SCMA469 Actuarial Statistics

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library(MASS)
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Chapter 1

Introduction to Stochastic Processes

The course will cover the probabilistic framework for stochastic models of real-world applications with emphasis on actuarial work. We will illustrate some practical actuarial problems for which we will develop mathematical models, tools and techniques for analysing and quantifying the uncertainty of the problems.

Here are some of the examples which will be covered later in the course.

1.1 Examples of real world processes

Example 1.1. ***(No claims discount systems (NCD))*** *A well-known model widely used by auto insurance companies is the* ***no claims discount system****, in which an insured receives a discount for a claim free year, while the insured is penalised by an additional premium when one or more accidents occur.*

An example of the NCD system in UK may be structured as follows:

Level	*7*	*6*	*5*	*4*	*3*	*2*	*1*
Premium	*100%*	*75%*	*65%*	*55%*	*45%*	*40%*	*33%*

The rules for moving between these levels are as follows:

- *For a claim-free year, a policyholder moves down 1 level.*
- *Levels 4-7:*
  - *For every one claim, the policyholder moves up 1 level or remains at level 7.*
  - *For every two or more claims, move to, or remains at, level 7.*
- *Levels 2-3:*
  - *For every one claim, move up 2 levels.*
  - *For every two claims, move up 4 levels.*
  - *For every three or more claims, move to level 7.*
- *Level 1:*
  - *For every one claim, move to level 4.*
  - *For every two claims, move to level 6.*
  - *For every three or more claims, move to level 7.*

The no claims discount system is a form of experience rating consisting of a finite number of levels (or classes), each with its own premium. The 7 levels are experience-rated as described above.

For the NCD model, questions of interest may include:

1. *For 10,000 policyholders at level 7, estimate the* expected numbers* at each discount level at a given time, or once stability has been achieved.
2. *What is the* probability* that a policyholder who is at a specific discount level (i.e. one of the levels 1-6) has no discount after 2 years?
3. *What is the* distribution* of being in one of the levels at time 5 years?
4. *Suppose a large number of people having the same claim probabilities take out policies at the same time. What is the proportion would you expect to be in each discount level in the long run?*

*What would be a suitable model to study the NCD system? As opposed to a **deterministic model** for which its outcomes are fixed, the outcomes of the NCD model are uncertain. It turns out that the NCD system can be studied within the framework of Markov chains, which are examples of stochastic processes. The use of matrix algebra provides a powerful tool to understand and analyse the processes.*

*The evolution of the states or levels can be described the random variables $X_0, X_1, X_2, \ldots$ and probability distributions, where X_n is the level of the policyholder at time n. In this example, the set of all states called the state space is discrete, which consists of seven levels, and the time variable is also discrete. This is an example of a **discrete time, discrete state space stochastic process**.*

Example 1.2. *(**Poisson processes**) Consider the number of claims that occur up to time t (denoted by N_t) from a portfolio of health insurance policies (or other types of insurance products). Suppose that the average rate of occurrence of claims per time unit (e.g. day or week) is given by λ.*

Here are some questions of interest:

1. *On average, 20 claims arrive every day, what is the probability that more than 100 claims arrive within a week?*
2. *What is the expected time until the next claim?*

In this example, the state space consists of all whole numbers $\{0, 1, 2, \ldots\}$, while the time variable is continuous. The process is a **continuous-time stochastic process with discrete state space**. The model used to model the insurance claims is an example of **Poisson processes**. The Poisson process is one of the most widely-used counting processes. Even thought we know that claims occur at a certain rate, but completely at random. Moreover, the timing between claims seem to be completely random.

Later, we will see that there are several ways to describe this process. One can focus on the number of claims that occur up to time t or the times between those claims when they occur. Many important properties of Poisson processes will be discussed.

Example 1.3. *(**Markov processes**) Suppose that we observe a total of n independent lives all aged between x and $x+1$. For life i, we define the following terms:*

- *$x + a_i$ is the age at which observation begins,* $0 \le a_i < 1$.
- *$x + b_i$ is the age at which observation ends, if life does not die,* $0 \le b_i < 1$.
- *$x + t_i$ is the age at which observation stops, by death or censoring.*
- *$d_i = 1$, if life i dies, otherwise $d_i = 0$, if life i censored.*

For example, consider the following mortality data on eight lives all aged between 70 *and* 71*.*

Life	a_i	b_i	d_i	t_i
1	0	1	1	0.25
2	0	1	1	0.75

Life	a_i	b_i	d_i	t_i
3	0	1	0	1
4	0.1	0.6	1	0.5
5	0.2	0.7	1	0.6
6	0.2	0.4	0	0.4
7	0.5	1	1	0.75
8	0.5	0.75	0	0.75

How would one use this dataset to estimate the probability that a life aged 70 dies before age $70+t$ or survives to at least age $70+t$, for $t \in [0,1)$?

In this example, we can represent the process by $\{X_t\}_{t\geq 0}$ with two possible states (alive or dead). This model is also an example of a ***continuous-time stochastic process with discrete state space****.*

Here, we illustrate three actuarial applications which can be modelled by some ***stochastic processes****. We should also emphasis that the outcome of one of the above processes is not fixed or uncertain. The course will provide important tools and techniques to analyse the problems with the goal of quantifying the uncertainty in the system.*

Chapter 2

Review of probability theory

2.1 Random variables

The dynamics of a stochastic process are describes by random variables and probability distributions. This section provides a brief discussion of the properties of random variables.

The probability theory is about random variables. Roughly speaking, a random variable can be regarded as an uncertain, numerical quantity (i.e. the value in $\mathbb{R}$) whose possible values depend on the outcomes of a certain random phenomenon. The random variable is usually denoted by a capital letter $X, Y, \ldots$, etc..

More precisely, let S be a sample space. A **random variable** X is a real-valued function defined on the sample space S,

$$X : S \to \mathbb{R}.$$

Hence, the random variable X is a function that maps outcomes to real values.

Example 2.1. *Two coins are tossed simultaneously and the outcomes are HH, HT, TH and TT. We can associate the outcomes of this experiment with the set $A = \{1, 2, 3, 4\}$, where $X(HH) = 1, X(HT) = 2, X(TH) = 3$ and $X(TT) = 4$. Assume each of the outcomes has an equal probability of 1/4. Here, we can associate a function P (known as a probability measure) defined on $S = \{HH, HT, TH, TT\}$ by

$$P(HH) = 1/4, P(HT) = 1/4, P(TH) = 1/4, P(TT) = 1/4.$$

A probability measure $P : \mathcal{A} \to [0, 1]$, where $\mathcal{A}$ is a collection of subsets of S, has the following properties

1. $0 \leq P(A), \quad A \subset S$.
2. $P(S) = 1$.
3. If $A_i \cap A_j = \emptyset$ for $i, j = 1, 2, \ldots$, and $i \neq j$ where $A_j \subset S$, then

$$P(\cup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} P(A_i).$$

Random variables can be discrete or continuous. If the range of a random variable is finite or countably infinite, then the random variable is a **discrete random variable**. Otherwise, if its range is an uncountable set, then it is a **continuous random variable**.

2.2 Probability distribution

The probability distribution of a random variable X is a function describing all possible values of X and their corresponding probabilities or the likelihood of obtaining those values of X. Functions that define the probability measure for a discrete or a continuous random variable are the **probability mass function (pmf)** and the **probability density function (pdf)**, respectively.

Suppose X is a discrete random variable. Then the function

$$f(x) = P(X = x)$$

that is defined for each x in the range of X is called the **probability mass function** (p.m.f) of a random variable X.

Suppose X is a continuous random variable with c.d.f F and there exists a nonnegative, integrable function f, $f : \mathbb{R} \to [0, \infty)$ such that

$$F(x) = \int_{-\infty}^{x} f(y)\, dy$$

Then the function f is called the **probability density function** (p.d.f) of a random variable X.

Examples of discrete and continuous random variables

The main quantities of interest in a portfolio of motor insurance are the number of claims arriving in a fixed time period and the sizes of those claims. Clearly, the number of claims can be describe by a discrete random variable, whose range is finite or countably infinite. On the other hand, the claim sizes can be describe by a continuous random variable defined over continuous sample spaces.

Example 2.2. *Let N denote the number of claims which arise up to a given time. The range of all possible values N is $\mathbf{N} \cup \{0\}$. Here N is an example of discrete random variable. We could model the number of claims by the Poisson family of distributions. Recall that a random variable N has a Poisson distribution with the parameter λ if its probability distribution is given by*

$$f(n) = e^{-\lambda} \frac{\lambda^n}{n!}, \quad \text{for } n = 0, 1, \dots .$$

Now suppose further that the number of claims N which arise on a portfolio in a week has a Poisson(λ) where $\lambda = 5$. Calculate the following quantities:

1. $\Pr(N \geq 6)$.
2. $\mathrm{E}[N]$.

1. $\Pr(N \geq 6) = 1 - \Pr(N \leq 5) = 1 - \sum_{n=0}^{5} f(n) = 0.3840393$.
2. Clearly, $\mathrm{E}[N] = \lambda = 5$.

In R, density, distribution function, for the Poisson distribution with parameter λ is shown as follows:

Distribution	Density function: $P(X = x)$	Distribution function: $P(X \leq x)$
Poisson	`dpois(x, lambda, log = FALSE)`	`ppois(q, lambda, lower.tail = TRUE, log.p = FALSE)`

When lower.tail is set to be TRUE (or default), probabilities are $P(X \leq x)$, otherwise, $P(X > x)$.

eyJsYW5ndWFnZSI6InIiLCJzYW1wbGUiOiJwcG9pcyg1LCBsYW1iZGEgPSA1LCBsb3dlci50YWlsID0gRkFMU0UpIn0=

Example 2.3. *Let X denote the claim sizes in a given time period. The range of all possible values X is the set of all non-negative numbers. Here X is an example of a continuous random variable. Suitable families of distributions which could be used to modelled claim sizes are "fat tails" distribution. They allow for possibilities of large claim sizes.*

Examples of fat-tailed distributions include

- *the Pareto distribution,*
- *the Log-normal distribution,*
- *the Weibull distribution with shape parameter greater than 0 but less than 1, and*
- *the Burr distribution.*

eyJsYW5ndWFnZSI6InIiLCJzYW1wbGUiOiJsaWJyYXJ5KGdncGxvdDIpXG5kZiA8LSBkYXRhLmZyYW1lKHg9c2VxKD

See https://dk81.github.io/dkmathstats_site/rvisual-cont-prob-dists.html for more details.

The course "SCMA 470 Risk Analysis and Credibility" provides more details about the loss distribution.

2.3 Conditional probability

A stochastic process can be defined as a collection or sequence of random variables. The concept of **conditional probability** plays an important role to analyse dependency between random variables in the process. Roughly speaking, conditional probability is the probability of seeing some event knowing that some other event has actually occurred.

Let A and B be two events (elements of $\mathcal{A}$). The conditional probability of event A given B denoted by $P(A|B)$ is defined as

$$P(A|B) = \frac{P(A \cap B)}{P(B)}.$$

Note that $P(A \cap B)$ is often called the joint probability of A and B, and $P(A)$ and $P(B)$ are often called the marginal probabilities of A and B, respectively.

The events A and B are independent if the occurrence of either one of the events does not affect the probability of occurrence of the other. More precisely, the events A and B are independent if

$$P(A \cap B) = P(A)P(B),$$

or equivalently,

$$P(A|B) = P(A).$$

2.4 Law of total probability

Suppose there are three events: A, B, and C. Events B and C are distinct from each other while event A intersects with both events. We do not know the probability of event A. However, partial information and dependencies between events can be used to calculate the probability of event A, i.e. we know the probability of event A under condition B and the probability of event A under condition C.

