa payable half-yearly.*

Solution: The coupon rate of 6% pa payable half-yearly means that bondholders will receive

$$\frac{6\%}{2} \times 100 = \text{ 3 every half-year.}$$

Example 2.2. *An investor purchases ฿95 for a 5-year fixed interest bond with face value (or nominal amount of) ฿100. The bond pays coupon of 6% pa half-yearly in arrear and the lump sum equal to the nominal amount in 5 years' time. The cashflows related to the payments of the bond can be shown as follows:*

Solution: Cashflows are given in the following table.

Time (year)	0	0.5	1	1.5	...	5
Cashflow	-95	3	3	3	...	$3 + 100$

Examples of bonds include

- domestic bonds issued in the domestic currency such as gilts issued by UK government and treasury bonds issued by US government.
- Eurobonds where an issuer sells the bond outside the domestic country.
- debenture bonds issued by corporations.

Example 2.3. *A company issues a 10-year bond, to be redeemed at 102%, with coupon of 6% pa payable half-yearly in arrears. The nominal amount of each bond is ฿100. What repayments are made?*

Solution: Cashflows are given in the following table.

Time (year)	0	0.5	1	1.5	...	9.5	10
Cashflow	P	3	3	3	...	3	$3 + 102$

Here the bond price in the table above is denoted by P.

The following questions may be asked:

- At what price should be paid by an investor to obtain a net yield of i per annum?
- Given the price of the fixed interest bond, what is the net yield per annum will be obtained?

2.1.2 Bond prices

Given a yield i%, the price of a bond can be computed by discounting all the future payments received net of any tax.

Example 2.4. *A tax-exempt investor buy the bond in Example 2.3 on its issue date. Calculate the price the investor should pay to obtain a yield of 9%.*

Solution: Appplying the Principle of Equivalence: by equate the present value of the future incomes at 9% with the unknown price P.

$$P = \frac{6}{2} a^{j}_{\overline{20}|} + 102(\frac{1}{1.09})^{20},$$

where $j = (1.09)^{(1/2)} - 1 = 0.044031$. This gives $P = 82.44$. It should be emphasised that the price in this example differs from the nominal amount of 100.

Example 2.5. *Refer to Examples 2.3 and 2.4. After the 10th coupon has been paid, the investor sells the bond. At that time the market yield on comparable 5-year bonds is 7% pa effective.*

1. *Calculate the price that he will obtain.*
2. *Calculate the yield that the first investor obtains if he paid ฿82.44 and received 10 coupon payments of ฿3 each and sold the bond after 5 years at the price of ฿97.75.*

Solution:

1. The remaining payments are shown in the table below.

Time (half-year)	10	11	12	13	...	19	20
Cashflow	–	3	3	3	...	3	$3 + 102$

Working in unit of half-year, we first calculate the effective rate per half-year, denoted by k, that is equivalent to $i = 7\%$.

$$k = (1.07)^{(1/2)} - 1 = 0.034408.$$

The price P' that he will obtain follows from the following equation.

$$P' = 3a_{\overline{10}|}^{k} + 102\left(\frac{1}{1.07}\right)^5 = 97.75.$$

2. The payments of the investor are shown in the table below.

Time (half-year)	0	1	2	3	...	9	10
Cashflow	-82.44	3	3	3	...	3	$3 + 97.75$

Let i denote the yield (per half-year) that the first investor obtains. Working at time 10, the equation of value of the cashflows is

$$f(i) = 82.44(1+i)^{10} - 3s_{\overline{10}|}^{i} - 97.75 = 0.$$

By linear interpolation, we have

$$f(0.05) = -1.1976, \quad f(0.06) = 10.3451,$$

and hence, the approximate of i is $i^* \approx 0.051038$. The annual yield is then approximately equal to $(1 + 0.051038)^2 - 1 = 10.468\%$

Notes

1. If the investor is not subject to taxation, the yield is called the *gross yield*.
2. The yield obtained by holding the bond until redemption is called *redemption yield*. This is quoted in financial newspapers.
3. If a bond is sold before redemption, the yield that the investor obtains is called *realised yield*. This yield depends on both the buying and selling prices.

Notes

1. There is an inverse relationship between the bond prices and yields.

2. The nominal amount of the loan is just a theoretical figure on which the coupon and redemption rates are based.
3. The amount of capital the borrower can actually raise on the issue date is the price that investors are willing to pay for the future income stream and also set by supply and demand.
4. The investors also consider the **credit risk** of the borrower. It is the risk that they might default on interest and capital payments.
5. The greater the credit risk, the higher the yield they will require.
6. The bonds can be traded on an exchange at any time until it is redeemed. The prices will depend on the remaining term to redemption and market conditions such as the yields obtainable on other investments.

2.1.3 No tax

A tax-exempt investor receives the full amount of the coupon and redemption payments. The price of an n year fixed interest bond which pays coupons of D per annum payable p thly in arrear and has redemption amount R is

$$P = Da^{(p)}_{\overline{n}|} + Rv^n$$

at rate i per annum.

In practice, we can calculate by using a suitable time period, for example a period of half a year as shown in the previous examples. Then the formula can be written as

$$P = \frac{D}{2}a_{\overline{2n}|} + Rv^{2n}$$

2.1.4 Income tax

Suppose an investor is subject to income tax at rate t_1 on the coupons, which is due at the time that the coupons are paid. Tax will affect both yields and bond prices. In general, the price of this bond, an n year fixed interest bond which pays coupons of D per annum payable p thly in arrear and has redemption amount R is

$$P = (1 - t_1)Da^{(p)}_{\overline{n}|} + Rv^n$$

at rate i per annum. Here the rate is called the *net yield*.

Example 2.6. *An investor liable to income tax at 30% buys a 15-year fixed interest bond which is redeemable at par and pays coupons of 8% pa half-yearly in arrear. Calculate the price the investor should pay to obtain a net yield of 9% pa.*

Solution: Coupon payments after tax are

$$(1 - t_1)D = (1 - 0.3)\frac{8\%}{2} \times 100 = 2.8.$$

The payments of the investor are shown in the table below.

Time (half-year)	0	1	2	3	...	29	30
Cashflow	P	2.8	2.8	2.8	...	2.8	$2.8 + 100$

The price the investor should pay to obtain a net yield of 9% pa can be calculated from

$$P = 2.8a_{\overline{30}|}^{j} + 100(\frac{1}{1+j})^{30} = 73.59,$$

where where $j = (1.09)^{(1/2)} - 1 = 0.044031$ per half-year effective.

2.1.5 Capital gains tax (CGT)

Capital gains tax is a tax levied on the capital gain made on the redemption of a bond (or the sale of the bond if sold earlier). The capital gain is simply the difference between

Note that **capital gain** refers to an increase in a capital asset's value and is considered to be realized when the asset is sold. A **capital loss** is incurred when there is a decrease in the capital asset value compared to an asset's purchase price.

Example 2.7. *An investor liable to the capital gains tax at 20% purchases two bonds*

- *Bond A for the price of 𝔹102 and*
- *Bond B for the price of 𝔹98.*

Calculate the capital gains tax on each bond if the investor sells them both one year later for 𝔹100 each.

Solution:

- Bond A: Tax payment is $0.2 \times \max\{100 - 102, 0\} = 0$ (i.e. capital loss)
- Bond B: Tax payment is $0.2 \times \max\{100 - 98, 0\} = 0.4$ (i.e. capital gain)

Here we assume that no 'relief', i.e. we cannot add the two bonds together and say we bought the two bonds for 𝔹200 and sold the bonds for 𝔹200.

Notes

1. Similar to income tax, the price of the bond is then calculated on the net redemption received after tax has been deducted.
2. When the purchase and sale (or redemption) prices are known, it it easy to calculate the yield.
3. In contrast, as the price depends on whether there is a capital gain and the capital gain depends on the price, it is not easy to calculate the price for a given redemption yield. There is a test to determine whether CGT is payable.

Example 2.8. *An investor liable to the capital gains tax at 20% purchases a 10-year bond with an annual coupon of 8% pa payable yearly in arrear, to be redeemed at par.*

1. *Calculate the redemption yield the investor obtain if the price is 𝔹96 per 𝔹100 nominal.*
2. *What price should the investor pay to obtain a yield of 7% pa effective?* (see note below)
3. *What price should the investor pay to obtain a yield of 9% pa effective?*

Note In this example, we can guess whether CGT is payable because we have calculated the price to yield 8.546% from the above question. Then we can use the appropriate equation to calculate the bond price and check whether or not our initial guess was correct.

2.2 Inflation

Inflation is a measure of the rate of change in the price of consumer goods and services, such as food and beverages, transportation and housing.

- In Thailand or US, it is measured with reference to a **consumer price index** (or CPI). In UK, it is measured in terms of **retail price index** (RPI).
- Bureau of Trade and Economic Indices reports the CPI on a monthly basis.
- High inflation means that prices increase quickly and hence **the purchasing power** significantly decreases.

Let $Q(t)$ be the CPI at time t. Then the rate of inflation per year denoted by $q(t)$ is

$$q(t) = \frac{Q(t)}{Q(t-1)} - 1.$$

The average inflation rate per year between time s and t, denoted by $\bar{q}$, satisfies

$$(1+\bar{q})^{t-s} = \frac{Q(t)}{Q(s)},$$

and hence

$$\bar{q} = \left(\frac{Q(t)}{Q(s)}\right)^{1/(t-s)} - 1.$$

Example 2.9. *A set of goods costs ฿98.25 in January 2013 and ฿101.44 in January 2018. Calculate the average inflation rate $\bar{q}$ over this period.*

Solution: The increase from Jan 2013 to Jan 2018 (5 years) is

$$\frac{101.44}{98.25} - 1 = 0.032.$$

The average of inflation rate $\bar{q}$ is given by

$$(1+\bar{q})^5 = 1.032.$$

Therefore $\bar{q} = 0.64\%$.

Therefore, ฿1 in January 2013 buys as much as ฿1.032 in January 2018.

