SCMA329 Practical Mathematical Financial Modeling

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Contents

1 Dates and date functions

Dates and date functions will be discussed in this section. Depending on the system setting, the dates used in this section corresponds to the U.S. setting, i.e. month/day/year. Thus, 7/1/2017 corresponds to 1 July 2017.

The following date formats are recognized by Excel:

- 7/1/2017, 7/1/17
- 7-1-2017, 7-1-17,
- 7-1/2017, 7-1/17,
- July 1, 2017

- 1-Jul-2017

if the date input is without the year, then it is the date on the current year, i.e. 7-1, 7/1, July 1, Jul 1, corresponds to 1 July 2018.

1.1 Inconsistent date entries

Example 1.1. Example 1. *What will be the output when we enter dates by using the following two digits for the years:*

1. *7/1/01*
2. *7/1/60*

Excel interprets two digit years between 00 and 29 as 21st century dates, and two digit years between 30 and 99 as 20th century dates.

1.2 Date serial number

Excel stores dates as sequential serial numbers so that they can be used in calculations. For example,

- 1 January 1900 is serial number 1,
- 1 January 2018 is serial number 43101 because it is 43101 days after 1 January 1900.

1.3 Date functions

The list of date functions can be seen by choosing

Formulas → Function Library → Function Library → Date & Time.

Function	**Description**
DATE	Returns the serial number of a particular date
DATEVALUE	Converts a date in the form of text to a serial number
DAY	Converts a serial number to a day of the month
DAYS	Returns the number of days between two dates
WEEKDAY	Converts a serial number to a day of the week
NETWORKDAYS	Returns the difference between two dates, excluding weekend days (Saturdays and Sundays)
WORKDAY	Returns a number that represents a date that is the indicated number of working days before or after
EDATE	Returns the serial number that represents the date that is the indicated number of months before or a
EOMONTH	Returns the serial number for the last day of the month that is the indicated number of months before
YEARFRAC	Returns a decimal value that represents fractional years between two dates

For a complete list, see the date and time functions available online. More detailed examples are also given in the Excel lab.

2 Working with Texts

In this section, we will see how Excel handles text strings, and how we use text functions to modify and manipulate text strings. First, we give a note regarding texts in Excel.

- A single cell can hold up to 32,000 characters. In case you need to display a lot of text in a worksheet, then use a text box (Choose Insert ⇒ Text ⇒ Text Box). It will be easier to edit texts in the text box than in cells.
- Sometimes, when you download numerical data from the internet or database, the imported values are treated as text, i.e. when you do a calculation with such data, you will get a #VALUE error. When a number is not treated as a number, there will be an error indicator. By clinking to expand a list of options, you can then convert it to the number.
- Another issue that you may encounter is about currency that uses different characters to separate thousands or decimals. https://en.wikipedia.org/wiki/Decimal_separator

2.1 Text functions

The following Excel functions can be used to modify text strings in the format you need. Alternatively, one may extract data by using the Convert Text To Columns Wizard (choose Data ⇒ Text To Columns).

1. RIGHT(text,[n]) returns the last n characters in a text string.
2. LEFT(text, [n]) returns the first n characters in a text string.
3. MID(text, start_num, num_chars) returns num_char characters from a text string, starting at start_num.
4. TRIM(text) removes all spaces from text except for single spaces between words. Use TRIM when text strings have irregular spacing.
5. LEN(text) returns the number of characters in a text string.
6. FIND(find_text, within_text, [start_num]) return the location at or after character start_num of the first character of find_text in within_text.
7. SEARCH(find_text, within_text, [start_num]) has the same syntax as FIND, but it is not case sensitive.
8. SUBSTITUTE(text, old_text, new_text, [instance_num]) is used to replace new_text for old_text in a text string. Here Instance_num is optional. It specifies which occurrence of old_text you want to replace with new_text. If you specify instance_num, only that instance of old_text is replaced. Otherwise, every occurrence of old_text in text is changed to new_text.

2.2 Character codes

Excel uses the standard ASCII character set. Therefore, each character has its own code. For example, to get the code number of "A", simply type

=CODE("A"), which returns the code number 65.

The CHAR function reverses the role of CODE function, i.e.

=CHAR(65), which returns the letter A. The input for the CHAR function should be a value between 1 and 255.

For the complete list of ASCII codes, please visit https://theasciicode.com.ar

Example 2.1. Example 1. *Create an Excel file to list all the first 255 ASCII codes. It is a good idea to compare the outputs with those from the website above.*

2.3 Determining whether two string are identical

In order to determine whether strings in cell A1 and A2 have the same contents, we use = A1 = A2, which returns either TRUE or FALSE. Note that the comparison is not case-sensitive.

Alternative, the function that provides an exact, case-sensitive comparison is EXACT function.