The total probability rule states that by using the two conditional probabilities, we can find the probability of event A, which is

$$P(A) = P(A \cap B) + P(A \cap C).$$

In general, suppose $B_1, B_2, \dots B_n$ be a collection of events that partition the sample space. Then for any event A,

$$P(A) = \sum_{i=1}^{n} P(A \cap B_i) = \sum_{i=1}^{n} P(A|B_i)P(B_i).$$

Example 2.4. *Suppose in a particular study area, the vaccination rate for the yearly flu virus is 70%. Suppose of those vaccinated, 10% of the residents still get the flu that year. Calculate the conditional probability of someone getting the flu in this area given that the person was vaccinated.*

Example 2.5. *You are an invester buying shares of a company. You have discovered that the company is planning to introduce a new project that is likely to affect the company's stock price. You have determined the following probabilities:*

- *There is a 80% probability that the new project will be launched.*
- *If a company launches the project, there is a 85% probability that the company's stock price will increase.*
- *If a company does not launch the project, there is a 30% probability that the company's stock price will increase.*

Calculate the probability that the company's stock price will increase.

2.5 Conditional distribution and conditional expectation

Let X and Y be two discrete random variables with joint probability mass function

$$f(x, y) = P(X = x, Y = y).$$

If X and Y are continuous random variables, the joint probability density function $f(x, y)$ satisfies

$$P(X \le x, Y \le y) = \int_{-\infty}^{x} \int_{-\infty}^{y} f(u, v)\, du\, dv.$$

When no information is given about the value of Y, the marginal probability density function of X, $f_X(x)$ is used to calculate the probabilities of events concerning X. However, when the value of Y is known, to find such probabilities, $f_{X|Y}(x|y)$, the conditional probability density function of X given that $Y = y$ is used and is defined as follows:

$$f_{X|Y}(x|y) = \frac{f(x, y)}{f_Y(y)}$$

provided that $f_Y(y) > 0$. The conditional mass function of X is defined in a similar manner.

$$P(X = x|Y = y) = \frac{P(X = x, Y = y)}{P(X = x)}.$$

Note also that the conditional probability density function of X given that $Y = y$ is itself a probability density function, i.e.

$$\int_{-\infty}^{\infty} f_{X|Y}(x|y)\, dx = 1.$$

Note that the conditional probability distribution function of X given that $Y = y$, the conditional expectation of X given that $Y = y$ can be as follows:

$$F_{Y|X}(x|y) = P(X \le x|Y = y) = \int_{-\infty}^{x} f_{X|Y}(t|y)\, dt$$

and

$$\mathrm{E}(X|Y=y) = \int_{-\infty}^{\infty} x f_{X|Y}(x|y)\,dx,$$

where $f_Y(y) > 0$.

Note that if X and Y are independent, then $f_{X|Y}$ coincides with f_X because

$$f_{X|Y}(x|y) = \frac{f(x,y)}{f_Y(y)} = \frac{f_X(x)f_Y(y)}{f_Y(y)} = f_X(x).$$

2.6 Central Limit Theorem

This section introduces the Central Limit Theorem, which is an important theorem in probability theory. It states that the mean of n independent and identically distributed random variables has an approximate normal distribution given a sufficiently large n. This applies to a collection of random variables from any distribution with a finite mean and variance. In summary we can use the Central Limit Theorem to extract probabilistic information about the sums of independent and identical random variables.

Let $X_1, X_2, ...$ be a sequence of i.i.d. random variables with a finite mean $\mathrm{E}[X_i] = \mu$ and finite variance $\mathrm{Var}[X_i] = \sigma^2$. Let Z_n be the normalised average of the first n random variables

$$\begin{aligned} Z_n &= \frac{\sum_{i=1}^{n} X_i/n - \mu}{\sigma/\sqrt{n}} \\ &= \frac{X_1 + X_2 + ... + X_n - n\mu}{\sigma\sqrt{n}}. \end{aligned}$$

Then Z_n converges in distribution to a standard normal distribution.

Chapter 3

Stochastic processes

Evolution of a random process is at least partially random, and each run the process leads to potentially a different outcome. It is of great interest to understand or model the behaviour of a random process by describing how different states, represented by random variables X's, evolve in the system over time. Just as probability theory is considered as the study of mathematical models of random phenomena, the theory of stochastic processes plays an important role in the study of time-dependent random phenomena. Stochastic processes can be used to represent many different random phenomena from different fields such as science, engineering, finance, and economics.

A **stochastic process** is a collection of random variables $\{X_t : t \in T\}$ defined on a common sample space, where

- t is a parameter running over some index set T, called the **time domain**.
- The common sample space of the random variables (the range of possible values for X_t) denoted by S is called the **state space** of the process.

1. The set of random variables may be dependent or need not be identically distributed.
2. Techniques used to study stochastic processes depend on whether the state space or the index set (the time domain) are discrete or continuous.

3.1 Classification of stochastic processes

Stochastic processes can be classified on the basis of the nature of their state space and index set.

1. **Discrete state space with discrete time changes** : No claims discount (NCD) policy: A car owner purchases a motor insurance policy for which the premium charged depends on the claim record. Let X_t denote the discount status of a policyholder with three levels of discount, i.e. $S = \{0, 1, 2\}$ corresponding to three discount levels of 0%, 20% and 40% and the time set is $T = \{0, 1, 2, ...\}$. Both time and state space are discrete.
2. **Discrete state space with continuous time changes** : In a health insurance system, an insurance company may classify policyholders as Healthy, Sick or Dead, i.e. $S = \{H, S, D\}$. The time domain can be taken as continuous, $T = [0, \infty)$.
3. **Continuous state space with continuous time changes** : Let S_t be the total amount claimed by time $t \in T$ where $T = [0, \infty)$ and the state space is $\mathbb{R}$. Both time and state space are continuous. Some continuous time stochastic process taking value in a continuous state space will be studied in Risk Analysis and Credibility course.

4. **Continuous state space with discrete time changes** : The outcomes of the above claim process S_t could be recorded continuously, however, we may choose to model the values only at discrete time, for e.g. the total claim amounts at the end of each day. This may due to the limitation of the measurement process (for e.g. expensive to measure). Hence, the time domain is discrete but the state space is continuous.

In case that claim amounts are recorded to the nearest baht or in satang, i.e. discrete state space, we could also approximate or model the process by using a discrete state space, rather than continuous.

3.2 Random walk: an introductory example

One of the simplest examples of a stochastic process is a simple random walk. Consider a simple model of the price of a stock measured in baht. For each trading day $n = 0, 1, 2, ...$, the stock price increases by 1 baht with probability p or decreases by 1 baht with probability $q = 1 - p$. Let X_n denote the stock price at day n and $X_0 = 100$. This simple model is called a simple random walk.

In the simple random walk process, time is discrete (as observed at the end of each day) and the state space is discrete. The stochastic model has an infinite number of outcomes known as **stochastic realisations or sample paths**. A **sample path** is then just the sequence of a particular set of experiments. Graphs of some stochastic realisations of the simple random walk with $p = 0.5$ and $a = 100$ are shown in Figure 3.1.

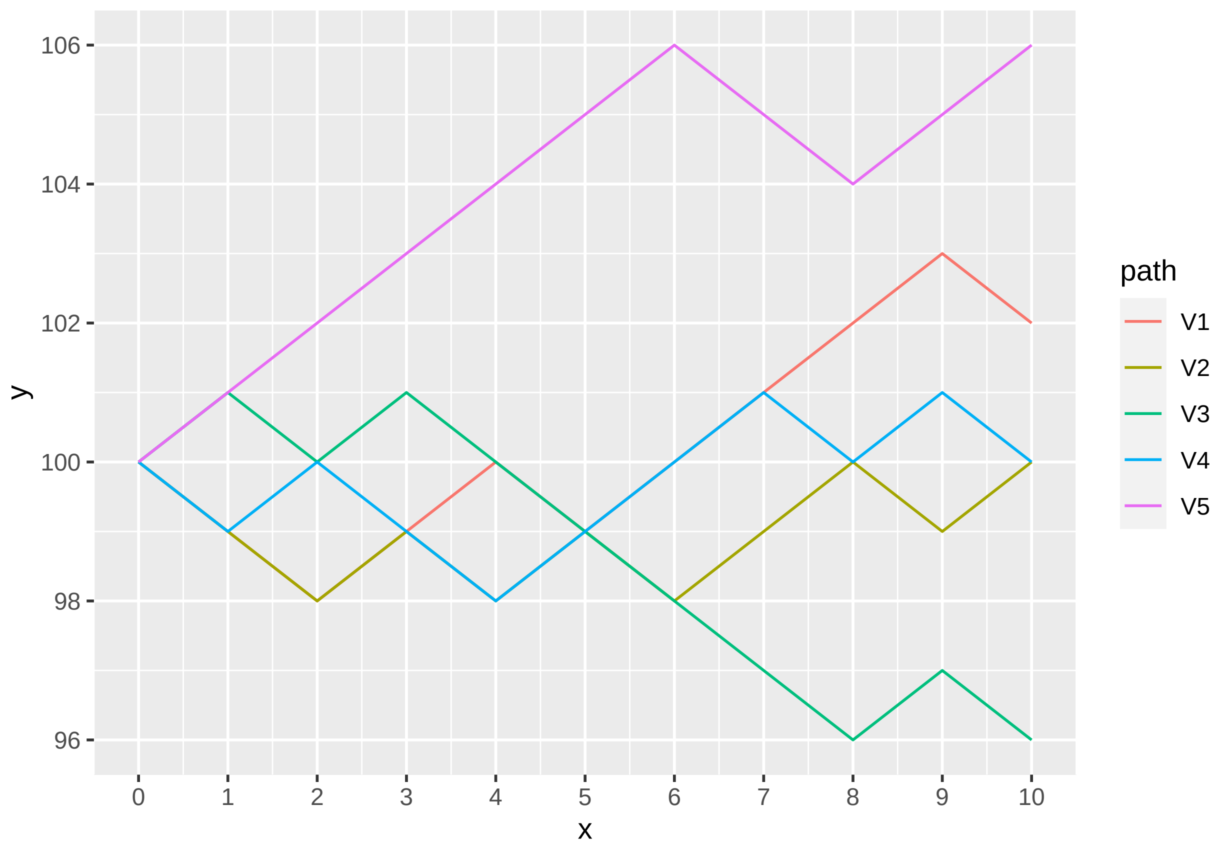


Figure 3.1: Some stochastic realisations of the simple random walk

We can use R to generate sample paths of this random walk.

eyJsYW5ndWFnZSI6InIiLCJzYW1wbGUiOiJsaWJyYXJ5KHRpZHlyKVxuc2V0LnNlZWQoMSlcbnAgPC0gMC41XG5aPC

A complete description of the simple random walk, observed as a collection of n random variables at time points $t_1, t_2, \dots, t_n$ can be specified by the joint distribution of $X_{t_1}, X_{t_2}, \dots, X_{t_n}$, i.e.

$$F(x_1, x_2, \dots, x_n) = \Pr(X_{t_1} \le x_1, X_{t_2} \le x_2, \dots, X_{t_n} \le x_n).$$

However, the **multidimensional distribution function cannot be easily written in a closed form** unless the random variables have a multivariate normal distribution. In practice, it is more convenient to deal with some stochastic processes via some simple **intermediary processes** or under some addition assumptions.

In general, a simple random walk X_n is a discrete-time stochastic process defined by

- $X_0 = a$ and
- for $n \ge 1$,

$$X_n = a + \sum_{i=1}^{n} Z_i, \text{ where } Z_i = \begin{cases} 1, & \text{with probability } p \\ -1, & \text{with probability } q = 1 - p. \end{cases}$$

1. When $p = 1/2$, the value of the process increases or decreases randomly by 1 unit with equal probability. In this case, the process is known as a **symmetric** random walk.
2. The (intermediary) process $\{Z_i : i \in \mathbb{N}\}$ is a sequence of independent identically distributed (i.i.d.) random variables. The process X_t themselves are neither independent nor identically distributed. This process Z_i is also known as **white noise process**.