The following table shows the monthly consumer price indices from January 2011 to December 2019. Source: http://www.price.moc.go.th/

Table 2.6: The monthly consumer price indices from January 2011 to December 2019.

Year	1	2	3	4	5	6	7	8	9	10	11	12
2011	91.93	92.3	92.75	94.03	94.35	94.47	94.64	95.05	94.73	94.91	95.12	94.66
2012	95.03	95.38	95.95	96.35	96.73	96.89	97.22	97.61	97.93	98.06	97.71	98.09
2013	98.25	98.46	98.52	98.68	98.92	99.07	99.17	99.16	99.32	99.49	99.59	99.73
2014	100.15	100.39	100.6	101.1	101.51	101.4	101.32	101.23	101.06	100.96	100.84	100.33
2015	99.74	99.86	100.03	100.05	100.22	100.32	100.25	100.03	99.98	100.18	99.86	99.47
2016	99.21	99.36	99.57	100.11	100.68	100.71	100.36	100.32	100.36	100.52	100.46	100.59
2017	100.75	100.79	100.33	100.49	100.64	100.66	100.53	100.64	101.22	101.38	101.45	101.37
2018	101.44	101.21	101.12	101.57	102.14	102.05	102.00	102.27	102.57	102.63	102.40	101.73
2019	101.71	101.95	102.37	102.82	103.31	102.94	103.00	102.80	102.90	102.74	102.61	102.62

Figure 2.1 shows the monthly consumer price indices from January 2011 to December 2019.

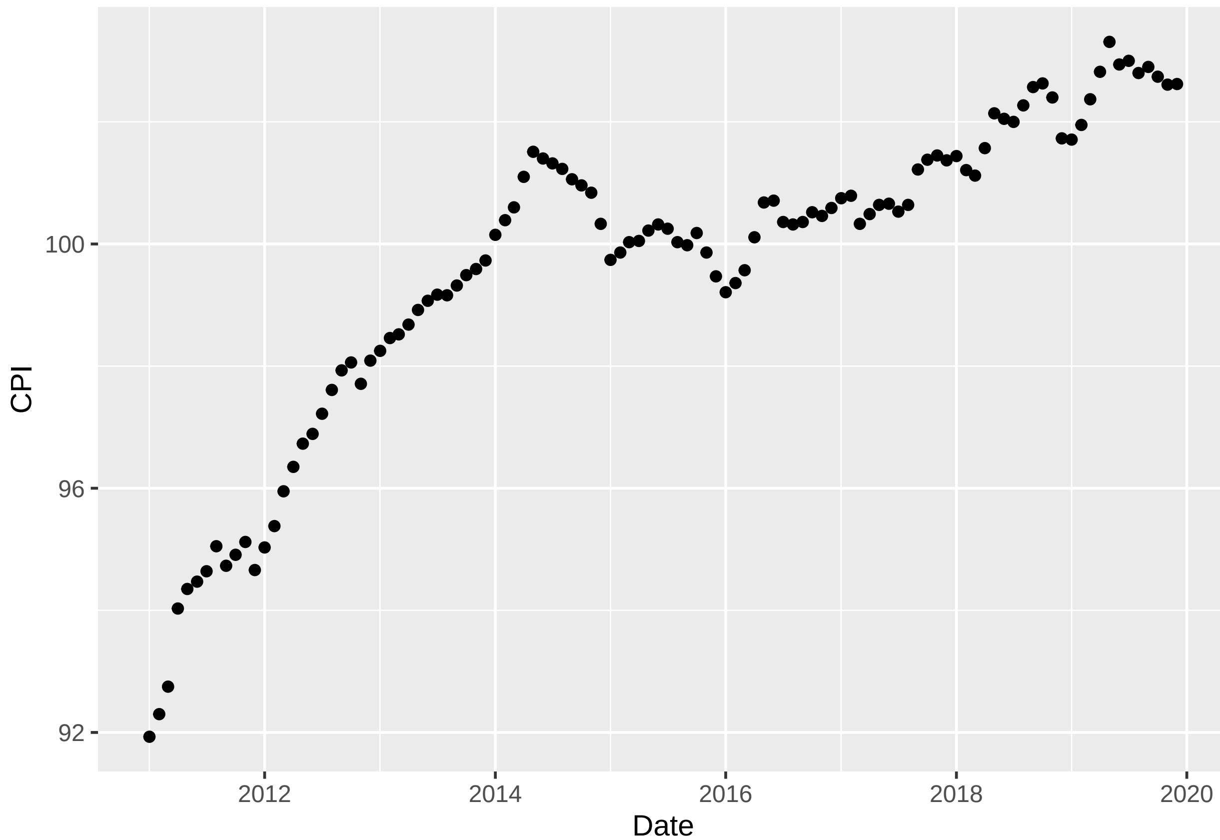

Figure 2.1: The monthly consumer price indices from January 2011 to December 2019

$$fig : digraph$$

Example 2.10. *An investment contract made on January 2018 promises to pay an investor of ฿10,000 in 5 years' time. Assume the average inflation rate at* $\bar{q} = 0.64\%$ *for the next 5 years.*

If a bowl of noodles costs ฿100 in 2018, then ฿10,000 could buy 100 bowls. How many bowls of noodles would the payment of ฿10,000 buy in the next 5 years?

Solution: *In Jan 2018, ฿100 buys as much as* $100(1.0064)^5 =$ *฿103.2 in Jan 2023.*

So ฿10000 in Jan 2023 could buy

$$\frac{10000}{103.2} \approx 96 \text{ bowls.}$$

Notice that we divide by $(1+\bar{q})^5$ *to calculate how much your money is worth at the end of the next five years.*

- *The quantity of goods that can be bought with 10,000 in January 2023 reduces from 100 to 96 bowls.*
- *The effect of inflation results in the reduction of the purchasing power of a unit of money in January 2023 compared to that in January 2018.*
- *The amount of ฿10,000 is referred to as the **monetary** (or **nominal**) payment in 5 years. This is the amount of money that change hands.*

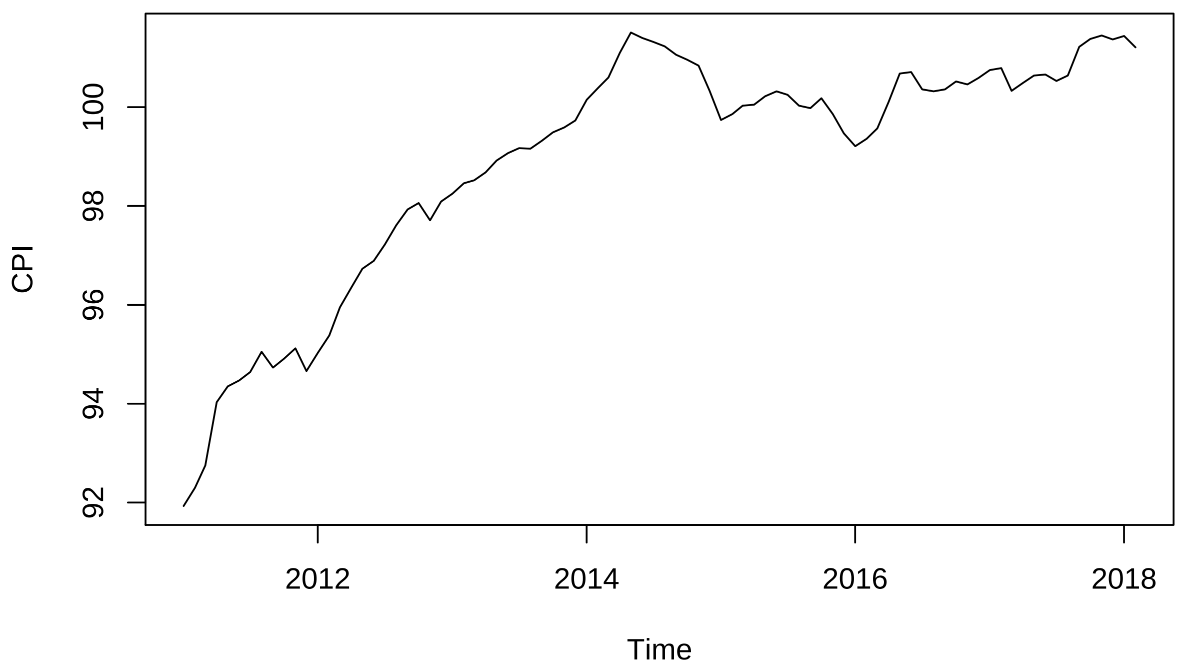

Figure 2.2: image

- *The **real payment of ฿10,000 (due at time 5 years) in time 0 unit** is*

$$\begin{aligned} 10000\frac{Q(0)}{Q(5)} &= 10000\frac{1}{(1.0064)^5} \\ &= 9686.05. \end{aligned}$$

  Here we have less purchasing power with your money at the end of the five years than you had at the start of the year.

- *The real payment (in time 0) is the purchase power of 10,000 paid in 5 years relative to today. **It is the amount of cash in hand at the end of the period reduced for the effects of inflation**.*

- *In general, ฿X at time t has the purchasing power relative to time s of*

$$X \cdot \frac{Q(s)}{Q(t)}.$$

2.2.1 Real rates of interest

The rate of interest which is calculated using monetary payments is called a **money (or monetary or nominal) rate of interest**.

The **real rate of interest** is calculated using real payments.

Example 2.11. *An investor deposits 100 at time 0 and receives 120 after one year.*

- *The monetary rate of interest effective is*

$$\frac{120}{100} - 1 = 20\%.$$

- *Suppose that the inflation rate over this one year period is 4%. Calculate the real payment of 120 at time 1 and the real rate of interest. After adjusting for the inflation rate, the real rate of interest can be calculated by first expressing both payments in units of the same purchasing power.*

  – *In term of time 0 money unit, the transaction is represented by*

    Here, the real payment of 120 due in 1 year in terms of time 0 unit is $120 \cdot \dfrac{1}{1.04} = 115.38$*. Hence the real rate of interest is 15.38%.*

  – *In term of time 1 money unit, the transaction is represented by*

    Similarly, we instead calculate the real payment of 100 relative to time 1, which gives $100 \cdot 1.04 = 104$*. The real rate of interest is*

$$\frac{120}{104} - 1 = 15.38\%,$$

    which is the same as the previous case.