2.4 Joining two or more cells

To join two or more cells, Excel uses an ampersand &. For example if the string “the effective interest rate per annum” is in A1 and the value of 5% (formatted value) is in B1. Then use the formula

= A1 & ” is ” & B1, which returns the effective interest rate per annum is 0.05.

A better solution is to use the TEXT function to format the value as text as follows.

= A1 & ” is ” & TEXT(B1,” 0.00%“) , which returns the effective interest rate per annum is 0.05.

Use **Home Format Format cells** to obtain the list of various text formats.

For example, if A1 contains the principle of $10,000, then

= “The principle is” & TEXT(A1, ” $#,##0”), which returns The principle is $10,000.

The following table gives examples that are format with TEXT function.

```
=TEXT(1234.567,"$#,##0.00")
Currency with a thousands separator and 2 decimals, like $1,234.57. Note that Excel rounds the value to

=TEXT(TODAY(),"MM/DD/YY")
Today's date in MM/DD/YY format, like 03/14/12

=TEXT(TODAY(),"DDDD")
Today's day of the week, like Monday

=TEXT(NOW(),"H:MM AM/PM")
Current time, like 1:29 PM

=TEXT(0.285,"0.0%")
Percentage, like 28.5%

=TEXT(4.34 ,"# ?/?")
Fraction, like 4 1/3

=TRIM(TEXT(0.34,"# ?/?"))
Fraction, like 1/3. Note this uses the TRIM function to remove the leading space with a decimal value.

=TEXT(12200000,"0.00E+00")
Scientific notation, like 1.22E+07

=TEXT(1234567898,"[<=9999999]###-####;(###) ###-####")
Special (Phone number), like (123) 456-7898

=TEXT(1234,"0000000")
Add leading zeros (0), like 0001234
```

The TEXT function will convert numbers to text. It is best practice to keep your original values in one cell, and formatted number in another cell. When you do calculation, you should refer to the cells containing original values.

3 Interest Rate, Present Values and Cashflows

3.1 Working with a single cashflow in Excel

3.1.1 How to calculate the future value of a single cashflow

Suppose an amount C is deposited in an account that pays a fixed interest at the rate of $i\%$ per time units. Then after t time units, the deposit will have accumulated to

$$C(1+i)^t.$$

In Excel, the function FV calculates the future value of a single investment (and also periodic constant payments) and a constant interest rate.

The syntax of the function is:

$$\text{FV(rate, nper, pmt, [pv], [type]),}$$

where

- rate is the interest rate per period
- nper is the number of periods over which the investment is made.
- pmt (used for an annuity type) is the payment made each period and cannot change over the life of the annuity.
- pv (optional) is an additional cash flow now (time 0)
- fv is an additional cash flow nper periods from now.
- type (optional) is an optional argument that defines whether the payment is made at the start or the end of the period:

  0 - the payment is made at the end of the period;

  1 - the payment is made at the start of the period.

  If the [type] argument is omitted, it takes on the default value of 0.

The following timeline illustrates the cashflows used for the FV function (assuming that the payments (pmt) are made in arrears).

Note For a single cash flow, we set pmt argument to be 0, as there are no ongoing payments after the initial investment.

Example 3.1. Example 1. *You plan to invest today with an interest rate of 3% per year effective. How much money will you accumulate at the end of 2 years?*

Solution: The accumulation of today in 2 years can be calculated by Excel as

$$\text{FV}(3\%, 2, 0, -100).$$

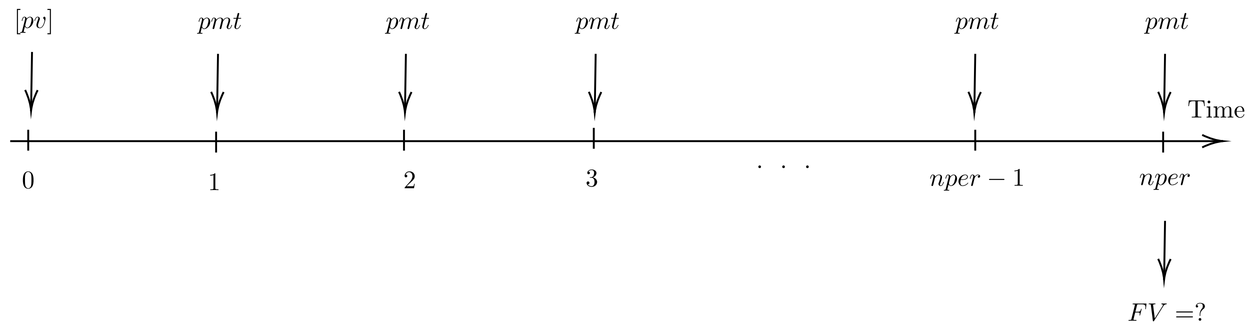

Figure 1: Timeline of cashflows for FV function

3.1.2 How to calculate the present value of a single cashflow

Similarly, the present value of a future cashflow C required at time t time units at a fixed interest of $i\%$ per time units can be calculated as

$$\frac{C}{(1+i)^t}.$$

In Excel, the function PV calculates the present value of a single investment (and also periodic constant payments) and a constant interest rate. The syntax of the function is:

PV(rate, nper, pmt, [fv], [type])

where fv is an additional cash flow nper periods from now.