Example 3.1. *Explain why the simple random process X_n is neither independent nor identically distributed.*

Solution: Suppose that $X_0 = 100$. Firstly, we will show that X_n is not independent. From definition, the process X_n can be written as

$$X_n = X_{n-1} + Z_n.$$

That is, the value of X_n is the previous value plus a random change of 1 or -1. Therefore, the value of the process depends on the previous value and they are not independent. For e.g.,

$$\Pr(X_2 = 102) > 0 \quad \text{but} \quad \Pr(X_2 = 102 | X_1 = 99) = 0.$$

The process X_n cannot be identically distributed. For e.g. X_1 can take the values of 99 and 101, while X_2 can take three different values of 98, 100 and 102.

Example 3.2. *Let*

$$\mu = \mathrm{E}[Z_i]$$
$$\sigma^2 = \mathrm{Var}[Z_i]$$

Calculate the expectation (μ) and variance (σ^2) of the random variable Z_i.

Solution:

$$\mu = \mathrm{E}[Z_i] = 1 \cdot p + (-1) \cdot q = p - q.$$

$$\begin{aligned} \sigma^2 &= \mathrm{Var}[Z_i] \\ &= \mathrm{E}[Z_i^2] - (\mathrm{E}[Z_i])^2 \\ &= 1 - (p-q)^2 = (p+q)^2 - (p-q)^2 \\ &= 4pq. \end{aligned}$$

Example 3.3. *Calculate the expectation and variance of the process X_n at time n.*

Solution:

$$\mathrm{E}[X_n] = \mathrm{E}[a + \sum_{i=1}^{n} Z_i] = a + n\mu.$$

$$\mathrm{Var}[X_n] = \mathrm{Var}[a + \sum_{i=1}^{n} Z_i] = n\sigma^2.$$

It should be noted that the variance of X_n increases with time.

Example 3.4. *For the random process, calculate*

$$\Pr(X_2 = 98, X_5 = 99 | X_0 = 100).$$

Solution: The process X_n must decrease on the first two days, which happens with probability $(1-p)^2$. Independently, it must then increases on another two days and decrease on one day (not necessarily in that order), giving three different possibilities. Each of these has probability $p^2(1-p)$. So

$$\Pr(X_2 = 98, X_5 = 99 | X_0 = 100) = (1-p)^2 \cdot 3p^2(1-p) = 3p^2(1-p)^3.$$

In what follows, we will see that exact calculations are possible for the simple random walk process. Note also that it is sufficient to understand the behaviour of the random walk when it starts at $X_0 = 0$.

Example 3.5. *For the random process with $X_0 = 0$, X_{2n} is always even and X_{2n+1} is always odd. Based on the binomial distribution, show that*

$$\begin{aligned}
\Pr(X_{2n} = 2m) &= \binom{2n}{n+m} p^{n+m} q^{n-m}, \quad -n \le m \le n \\
\Pr(X_{2n+1} = 2m+1) &= \binom{2n+1}{n+m+1} p^{n+m+1} q^{n-m}, \quad -n-1 \le m \le n.
\end{aligned}$$

Solution:

Let A denote the number of $+1$ and B denote the number of -1. Then $A + B = 2n$ and $X_{2n} = A - B$ (i.e. the position at time $2n$). Hence,

$$\begin{aligned}
\Pr(X_{2n} = 2m) &= \Pr(A - B = 2m) \\
&= \Pr(A - (2n - A) = 2m) = \Pr(2A - 2n = 2m) = \Pr(A = m + n) \\
&= \binom{2n}{n+m} p^{n+m} q^{2n-(n+m)}, \quad -n \le m \le n \\
&= \binom{2n}{n+m} p^{n+m} q^{n-m}, \quad -n \le m \le n.
\end{aligned}$$

Chapter 4

Discrete-time Markov chains

Recall the simple random walk model of the price of the stock. Suppose the stock price for the first four days are

$$(X_0, X_1, X_2, X_3) = (100, 99, 98, 99).$$

Based on this past information, what can we say about the price at day 4, X_4? Although, we completely know the whole past price history, the only information relevant for predicting their future price is the price on the previous day, i.e. X_3. This means that

$$\Pr(X_4 = j | X_0 = 100, X_1 = 99, X_2 = 98, X_3 = 99) = \Pr(X_4 = j | X_3 = 99).$$

Given the current price X_3, the price X_4 at day 4 is independent of the history prices X_0, X_1, X_2. The sequence of stock prices $X_0, X_1, \dots, X_n$ is an example of a **Markov chain**.

A Markov process is a special type of stochastic processes with the property that the future evolution of the process depends only on its current state and not on its past history. That is given the value of X_t, the values of X_s for $s > t$ do not depend on the values of X_u for $u < t$. This property is called the **Markov property**.

A **discrete-time Markov chain** is a discrete-time stochastic process that satisfies the Markov property:

$$\Pr(X_{n+1} = j | X_n = i, X_{n-1} = i_{n-1}, \dots, X_0 = i_0) = \Pr(X_{n+1} = j | X_n = i),$$

for all time points n and all states $i_0, i_1, \dots, i_{n-1}, i, j$. It is convenient to assume that the state space of the Markov chain is a subset of non-negative integers, i.e. $S \subseteq \{0, 1, \dots\}$.

Example 4.1. *A process with independent increments has the Markov property.*

Solution:

Recall the following definitions. An increment of a process is the amount by which its value changes over a period of time, for e.g. $X_{t+u} - X_t$ where $u > 0$.

A process X_t is said to have independent increments if for all t and every $u > 0$, the increment $X_{t+u} - X_t$ is independent of all the past of the process $\{X_s : 0 \le s \le t\}$).

In order to show that a process with independent increments has the Markov property, we proceed as follows: for all times $s_1 < s_2 < \dots < s_n < s < t$

$$\begin{aligned}
\Pr(X_t \in A | X_{s_1} = x_1, X_{s_2} = x_2, \dots, X_{s_n} = x_n, X_s = x) &= \Pr(X_t - X_s + x \in A | X_{s_1} = x_1, X_{s_2} = x_2, \dots, X_{s_n} = x_n, X_s = x) \\
&= \Pr(X_t - X_s + x \in A | X_s = x) \text{(by independence of the past)} \\
&= \Pr(X_t \in A | X_s = x).
\end{aligned}$$

Note The random walk process has the Markov property.

4.1 One-step transition probabilities

The conditional probability that X_{n+1} is in state j given that X_n is in state i is called **one-step transition probability** and is denoted by

$$\Pr(X_{n+1} = j | X_n = i) = p_{ij}^{n,n+1}.$$

Note that the transition probabilities depend not only on the current and future states, **but also on the time of transition** n.

If the transition probabilities $p_{ij}^{n,n+1}$ in a Markov chain do not depend on time n, the Markov chain is said to be **time-homogeneous or stationary or simply homogeneous**. Then

$$p_{ij}^{n,n+1} = \Pr(X_{n+1} = j | X_n = i) = \Pr(X_1 = j | X_0 = i) = p_{ij}.$$

Otherwise, it is said to be **nonstationary** or **nonhomogeneous**.

Unless stated otherwise, it shall be assumed that the Markov chain is stationary. The matrix P whose elements are p_{ij} is called the **transition probability matrix** of the process.

$$P = \begin{bmatrix} p_{11} & p_{12} & p_{13} & \cdots \\ p_{21} & p_{22} & p_{23} & \cdots \\ p_{31} & p_{32} & p_{33} & \cdots \\ \vdots & \vdots & \vdots & \vdots \end{bmatrix}.$$

Note that the elements of the matrix P satisfies the following properties:

$$0 \leq p_{ij} \leq 1, \quad \text{and} \quad \sum_{j \in S} p_{ij} = 1,$$

for all $i, j \in S$. A matrix that satisfies these properties is called a **stochastic matrix**.

Example 4.2. *No claims discount (NCD) policy: Let X_n be the discount status of a policyholder at time n. There are three levels of discount, i.e. $S = \{0, 1, 2\}$ corresponding to three discount levels of 0, 20% and 40%. The following rules are assumed:*

- *For a claim-free year, the policyholder moves up a level or remains in state 2 (the maximum discount state).*
- *If there is at least one claim, the policyholder moves down one level or remains in state 0.*

Suppose also that the probability of a claim-free year is p and is independent of n. The transition probability matrix is given by

$$P = \begin{bmatrix} 1-p & p & 0 \\ 1-p & 0 & p \\ 0 & 1-p & p \end{bmatrix}.$$

The transition diagram is illustrated in the following figure.

The following questions are of interest.

1. *What is the probability*
$$\Pr(X_0 = i_0, X_1 = i_1, X_2 = i_2, \dots, X_n = i_n)?$$

2. *What is the probability*
$$\Pr(X_{n+1} = i_{n+1}, X_{n+2} = i_{n+2}, \dots, X_{n+m} = i_{n+m} | X_n = i_n, X_{n-1} = i_{n-1}, \dots, X_0 = i_0)?$$

3. *What is the probability of transferring from state i to state j in n steps*
$$\Pr(X_{m+n} = j | X_m = i)?$$

4. *What is the long-term behavior of the Makov chain, i.e.* $\lim_{n\to\infty} \Pr(X_n = j), j = 0, 1, 2$ *given that* $\Pr(X_0 = 0)$.

Later we will apply matrix algebra to compute these types of probabilities and long-term probabilities.

Example 4.3. *In a health insurance system, at the end of each day an insurance company classifies policy-holders as Healthy, Sick or Dead, i.e.* $S = \{H, S, D\}$. *The following transition matrix* P *for a healthy-sick-dead model is given by*

$$P = \begin{bmatrix} p_{11} & p_{12} & p_{13} \\ p_{21} & p_{22} & p_{23} \\ 0 & 0 & 1 \end{bmatrix}.$$

The transition diagram is shown below.

It turns out that the probabilistic description of the Markov chain is completely determined by its transition probability matrix and its initial probability distribution X_0 at time 0.