2.2.2 Real yields

It is often useful to look at the rate of return earned on an investment after taking into account of inflation. As analogous to the real rate of interest, a **real yield** is calculated using real payments, which can be obtained by expressing payments in units of the same purchasing power **at some specific date**.

Example 2.12. *A 5-year bond with nominal value of ฿100 was issued in January 2013. The coupon rate was 8% p.a. payable yearly in arrears. Redemption was at par after 5 years. The bond was issued at 100%. Calculate the yield to a non-tax paying investor*

1. *in monetary terms* ***Solution:***

   The transaction together with the inflation indices $Q(t)$ at time t is shown as follows:

   Clearly, the monetary rate of return on this transaction is 8%. This is because the investor receives the interest payment of ฿8 at the end of each year plus the initial capital of ฿100 at the end of five years.

   Alternatively, one can solve for the monetary rate of return from the following equation of value

   $$f(i) = -100 + 8a^{i}_{\overline{5}|} + 100\frac{1}{(1+i)^5} = 0.$$

2. *in real terms with reference to the CPI.*

   Taking into account of the inflation rates, we calculate the real payment in term of time 0 unit by dividing the monetary amounts by the ***proportional*** *increase in the inflation index from 0 to t.*

   The real yield i' p.a. effective solve the equation of value as follows:

   $$f(i') = -100 + 7.85v + 7.88v^2 + 7.91v^3 + 7.80v^6 + 104.6v^5 = 0,$$

   which gives $i' \approx 7.30\%$ by the linear interpolation.

In general, the real yield i' for a series of cashflows $C(t_1), C(t_2), \dots, C(t_n)$, given associated inflation index $Q(t_k)$ for $k = 1, \dots, n$, can be obtained in terms of time 0 money units as

$$\sum_{k=1}^{n} C(t_k)\frac{Q(0)}{Q(t_k)}\frac{1}{(1+i')^{t_k}} = 0.$$

This is equivalent to

$$\sum_{k=1}^{n} C(t_k)\frac{1}{Q(t_k)}\frac{1}{(1+i')^{t_k}} = 0.$$

Therefore, **the real yield is independent of the date the payment units are adjusted to**.

2.2.3 Calculating real yields given constant inflation assumptions

For future cashflows, the inflation index will not be known. Suppose we assume a constant rate of inflation q p.a. The cashflows $C(t_k)$ at time t_k have the purchasing power at time 0 (or real payments relative to time 0)

$$C(t_k)\cdot\frac{Q(0)}{Q(t_k)} = C(t_k)\cdot\frac{Q(0)}{Q(0)(1+q)^{t_k}} = C(t_k)\cdot\frac{1}{(1+q)^{t_k}}, \quad k = 1, \dots, n.$$

The relation between the real yield i', the constant rate of inflation q and the monetary yield i can be obtained as follows: From the equation of value,

$$\begin{aligned} 0 &= \sum_{k=1}^{n} C(t_k)\frac{Q(0)}{Q(t_k)}\frac{1}{(1+i')^{t_k}} \\ &= \sum_{k=1}^{n} C(t_k)\cdot\frac{1}{(1+q)^{t_k}}\cdot\frac{1}{(1+i')^{t_k}} \end{aligned}$$

With no inflation adjustment, the monetary rate of return i satisfies

$$0 = \sum_{k=1}^{n} C(t_k)\cdot\frac{1}{(1+i)^{t_k}}.$$

Therefore, if **we assume a constant rate of inflation** q p.a., then the following relation holds:

$$(1+i) = (1+q)(1+i').$$

This provides the relationship between the real yield i', the monetary yield i and the inflation rate q.

2.2.4 Index-linked securities

An index-linked security is an investment security in which interest payments and the redemption are adjusted in line with inflation index values by linking the payments to the Consumer Price Index (CPI). The reasons for these types of security are

- to protect investors against inflation risk, and
- to help pension funds to provide index-link benefits so that the index-link liability can be matched with the index-link asset.

Example 2.13. *Consider an index-link bond of a nominal of ฿100 issued at time t_0, bearing an annual coupon of C% payable m times a year and a redemption is at R%. Then per ฿100 nominal, the monetary amount (actual cashflow) of an interest payment $D(t_k)$ at time t_k is*

The monetary amount of the redemption amount at time t_n is

Example 2.14. *An investor purchased a 3-year index-linked bond in January 2015. The investor received payments at the end of each year plus a final redemption amount, all of which were adjusted in line with the CPI values reported in Table 2.6. Calculate the actual payments received by the investor.*

Note In practice, due to delays in calculating the index, the payments (or cashflows) will be adjusted based on the inflation index value from an earlier period.

Let s denote the indexation time lag. The payments are adjusted with reference to inflation index value at time s (months) before the payment is made. Then the monetary amount of an interest payment $D(t_k)$ per ฿100 nominal at time t_k is

$$D(t_k) = 100\frac{C}{m} \cdot \frac{Q(t_k - \frac{s}{12})}{Q(t_0 - \frac{s}{12})}$$

and the monetary amount of redemption at time t_n is

$$D(t_n) = 100R \cdot \frac{Q(t_n - \frac{s}{12})}{Q(t_0 - \frac{s}{12})}.$$

The term $Q(t_0 - \frac{s}{12})$ is called the base inflation figure (the base CPI figure).

Example 2.15. *Repeat Example2.14 for a 3-year index linked bond. The indexation adjustments are made according to the CPI three months before each payment, i.e. $s = 3$ months.*

Example 2.16. *In January 2015, the government issued an index-linked bond of term 10 years. Coupons are payable half-yearly in arrears, and the annual nominal coupon rate is 4%. The coupons and redemption amount are adjusted with reference to the inflation index value 3 months before the payment is made.*

Assume the constant inflation rate from February 2018 is 2% p.a.

1. *Find the base CPI figure (i.e. it is the October 2014 CPI which is 3 months before the issue date).*
2. *Calculate the actual payments received by the investor.*
3. *Assume that the price of ฿100 nominal of this index-linked bond in January 2018 (after the January 2018 coupon payment) is ฿ . Calculate the monetary yield that an investor who purchased the bond in January 2018 (after the January 2018 coupon payment) will obtained.*
4. *Calculate the real yield for this investor under the above assumptions.*

Chapter 3

Processes for stock prices

In this chapter, we will consider some models (or processes) that are used to model stock price. The prices of stocks are continuous in time and value, which are unpredictable. However, we assume that stock prices follow a type of models (which are stochastic processes) known as **geometric Brownian motion**. This is one of the main tools in mathematical financial used in an analysis of derivatives. You will study in more details in the course "SCMA 459 Investment Science II".

We will first begin with the brief introduction of the **force of interest** (or **continuous compounding**). The process for a stock price and an algorithm to simulate stock prices will be presented after main concepts of random walks and Brownian motion are given.

3.1 Force of interest

Given an annual effective rate i, a principal of 1 accumulates to $(1+i)$ at the end of the first year. The following formula provides the **equivalent nominal rates**, $i^{(n)}$, of interest convertible n times per year,

$$\left(1+\frac{i^{(n)}}{n}\right)^n = (1+i),$$

which gives

$$i^{(n)} = n\left((1+i)^{1/n} - 1\right).$$

Note that $\dfrac{i^{(n)}}{n}$ is the effect interest rate applied for each n-th of a year. For e.g. if $n=2$, then $\dfrac{i^{(2)}}{2}$ is the half-yearly effect interest rate equivalent to i.

Example 3.1. *Given the annual effective rate $i = 4\%$,*

1. *calculate the equivalent nominal rates of interest convertible n times a year for $n = 2, 4, 12, 52,$ and 365,*
2. *find the limit of $i^{(n)}$ as n tends to infinity, and*
3. *plot the graph of $i^{(n)}$ as a function of n.*

Notes

1. Using the result from calculus, we can show that

$$\lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n = \exp(x).$$

2. As n tends to infinity, a rate of $i^{(n)}$ compounding n times a year converges to a constant, which will be denoted by δ, i.e. $i^{(n)} \to \delta$ as $n \to \infty$.
3. The above observations imply following relation:

$$\lim_{n\to\infty} \left(1 + \frac{i^{(n)}}{n}\right)^n = \exp(i^{(\infty)}) = \exp(\delta),$$

   where we write $i^{(\infty)} = \delta$.
4. The constant δ is known as the **force of interest** or also known as a **continuous compounded rate**. It is the interest rate paid continuously throughout the period. It is a constant **force** that causes the initial investment to grow.
5. The following relationship between the force of interest and the annual effective rate of interest holds:

$$1 + i = \exp(\delta) \quad \text{ or } \delta = \ln(1 + i).$$

Example 3.2. *฿200 is invested in an account which pays a force of interest of 4% pa. Calculate the amount in the account after 3 years.*

Example 3.3. *A payment of ฿500 is due in 5 years' time. Calculate the present value of this payment at a force of interest of 4% pa.*

For a force of interest δ, the accumulation at time t of 1 unit paid at time s is

$$A(s,t) = \exp(\delta(t-s)).$$

The present value at time s of 1 unit paid at time t is

$$V(s,t) = \exp(-\delta(t-s)).$$

Notes

1. In general, the force of interest $\delta(t)$ per annum at time t is a function of t.
2. Given the force of interest per annum $\delta(t)$, the accumulation at time t of 1 unit paid at time s is

$$A(s,t) = \exp\left(\int_s^t \delta(s)\,ds\right).$$

   Also, the present value at time s of 1 unit paid at time t is

$$V(s,t) = \exp\left(-\int_s^t \delta(s)\,ds\right).$$

3. The more detailed discussion regarding the force of interest and continuous cashflows will be given in the course SCMA 361 Theory of Interest.