Example 3.2. Example 2. *How much should you deposit into the account with an interest of 8% so that 10 years from now its value would be ?*

Solution: The present value of today in 10 years can be calculated by Excel as

$$\mathrm{PV}(8\%, 10, 0, -1000).$$

3.2 Present values of a series of cashflows

Consider a series of cashflows defined by

1. the times of payments (cashflows), denoted by $t_1, t_2, \ldots$, and
2. the amount of payments, denoted by C_r (in short for C_{t_r}), which will be paid at time t_r, for $r = 1, 2, \ldots$. The amounts can be positive or negative

The present value at any time t of this series of cashflow is

$$PV(t) = \sum_{r=1}^{\infty} C_r (1+i)^{t-t_r} = \sum_{r=1}^{\infty} C_r v^{t_r - t}$$

where i is the effective rate of interest and $v = 1/(1+i)$.

Notes 1. At a fixed effective rate of interest, the original series of cashflows is equivalent to a single payment of amount $PV(t)$ at time t.

2. If two different series of cashflows have the same PV at one time at a given effective rate of interest, then they have the same PV at any time at that effective rate of interest.

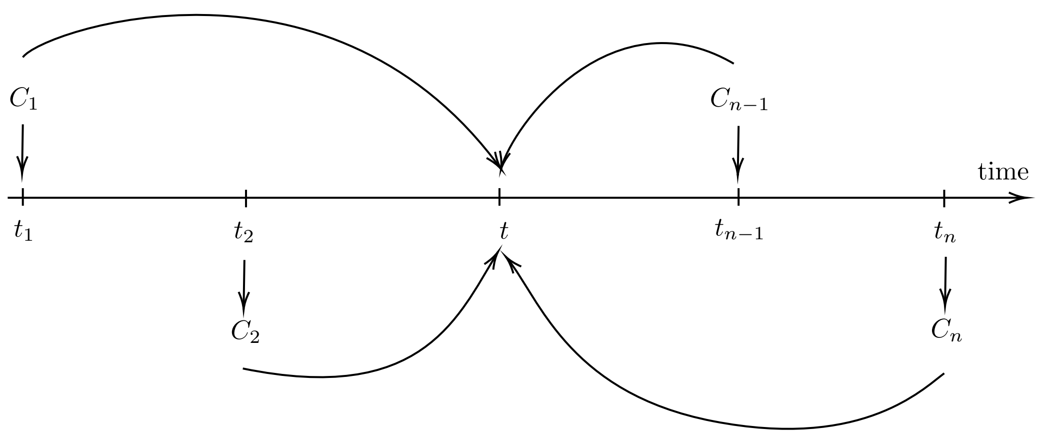

Figure 2: Timeline of a series of cashflows

3.2.1 Level Annuities certain

An *annuity* is a regular series of payments (cashflows). When the payments are certain which are payable for a definite period of time, we call it an *annuity certain.*

- If the payments are made at the end of each time period, they are paid *in arrear*.
- Otherwise, payments are made at the beginning of each time period, they are pain *in advance*.
- An annuity paid in advance is also known as an *annuity due*
- If each payment is for the same amount, this is a *level* annuity.

Example 3.3. Example 3. *Let i be the constant effective rate of interest per time unit. In Excel, one can calculate the accumulated value of a level annuity certain having cashflow of pmt unit at the end of each of the next n time units by*

$$FV(i\%, n, pmt, 0).$$

The cashflows of this annuity is shown in the timeline below.

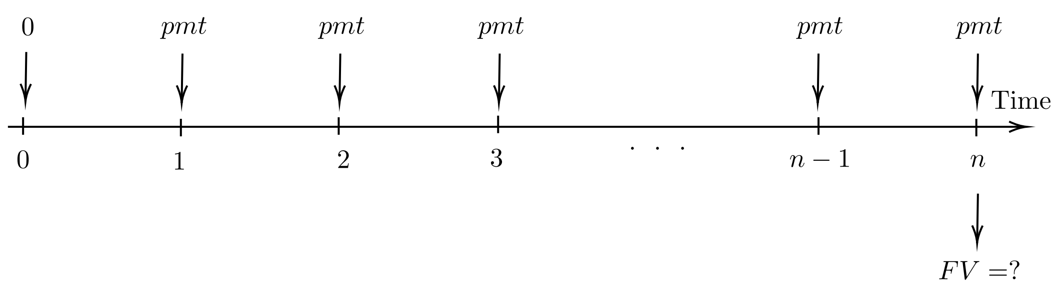

Figure 3: Level Annuity Certain

Note *The last argument of the above syntax, [pv] (optional) which is an additional cash flow now (at time 0), has been set to 0.*