Example 4.4. *By using the definition of conditional probabilities, show that*

$$\Pr(X_0 = i_0, X_1 = i_1, X_2 = i_2, \dots, X_n = i_n) = \mu_{i_0}\, p_{i_0 i_1} \cdots p_{i_{n-2} i_{n-1}}\, p_{i_{n-1} i_n},$$

where $\mu = \mu^{(0)}$ *is the distribution of initial random variable* X_0, *i.e.* $\mu_i = \Pr(X_0 = i)$ *(the probability mass function of* X_0*).*

Solution:

$$\begin{aligned}
&\Pr(X_0 = i_0, X_1 = i_1, \dots, X_n = i_n) \\
&= \Pr(X_0 = i_0, X_1 = i_1, \dots, X_{n-1} = i_{n-1}) \cdot \Pr(X_n = i_n | X_0 = i_0, X_1 = i_1, \dots, X_{n-1} = i_{n-1}) \\
&= \Pr(X_0 = i_0, X_1 = i_1, \dots, X_{n-1} = i_{n-1}) \cdot \Pr(X_n = i_n | X_{n-1} = i_{n-1}) \\
&= \Pr(X_0 = i_0, X_1 = i_1, \dots, X_{n-1} = i_{n-1}) \cdot p_{i_{n-1} i_n} \\
&\quad \vdots \\
&= \mu_{i_0}\, p_{i_0 i_1} \cdots p_{i_{n-2} i_{n-1}}\, p_{i_{n-1} i_n}.
\end{aligned}$$

Example 4.5. *Show that*

$$\begin{aligned}
\Pr(X_{n+1} = i_{n+1}, X_{n+2} = i_{n+2}, \dots, X_{n+m} = i_{n+m} | X_n = i_n, X_{n-1} = i_{n-1}, \dots, X_0 = i_0) \\
= \Pr(X_{n+1} = i_{n+1}, X_{n+2} = i_{n+2}, \dots, X_{n+m} = i_{n+m} | X_n = i_n) \\
= p_{i_n i_{n+1}} \cdots p_{i_{n+m-2} i_{n+m-1}}\, p_{i_{n+m-1} i_{n+m}}.
\end{aligned}$$

Solution:

$$\begin{aligned}
&\Pr(X_{n+1} = i_{n+1}, X_{n+2} = i_{n+2}, \dots, X_{n+m} = i_{n+m} | X_n = i_n, X_{n-1} = i_{n-1}, \dots, X_0 = i_0) \\
&= \Pr(X_{n+1} = i_{n+1}, X_{n+2} = i_{n+2}, \dots, X_{n+m} = i_{n+m} | X_n = i_n) \\
&= \Pr(X_{n+2} = i_{n+2}, \dots, X_{n+m} = i_{n+m} | X_{n+1} = i_{n+1}, X_n = i_n) \cdot \Pr(X_{n+1} = i_{n+1} | X_n = i_n) \\
&= \Pr(X_{n+2} = i_{n+2}, \dots, X_{n+m} = i_{n+m} | X_{n+1} = i_{n+1}) \cdot p_{i_n i_{n+1}} \\
&\quad \vdots \\
&= \Pr(X_{n+m} = i_{n+m} | X_{n+m-1} = i_{n+m-1}) \cdots \Pr(X_{n+2} = i_{n+2} | X_{n+1} = i_{n+1}) \cdot p_{i_n i_{n+1}} \\
&= p_{i_{n+m-1} i_{n+m}} \cdot p_{i_{n+m-2} i_{n+m-1}} \cdots p_{i_n i_{n+1}}.
\end{aligned}$$

Note More general probabilities of the possible realisations of the process can be calculated by summing the probabilities of elementary elements of these forms.

Example 4.6. *For the NCD system defined on the state space $S = \{0, 1, 2\}$ as given in Example 4.2 , suppose that the probability of a claim-free year $p = 3/4$, and the distribution of the initial discount rate $\mu = (0.5, 0.3, 0.2)$. Find the following:*

1. $\Pr(X_0 = 2, X_1 = 1, X_2 = 0)$.
2. $\Pr(X_1 = 1, X_2 = 0|X_0 = 2)$.
3. $\Pr(X_{10} = 2, X_{11} = 1, X_{12} = 0)$.
4. $\Pr(X_{11} = 1, X_{12} = 0|X_{10} = 2)$.

Solution: The corresponding transition matrix is

$$P = \begin{bmatrix} 1/4 & 3/4 & 0 \\ 1/4 & 0 & 3/4 \\ 0 & 1/4 & 3/4 \end{bmatrix}.$$

1. Denote $\mu = (\mu_1, \mu_2, \mu_3) = (0.5, 0.3, 0.2)$

$$\begin{aligned} \Pr(X_0 = 2, X_1 = 1, X_2 = 0) &= \Pr(X_0 = 2, X_1 = 1) \cdot \Pr(X_2 = 0|X_0 = 2, X_1 = 1) \\ &= \Pr(X_0 = 2, X_1 = 1) \cdot \Pr(X_2 = 0|X_1 = 1) \quad \text{(by Markov property)} \\ &= \Pr(X_0 = 2) \cdot \Pr(X_1 = 1|X_0 = 2) \cdot \Pr(X_2 = 0|X_1 = 1) \quad \text{(again by Markov property)} \\ &= \mu_3 p_{32} p_{21} = 0.2 \cdot (1/4) \cdot (1/4) = 1/80. \end{aligned}$$

Alternatively, it follows from Example 4.5 that,

$$\Pr(X_0 = 2, X_1 = 1, X_2 = 0) = \mu_3 p_{32} p_{21} = 0.2 \cdot (1/4) \cdot (1/4) = 1/80.$$

2.

$$\begin{aligned} \Pr(X_1 = 1, X_2 = 0|X_0 = 2) &= \Pr(X_1 = 1|X_0 = 2) \cdot \Pr(X_2 = 0|X_1 = 1, X_0 = 2) \\ &= \Pr(X_1 = 1|X_0 = 2) \cdot \Pr(X_2 = 0|X_1 = 1) \quad \text{(by Markov property)} \\ &= p_{32} p_{21} = (1/4) \cdot (1/4) = 1/16. \end{aligned}$$

3. Following conditional probability, the Markov property, and time-homogeneity (to be discussed later) results in

$$\begin{aligned} \Pr(X_{10} = 2, X_{11} = 1, X_{12} = 0) &= \Pr(X_{10} = 2) \cdot \Pr(X_{11} = 1, X_{12} = 0|X_{10} = 2) \\ &= \Pr(X_{10} = 2) \cdot \Pr(X_{11} = 1|X_{10} = 2) \cdot \Pr(X_{12} = 0|X_{10} = 2, X_{11} = 1) \\ &= \Pr(X_{10} = 2) \cdot \Pr(X_{11} = 1|X_{10} = 2) \cdot \Pr(X_{12} = 0|X_{11} = 1) \\ &= \Pr(X_{10} = 2) \cdot \Pr(X_1 = 1|X_0 = 2) \cdot \Pr(X_1 = 0|X_0 = 1) \\ &= \Pr(X_{10} = 2) \cdot p_{32} p_{21} = 0.6922 \cdot (1/4) \cdot (1/4) = 0.0433. \end{aligned}$$

Later we will show that $\Pr(X_{10} = 2) = (\mu P^{10})_3 = 0.6922$ (here $(\mu P^{10})_i$ denotes the i-th entry of the vector μP^{10}.

4. From conditional probability, the Markov property, and time-homogeneity, it follows that

$$\begin{aligned} \Pr(X_{11} = 1, X_{12} = 0|X_{10} = 2) &= \Pr(X_{11} = 1|X_{10} = 2) \cdot \Pr(X_{12} = 0|X_{11} = 1, X_{10} = 2) \\ &= \Pr(X_{11} = 1|X_{10} = 2) \cdot \Pr(X_{12} = 0|X_{11} = 1) \quad \text{(by Markov property)} \\ &= \Pr(X_1 = 1|X_0 = 2) \cdot \Pr(X_1 = 0|X_0 = 1) \quad \text{(by time-homogeneity)} \\ &= p_{32} p_{21} = (1/4) \cdot (1/4) = 1/16. \end{aligned}$$

4.2 The Chapman-Kolmogorov equation and n-step transition probabilities

The n-step transition probability denoted by $p_{ij}^{(n)}$ is the probability that the process goes from state i to state j in n transitions, i.e.

$$p_{ij}^{(n)} = \Pr(X_{m+n} = j | X_m = i).$$

Note that for homogeneous process, the left hand side of the above equation does not depend on m. Suppose that the transition from state i at time m to state j at time $m+n$ (i.e. in n steps), going via state k in l steps. One needs to examine all possible paths (from i to k and then k to j) and hence the n-step transition probability $p_{ij}^{(n)}$ can be expressed as the sum of the product of the transition probabilities $p_{ik}^{(l)}\, p_{kj}^{(n-l)}$.

$$p_{ij}^{(n)} = \sum_{k\in S} p_{ik}^{(l)} p_{kj}^{(n-l)}, \quad 0 < l < n$$

To derive the Chapman-Kolmogorov equation, we proceed as follows:

$$\begin{aligned}
p_{ij}^{(n)} &= \Pr(X_n = j | X_0 = i) \\
&= \sum_{k\in S} \Pr(X_n = j, X_l = k | X_0 = i) \\
&= \sum_{k\in S} \Pr(X_n = j | X_l = k, X_0 = i) \cdot \Pr(X_l = k | X_0 = i) \\
&= \sum_{k\in S} \Pr(X_n = j | X_l = k) \cdot \Pr(X_l = k | X_0 = i) \\
&= \sum_{k\in S} p_{kj}^{(n-l)} p_{ik}^{(l)}, = \sum_{k\in S} p_{ik}^{(l)} p_{kj}^{(n-l)}, \quad 0 < l < n.
\end{aligned}$$

This result is known as the Chapman-Kolmogorov equation. This relation can be expressed in terms of matrix multiplication as

$$P^n = P^l P^{n-l}.$$

The n-**step transition probabilities** $p_{ij}^{(n)}$ are the ij elements of P^n.

Example 4.7. *For the NCD system given in Example 4.2, suppose that $p = 3/4$, the probability of a claim-free year and the initial discount level of a policyholder is 1 (with 20% discount).*

1. *Calculate the probability of starting with a discount level of 20% and ending up 3 years later at the same level.*
2. *Calculate the policyholder's expected level of discount after 3 years.*

Solution: The transition matrix is

$$P = \begin{bmatrix} 1/4 & 3/4 & 0 \\ 1/4 & 0 & 3/4 \\ 0 & 1/4 & 3/4 \end{bmatrix},$$

and

$$P^3 = \begin{bmatrix} 7/64 & 21/64 & 9/16 \\ 7/64 & 3/16 & 45/64 \\ 1/16 & 15/64 & 45/64 \end{bmatrix}.$$

1. The probability of starting with a discount level of 20% and ending up 3 years later at the same level is equal to $p_{11}^{(3)} = 3/16$, which is the element in the second row and second column of the matrix P^3 (not to be confused with the indices used) .

2. The policy's expected level of discount after 3 years is

$$\begin{aligned}\mathrm{E}[X_3|X_0=1] &= \sum_{j=0}^{2} j\cdot \Pr(X_3=j|X_0=1)\\ &= 0\cdot(7/64)+1\cdot(3/16)+2\cdot(45/64)\\ &= 51/32 = 1.59375.\end{aligned}$$

4.3 Distribution of X_n

Let $\mu^{(n)}$ be the vector of probability mass function of X_n, i.e.

$$\mu^{(n)} = (\mu_1, \mu_2, ...),$$

where $\mu_i = \Pr(X_n = i)$. It follows that

$$\mu^{(n+1)} = \mu^{(n)}P$$

and, in general,

$$\mu^{(n+m)} = \mu^{(n)}P^m.$$

Example 4.8. Consider the following questions:

1. Show that
$$Pr(X_1 = i) = \sum_{k\in S} \mu_k p_{ki} = (\mu P)_i,$$
which is the ith element of the vector μP. Here $\mu = \mu^0$ is the distribution of initial random variable X_0 with $\mu_i = \Pr(X_0 = i)$.

2. In general, show that the distribution of X_n is given by
$$Pr(X_n = i) = (\mu P^n)_i.$$

Solution:

1.
$$\begin{aligned}\Pr(X_1=i) &= \sum_{k\in S}\Pr(X_1=i|X_0=k)\cdot\Pr(X_0=k)\\ &= \sum_{k\in S}\mu_k\cdot p_{ki}\\ &= (\mu P)_i.\end{aligned}$$

Example 4.9. *The simple weather pattern can be classified into three types including rainy (R), cloudy (C) and sunny (S). The weather is observed daily. The following information is provided.*

- *On any given rainy day, the probability that it will rain the next day is 0.7; the probability that it will be cloudy the next day 0.2.*
- *On any given cloudy day, the probability that it will rain the next day is 0.75; the probability that it will be sunny the next day 0.1.*
- *On any given sunny day, the probability that it will rain the next day is 0.2; the probability that it will be sunny the next day 0.4.*

The weather forecast for tomorrow shows that there is a 40% chance of rain and a 60% chance of cloudy. Find the probability that it will sunny 2 days later.