The diagram below shows the accumulation at time t of 1 unit paid at time s at the annual effective rate i and the force of interest $\delta(t)$.

Example 3.4. *A payment of ฿1000 is due in 1 years' time. Calculate the accumulated amount at time 1 of this payment at a force of interest of $\delta(t)$ pa, where*

$$\delta(t) = a + bt + ct^2,$$

a, b and c are given constants.

3.2 Stochastic processes

It is of great interest to model the behaviour of a system by describing how different states, describe by random variables X's, in the system evolve with time.

Stochastic processes can be used to represent many different phenomena from various areas including science, engineering, finance, and economics. As opposed to a deterministic model whose outcome is fixed, the outcome of a stochastic process is not certain, the stochastic process is simply a collection of random variables defined as follows:

A **stochastic process** is a collection of random variables $\{X_t : t \in T\}$, where

- t is a parameter running over some index set T, called the **time domain**.
- The common sample space of the random variables (the range of possible values for X_t) denoted by S is called the **state space** of the process.

Notes

1. The set of random variables may be dependent or need not be identically distributed.
2. Techniques used to study stochastic processes depend on whether the state space or the index set (the time domain) are discrete or continuous.
3. Topics on stochastic processes will be discussed in the course "SCMA 469 Actuarial Statistics".

3.2.1 A simple random walk

Consider a simple model of the price of a stock measured in units of Thai baht. For each trading day $n = 0, 1, 2, ...$, the stock price increases by 1 baht with probability p or decreases by 1 baht with probability $q = 1 - p$.

Let X_n denote the stock price at day n and $X_0 = ฿100$. This simple model is called a **simple random walk**.

In general, a simple random walk X_n is a discrete-time stochastic process defined by

- $X_0 = a$ and
- for $n \geq 1$,

$$X_n = a + \sum_{i=1}^{n} Z_i, \text{ where } Z_i = \begin{cases} 1, & \text{with probability } p \\ -1, & \text{with probability } q = 1 - p. \end{cases}$$

Note When $p = 1/2$, the value of the process increases or decreases randomly by 1 unit with equal probability. In this case, the process is known as a **symmetric** random walk.

Example 3.5. *Calculate the expectation and variance of the random variable Z_i.*

Solution:

$$\mu = \mathrm{E}[Z_i] = 1 \cdot p + (-1) \cdot q = p - q.$$

$$\begin{aligned} \sigma^2 &= \mathrm{Var}[Z_i] \\ &= \mathrm{E}[Z_i^2] - (\mathrm{E}[Z_i])^2 \\ &= 1 - (p-q)^2 = (p+q)^2 - (p-q)^2 \\ &= 4pq. \end{aligned}$$

Example 3.6. *Calculate the expectation and variance of the process X_n at time n.*

Solution:

$$\mathrm{E}[X_n] = \mathrm{E}[a + \sum_{i=1}^{n} Z_i] = a + n\mu.$$

$$\mathrm{Var}[X_n] = \mathrm{Var}[a + \sum_{i=1}^{n} Z_i] = n\sigma^2.$$

It should be noted that the variance of X_n increases with time.

The next example shows how to use a coin to simulate price paths of a stock.

Example 3.7. *We flip the coin once to represent each trading day. If the coin comes up head then the stock price goes up by 1 baht; if it comes up tail then the stock price goes down by 1 baht. Assume that the initial stock price is $X_0 = 100$. Let us flip the coin 20 times and then draw the graph of the stock price against time (in day). Repeat this process 4 more times.*

Solution: See the Excel worksheet.

3.2.2 Sample paths

In the simple random walk process, time is discrete (as observed at the end of each day) and the state space is discrete. The stochastic model has an infinite number of **stochastic realisations**. A **sample path** is then just the sequence of a particular set of experiments. Graphs of some stochastic realisations of the simple random walk with $p = 0.5$ and $a = 100$ are shown in Figure are shown in Figure 3.1.

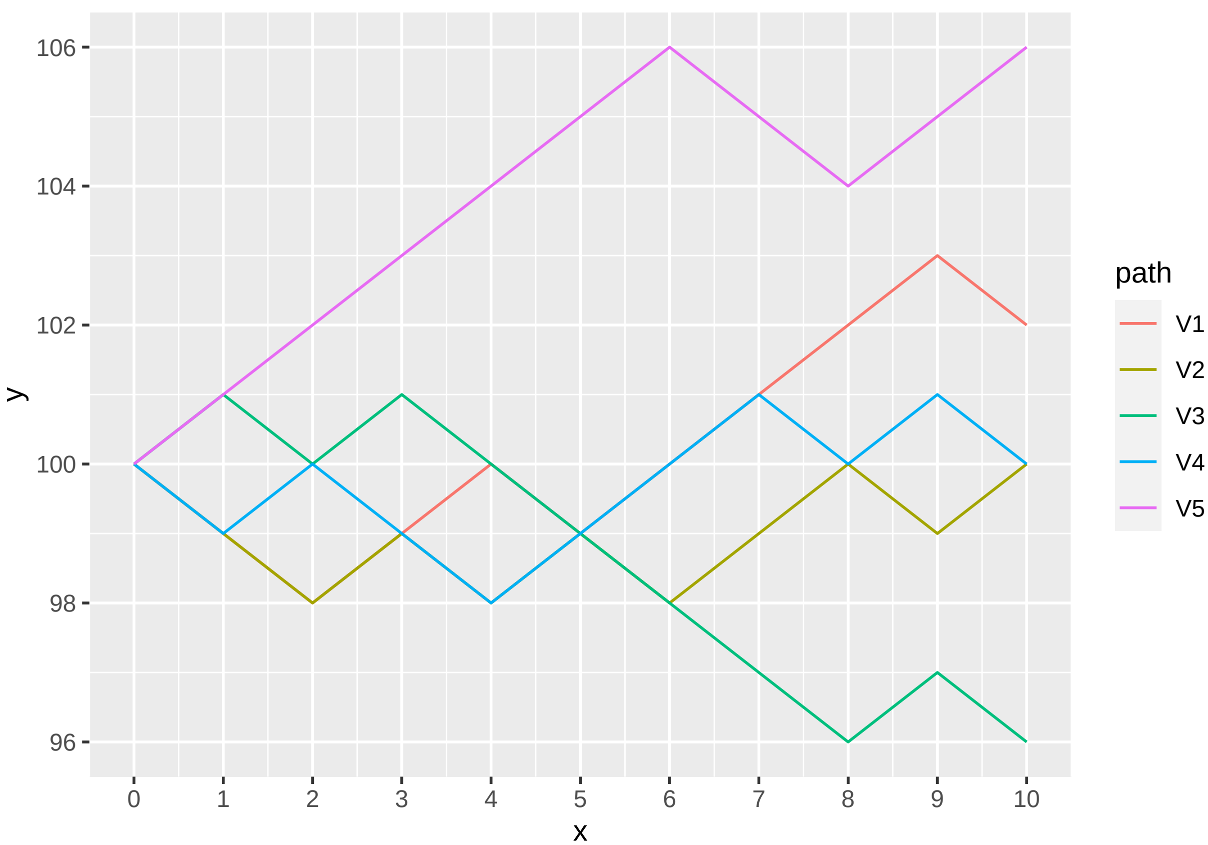

Figure 3.1: Some stochastic realisations of the simple random walk

Example 3.8. *For the random process, calculate*

$$\Pr(X_2 = 98, X_5 = 99 | X_0 = 100).$$

Solution: The process X_n must decrease on the first two days, which happens with probability $(1-p)^2$. Independently, it must then increases on another two days and decrease on one day (not necessarily in that order), giving three different possibilities. Each of these has probability $p^2(1-p)$. So

$$\Pr(X_2 = 98, X_5 = 99 | X_0 = 100) = (1-p)^2 \cdot 3p^2(1-p) = 3p^2(1-p)^3.$$

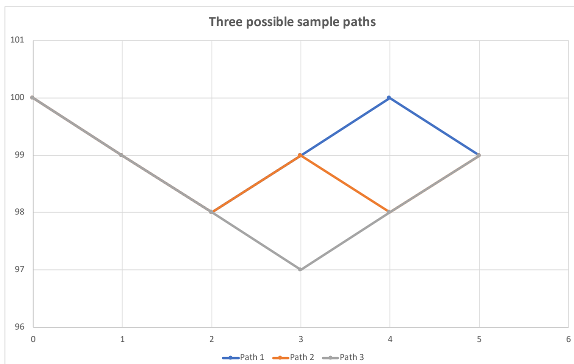

Figure 3.2: Three possible sample paths

3.2.3 Simulating a simple random walk

To simulate a sample path of the simple random walk process X_n for $n = 0, 1, \ldots, N$, we proceed as follows:

1. Generate N random samples from the discrete distribution of the random variable Z_i, $i = 1, \ldots, N$, as follows:

    1. We first generate random numbers from the uniform distribution $U(0, 1)$. Denote these numbers by U_i.

    2. If $U_i \leq p$, then set $Z_i = 1$. Otherwise, $Z_i = -1$

2. The value X_n of the random walk process at time n, $1 \leq n \leq N$, can then be calculated from

$$X_n = a + \sum_{i=1}^{n} Z_i.$$

Notes

1. Excel provides two function **RAND()** and **RANDBETWEEN(a,b)** to generate a random number between 0 and 1 and a random integer between a and b.

2. Excel also provides Random Number Generation to draw sample numbers from some specified distributions (to be discussed in Excel Lab).