Example 3.4. Example 4. *Let i be the constant effective rate of interest per time unit. In Excel, one can calculate the present value at time 0 of a level annuity certain having cashflow of pmt unit at the end of each* *of the next n time units by*

$$PV(i\%, n, pmt, 0).$$

The timeline of these cashflows is shown in the figure below:

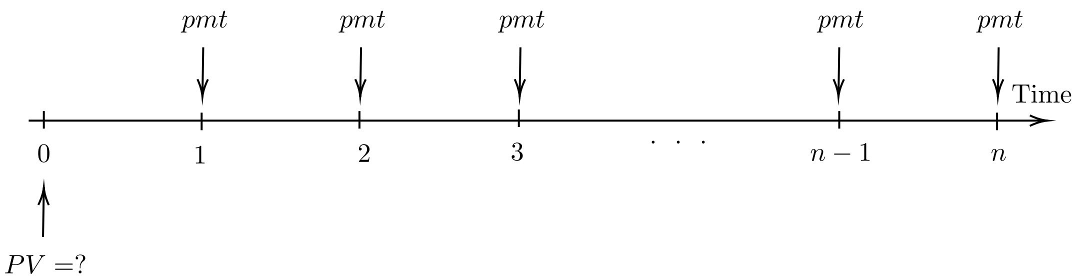

Figure 4: Level Annuity Certain

Note The last argument of the above syntax, [fv] (optional) which is an additional cash flow at time n, has been set to 0.

Example 3.5. Example 5. *Given the effective rate of interest of* 8% *p.a., use Excel to calculate*

1. *the accumulation at 12 years of payable yearly in arrear for the next 12 years.*
2. *the present value now of ,000 payable yearly in arrear for the next 6 years.*
3. *the present value now of ,000 payable half-yearly in arrear for the next 12.5 years.*

Example 3.6. Example 6. *Let* $i = 4\%$ *effective per time unit. Cashflows are given as follows:*

- $C_1 = 200$ *at time* $t_1 = 1$.
- $C_2 = 300$ *at time* $t_2 = 3$.
- $C_3 = -100$ *at time* $t_3 = 5$.
- $C_4 = -50$ *at time* $t_4 = 6$.

Develop the model using Excel to calculate

1. *the accumulation at time* $t = 7$.
2. *the present value at time* $t = 0$.
3. *the present value at time* $t = 4$.

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4 Tutorial 1

1. Calculate the following accumulation:
    1. Accumulate \$5,000 for 4 years at 7.5% per annum effective.
    2. Accumulate \$800 for 2.7 years at 3% per quarter-year effective.
    3. Accumulate \$10,000 for 27 months at 4.25% per half-year effective.
2. Calculate the present values on 1 January 2015 of the following payments at the given rates of interest:

   1. \$1,000 on 1 January 2016, at 7.5% per annum effective.
   2. \$100 on 1 October 2016, at 3% per quarter-year effective.
   3. \$10,000 on 1 April 2016, at 4.25% per half-year effective.

3. 1. If the effective rate of interest is 4% per annum, calculate the effective rate of interest per month?
   2. If the effective rate of interest is 6.5% per half-year, calculate the effective rate of interest per quarter-year?

4. The effective rate of interest per annum was 4% during 2015, 5% during 2016 and 6% thereafter.
   1. Calculate the accumulation of \$500 from 1 January 2015 to 1 January 2018.
   2. Calculate the accumulation of \$2000 from 1 April 2015 to 1 October 2017.
   3. Calculate the accumulation factor from 1 January 2015 to 1 January 2018.

5. You deposit \$ 3000 to an account that earn 2.5% compounded annually. How much will you have in three years?

6. A person borrows a sum of \$5,000 and agrees to pay this back at the end of 1 year with interest calculated at an effective rate of 10% per annum. Calculate the amount to be repaid for the loan.

7. You want to have \$1000 in 2 years and \$2000 in 4 years. How much should you deposit now into an account earning the effective rate of 5.75% semiannually?

8. Katy deposits 100 into a saving account which pays interest at i **per quarter** effective.

   At the same time, Taylor deposits 500 into a different saving account which pays a simple interest at an annual rate of i.

   During the last 3 months of the 4th year, they both earn the same amount of interest. Calculate i.

9. An ordinary annuity is a series of equal payments made at the end of consecutive periods over a fixed length of time. Draw a timeline for the following annuity having cashflow of 1 unit at the end of each of the next n time units.