Solution: As the ordered state of the chain is R, C, S, the initial distribution is $\mu = (0.4, 0.6, 0)$. The transition matrix P is given by

$$P = \begin{bmatrix} 0.7 & 0.2 & 0.1 \\ 0.75 & 0.15 & 0.1 \\ 0.2 & 0.4 & 0.4 \end{bmatrix},$$

and

$$P^2 = \begin{bmatrix} 0.66 & 0.21 & 0.13 \\ 0.6575 & 0.2125 & 0.13 \\ 0.52 & 0.26 & 0.22 \end{bmatrix}.$$

This gives

$$\mu \cdot P^2 = (0.6585, 0.2115, 0.13).$$

Hence, the desired probability of sunny is

$$\Pr(X_2 = S) = (\mu \cdot P^2)_S = (\mu \cdot P^2)_3 = 0.13.$$

4.4 Joint Distribution

Let $X_0, X_1, ...$ be a Markov chain with transition matrix P and initial distribution μ. For all $0 \le n_1 \le n_2 < \cdots < n_{k-1} < n_k$ and states $i_1, i_2, \ldots, i_{k-1}, i_k$,

$$P(X_{n_1} = i_1, X_{n_2} = i_2, \ldots, X_{n_k} = i_{k1}1, X_{n_k} = i_k) = (\mu P^{n_1})_{i_1} (P^{n_2 - n_1})_{i_1 i_2} \cdots (P^{n_k - n_{k-1}})_{i_{k-1} i_k}.$$

From the above result, the joint probability is obtained from just the initial distribution μ and the transition matrix P.

Example 4.10. *In Example 4.9, on Sunday, the chances of rain, cloudy and sunny have the same probabilities. Find the probability that it will be sunny on the following Wednesday and Friday, and cloudy on Saturday.*

Solution: We are given that $\mu = (1/3, 1/3, 1/3)$. From

$$P^3 = \begin{bmatrix} 0.645500 & 0.215500 & 0.139 \\ 0.645625 & 0.215375 & 0.139 \\ 0.603000 & 0.231000 & 0.166 \end{bmatrix},$$

the required probability is

$$\begin{aligned} \Pr(X_3 = S, X_5 = S, X_6 = C) &= (\mu \cdot P^3)_S \cdot P^2_{SS} \cdot P_{SC} \\ &= (\mu \cdot P^3)_3 \cdot P^2_{33} \cdot P_{32} \\ &= 0.148 \cdot 0.22 \cdot 0.4 = 0.013024. \end{aligned}$$

4.5 Random walk with absorbing and reflecting barrier(s)

Example 4.11. *A one-dimensional random walk $\{X_n\}$ is defined on a finite or infinite subset of integers in which the process in state i can either stay in i or move to its neighbouring states $i-1$ and $i+1$. Suppose that given that $X_n = i$ at time n,*

- *the probability of moving to state $i+1$ is p_i,*
- *the probability of remaining in state i is r_i, and*
- *the probability of moving to state $i-1$ is q_i,*

where $p_i + q_i + r_i = 1$ for all i.

1. *Write down the transition matrix.*
2. *Show that the random walk has Markov property.*

Solution: 1. The transition diagram and the transition matrix are infinite:

$$P = \begin{bmatrix} \ddots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & q_{-1} & r_{-1} & p_{-1} & \cdots & \cdots & \cdots \\ \cdots & \cdots & q_0 & r_0 & p_0 & \cdots & \cdots \\ \cdots & \cdots & \cdots & q_1 & r_1 & p_1 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \ddots \end{bmatrix}.$$

2. Clearly, the Markov property holds because

$$\begin{aligned} &\Pr(X_{n+1} = k | X_n = i, X_{n-1} = i_{n-1}, \dots, X_0 = i_0) \\ &= \Pr(X_{n+1} = k | X_n = i) = \begin{cases} p_i, & k = i+1 \\ r_i, & k = i \\ q_i, & k = i-1 \\ 0, & \text{otherwise.} \end{cases} \end{aligned}$$

Example 4.12. *The random walk can be used to model the fortune of a gambler. The gambler bets per game and the probability of winning is p and the probability of losing is q where $p+q=1$. In addition, the gambler is ruined (or goes broke) if he reaches state 0, and also stops the game if he reaches state N. Therefore, the state space is $S = \{0, 1, \dots, N\}$. Note that*

$$p_{00} = 1 \text{ and } p_{NN} = 1.$$

*The states 0 and N are referred to as **absorbing boundaries (absorbing states)** and the remaining states $1, 2, \dots, N-1$ are **transient**. Roughly speaking, if a state is known as transient if there is a possibility of leaving the state and never returning.*

Solution: The transition diagram and the transition matrix of the simple random walk with absorbing boundaries (states) are given as follows:

$$P = \begin{bmatrix} 1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ q & 0 & p & 0 & \cdots & 0 & 0 & 0 \\ 0 & q & 0 & p & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & q & 0 & p \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 1 \end{bmatrix}.$$

1. In general, a state i is called **absorbing** if $p_{ii} = 1$
2. The simple random walk as given in Example 4.12 can be modified so that whenever the process is in state 0 (or state N),
   - the probability of remaining in state 0 is α, and
   - the probability of moving to state 1 is $1 - \alpha$.

   In this case, the state 0 is referred to as a **reflecting barrier** for the chain. The process might be used to model the fortune of an individual when negative fortune is reset to zero.

4.6 An example of nonhomogeneous Markov chain

In this section, we give an example of a discrete-time nonhomogeneous Markov chain. Again, without stated otherwise, we shall assume that the discrete-time Markov chains are homogeneous.

Example 4.13. *(Adapted from W.J.Stewart) A Markov chain $X_0, X_1, \ldots$ consists of two states $\{1, 2\}$. At time step n, the probability that the Markov chain remains in its current state is given by*

$$p_{11}(n) = p_{22}(n) = 1/n,$$

while the probability that it changes state is given by

$$p_{12}(n) = p_{21}(n) = 1 - 1/n.$$

1. *Draw a transition diagram of the Markov chain.*
2. *Write down the transition matrix.*
3. *Calculate* $\Pr(X_5 = 2, X_4 = 2, X_3 = 1, X_2 = 1 | X_1 = 1)$.

Solution: 1. The transition diagram and the transition matrix are dependent of the time step n, and are given as follows:

$$P(n) = \begin{bmatrix} \frac{1}{n} & \frac{n-1}{n} \\ \frac{(n-1)}{n} & \frac{1}{n} \end{bmatrix}.$$

2. The probability of taking a particular part can be calculated by

$$\begin{aligned} \Pr(X_5 = 2, X_4 = 2, X_3 = 1, X_2 = 1 | X_1 = 1) &= p_{11}(1) \cdot p_{11}(2) \cdot p_{12}(3) \cdot p_{22}(4) \\ &= 1 \cdot 1/2 \cdot 2/3 \cdot 1/4 = 1/12. \end{aligned}$$

Other paths lead to state 2 after four transitions, and have different probabilities according to the route they follow. What is important is that, no matter which route is chosen, once the Markov chain arrives in state 2 after four steps, the future evolution is specified by $P(5)$, and not any other $P(i), i \leq 4$.

4.7 Simulation

Simulation is a powerful tool for studying Markov chains. For many Markov chains in real-world applications, state spaces are large and matrix methods may not be practical.

A Markov chain can be simulated from an initial distribution and transition matrix. To simulate a Markov sequence $X_0, X_1, \ldots$, simulate each random variable sequentially conditional on the outcome of the previous variable. That is, first simulate X_0 according to the initial distribution. If $X_0 = i$, then simulate X_1 from the $i - th$ row of the transition matrix. If $X_1 = j$, then simulate X_2 from the $j - th$ row of the transition matrix, and so on.

```
<!-- [frame=single, escapeinside={(*}{*)}, caption={Algorithm for Simulating a Markov Chain}] -->

Algorithm for Simulating a Markov Chain

Input: (i) initial distribution  (*$\boldsymbol{\mu}$*), (ii) transition matrix (*$P$*), (iii) number o
Output: (*$X_0, X_1, \ldots , X_n$*)
Algorithm:
    Generate (*$X_0$*) according to (*$\boldsymbol{\mu}$*)
```

```
FOR (*$i = 1, \ldots ,n$*)
    Assume that (*$X_{i-1} = j$*)
    Set (*$\boldsymbol p = j-$*)th row of (*$P$*)
    Generate (*$X_i$*) according to (*$\boldsymbol p$*)
END FOR
```

4.8 Monte Carlo Methods

Monte Carlo methods are simulation-based algorithms that rely on generating a large set of samples from a statistical model to obtain the behaviour of the model and estimate the quantities of interest. For a large sample set of a random variable representing a quantity of interest, the law of large numbers allows to approximate the expectation by the average value from the samples.

Consider repeated independent trials of a random experiment. We will need to generate a large number of samples $X_1, X_2, ...$ from the model. A Monte Carlo method for estimating the expectation $\mathrm{E}(X)$ is a numerical method based on the approximation

$$\mathrm{E}(X) \approx \frac{1}{N}\sum_{i=1}^{N} X_i,$$

where $X_1, X_2, ...$ are i.i.d. with the same distribution as X.

While computing expectations and computing probabilities at first look like different problems, the latter can be reduced to the former: if X is a random variable, we have

$$\Pr(X \in A) = \mathrm{E}(1_A(X)).$$

Using this equality, we can estimate $\Pr(X \in A)$ by

$$\Pr(X \in A) = \mathrm{E}(1_A(X)) = \frac{1}{N}\sum_{i=1}^{N} 1_A(X_i).$$

Recall that the indicator function of the set A is the defined as

$$1_A(x) = \begin{cases} 1, & \text{if } x \in A \\ 0, & \text{otherwise.} \end{cases}$$

The following user-defined function in Excel can be used to simulate random numbers from a discrete distribution.

```
<!-- [frame=single, escapeinside={(*}{*)}, caption={A user-defined function in Excel to simulate random
<!-- }] -->

A user-defined function in Excel to simulate random numbers from a discrete distribution.
Public Function Discrete(value As Variant, prob As Variant)

Dim i As Integer
Dim cumProb As Single
Dim uniform As Single
Randomize
'Randomize Statement
'Initializes the random-number generator.
```

```
Application.Volatile

' This example marks the user-defined function Discrete as volatile.
' The function will be recalculated when any cell in any workbook
' in the application window changes value worksheet.

uniform = Rnd
cumProb = prob(1)
i = 1
Do Until cumProb > uniform
    i = i + 1
    cumProb = cumProb + prob(i)
Loop
Discrete = value(i)

End Function
```

```r
library(expm)
```

```
## Loading required package: Matrix

##
## Attaching package: 'Matrix'

## The following objects are masked from 'package:tidyr':
##
##     expand, pack, unpack

##
## Attaching package: 'expm'

## The following object is masked from 'package:Matrix':
##
##     expm
```

```r
library(markovchain)
```

```
## Package:  markovchain
## Version:  0.8.6
## Date:     2021-05-17
## BugReport: https://github.com/spedygiorgio/markovchain/issues
```

```r
library(diagram)
```

```
## Loading required package: shape
```