Simulating a simple random walk

Initial state (a)	100
Probability (p)	0.5

Time (n)	Random draw from U(0,1)	Z	X_n
0			100
1	0.53	-1	99
2	0.65	-1	98
3	0.30	1	99
4	0.50	-1	98
5	0.23	1	99
6	0.83	-1	98
7	0.26	1	99
8	0.36	1	100
9	0.48	1	101
10	0.64	-1	100
11	0.31	1	101
12	0.04	1	102
13	0.51	-1	101
14	0.77	-1	100
15	0.78	-1	99
16	0.83	-1	98

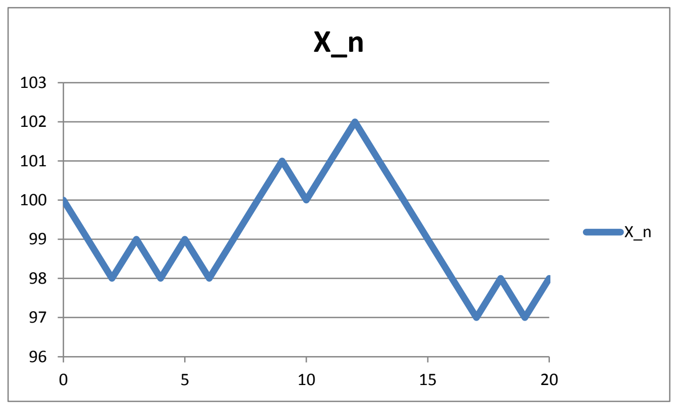

Figure 3.3: Simulating a sample path of a simple random walk

3.3 Monte Carlo Methods

Monte Carlo methods are simulation-based algorithms that rely on generating a large set of samples from a statistical model to obtain the behaviour of the model and estimate the quantities of interest. For a large sample set of a random variable representing a quantity of interest, the law of large numbers allows to approximate the expectation by the average value from the samples.

Consider repeated independent trials of a random experiment. We will need to generate a large number of samples $X_1, X_2, ...$ from the model. A Monte Carlo method for estimating the expectation $\mathrm{E}[X]$ is a numerical method based on the approximation

$$\mathrm{E}[X] \approx \frac{1}{N}\sum_{i=1}^{N} X_i,$$

where $X_1, X_2, ...$ are i.i.d. with the same distribution as X.

While computing expectations and computing probabilities at first look like different problems, the latter can be reduced to the former: if X is a random variable, we have

$$\Pr(X \in A) = \mathrm{E}[\mathbb{1}_A(X)].$$

Using this equality, we can estimate $\Pr(X \in A)$ by

$$\Pr(X \in A) = \mathrm{E}[\mathbb{1}_A(X)] = \frac{1}{N}\sum_{i=1}^{N} \mathbb{1}_A(X_i).$$

Recall that the indicator function of the set A is the defined as

$$\mathbb{1}_A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{otherwise.} \end{cases}$$

Example 3.9. *Using Monte Carlo estimation, approximate the expectation* $\mathrm{E}[X]$ *where* $X \sim \mathcal{N}(0,1)$*, and estimate the percent of values lie within two standard deviation of the mean of* X

Solution: We generate samples $X_1, X_2, ..., X_N$ from the model X for large N, for example for $N = 10000$, we can use estimates such as

$$\mathrm{E}[X] \approx \frac{1}{N}\sum_{i=1}^{N} X_i,$$

and

$$\mathrm{Var}[X] \approx \frac{1}{N-1}\sum_{i=1}^{N} (X_i - \bar{X})^2.$$

3.4 The Markov Property

A Markov process is a special type of stochastic processes with the property that the future evolution of the process depends only on its current state and not on its past history.

That is given the value of X_t, the values of X_s for $s > t$ do not depend on the values of X_u for $u < t$. This property is called the **Markov property**.

A **Markov process** is any stochastic process that satisfies the Markov property

Notes

1. Stock prices are usually assumed to follow a Markov process where only the current stock price is relevant for predicting future prices, and thus the past values are irrelevant
2. Predictions in the future is uncertain, and must be expressed in terms of probability distributions. Hence, the probability distribution of the future stock price is independent of the particular price path in the past.
3. **Time series models** are widely used in economics, business and engineering to predict the variability of a variable over time, where past values are used as the input variables for the model.

3.5 Wiener processes

We start with the Markov process that is a building block for the geometric Brownian motion. Analogously to the random walk process, it is defined in terms of some simple intermediary processes, that is the increment process Z_i. The **Wiener process** is a Markov process which will be used to **model the change in the stock prices**.

The Wiener process is a (**continuous-time stochastic**) process which can be described by a variable W having the following properties:

- We assume that the current value is 0 (i.e. $W(0) = 0$) and **the change in its value during a year** (i.e. $W(1) - W(0)$) is $N(0, 1)$ where $N(m, v)$ is a probability distribution normally distributed with mean m and variance v.

  Notes The following properties of the normal distributions are useful.

  1. Given $X \sim N(\mu, \sigma^2), Y = a + bX$, it follows that
  $$Y \sim N(a + b \cdot \mu, b^2\sigma^2).$$
  2. Given $X \sim N(\mu_X, \sigma_X^2), Y \sim N(\mu_Y, \sigma_Y^2)$ with X, Y independent, it follows that
  $$X + Y \sim N(\mu_X + \mu_Y, \sigma_X^2 + \sigma_Y^2).$$
  3. It is well known that if two i.i.d. random variables are normally distributed, their sum is also normally distributed.

     The converse is also true.

     That is, suppose X and Y are two i.i.d. random variables such that $X+Y$ is normal. It necessarily the case that X and Y are also normal.

- The probability distribution of the change in the value of the variable during any time period of length T is $N(0, T)$. For example,
  - The change in 2 years is the sum of two normal distributions, each of which has mean of 0 and variance of 1. **Solution:**
  $$\begin{aligned} W(2) - W(0) &= (W(2) - W(1)) + (W(1) - W(0)) \\ &\sim X + Y \text{ where } X, Y \text{ are independent } N(0, 1) \text{ distributions.} \\ &\sim N(0, 2). \end{aligned}$$

    The Markov property implies that the two distributions are **independent**. Hence the sum of two independent normal distributions is a normal distribution where the mean is the sum of the means and the variance is the sum of the variances.

    The the probability distribution of the change in the value of the variable in 2 years is $N(0, 2)$.

– For the change in 6 months, the variance of the change in the value of the variable during 1 year (which is 1) equals to the sum of the variance of the change in the first 6 months and the second 6 months. We also assume that the two distributions of the six-months period are the same (or **identical**). As a result, the the probability distribution of the change in the value of the variable during 6 months is $N(0, 0.5)$.

Solution:

$$\begin{aligned} W(1) - W(0) &= (W(1) - W(0.5)) + (W(0.5) - W(0)) \\ &\sim U + V \text{ where } U, V \text{ are identical and independent} \end{aligned}$$

Since $W(1) - W(0) \sim N(0, 1)$, it follows that U and V are both normally distributed $N(0, 0.5)$.

Example 3.10. *Given that $W(0) = a$, what is the probability distribution of the value of W at 3 months, i.e. $W(0.25)$?*

Solution:

$$\begin{aligned} W(0.25) &= W(0) + (W(0.25) - W(0)) \\ &\sim a + N(0, 0.25) \\ &\sim N(a, 0.25). \end{aligned}$$

The mean and the variance of $W(0.25)$ is a and 0.25, respectively.

More precisely, a **standard Brownian motion** or **standard Wiener process** over $[0, T]$ is a random variable $W(t)$ that depends continuously on $t \in [0, T]$ and satisfies the following conditions:

1. $W(0) = 0$ (with probability 1).
2. For $0 \leq s < t \leq T$, the **change (or increment)** $W(t) - W(s)$ is normally distributed with mean zero and variance $t - s$.
3. For $0 \leq s < t < u < v \leq T$, the change (or increment) $W(t) - W(s)$ and $W(v) - W(u)$ are independent.

Notes

1. The second condition above implies that the change $W(t) - W(s) \sim \sqrt{t-s}N(0, 1)$, where $N(0, 1)$ denotes a normally distributed random variable with mean zero and unit variance.
2. The uncertainty about the value of the standard Brownian motion measured in terms of its standard deviation increases as the square root of how far we are looking ahead, i.e. the square root of the length of the period considered.

3.5.0.1 Simulating a standard Brownian motion

Recall that the standard Brownian motion is a continuous process. We can simulate it by discretising the time domain (the interval $[0, T]$) . The algorithm can be summarised as follows:

1. We set $\Delta t = T/N$ for some positive inter N and let W_j denote $W(t_j)$ with $t_j = j \cdot \Delta t$
2. Condition 1 implies that $W_0 = 0$ with probability 1, and conditions 2 and 3 tells us that

$$W_j = W_{j-1} + \Delta W_j,\ j = 1, 2, \dots, N,$$

   where each ΔW_j is an independent random variable of the form $\sqrt{\Delta t}N(0, 1)$.

Note

1. How can I simulate values of a normal random variable?

   (Excel) If you type in any cell the formula =NORM.INV(rand(),mu,sigma), you will generate a simulated value of a normal random variable having a mean mu and a standard deviation sigma (why?).

2. In ordinary calculus, the notation
$$dx = adt$$
indicates that $\Delta x = a\Delta t$ in the limit as $\Delta t \to 0$. This means that the change in x per unit of time is constant, i.e. equal to a.

3. In stochastic calculus, we use dW as a Wiener increment which is the limit of ΔW_j as $\Delta t \to 0$.

4. We see that the Wiener process has very jagged (irregular) sample paths.

Simulating standard Brownian Motion

Simulate W(t) over [0,T]

T	5
N	100
delta t	0.05
W0	0

t	RAND()	Sample number from N(0,1)	W_t
0.00			0.00
0.05	0.72	0.60	0.13
0.10	0.15	-1.03	-0.10
0.15	0.27	-0.62	-0.23
0.20	0.68	0.46	-0.13
0.25	0.49	-0.02	-0.14
0.30	0.96	1.78	0.26
0.35	0.90	1.31	0.55
0.40	0.09	-1.32	0.26
0.45	0.31	-0.49	0.15
0.50	0.77	0.75	0.31
0.55	0.79	0.81	0.50
0.60	0.00	-2.71	-0.11
0.65	0.12	-1.18	-0.37
0.70	0.64	0.37	-0.29
0.75	0.07	-1.45	-0.62
0.80	0.34	-0.41	-0.71
0.85	0.11	-1.20	-0.98
0.90	0.78	0.78	-0.80

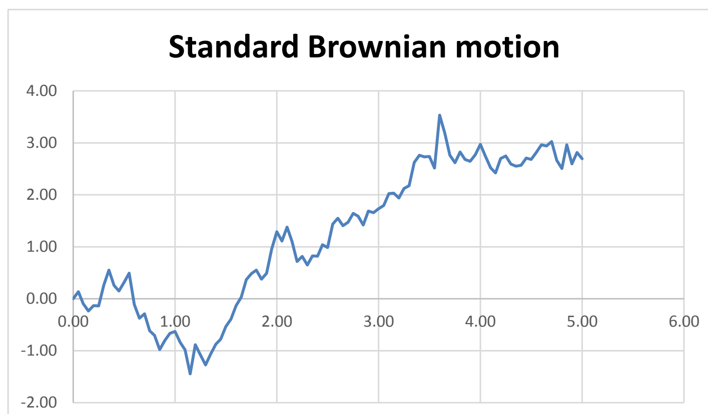

Figure 3.4: Simulating a sample path of the standard Brownian motion

In the following example, we do not assume that $W_0 = 0$ with probability 1.