10. Draw a timeline to illustrate this insurance benefit: Whole Life Insurance - payable immediately on death - has following conditions:
    - death benefit (sum insured) of 1
    - payable immediately on the death
    - of an individual currently aged x
    - for death occurring any time in the future.

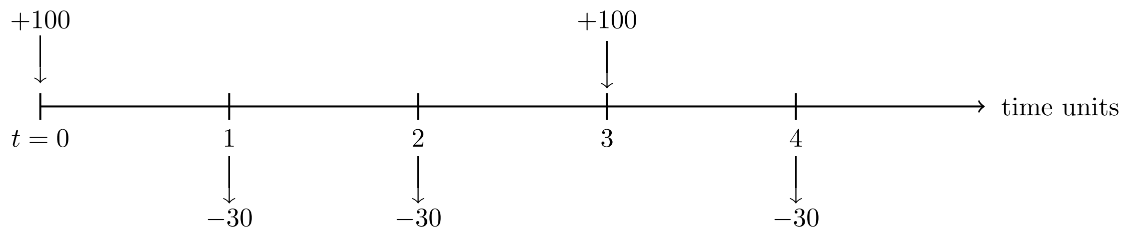
11. (Excel) It is a good exercise to check whether the Excel worksheet you have developed so far for calculating the present value and future value can be applied to the questions in this Tutorial. What would you do to improve the Excel worksheet that can be applied to a more general scenario?

5 Tutorial 2

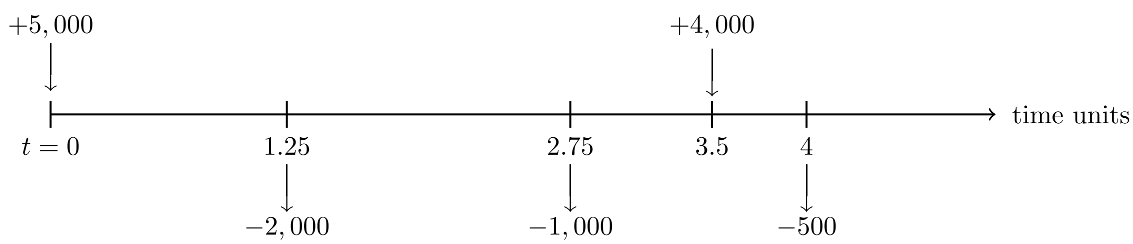
1. Starting at 1 January 2015, the effective rate of interest per annum was 3% per quarter-year for 9 months, 4% per half-year for 15 months and and 2% per month thereafter.
   1. Calculate the accumulation factor from 1 January 2015 to 1 January 2018.
   2. Calculate the accumulation of \$5,000 from 1 July 2015 to 1 October 2017.
   3. Calculate the accumulation of \$100 from 1 March 2016 to 1 August 2018.

    4. Calculate the present value at 1 January 2015 of \$ 25,000 receivable on 1 July 2016.
    5. Calculate the present value at 1 April 2015 of \$ 8,000 receivable on 1 October 2017.
    6. Calculate the discount factor from 1 July 2015 to 1 October 2016.

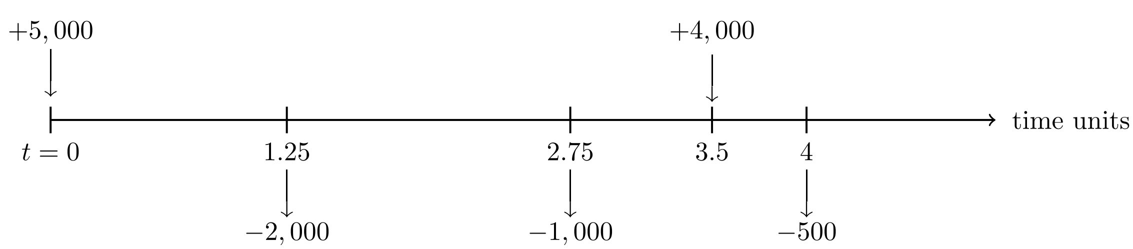
2. The effective rate of interest is 7.25% per time unit. Cashflows are shown in the following time line.
    1. Calculate the accumulation at time time t = 4 units of these cashflows.
    2. Calculate the accumulation at time time t = 8 units of these cashflows.
    3. Calculate the present value at time time t = 0 units of these cashflows.



3. The effective rate of interest is 6% per time unit. Cashflows are shown in the following time line.
    1. Calculate the accumulation at time time t = 5 units of these cashflows.
    2. Calculate the value at time time t = 2 units of these cashflows.
    3. Calculate the present value at time time t = 0 units of these cashflows.



4. The effective rate of interest per annum was 4% during 2015, 3% per half-year until 1 October 2017 and 1.5% per month thereafter. Cashflows are shown in the following time line.
    1. Calculate the accumulation on 1/1/2019 of these cashflows.
    2. Calculate the present value on 1/1/2015 of these cashflows.
    3. Calculate the value at time time 1/7/2017 of these cashflows.