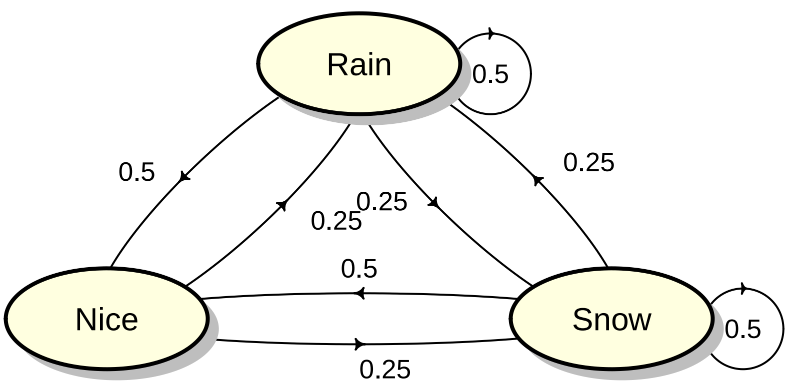
```r
library(pracma)
```

```
##
## Attaching package: 'pracma'

## The following objects are masked from 'package:expm':
##
##     expm, logm, sqrtm

## The following objects are masked from 'package:Matrix':
##
##     expm, lu, tril, triu
```

```r
stateNames <- c("Rain","Nice","Snow")
Oz <- matrix(c(.5,.25,.25,.5,0,.5,.25,.25,.5),
             nrow=3, byrow=TRUE)
row.names(Oz) <- stateNames;
colnames(Oz) <- stateNames
Oz
```

```
##      Rain Nice Snow
## Rain 0.50 0.25 0.25
## Nice 0.50 0.00 0.50
## Snow 0.25 0.25 0.50
```

```r
plotmat(Oz,pos = c(1,2),
        lwd = 1, box.lwd = 2,
        cex.txt = 0.8,
        box.size = 0.1,
        box.type = "circle",
        box.prop = 0.5,
        box.col = "light yellow",
        arr.length=.1,
        arr.width=.1,
        self.cex = .4,
        self.shifty = -.01,
        self.shiftx = .13,
        main = "")
```




```r
Oz3 <- Oz %^% 3
round(Oz3,3)
```

```
##       Rain  Nice  Snow
## Rain 0.406 0.203 0.391
## Nice 0.406 0.188 0.406
## Snow 0.391 0.203 0.406
```

```r
u <- c(1/3, 1/3, 1/3)
round(u %*% Oz3,3)
```

```
##       Rain  Nice  Snow
## [1,] 0.401 0.198 0.401
```

We can use R to generate sample paths of a Markov chain. We first load the library `markovchain` package. See https://cran.r-project.org/web/packages/markovchain/vignettes/an_introduction_to_markovchain_package.pdf for more details.

eyJsYW5ndWFnZSI6InIiLCJzYW1wbGUiOiJsaWJyYXJ5KG1hcmtvdmNoYWluKVxud2VhdGhlclN0YXRlcyA8LSBjKFwi

The following user-defined function in Excel can be used to simulate random numbers from a discrete distribution.

```
[frame=single, escapeinside={(*}{*)}, caption={A user-defined function in Excel to simulate random numbe
}]
Public Function Discrete(value As Variant, prob As Variant)

Dim i As Integer
Dim cumProb As Single
Dim uniform As Single
Randomize
'Randomize Statement
'Initializes the random-number generator.

Application.Volatile

' This example marks the user-defined function Discrete as volatile.
' The function will be recalculated when any cell in any workbook
' in the application window changes value worksheet.

uniform = Rnd
cumProb = prob(1)
i = 1
Do Until cumProb > uniform
    i = i + 1
    cumProb = cumProb + prob(i)
Loop
Discrete = value(i)

End Function
```

Example 4.14. *(R or Excel) A gambler starts with and plays a game where the chance of winning each round is 60%. The gambler either wins or loses on each round. The game stops when the gambler either gains or goes bust.*

1. *Develop an Excel worksheet or create an R code to simulate 50 steps of the finite Markov chain of the random walk X_n given in Example 4.12. Repeat the simulation 10 times. How many of your simulations end at 0.*
2. *Use the results from the simulations to estimate the mean and variance of X_5.*
3. *Use the results from the simulations to estimate the probability that the gambler is eventually ruined.*

Example 4.15. *(R or Excel) A Markov chain $X_0, X_1, \ldots$ on states $\{1,2\}$ has the following transition matrix*

$$P = \begin{bmatrix} 1-a & a \\ b & 1-b \end{bmatrix},$$

where $0 < a, b < 1$.

1. *Use either Excel or R to estimate the long-term distribution of the Markov chain. (Hint: consider the n-step transition matrix for several increasing values of n). Comments on the results obtained.*

Note *Later we will see that in many cases, a Markov chain exhibits a long-term limiting behaviour. The chain settles down to an equilibrium distribution, which is independent of its initial state.*

2. *Use simulations to estimate the long-term probability that a Markov chain hits each of the states. (Hint: simulate the Markov chain 1000 steps and calculate the proportion of visits to each state)*

4.9 Classification of states

Throughout this section $\{X_n\}_{n\geq 0}$ is a time homogeneous Markov chain with state space S and transition matrix $P = (p_{ij})_{i,j\in S}$.

For any $i, j \in S$,

- The state j can be **reached** from the state i, denoted by $i \to j$ if there is a nonzero probability $p_{ij}^{(n)} > 0$ for some $n \geq 0$.
- The states i and j are said to **communicate**, or to be **in the same class**, and denoted by $i \leftrightarrow j$, if $i \to j$ and $j \to i$.

Note Note that $i \to j$ if and only if there exist states $k_1, k_2, \dots, k_r$ such that

$$p_{ik_1} p_{k_1 k_2} \dots p_{k_r j} > 0,$$

i.e. it is not necessary that j can be reached from the state i in one single step.

- The relation $\leftrightarrow$ is an equivalence relation and partition the state space S into equivalence classes, which are known as **classes (or communication classes)** of the Markov chain. Thus in any class all the states communicate, but none of them communicates with any state outside the class.

Additional properties for a communication class are defined as follows:

- The class C is said to be **closed** if it is impossible to reach any state outside C from any state in C, i.e. escape from C is impossible. Otherwise, the class C is said to be **non-closed**, i.e. escape from C is possible.
- If the entire state space S is only one communication class (all states communicate), then it is necessarily closed and the Markov chain is said to be **irreducible**. Otherwise, the Markov chain is said to be **reducible**.
- A closed class consisting of a single state is an **absorbing state**.

Example 4.16. *Consider each of the following Markov chains:*

(a) *NCD system (Example 4.2),*

(b) *the health insurance system (Example 4.3), and*

(c) *a simple random walk (Example 4.12),*

Identify the communication classes. Is the Markov chain irreducible?

Solution: It is a good practice to use transition diagram and also verify the answers.

(a) Every two states are intercommunicating, so $\{0, 1, 2\}$ is a single closed class, and hence the Markov chain is irreducible. This is because $0 \to 1$ and $1 \to 0$ ($p_{01} > 0$ and $p_{10} > 0$), and $1 \to 2$ and $2 \to 1$ ($p_{12} > 0$ and $p_{21} > 0$).

(b) There are two classes of intercommunicating states, $O = \{H, S\}$ is non-closed, and $C = \{D\}$ is closed (and also an absorbing state). Clearly, the Markov chain is not irreducible. This is because

   - $p_{DD} = 1$, i.e. C is a class.
   - O is open class because $p_{HS} > 0$, $p_{SH} > 0$, so this is a class, and for example $p_{HD} > 0$ but $p_{DH}^{(n)} = 0$ for all n (i.e. one cannot leave C starting from the state D), so O is an open class.

(c) The simple random walk with absorbing boundaries has three classes, $\{1, 2, \dots, N-1\}$ is non-closed class, $\{0\}$ and $\{N\}$ are two closed classes.

Example 4.17. *A Markov chain with state space $S = \{1, 2, 3, 4, 5\}$ has the following transition matrix:*

$$P = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1/5 & 1/5 & 1/5 & 1/5 & 1/5 \\ 1/3 & 1/3 & 0 & 1/3 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1/2 & 1/2 \end{bmatrix}.$$

1. *Draw a transition diagram.*
2. *Identify the communication classes. Is the Markov chain irreducible?*

Solution: There are two closed classes $C_1 = \{1\}$ and $C_2 = \{4, 5\}$ and one non-closed class $O = \{2, 3\}$. This is because

- C_1 is closed because $p_{11} = 1$.
- C_2 is a class because $4 \to 5$ ($p_{45} > 0$) and $5 \to 4$ ($p_{54} > 0$), and is closed because $p_{ij} = 0$ for all $i \in C_2$ and $j \notin C_2$.
- O is a class because $2 \to 3$ ($p_{23} > 0$) and $3 \to 2$ ($p_{32} > 0$), and is non-closed because $p_{21} > 0$, but $p_{11} = 1$.

4.10 Absorption probabilities and expected time to absorption

For the random walk with absorbing boundaries (i.e. 0 and N), two questions arises, in which state, 0 or N is the process eventually absorbed (or trapped) and on the average how long does it take to reach one of these absorbing states? We first define the following terms which applies to the random walk process with absorbing boundaries.

The time of absorption T is defined as

$$T = \min\{n \geq 0 | X_n = 0 \text{ or } X_n = N\}$$

and the **probability of eventually absorption** in state 0 is given by

$$u_i = \Pr\{X_T = 0 | X_0 = i\}, \text{ for } i = 1, 2, \dots, N - 1.$$

The **mean time to absorption** of the process is given by

$$\mathrm{E}[T | X_0 = i] \text{ for } i = 1, 2, \dots, N - 1.$$

4.11 First step analysis

First step analysis allows us to evaluate quantities of interest from the Markov chain, for e.g. the absorption probabilities and the mean duration until absorption. The method is based on considering all possibilities at the end of the first transition and then apply the law of total probability to formulate equations involved all unknown quantities. We illustrate how to use the first step analysis in the following Markov chain.

Example 4.18. *Consider the Markov chain with state space $S = \{0, 1, 2\}$ and transition probability matrix given by*

$$P = \begin{bmatrix} 1 & 0 & 0 \\ p_{10} & p_{11} & p_{12} \\ 0 & 0 & 1 \end{bmatrix}.$$

Solution: The classes and types are as follows:

- Two closed classes are $C_1 = \{0\}$ and $C_2 = \{2\}$.
- $\{1\}$ is a non-closed class.

Let us consider the problem of evaluating the absorption probabilities. For any closed class C, define

$$u_i^C = \Pr(\text{Markov chain eventually absorbed in } C|X_0 = i).$$

Clearly, the absorption probabilities also depend on the initial states. A vector of absorption probabilities is then given by $\mathbf{u}^C = (u_i^C)_{i\in S}$ We suppress the superscript C and simply write $u_i^C = u_i$ and $\mathbf{u}^C = \mathbf{u}$.

Consider the closed class $C_1 = \{0\}$. We have

$$\begin{aligned}
u_0 &= \Pr(\text{Markov chain eventually absorbed in } C_1|X_0 = 0) = 1,\\
u_2 &= \Pr(\text{Markov chain eventually absorbed in } C_1|X_0 = 2) = 0,\\
u_1 &= \Pr(\text{Markov chain eventually absorbed in } C_1|X_0 = 1) = u_1.
\end{aligned}$$

By considering the first transition from state 1 to either state 0, 1 and 2, and using the Markov property, the law of total probability gives

$$\begin{aligned}
u_1 &= \Pr(\text{Markov chain eventually absorbed in } C_1|X_0 = 1)\\
&= \sum_{k=0}^{2} \Pr(\text{Markov chain eventually absorbed in } C_1|X_0 = 1, X_1 = k)\Pr(X_1 = k|X_0 = 1)\\
&= \sum_{k=0}^{2} \Pr(\text{Markov chain eventually absorbed in } C_1|X_1 = k)\Pr(X_1 = k|X_0 = 1)\\
&= (p_{10})\cdot u_0 + (p_{11})\cdot u_1 + (p_{12})\cdot u_2\\
&= (p_{10})\cdot 1 + (p_{11})\cdot u_1 + (p_{12})\cdot 0.
\end{aligned}$$

Solving for u_1 gives

$$u_1 = u_1^{C_1} = \frac{p_{10}}{1-p_{11}} = \frac{p_{10}}{p_{10}+p_{12}}.$$

Note Similarly, we have

$$\begin{aligned}
u_0 &= p_{00}\cdot u_0 + p_{01}\cdot u_1 + p_{02}\cdot u_2\\
u_1 &= p_{10}\cdot u_0 + p_{11}\cdot u_1 + p_{12}\cdot u_2\\
u_2 &= p_{20}\cdot u_0 + p_{21}\cdot u_1 + p_{22}\cdot u_2,
\end{aligned}$$

where the first and the last equations reduce to $u_0 = u_0$ and $u_2 = u_2$, respectively. In general, for a closed class C, the vector of absorption probabilities $\mathbf{u}$ satisfies the following system of linear equations:

1. $\mathbf{u} = P\mathbf{u}$ (here $\mathbf{u}$ is treated as a column vector),
2. $u_i = \Pr(\text{Markov chain eventually absorbed in } C)|X_0 = i) = 1$ for all $i \in C$, and
3. $u_i = \Pr(\text{Markov chain eventually absorbed in } C)|X_0 = i) = 0$ for all i in any other close classes.