Example 3.11. *Consider a Wiener process denoted by $W(t)$ with $W(0) = a$ with probability 1. The time is measured in years.*

1. *What is the probability distribution of the value of the variable at the end of one year, i.e. $W(1)$?*

2. *What is the probability distribution of the value of the variable, i.e. $W(4)$? and what is the mean and the standard deviation of $W(4)$.*

What is the probability distribution of the change in the value of W during 3 months, i.e. $W(0.25) - W(0) = W(0.25)$?

3.6 Generalised Wiener process (Brownian Motion with Drift)

In this section, we will consider a variant of the standard Wiener process, which is modified by introducing the drift and variance parameters. This new process is known as a **generalised Wiener process** or **Brownian motion with drift**.

More precisely, a generalised Wiener process for a variable X is defined in terms of dW as

$$dX = adt + bdW,$$

where a is a drift parameter and b^2 is a variance parameter.

Note The process defined above is also known as a **linear stochastic differential equation**.

- The term adt (deterministic term) implies that X has an **expected drift rate** of a per unit of time (or the mean change per unit time). To see this, without the stochastic or random component bdW, the equation is simply an ordinary differential equation

  $$dX = adt \quad \text{or} \quad \frac{dX}{dt} = a.$$

  The solution to this equation with an initial condition $X(0) = X_0$ is

  $$X = X_0 + at.$$

  Clearly, for each time unit the variable X increases by an amount a.

- The stochastic or random component bdW, measured in terms of the variance of the change in X, adds **uncertainty**, **noise** or **variability** to the path followed by X. The term b^2 is referred to as the **variance rate**.

The following example shows that properties of the generalised Wiener processcan be obtained from the properties of the standard Wiener process.

Example 3.12. *Calculate the probability that the generalised Wiener process $X(t)$ with the drift parameter $a = 0.5$ and the variance $b^2 = 0.16$ takes the values between 3 and 4 at time $t = 4$. Assume that its initial value $X(0) = 1$.*

Solution: The change of process over 4 units of time is given by

$$X(t) - X(0) = 0.5 \cdot t + 0.4 \cdot (W(t) - W(0)) = 0.5 \cdot t + 0.4 \cdot W(t).$$

The required probability is

$$\begin{aligned}
\Pr(3 \le X(4) \le 4) &= \Pr(3 \le X(0) + 0.5 \cdot 4 + 0.4 \cdot W(4) \le 4) \\
&= \Pr(0 \le W(4) \le 2.5) \\
&= \Pr(0 \le 2Z \le 2.5), \text{ where } Z \sim N(0, 1) \\
&= \Pr(0 \le Z \le 1.25) = 0.3944
\end{aligned}$$

Here we can use the standard normal distribution table to obtain the probability, or use the Excel function NORM.S.DIST(z, TRUE), which returns the cumulative distribution of the standard normal distribution.

Notes

1. The standard Wiener process has the expected drift rate of zero, $a = 0$ and unit variance rate (noise or variability) $b^2 = 1$. For the generalised Wiener process, the amount of this noise or variability is b times a Wiener process. For **each time unit**, the Wiener increment dW follows the standard normal distribution $N(0,1)$. Therefore, the variance per time unit time of the process X is equal to b^2.

2. The change in the value of the process X in any time interval t is normal distributed with mean of at and variance of b^2t, i.e. for $s, t > 0$,

$$X(s+t) - X(s) \sim N(at, b^2t).$$

   This follows from the definition of the process:

$$\begin{aligned} X(s+t) - X(s) &= \Delta X = a\Delta t + b\Delta W \\ &= a(s+t-s) + b\Delta W = at + b(W(s+t) - w(s)) \\ &\sim at + b \cdot N(0,t) = N(at, b^2t), \end{aligned}$$

   since $W(s+t) - w(s) \sim N(0,t)$.

3. In general, the drift and variance rates of a stochastic process may depend on both X and t. The process is called an Itô process and can be written as

$$dX = a(X,t)dt + b(X,t)dW,$$

   where a and b are functions of X and t.

Example 3.13. *Consider the cash position of a company measured in millions of Thai baht. Assume that it follows a generalised Wiener process with a drift of 10 per year and a variance rate of 400 per year. Let* $X(0) = 50$*.*

1. *What is the distribution of the process at the end of the first year,* $X(1)$*?*
2. *What is the distribution of the process at the end of 6 months,* $X(1/2)$*?*
3. *What is the probability that the cash position will be at least 60 million baht at the end of 6 months?*

Solution:

1. At the end of the first year, we know that

$$X(1) - X(0) \sim N(a \cdot 1, b^2 \cdot 1),$$

   where $a = 10$ and $b^2 = 400$. It follows that

$$X(1) \sim N(X(0) + a, b^2) = N(60, 400).$$

2. Similarly, at the end of 6 months,

$$X(1/2) - X(0) \sim N(a \cdot \frac{1}{2}, b^2 \cdot \frac{1}{2}),$$

   or

$$X(1/2) \sim N(55, 200).$$

3. The required probability is

$$\begin{aligned} \Pr(X(1/2) \geq 60) &= \Pr(Z \geq \frac{60-55}{\sqrt{200}}) \\ &= \Pr(Z \geq 0.35) \\ &= 1 - \text{NORM.S.DIST(0.35, TRUE)} \\ &= 1 - 0.638 = 0.362. \end{aligned}$$

3.6.0.1 Simulating a generalised Wiener process

Simulating a generalised Wiener process over $[0, T]$ is similar to a standard Wiener process.

1. We first discretise the interval by let $\Delta t = T/N$ for some positive integer N and $t_j = j \cdot \Delta t$.
2. Denote $X(t_j)$ by X_j. The value X_j can be approximated by

$$X_j = X_{j-1} + a\Delta t + b\Delta W_j,$$

where each $\Delta W_j = W(t_j) - W(t_{j-1})$ is an independent random variable of the form $\sqrt{\Delta t}N(0,1)$.

Simulating a generalised Weiner process

Simulate X(t) over [0,T] where dX = a dt + b d W

T	5
N	100
delta t	0.05
X0	0
a	0.5
b	2

t	rand()	sample number from N(0,1)	X_t	X_t without noise
0.00			0.00	0.00
0.05	0.44	-0.14	-0.04	0.03
0.10	0.27	-0.61	-0.28	0.05
0.15	0.53	0.07	-0.23	0.08
0.20	0.55	0.12	-0.15	0.10
0.25	0.08	-1.38	-0.74	0.13
0.30	0.13	-1.13	-1.22	0.15
0.35	0.49	-0.01	-1.20	0.18
0.40	0.63	0.34	-1.02	0.20
0.45	0.32	-0.46	-1.20	0.23
0.50	0.39	-0.28	-1.31	0.25
0.55	0.25	-0.68	-1.59	0.28
0.60	0.99	2.53	-0.43	0.30
0.65	0.46	-0.09	-0.45	0.33
0.70	0.47	-0.07	-0.46	0.35
0.75	0.90	1.26	0.13	0.38

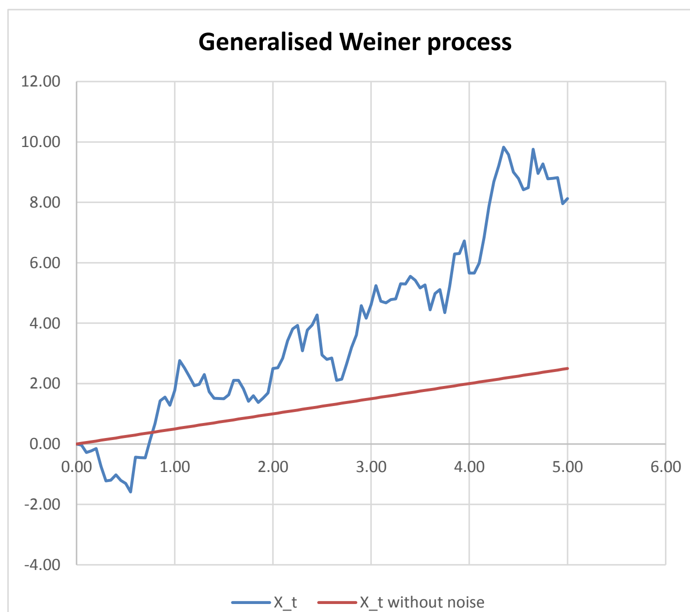

Figure 3.5: Simulating a sample path of the generalised Wiener process with $a = 0.5$ and $b = 2$.

One-Period Simple Return

Let P_t be the price of an asset at time index t. Holding the asset for one period from date t-1 to date t would result in a simple gross return

$$1 + R_t = \frac{P_t}{P_{t-1}} \text{ or } P_t = P_{t-1}(1 + R_t).$$

Continuously Compounded Return The natural logarithm of the simple gross return of an asset is called the continuously compounded return or log return

$$r_t = \ln(1 + R_t) = \ln \frac{P_t}{P_{t-1}}.$$

3.7 The process for a stock price

We will discuss a simple model (or a stochastic process) for the price of a non-dividend-paying stock. The model is known as **geometric Brownian motion**. We will show that this stochastic model assumes or implies the following properties for stock prices.