5. (Excel) It is a good exercise to check whether the Excel worksheet you have developed so far for calculating the present value and future value can be applied to the questions in this Tutorial. What would you do to improve the Excel worksheet for a more general scenario?

6 Tutorial 3

1. Calculate the present value now of an annuity payable monthly in advance. The annual amount of the annuity will be \$ 2,400 for the first 10 years and \$ 3,600 for the next 15 years, after which payment will cease. Assume that the effective rate of interest is 2% per annum.

2. Assume that the effective rate of interest will be 3% for 5 years from now, 4% for the next 5 years and 5% thereafter. Calculate the following values:

    1. The present value of an annuity of \$ 1,000 per annum, payable in arrear for 15 years.

    2. The present value of an annuity due of \$ 500 per annum, payable at the beginning of the year for 20 years.

    3. The accumulation value of an increasing annuity payable yearly in arrear for 30 years. The first annual payment is \$ 100, and payments will be increase by \$ 100 each year.

    4. The accumulation value of an increasing annuity payable yearly in advance for 18 years. The first annual payment is \$ 1,000, and payments will be increase by 2% each year (compound).

    5. The present value of an annuity of \$ 200, payable in arrear for 10 years and deferred for 3 years.

3. You borrow \$ 240,000 from a bank to be repaid by the end of 5 years. Assume that the interest rate is 4% per annum. Consider the following four possible options for the loan to be repaid.

    1. Calculate the amount of the repayments to repay if you choose to repay the loan as late as possible.

    2. You may choose to repay interest only during the 5 years term of loan and repay the capital at the end of the term. Calculate interest to be repaid and draw the timeline to illustrate the cashflows for the repayment of the loan.

    3. Calculate the amount X of level instalments to repay the loan which will be paid at the end of each year for 5 years and draw the timeline to illustrate the cashflows for the repayment of the loan.

    4. Calculate the amount Y of level instalments to repay the loan which will be paid at the end of each month for 5 years and draw the timeline to illustrate the cashflows for the repayment of the loan. **Instalment** is a sum of money due as one of several equal payments for something, spread over an agreed period of time.

4. A person now age 30 has received a pension from a company. When he retires at age 60, he will be paid on each birthday from the 60 to the 85th inclusive. The first annual payment will be half of his salary when he retires, and payments will then increase by 2% compounding each year. Currently, he receive a salary of \$ 20,000 and will increase by 3% each year compounding in line with inflation. Assume that the effective rate of interest will be 4% for the next 20 years and 5% thereafter. Calculate the present value now of this pension.

5. (Excel) Use Excel worksheet you have developed so far to calculate the results from the questions in this Tutorial.

7 Solutions to Tutorial 1

1. The solutions to each question are as follows:

    1. $5000(1.075)^4 = 6677.345703$
    2. Let i% be the annual rate effective equivalent to 3% per quarter-effective, $i = (1.03)^4 - 1$. Hence, the accumulation is
    $$800(1+i)^{2.7} = 800(1.03)^{4\times 2.7} = 1100.859802.$$

   3. Let $j\%$ be the monthly rate effective equivalent to 4.25% per half-year effective, $j = (1.0425)^{2/12} - 1$. Hence, the accumulation is

$$10000(1.0425)^{(2/12)\times 27} = 12059.86056.$$

2. The solutions to each question are as follows:
   1. $\frac{1000}{1.075} = 930.232558$.
   2. $\frac{100}{(1.03)^7} = 81.309151$.
   3. Let $j\%$ be the quarterly rate effective equivalent to 4.25% per half-year effective, $j = (1.0425)^{2/4} - 1$. Hence, the present value is

$$10000 \times (1+j)^{-5} = 10000(1.0425)^{-(5/2)} = 9011.764643.$$

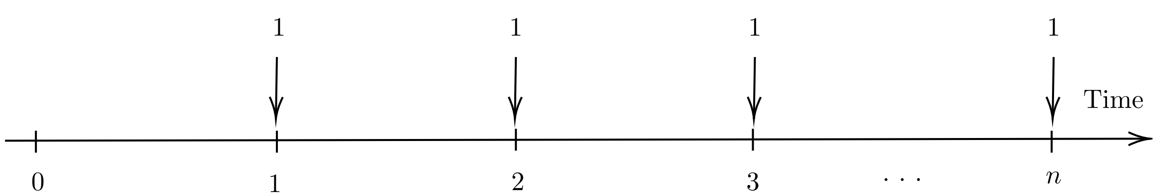
3. The solutions to each question are as follows:
   1. 0.3274%
   2. 3.1988%
4. The solutions to each question are as follows:
   1. $500(1.04)(1.05)(1.06)$ =578.76
   2. $2000(1.04)^{3/4}(1.05)(1.06)^{3/4} = 2259.299$
   3. $(1.04)(1.05)(1.06) = 1.15752$
5. The account balance in 3 years is $3000(1.025)^3 = 3230.67$.
6. The amount to be repaid for the loan is $5000(1.1) = 5500$.
7. Let X be the amount to be deposited now.