Example 4.19. *In this example, consider the closed class $C_2 = \{2\}$. Find the absorption probabilities $u_0^{C_2}, u_1^{C_2}$ and $u_2^{C_2}$. Comment on these results.*

Solution: For the closed class C_2, we proceed in the same way as in the closed class C_1. Let $\mathbf{u} = \mathbf{u}^{C_2} = (u_0, u_1, u_2)^T$ be the vector of absorption probabilities in the closed class $C_2 = \{2\}$ with $u_0 = 0$ and $u_2 = 1$. It follows that

$$\begin{aligned}
u_1 &= p_{10}\cdot u_0 + p_{11}\cdot u_1 + p_{12}\cdot u_2\\
&= p_{11}\cdot u_1 + p_{12}.
\end{aligned}$$

Hence, $u_1 = \frac{p_{12}}{1-p_{11}} = \frac{p_{12}}{p_{10}+p_{12}}$. It should be emphasised that

$$\mathbf{u}^{C_1} + \mathbf{u}^{C_2} = \mathbf{1}.$$

Notes 1. For any initial state i, the sum of the absorption probabilities over all closed classes is 1 (as verified in Example ExampleÂ 19). In particular, when a Markov chain has two closed classes C_1 and C_2, $\mathbf{u}^{C_2} = \mathbf{1} - \mathbf{u}^{C_1}$.

2. In the case when S is finite or when the set of states in non-closed classes is finite, the vector $\mathbf{u}$ is the unique solution of the above system of linear equations.

Example 4.20. *Consider the Markov chain defined in Example 4.17. Find the absorption probabilities in the closed class $C_1 = \{1\}$ and $C_2 = \{4, 5\}$.*

Solution: Here $C_1 = \{1\}$ and $C_2 = \{4, 5\}$ are closed classes and $0 = \{2, 3\}$ is an open class.

Let $\mathbf{u} = \mathbf{u}^{C_1}$ be the vector of absorption probabilities in the closed class $C_1 = \{1\}$. Write $\mathbf{u} = (u_1, u_2, \dots, u_5)^T$ and $u_1 = 1$ and $u_4 = u_5 = 0$, From $\mathbf{u} = P \cdot \mathbf{u}$,

$$\begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_5 \end{pmatrix} = P \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_5 \end{pmatrix} \text{gives}$$

$$\begin{aligned} u_2 &= \frac{1}{5} + \frac{1}{5}u_2 + \frac{1}{5}u_3 \\ u_3 &= \frac{1}{3} + \frac{1}{3}u_2. \end{aligned}$$

Solving the linear system for u_2 and u_3 yields $u_2 = 4/11$ and $u_3 = 5/11$. Hence, the absorption probabilities in the closed class C_1 is

$$\mathbf{u} = (1, 4/11, 5/11, 0, 0)^T.$$

In addition, since there are two closed classes, $\mathbf{u}^{C_2} = \mathbf{1} - \mathbf{u}^{C_1} = (0, 7/11, 6/11, 1, 1)^T$.

4.12 The expected time to absorption

The expected time to absorption can be determined by analysing all possibilities occurring in the first step. We again consider the process defined in Example @ref(exm:eg_absorption) on the set $\{0, 1, 2\}$ with the transition matrix

$$P = \begin{bmatrix} 1 & 0 & 0 \\ p_{10} & p_{11} & p_{12} \\ 0 & 0 & 1 \end{bmatrix}.$$

The time of absorption T is defined as

$$T = \min\{n \geq 0 | X_n = 0 \text{ or } X_n = 2\}$$

and the mean time to absorption of the process is given by $v = \mathrm{E}[T|X_0 = 1]$.

The following observations can be made:

1. The absorption time T is always at least 1.
2. If either $X_1 = 0$ or $X_1 = 2$, then no further steps are required.
3. If $X_1 = 1$, then the process is back at its starting point and on the average v additional steps are required for absorption.

Weighting all these possibilities by their respective probabilities, we obtain the following equation

$$\begin{aligned} v &= 1 + p_{10} \cdot 0 + p_{11} \cdot v + p_{12} \cdot 0 \\ &= 1 + p_{11} \cdot v, \end{aligned}$$

which results in

$$v = \frac{1}{1 - p_{11}}.$$

Example 4.21. *Consider the Markov chain with state space $S = \{0, 1, 2, 3\}$ and transition probability matrix given by*

$$P = \begin{bmatrix} 1 & 0 & 0 & 0 \\ p_{10} & p_{11} & p_{12} & p_{13} \\ p_{20} & p_{21} & p_{22} & p_{23} \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Let T be the time of absorption defined by

$$T = \min\{n \geq 0 | X_n = 0 \text{ or } X_n = 3\}$$

*and the **absorption probabilities** given by*

$$u_i = \Pr\{X_T = 0 | X_0 = i\}, \text{ for } i = 1, 2$$

and the mean time to absorption of the process is given by

$$v_i = \mathrm{E}[T | X_0 = 1], \text{ for } i = 1, 2.$$

Calculate the absorption probabilities and the mean time to absorption.

Solution: There are 2 closed classes including $C_1 = \{0\}$, and $C_2 = \{3\}$, and one non-closed class $O = \{1, 2\}$. Here,

$$u_i = u_i^{C_1} = \Pr\{X_T = 0 | X_0 = i\} = \Pr(\text{Markov chain eventually absorbed in } C_1 | X_0 = i), \text{ for } i = 1, 2$$

By conditioning on the first step from state i and using the Markov property, we have

$$u_i^{C_1} = \sum_{j \in S} p_{ij} u_j^{C_1}.$$

Clearly, $u_0^{C_1} = 1$ and $u_3^{C_1} = 0$. In particular, we have

$$\begin{aligned} u_1 &= p_{10} \cdot 1 + p_{11} \cdot u_1 + p_{12} \cdot u_2 \\ u_2 &= p_{20} \cdot 1 + p_{21} \cdot u_1 + p_{22} \cdot u_2, \end{aligned}$$

which can also be obtained from the matrix equation $\mathbf{u} = P\mathbf{u}$, where $\mathbf{u} = (1, u_1, u_2, 0)^T$. The solution to the system of linear equations is

$$\begin{aligned} u_1 &= \frac{p_{10}(p_{22} - 1) - p_{12}p_{20}}{p_{11}(-p_{22}) + p_{11} + p_{12}p_{21} + p_{22} - 1}, \\ u_2 &= \frac{(p_{11} - 1)p_{20} - p_{10}p_{21}}{p_{11}(-p_{22}) + p_{11} + p_{12}p_{21} + p_{22} - 1}. \end{aligned}$$

Similarly, the mean time to absorption also depends on the starting state. By the first step analysis, we have for $v_i = \mathrm{E}[T | X_0 = 1]$,

$$\begin{aligned} v_1 &= 1 + p_{11} \cdot v_1 + p_{12} \cdot v_2 \\ v_2 &= 1 + p_{21} \cdot v_1 + p_{22} \cdot v_2. \end{aligned}$$

Here the absorption time T is always at least 1. If either $X_1 = 0$ or $X_1 = 3$, then no further steps are required. On the other hand, if $X_1 = 1$ or $X_1 = 2$, then the process will require additional steps, and on the average, these are v_1 and v_2. Weighting these two possibilities, i.e. whether $X_1 = 1$ or $X_1 = 2$, by their respective probabilities and summing according to the law of total probability result in the above system of equations.

Solving the equations for v_1 and v_2 give the mean time to absorption

$$v_1 = \frac{-p_{12} + p_{22} - 1}{p_{11}(-p_{22}) + p_{11} + p_{12}p_{21} + p_{22} - 1},$$
$$v_2 = \frac{p_{11} - p_{21} - 1}{p_{11}(-p_{22}) + p_{11} + p_{12}p_{21} + p_{22} - 1}.$$

4.13 The long-term distribution of a Markov chain

In this section, we present another important property concerning limiting behaviour of P^n as $n \to \infty$ and hence the long-term distribution of a Markov chain satisfying some certain conditions. In particular, some Markov chains will converge to an equilibrium (limiting) distribution, which is independent of its initial state.

We also assume that the Markov chain with **a single closed class** S.

4.14 Stationary and limiting distributions for a single closed class

Stationary distributions

Throughout this section, we consider a Markov chain whose transition probability matrix is P and state space S is a single close class. Then S is necessarily closed and hence irreducible.

A probability distribution $\pi = (\pi)_{i \in S}$ on S is **stationary** if the following conditions hold (here π is a row vector):

1. $\pi_j = \sum_{i \in S} \pi_i p_{ij}$ or equivalently $\pi P = \pi$,
2. $\pi_j \geq 0$,
3. $\sum_{j \in S} \pi_j = 1$.

Notes 1. For any stationary distribution π, for all $n \geq 1$,

$$\pi P^n = \pi.$$

Therefore, if we take π as the initial probability distribution, i.e. $\Pr(X_0 = i) = \pi_i$, then then the distribution of X_n is also π , i.e. $\Pr(X_n = i) = \pi_i$

2. The probability distribution π is said to be an invariant probability distribution.
3. The most important property concerning the stationary distribution(which will be made formal) is that it gives the **long-term (limiting) distribution** of a Markov chain. In addition, π_j also gives **the long run mean fraction of time** that the process $\{X_n\}$ is in state j.

Example 4.22. *A Markov chain $X_0, X_1, ...$ on states $\{1, 2\}$ has the following transition matrix*

$$P = \begin{bmatrix} 1-a & a \\ b & 1-b \end{bmatrix},$$

where $0 < a, b < 1$.

1. *Show that*
$$P^n = \frac{1}{a+b}\begin{bmatrix} b & a \\ b & a \end{bmatrix} + \frac{(1-a-b)^n}{a+b}\begin{bmatrix} a & -a \\ -b & b \end{bmatrix}.$$
2. *Show that the stationary probability distribution is*
$$\pi = \left(\frac{b}{a+b}, \frac{a}{a+b}\right).$$
3. *Show that*
$$\lim_{n\to\infty} p_{ij}^{(n)} = \pi_j > 0, \text{ for } j \in \{1,2\}.$$

Solution: The solutions can be found from the tutorial.