1. The process is **continuous in time and value**.
2. It possesses the **Markov property**, i.e. future prices depend only on its the current stock price and not on its the past historical values.
3. The **proportional return on a stock** over a short time interval is normally distributed.
4. The **price of a stock** is lognormally distributed.
5. The **continuously compounded rate of return per year** for a stock is normally distributed.

Example 3.14. *Explain why a generalised Wiener (GW) process is inappropriate to model a stock price.*

Solution: The GW model assumes that the expected change (not the return) is independent of the stock price, which does not explain a key aspect of stock prices. The expected percentage return (not the change in stock price) required by an investor in a short period of time, Δt, should be the same regardless of the stock price.

In particular, when either the stock price is ฿100 or ฿1000, the investor would require the same expected return.

3.7.1 Model Assumptions of the Geometric Brownian Motion

The geometric Brownian motion also consists of two components:

1. **The deterministic component:** The model assumes that the expected drift rate in S (the mean change in S per unit of time) is proportion to the price S, i.e. equal to μS, for some constant parameter μ. As a result, the expected increase in S in a short time interval Δt is $\mu S \Delta t$. Therefore, without uncertainty, the model can be expressed as

$$\Delta S = \mu S \Delta t$$

As $\Delta t \to 0$, the model can be described by an ordinary differential equation

$$\frac{dS}{dt} = \mu S,$$

and the solution to this differential equation is

$$S_T = S_0 \exp(\mu T),$$

where S_0 and S_T are the stock prices at time 0 and time T.

Note When there is no uncertainty, the stock price grows at a continuously compounded rate of μ per unit of time (generally per year). If the price of the stock today is S_0, then its price S_T at time T is $S_T = S_0 \exp(\mu T)$.

2. **The stochastic component:**

   An additional assumption of the model is that the variability of the percentage return ($\Delta S/S$) in a short time period Δt is the same regardless of the stock price. This means that the uncertainty of the percentage return when the stock price is ฿100 is the same as that when the stock price is ฿250.

   This assumption adds stochastic or random component into the model.

   As a result, by adding the stochastic or random component to the deterministic component, the **geometric Brownian motion** can be expressed as

$$dS = \mu S dt + \sigma S dW$$

   where W is a standard Brownian motion (or a Weiner process).

Notes

1. The geometric Brownian motion is the continuous-time version of a random walk.
2. The discrete-time version of the geometric Brownian motion is

$$\frac{\Delta S}{S} = \mu \Delta t + \sigma \sqrt{\Delta t} \epsilon,$$

   where ϵ has a standard normal distribution (with zero mean and unit variance) The parameter μ is the **expected rate of return per unit of time** from the stock, and σ is the **volatility** of the stock price.

3.7.1.1 Proportional return on stocks are normally distributed

It follows from the definition of the the geometric Brownian motion that the proportional return on the stock over a short time interval is normally distributed with a mean $\mu\Delta t$ and variance of $\sigma^2 \Delta t$. In particular,

$$\frac{\Delta S}{S} = N(\mu \Delta t, \sigma^2 \Delta t).$$

Example 3.15. *Consider a stock that pays no dividends, provide an expected return of 20% per annum with continuous compounding and has a volatility of 40% per annum. Assume that the current stock price is ฿20 and ϵ sampled from the standard normal distribution is 0.6.*

1. *Calculate the change of the stock price over 1 day, assuming that there are 250 trading days per year.*
2. *Calculate the stock price over 10 days if the random samples for ϵ are given in Figure 3.6.*

Solution:

1. Given that $\mu = 0.2$ and $\sigma = 0.4$ per annum, the process of the stock price is

$$\frac{dS}{S} = \mu dt + \sigma dW.$$

   Here S is the stock price at a particular time. If ΔS is the change in the stock price in the next small interval Δt of time, then the discrete-time version of the model is

$$\begin{aligned} \frac{\Delta S}{S} &= \mu \Delta t + \sigma \sqrt{\Delta t} \epsilon \\ &= 0.2 \Delta t + 0.4 \sqrt{\Delta t} \epsilon, \end{aligned}$$

where ϵ has a standard normal distribution.

For a time interval of 1 day or 0.004 years,

$$\frac{\Delta S}{S} = 0.2 \cdot 0.004 + 0.4\sqrt{0.004}\epsilon.$$

If the current stock price is 20 and $\epsilon = 0.6$, then

$$\Delta S = 20(0.2 \cdot 0.004 + 0.4\sqrt{0.004}0.6) = 0.3196,$$

and $S(0.004) = S(0) + \Delta S = 20.3196$.

2. Leave as an exercise.

Simulating a geometric Brownian motion

Simulate S(t) over [0,T] where dS = mu S dt + sigma S d W

T	40
N	1000
delta t	0.04
S0	20
mu	0.2
sigma	0.4

t	rand()	sample number from N(0,1)	S_t	S_t without noise
0.000			20.000	20.000
0.040	0.315	-0.481	19.390	20.160
0.080	0.961	1.764	22.281	20.321
0.120	0.330	-0.441	21.673	20.484
0.160	0.650	0.384	22.513	20.648
0.200	0.850	1.038	24.563	20.813
0.240	0.501	0.003	24.764	20.979
0.280	0.339	-0.415	24.139	21.147
0.320	0.241	-0.702	22.978	21.316
0.360	0.149	-1.040	21.250	21.487
0.400	0.793	0.816	22.808	21.659
0.440	0.996	2.677	27.874	21.832
0.480	0.393	-0.271	27.494	22.007
0.520	0.396	-0.264	27.132	22.183

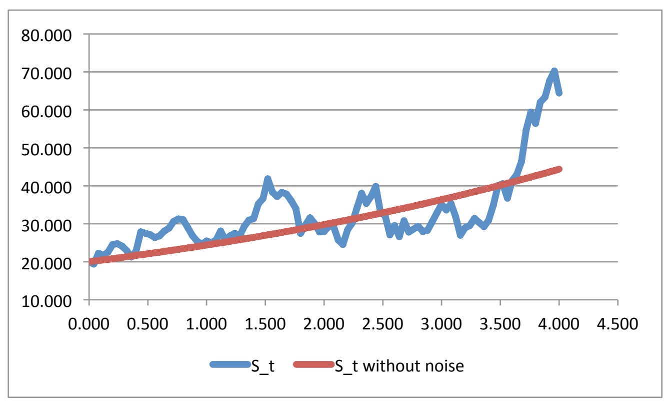

Figure 3.6: Simulating a sample path of a geometric Brownian motion with $\mu = 0.2$ and $\sigma = 0.4$.

Notes

1. The parameter μ is the annualised expected return in a short period of time. It should depend on the risk of the return from the stock, and should also depend on the level of interest rates in the market. The higher the interest rates, the higher the expected return.

2. The value of a (financial) derivative, for example options, is independent of μ. So we do not need to concern with the estimation of μ.

3. The parameter σ, the stock price volatility, is important to determine the value of (financial) derivatives. We will discuss how to estimate σ.

4. The standard deviation of the proportional change in the stock price in a **small interval** Δt is $\sigma\sqrt{\Delta t}$. For a **relatively long period** of time T, it is approximately equal to $\sigma\sqrt{T}$.

   As a result, **the volatility σ can be interpreted as the standard deviation of the change in the stock price in 1 year**.

Stock prices are lognormally distributed

Using Itô lemma (from stochastic calculus), it follows that stock prices are lognormally distributed:

$$\ln \frac{S_T}{S_0} \sim N\left((\mu - \frac{\sigma^2}{2})T, \sigma^2 T\right)$$

and

$$\ln S_T \sim N\left(\ln S_0 + (\mu - \frac{\sigma^2}{2})T, \sigma^2 T\right)$$

where S_T is the stock price at time T and S_0 is the stock price at time 0. Here $\ln S_T$ is normally distributed and hence S_T has a lognormal distribution.

Note

1. The logarithm of a price follows a generalised Wiener process with drift rate $\mu - \sigma^2/2$ and variance rate σ^2.

2. The mean of the distribution for the change in logarithm of price over each time unit (i.e. log return) is not μ but $\mu - \sigma^2/2$.

3. Why do stock prices follow a lognormal random variable? Let X_i denotes the percentage change in the stock price during day i. Then the stock price at day n satisfies

   $$S_n = S_0 \cdot X_1 \cdots X_n$$

   or

   $$\ln(S_n) = \ln(S_0) + \ln(X_1) + \cdots + \ln(X_n).$$

   Suppose that the changes in prices on different days are (identical) independent random variables. Then the central limit theorem implies that the sum of those independent random variables will be a normal random variable. Hence, the stock prices are lognormally distributed.

4. Recall that if the random variable X is lognormally distributed, then $Y = \ln(X)$ has a normal distribution:

   $$X \sim \text{Lognormal}(\mu, \sigma^2) \Longleftrightarrow \ln(X) \sim N(\mu, \sigma^2).$$

   $\text{E}[X] = e^{\mu + \frac{1}{2}\sigma^2}$, $\text{Var}[X] = e^{2\mu+\sigma^2}\left(e^{\sigma^2 - 1}\right)$.