$$X = \frac{1000}{(1.0575)^4} + \frac{2000}{(1.0575)^8} = 2078.36.$$

8. At time 3.75 years, Katy has a balance of $100(1+i)^{15}$. The interest on this balance over the next 3 months is $100(1+i)^{15} \cdot i$. Taylor earns simple interest on the original amount which is equal to $500i \cdot \frac{3}{12}$. Therefore, we solve for i from the following equation:

$$100(1+i)^{15} \cdot i = 500i \cdot \frac{3}{12},$$

   which gives $i = 0.014987$.
9. The timeline for the following annuity having cashflow of 1 unit at the end of each of the next n time units is given in the figure below:



10. (More details in the course "Life Contingencies I"). We need to define a random variable T_x = the remaining future life time of a life aged x.

The quantity of interest is the present value of the death benefit assuming the interest rate of $i\%$ p.a. effective. It is also a random variable,

$$PV = \frac{1}{(1+i)^{T_x}}.$$

It turns out that the premium rate of this whole life insurance is $E[PV]$, the expected value of the present value, PV.

8 Solutions to Tutorial 2

1. The solutions to each question are as follows:
    1. $(1.03)^3(1.04)^{2.5}(1.02)^{12} = 1.528611$
    2. $5000A(0.5, 2.75) = 5000(1.03)(1.04)^{2.5}(1.02)^9 = 6788.786068$
    3. We first find the rate $j\%$ per month effective that is equivalent to the rate of 4% per half-year effective.

       $$j = (1.04)^{1/6} - 1 = 0.00656.$$

       The accumulated value is $100A(1 + 2/12, 3 + 7/12) = 100(1 + j)^{10}(1.02)^{19} = 155.522118$.
    4. $25000V(0, 1.5) = \frac{25000}{(1.03)^3(1.04)^{1.5}} = 21571.39968$.
    5. $8000V(0.25, 2.75) = \frac{8000}{(1.03)^2(1.04)^{2.5}(1.02)^9} = 5720.455921$.
    6. $V(0.5, 1.75) = \frac{1}{(1.03)(1.04)^2} = 0.897627$.
2. The solutions to each question are as follows:
    1. With $i = 7.25\%$ per time period, $V(4) = 100(1+i)^4 - 30(1+i)^3 - 30(1+i)^2 + 100(1+i) - 30 = 138.041762$.
    2. $V(8) = V(4)\cdot(1+i)^4 = 182.641597$.
    3. $PV(0) = V(4)\cdot(1+i)^{-4} = 104.332903$.
3. The solutions to each question are as follows:
    1. With \$ = 6%\$ per time period, $V(5) = 5000(1+i)^5 - 2000(1+i)^{3.75} - 1000(1+i)^{2.25} + 4000(1+i)^{1.5} - 500(1+i) = 6897.948585$.
    2. $V(2) = V(5)(1+i)^{-3} = 5791.650645$.
    3. $PV(0) = V(5)(1+i)^{-5} = 5154.548456$.
4. The solutions to each question are as follows:
    1. $V(1/1/2019) = 100(1.04)(1.03)^{3.5}(1.015)^{15} - 30(1.03)^{3.5}(1.015)^{15} - 30(1.03)^{1.5}(1.015)^{15} + 100(1.015)^{12} - 30 = 152.955693$.
    2. $PV(0) = V(1/1/2015) = \frac{152.955693}{(1.04)(1.03)^{3.5}(1.015)^{15}} = 106.074596$.
    3. $V(1/7/2017) = PV(0)(1.04)(1.03)^3 = 120.546998$.

9 Solutions to Tutorial 3

1. Let j be the effective rate per month equivalent to $i = 2\%$. We have

   $$j = (1.02)^{1/12} - 1 = 0.001652.$$

   Hence,

   $$PV(0) = 200\ddot{a}^{j}_{120} + 300\ddot{a}^{j}_{180}\left(\frac{1}{1.02}\right)^{10} = 60148.03.$$
2. The solutions to each question are as follows:

1. The cashflows have been splitted into three periods: (a) from time point 0-5, (b) 5-10 and (c) time point 10 onward.

$$PV(0) = 1000(a_{5}^{3\%} + 1.03^{-5}a_{5}^{4\%} + 1.03^{-5}1.04^{-5}a_{5}^{5\%}) = 11489.49$$

2. We have

$$PV(0) = 500(\ddot{a}_{5}^{3\%} + 1.03^{-5}\ddot{a}_{5}^{4\%} + 1.03^{-5}1.04^{-5}\ddot{a}_{10}^{5\%}) = 7229.67$$

3. The accumuated value is

$$100(Is)_{5}^{3\%}(1.04)^5(1.05)^{20} + \left[100(Is)_{5}^{4\%} + 500s_{5}^{4\%}\right](1.05)^{20} + \left[100(Is)_{20}^{5\%} + 1000s_{20}^{5\%}\right] = 78929.01$$

4. Let $i_1 = 3\%, i_2 = 4\%$ and $i_3 = 5\%$. The accumuated value is given by

$$\begin{aligned}
V(18) &= \left[1000(1+i_1)^5 + 1000(1.02)(1+i_1)^4 + 1000(1.02)^2(1+i_1)^3 + ... + 1000(1.02)^4(1+i_1)\right](1.04)^5(1.05)^8 \\
&+ \left[1000(1.02)^5(1+i_2)^5 + 1000(1.02)^6(1+i_2)^4 + 1000(1.02)^7(1+i_2)^3 + ... + 1000(1.02)^9(1+i_2)\right](1.05)^8 \\
&+ \left[1000(1.02)^{10}(1+i_3)^8 + 1000(1.02)^{11}(1+i_3)^7 + 1000(1.02)^{12}(1+i_3)^6 + ... + 1000(1.02)^{17}(1+i_3)\right] \\
&= 1000(1.02)^5\left[\left(\frac{1+i_1}{1.02}\right)^5 + \left(\frac{1+i_1}{1.02}\right)^4 + ... + \left(\frac{1+i_1}{1.02}\right)\right](1.04)^5(1.05)^8 \\
&+ 1000(1.02)^{10}\left[\left(\frac{1+i_2}{1.02}\right)^5 + \left(\frac{1+i_2}{1.02}\right)^4 + ... + \left(\frac{1+i_2}{1.02}\right)\right](1.05)^8 \\
&+ 1000(1.02)^{18}\left[\left(\frac{1+i_3}{1.02}\right)^8 + \left(\frac{1+i_3}{1.02}\right)^7 + ... + \left(\frac{1+i_3}{1.02}\right)\right]
\end{aligned}$$

Let $1 + j_1 = \frac{1+i_1}{1.02}$. Then, $j_1 = 0.009804$ and

$$\left[\left(\frac{1+i_1}{1.02}\right)^5 + \left(\frac{1+i_1}{1.02}\right)^4 + ... + \left(\frac{1+i_1}{1.02}\right)\right] = \frac{(1+j_1)^5 - 1}{j_1/(1+j_1)} = 5.148995.$$

Let $1 + j_2 = \frac{1+i_2}{1.02}$. Then, $j_2 = 0.019608$ and

$$\left[\left(\frac{1+i_2}{1.02}\right)^5 + \left(\frac{1+i_2}{1.02}\right)^4 + ... + \left(\frac{1+i_2}{1.02}\right)\right] = 5.301921.$$

Let $1 + j_3 = \frac{1+i_3}{1.02}$. Then, $j_3 = 0.029412$ and

$$\left[\left(\frac{1+i_3}{1.02}\right)^8 + \left(\frac{1+i_3}{1.02}\right)^7 + ... + \left(\frac{1+i_3}{1.02}\right)\right] = 9.134790.$$

Therefore, $V(18) = 32814.45$.

5. The present value is

$$\begin{aligned}
PV(0) &= \left(\frac{200}{(1.03)^4} + \frac{200}{(1.03)^5}\right) + 200a_{5}^{0.04}(1.03)^{-5} + 200a_{3}^{0.05}(1.03)^{-5}(1.04)^{-5} \\
&= 350.2192 + 768.0362 + 386.1574 = 1504.413
\end{aligned}$$

3. The solutions to each question are as follows:

    1. $240000(1.04)^5 = 291996.7$
    2. The interest amounts are $0.04 \times 240000 = 9600$.

3. By the Principle of Equivalence, we have

$$240000 = Xa_{5}^{0.04}.$$

This gives X = 53910.51.

4. Level installments are payable monthly, which follows

$$240000 = Ya_{60}^{j},$$

where $j = (1.04)^{1/12} - 1$. This gives Y = 4412.23.

4. The person retires in 30 years, when his salary is expected to be $20000 \times (1.03)^{30} = 48545.25$. The first payment will be half of this which is equal to 24272.62. The present value at age 60 of his pension is

$$24272.62 \times \ddot{a}_{26}^{0.029412} = 449717.9$$

(the precise value is 449719.051954). Here we use $\frac{1.05}{1.02} = 1.029412$ and the annuity is paid from the 60th to the 85th birthday inclusive so there are 26 payments made in advance. Therefore, the present value of this at age 30 is

$$449717.9 \times (1.05)^{-10} \times (1.04)^{-20} = 126002.9.$$

(the precise value is 126003.181173)