Example 3.16. *Consider a stock that pays no dividends, provide an expected return of 20% per annum with continuous compounding and has a volatility of 40% per annum. Assume that the current stock price is ฿20.*

1. *Find the probability distribution of the logarithm of the stock price S_T in 3 months' time.*

2. *Calculate the mean and the standard deviation of $\ln S_T$ in 3 months' time.*

3. *Find the 95% confidence interval of S_T in 3 months' time.*

Solution:

1. Given that $\mu = 0.2$, $\sigma = 0.4$ per annum and $S_0 = 20$, the probability distribution of $\ln S_T$ in 3 months' time ($T = 0.25$ years) is

$$\begin{aligned}\ln S_T &\sim N\left(\ln S_0 + (\mu - \frac{\sigma^2}{2})T, \sigma^2 T\right)\\ &= N\left(\ln 20 + (0.2 - \frac{0.4^2}{2})0.25, 0.4^2 \cdot 0.25\right) = N(3.026, 0.04).\end{aligned}$$

2. It follows that for $T = 0.25$ years,

$$\mathrm{E}[\ln S_T] = 3.026, \text{ and } \mathrm{SD}(\ln S_T) = \sqrt{0.04} = 0.2.$$

3. The 95% CI of $\ln S_T$ for $T = 0.25$ years, is

$$\begin{aligned}3.026 - 1.96 \cdot 0.2 < \ln S_T < 3.026 + 1.96 \cdot 0.2\\ 2.634 < \ln S_T < 3.418.\end{aligned}$$

Hence,

$$\begin{aligned}e^{2.634} < S_T < e^{3.418}\\ 13.93 < S_T < 30.51.\end{aligned}$$

There is a 95% probability that the stock price in in 3 months' time will lie between 13.93 and 30.51.

Note It should be noted that

$$\begin{aligned}\mathrm{E}[S_T] &= S_0 e^{\mu T} = 20e^{0.2 \cdot 0.25} = 21.025\\ &\neq e^{\mathrm{E}[\ln S_T]} = 20.615.\end{aligned}$$

In particular, $\ln \mathrm{E}[S_T] \neq \mathrm{E}[\ln S_T]$.

Notes

1. A variable that has a lognormal distribution can take any value between zero and infinity. It is not symmetrical unlike the normal distribution.

2. From the properties of the lognormal distribution, the expected value of the stock price is given by (exercises)

$$\mathrm{E}[S_T] = S_0 \exp(\mu T)$$

and the variance of S_T is

$$\mathrm{Var}[S_T] = S_0^2 \exp(2\mu T)(\exp(\sigma^2 T) - 1).$$

Example 3.17. *Consider a stock that pays no dividends, provide an expected return of 20% per annum with continuous compounding and has a volatility of 40% per annum. Assume that the current stock price is ฿20. Calculate the mean and the standard deviation of the stock price S_T in 1 year.*

Solution: The mean of S_1 is

$$\mathrm{E}[S_1] = 20 \exp(0.2 \cdot 1) = 24.428.$$

and the variance of S_T is

$$\mathrm{Var}[S_1] = 20^2 \exp(2 \cdot 0.2 \cdot 1)(\exp(0.4^2 \cdot 1) - 1) = 103.539.$$

and

$$\mathrm{SD}(S_1) = \sqrt{103.539} = 10.175.$$

3.7.2 More accurate simulation : the lognormal model

Because T can be any time interval, we can use the lognormal distribution of stock prices to simulate the price of a stock at time $t+\Delta t$ given its price at time t. This provides accurately simulated points along a typical path, regardless of the size of Δt.

The following equations can be use to simulate stock price and estimate volatility:

$$\ln\left(\frac{S_{t+\Delta t}}{S_t}\right) = (\mu - \frac{1}{2}\sigma^2)\Delta t + \sigma\epsilon\sqrt{\Delta t}$$

and

$$S_{t+\Delta t} = S_t \exp((\mu - \frac{1}{2}\sigma^2)\Delta t + \sigma\epsilon\sqrt{\Delta t})$$

Note The above equations follows from the fact that the process $\ln S$ satisfies the following stochastic differential equation

$$d\ln S = (\mu - \frac{1}{2}\sigma^2)\,dt + \sigma\,dW.$$

3.7.3 The distribution of the rate of return

The lognormal property of stock prices can be used to show that the probability distribution of the **continuously compounded rate of return per year** for a stock is **normally distributed**.

Recall that with the continuously compounded rate δ of return per year, the accumulated value at time T of the stock S_0 at time 0 is

$$S_T = S_0\exp(\delta T),$$

or

$$\delta = \frac{1}{T}\ln\frac{S_T}{S_0}.$$

Hence, it follows that

$$\delta \sim N\left(\mu - \frac{1}{2}\sigma^2, \frac{\sigma^2}{T}\right).$$

Notes

1. The standard deviation of δ declines as we consider longer time intervals. This means that
   - if we hold a stock for a short time, our actual return may vary significantly from the expected return,
   - but the longer we holder a stock, the more likely we are to earn a return close to the expect return.
2. When $T = 1$, the standard deviation of the stock return equal σ. As been seen earlier, the **volatility of a stock is the standard deviation of the distribution of its continuously compounded return per year**.

3.7.4 The two expected returns (this section may be skipped)

As we have seen, there are two different expected returns of stocks, μ and $\mu - \sigma^2/2$.

- Equation

  $$propReturn$$

  shows that the expected percentage change (or proportional change) in the stock price in a very short period of time, Δt is $\mu\Delta t$.

Hence, μ is the **expected rate of return per year for the stock for infinitesimal interval of time**.

- Whereas, $\mu - \sigma^2/2$ is the **expected continuously compounded rate of return per year for finite intervals of time** (like days or years), i.e. realised over a period of time of length T (as shown in Equation

$$returnRate$$

).

Notes

1. Recall that the expected price of a stock grows at the continuously compounded rate of μ and we can write
$$\mathrm{E}[S_T] = S_0 \exp(\mu T).$$
2. However, the expected continuously compounded rate of return per year for a stock is $\mu - \sigma^2/2$. This is not the same as the rate of return we would calculate from the expected future price of stock return. μ is not the expected continuously compounded return on the stock.
3. As a result, we cannot get the expected future price of the stock by growing its current price at the expected continuously compounded rate of return per year.
4. If two stocks have the same price today and the same μ, then their expected prices at any time in the future will be the same. The stock with the higher volatility will have a lower continuously compounded expected rate of return.

3.7.5 Estimating volatility

There are many different methods to estimate the volatility of a stock. Here the method is based on a stock's historical volatility calculated from its daily prices for a certain number of days. As we have seen, the simulations of a stock price can be simulated by using the following equation

$$\ln\left(\frac{S_{t+\Delta t}}{S_t}\right) = (\mu - \frac{1}{2}\sigma^2)\Delta t + \sigma\epsilon\sqrt{\Delta t}.$$

The estimation of the volatility of the stock price can be done as follows: Let $n+1$ denote the number of observations and τ to be the length of observation time interval in years.

1. Given the stock price S_i at end of the ith interval for $i = 0, 1, \ldots, n$, we define
$$u_i = \ln\left(\frac{S_i}{S_{i-1}}\right), \qquad \text{for } i = 0, 1, \ldots, n$$
2. The estimates of the mean and standard deviation of the **daily returns** are given by
$$\begin{aligned}\bar{u} &= \frac{1}{n}\sum_{i=1}^{n} u_i \\ s &= \sqrt{\frac{1}{n-1}\sum_{i=1}^{n}(u_i - \bar{u})^2} \\ &= \sqrt{\frac{1}{n-1}\sum_{i=1}^{n} u_i^2 - \frac{1}{n(n-1)}\left(\sum_{i=1}^{n} u_i\right)^2}\end{aligned}$$

3. Because we are using observations at intervals of τ measured in years, the estimate of the **annualised volatility** is given by

$$\hat{\sigma} = \frac{s}{\sqrt{\tau}}.$$

The **standard error of this estimate** is approximately $\hat{\sigma}/\sqrt{2n}$.

For example, if we use daily prices and assume that there are 250 trading days in a year, then the time interval is 1/250 years. To annualise the daily volatility, we have to multiply s by $\sqrt{250}$.

Note It follows from

$$returnRate$$

that the estimated expected return $\hat{\mu}$ is given by

$$\hat{\mu} = \frac{\bar{u}}{\tau} + \frac{\hat{\sigma}^2}{2} = \frac{\bar{u}}{\tau} + \frac{s^2}{2\tau}.$$

Example 3.18. *Verify the calculation using the stock price historical data from the table below.*

Solution: See an accompanied Excel file.

Computation of volatility

Date	Closing Price (S_i)	Price relative (S_i/S_i-1)	Daily return (u_i)	u_i^2
3/15/2018	1816.08			
3/16/2018	1811.76	0.998	-0.002	0.000006
3/19/2018	1799.79	0.993	-0.007	0.000044
3/20/2018	1799.84	1.000	0.000	0.000000
3/21/2018	1801.43	1.001	0.001	0.000001
3/22/2018	1798.55	0.998	-0.002	0.000003
3/23/2018	1794.21	0.998	-0.002	0.000006
3/26/2018	1801.1	1.004	0.004	0.000015
3/27/2018	1802.58	1.001	0.001	0.000001
3/28/2018	1784.99	0.990	-0.010	0.000096
3/29/2018	1766.92	0.990	-0.010	0.000104
3/30/2018	1776.26	1.005	0.005	0.000028
4/2/2018	1782.28	1.003	0.003	0.000011
4/3/2018	1765.24	0.990	-0.010	0.000092
4/4/2018	1724.98	0.977	-0.023	0.000532
4/5/2018	1739.92	1.009	0.009	0.000074
4/9/2018	1751.27	1.007	0.007	0.000042
4/10/2018	1760.95	1.006	0.006	0.000030
4/11/2018	1763.22	1.001	0.001	0.000002
4/12/2018	1767.17	1.002	0.002	0.000005
4/17/2018	1755.53	0.993	-0.007	0.000044

n	20
Sum u_i	-0.0339
Sum u_i^2	0.0011
Estimate of the sd of daily return	0.753%
an estimate for the volatility per annum	11.907%
SE of this estimate	1.883%

Figure 3.7: Computation of volatility

3.8 Chapter TWO

Chapter 4

Chapter THREE

Chapter 5

Applications

Some *significant* applications are demonstrated in this chapter.

5.1 DataCamp Light

By default, `tutorial` will convert all R chunks.

eyJsYW5ndWFnZSI6InIiLCJzYW1wbGUiOiJhIDwtIDJcbmIgPC0gM1xuYSArIGIifQ==