Proportion of Time in Each State

The limiting distribution provides the long-term behaviour of the Markov chain, i.e. it is the long-term probability that a Markov chain hits each state. In this section, it can be shown that it also gives the long-term proportion of time that the chain visits each state. Let us consider a Markov chain $X_0, X_1, \ldots$ whose transition probability matrix is P and its limiting distribution is π. Note that the limiting distribution for the Markov chain satisfies
$$\lim_{n\to\infty} p_{ij}^{(n)} = \pi_j.$$

For each state j, define indicator random variable
$$I_k = \begin{cases} 1, & \text{if } X_k = j \\ 0, & \text{otherwise,} \end{cases},$$

for $k = 0, 1, \ldots$. Hence, the number of times that the Markov chain visits j in the first n steps is given by $\sum_{k=0}^{n-1} I_k$ and the expected long-term proportion of time that the chain visits state j given that its initial state is i is
$$\begin{aligned}
\lim_{n\to\infty} \mathrm{E}\left(\frac{1}{n}\sum_{k=0}^{n-1} I_k \,|\, X_0 = i\right) &= \lim_{n\to\infty} \frac{1}{n}\sum_{k=0}^{n-1} \mathrm{E}(I_k \,|\, X_0 = i) \\
&= \lim_{n\to\infty} \frac{1}{n}\sum_{k=0}^{n-1} \Pr(X_k = j | X_0 = i) \\
&= \lim_{n\to\infty} \frac{1}{n}\sum_{k=0}^{n-1} p_{ij}^{(k)} \\
&= \lim_{n\to\infty} p_{ij}^{(n)} = \pi_j.
\end{aligned}$$

Here we use the fact that if the sequence of numbers converges to a limit, i.e. $x_n \to x$ as $n \to \infty$, then the sequence of partial averages also converges to that limit, i.e. $(x_1 + x_2 + \cdots x_n)/n \to x$ as $n \to \infty$. This result is known as Cesaro's lemma.

Example 4.23. *Recall from Example 4.9, the simple weather pattern can be classified into three types including rainy (R), cloudy (C) and sunny (S). The weather is observed daily and can be modelled by the Markov transition matrix*
$$P = \begin{bmatrix} 0.7 & 0.2 & 0.1 \\ 0.75 & 0.15 & 0.1 \\ 0.2 & 0.4 & 0.4 \end{bmatrix}.$$
It can be shown that the stationary distribution and also the limiting distribution of the Markov chain is
$$\pi = (94/147, 32/147, 1/7)$$
which gives the proportions of visits to rainy, cloudy and sunny states are 94/147, 32/147, 1/7, respectively.

The method of finding the stationary distribution

To find the stationary distribution, we simply solve the linear equations $\pi P = \pi$ (note that one of the equations can be discarded), together with the condition $\sum_{j \in S} \pi_j = 1$.

Example 4.24. *For the NCD process in Example 4.2, the Markov chain has the following transition probability matrix*

$$P = \begin{bmatrix} 1-p & p & 0 \\ 1-p & 0 & p \\ 0 & 1-p & p \end{bmatrix}.$$

Find the stationary probability distribution of this chain.

Solution: Denote the stationary probability distribution by $\pi = (\pi_1, \pi_2, \pi_3)$. From $\pi P = \pi$ and $\pi_1 + \pi_2 + \pi_3 = 1$,

$$(\pi_1, \pi_2, \pi_3) \begin{bmatrix} 1-p & p & 0 \\ 1-p & 0 & p \\ 0 & 1-p & p \end{bmatrix} = (\pi_1, \pi_2, \pi_3),$$

which is equivalent to

$$\begin{aligned} (1-p)\pi_1 + (1-p)\pi_2 &= \pi_1 \\ p\pi_1 + (1-p)\pi_3 &= \pi_2 \\ p\pi_2 + p\pi_3 &= \pi_3 \end{aligned}$$

By discarding one of the equations and adding the condition that $\pi_1 + \pi_2 + \pi_3 = 1$, one can solve for π_1, π_2, π_3:

$$\pi_1 = \frac{(p-1)^2}{p^2 - p + 1}, \quad \pi_2 = \frac{-p^2 + p}{p^2 - p + 1}, \quad \pi_3 = \frac{p^2}{p^2 - p + 1}.$$

Note 1. In the above two examples, it can be shown that

$$\lim_{n \to \infty} p_{ij}^{(n)} = \pi_j > 0, \text{ for } j \in S,$$

or, in terms of the Markov chain $\{X_n\}$,

$$\lim_{n \to \infty} \Pr(X_n = j | X_0 = i) = \pi_j > 0, \text{ for } j \in S.$$

This means that in the long run (as $n \to \infty$), the probability of finding Markov chain in state j is approximately π_j **no matter in which state the chain began at time 0**. This property holds for some Markov chains which satisfy **"certain conditions"**.

4.15 Sufficient conditions for the long-run behaviour of a Markov chain

In what follows, we will establish a set of sufficient conditions for the long-run behaviour of a Markov chain. Two important results are stated without proof.

Theorem 4.1. ***Theorem 1.*** A Markov chain with a finite state space has at least one stationary probability distribution.

Theorem 4.2. ***Theorem 2.*** An irreducible Markov chain with a finite state space has a unique stationary probability distribution.

Example 4.25. *The simple random walks X_n on $S = \{\dots, -2, -1, 0, 1, 2, \dots\}$ is defined as*

$$X_n = X_0 + \xi_1 + \xi_2 + \dots + \xi_n,$$

where the random variables ξ_j are independent identically distributed with

$$\Pr(\xi = 1) = p, \quad \Pr(\xi = -1) = 1 - p.$$

We can check that this Markov chain is irreducible. However, the state space S is infinite. It can be checked directly from the equations $\pi P = \pi$ that there is ***no*** *stationary distribution (given as an exercise).*

Solution: First we know that the entries of a stationary distribution sum to one. Suppose the contrary that there is a stationary distribution π for the simple random walk. Then by spatial invariance of the simple random walk, π_i is constant for all $i \in S$ and also $\sum_{i \in S} \pi_i = 1$, which is impossible because S is infinite. Hence, there is **no** stationary distribution for the simple random walk.

4.16 Limiting distributions

One of the important properties of stationary distributions is that the distribution of the Markov chain satisfying certain conditions converges to the stationary distribution. This result provides the long-term behaviour of the Markov chain. In order to state the main result of this section, we need to introduce another concept, namely the period of a state.

A state i is said to be **periodic** with period $d > 1$ if a return to i is possible only in a number of steps that is a multiple of d. Equivalently, the period d is the greatest common divisor of all integers n for which $p_{ii}^{(n)} > 0$. If the greatest common divisor is 1, the state has period 1 and is said to be **aperiodic**.

A Markov chain in which each state has period 1 is called **aperiodic**. Most Markov chains in applications are aperiodic.

Example 4.26. *Is the NCD system in Example 4.2 aperiodic?*

Solution: The entire state space of the NCD system is a single class, and is necessarily closed. Note also that $p_{00} > 0$ (and also $p_{22} > 0$), i.e. the state 0 (and state 2) has an arrow back to itself. Consequently, it is aperiodic because a return to this state is possible in any number of steps (or the system can remain in this state in any length of time).

Similarly, a return to state 1 is possible in $2, 3, \dots$ steps. Therefore, the NCD system is aperiodic.

Notes 1. Periodicity is a class property, i.e. all states in one class have the same period (or if $i \leftrightarrow j$, then i and j have the same period).

2. The Markov chain $\{X_n\}_{n \geq 0}$ is aperiodic if and only if there exists some $n > 0$ such that

$$p_{ij}^{(n)} > 0 \text{ for all } i, j \in S.$$

Such Markov chain is also called **regular**.

Example 4.27. *In the random walk model on a finite state space $S = \{0, 1, \dots, N\}$ with absorbing boundaries in Example 4.12, determine the period of each state.*

Solution: The transition diagram and the transition matrix of the simple random walk with absorbing boundaries (states) are given as follows:

The simple random walk with absorbing boundaries has three classes, $O = \{1, 2, \dots, N-1\}$ is non-closed class, $C_1 = 0$ and $C_2 = N$ are two closed classes. For each state i in O, $p_{ii}^{(2n)} > 0$ and $p_{ii}^{(2n+1)} = 0$ for $n = 1, 2, \dots$. Therefore, the state i in this open communication class has period 2.

On the other hand, $p_{00} = 1$ (and also $p_{NN} = 1$) and hence, the states 0 and N are aperiodic because a return to each of these states is possible in any number of steps (or the system can remain in this state in any length of time).

Example 4.28. *In the random walk model on an infinite state space* $S = \{\dots, -2, -1, 0, 1, 2, \dots\}$ *in Example 4.25, determine the period of each state.*

Solution: The entire state space is a single class. Note also that $p_{ii}^{(2n)} > 0$ and $p_{ii}^{(2n+1)} = 0$ for $n = 1, 2, \dots$. Therefore each state in the random walk on an infinite set S is periodic with period 2.

Example 4.29. *Suppose the states of the system are* $\{1, 2, 3, 4\}$ *and the transition matrix is*

$$P = \begin{bmatrix} 0 & 1/2 & 0 & 1/2 \\ 1/4 & 0 & 3/4 & 0 \\ 0 & 1/3 & 0 & 2/3 \\ 1/2 & 0 & 1/2 & 0 \end{bmatrix}.$$

Determine the period of each state.

Solution: The entire state space is a single class. Note also that $p_{ii}^{(2n)} > 0$ and $p_{ii}^{(2n+1)} = 0$ for $n = 1, 2, \dots$. Therefore each state in the state space S is periodic with period 2.

Example 4.30. *A Markov chain* $X_0, X_1, \dots$ *on states* $\{1, 2\}$ *has the following transition matrix*

$$P = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},$$

1. Find the stationary distribution(s) of this Markov chain.*
2. *Describe the long-term behaviour of the Markov chain. Does the distribution of the chain tend to the stationary distribution(s) found in 1.*

Solution: 1. To find the stationary distribution $\pi = (\pi_1, \pi_2)$, we need to solve

$$(\pi_1, \pi_2) \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = (\pi_1, \pi_2),$$

and

$$\pi_1 + \pi_2 = 1.$$

This gives $\pi = (1/2, 1/2)$.

2. There is an equal chance of being in either state. Note that for any initial probability distribution $\mu = (\mu_1, \mu_2)$ with $\mu_1 + \mu_2 = 1$, we have

$$\mu \cdot P = \mu \cdot P^3 = \mu \cdot P^5 = \dots = (\mu_2, \mu_1)$$

and

$$\mu P^2 = \mu P^4 = \mu P^6 = \dots = (\mu_1, \mu_2).$$

The process does not settle down to an equilibrium position. Note also that the chain is not aperiodic, i.e. each state is periodic of period 2. The process does not conform to stationary in the long run.

Example 4.31. *A Markov chain* $X_0, X_1, \dots$ *on states* $\{1, 2\}$ *has the following transition matrix*

$$P = \begin{bmatrix} 1/3 & 2/3 \\ 2/3 & 1/3 \end{bmatrix},$$

Answer the same questions as given in the Example 4.30

Solution: 1. The process is finite and irreducible, so a unique stationary distribution exists. To find the stationary distribution $\pi = (\pi_1, \pi_2)$, we need to solve

$$(\pi_1, \pi_2) \begin{bmatrix} 1/3 & 2/3 \\ 2/3 & 1/3 \end{bmatrix} = (\pi_1, \pi_2),$$

and

$$\pi_1 + \pi_2 = 1.$$

Solving the system of linear equations gives $\pi = (1/2, 1/2)$.

2. The chain is aperiodic. Therefore, according to the results from Example 4.22, if follows that

$$\lim_{n\to\infty} p_{i1}^{(n)} = 1/2 > 0, \text{ and } \lim_{n\to\infty} p_{i2}^{(n)} = 1/2,$$

which is independent of i. This is contrast to the process given in ExampleÂ ExampleÂ 30, i.e. the process in this example reaches the stationary probability distribution in the long run.

4.17 Main result

The main result in this section can be stated as follows:

Theorem 4.3. ***Theorem 3****.* Let P be the transition probability matrix of a homogeneous discrete-time Markov chain $\{X_n\}_{n\geq 0}$. If the Markov chain is

- finite,
- irreducible and
- aperiodic,

then there is the unique probability distribution $\pi = (\pi_j)_{j\in S}$ such that

$$\lim_{n\to\infty} p_{ij}^{(n)} = \pi_j > 0, \text{ for any } j \in S$$

and

$$\sum_{j\in S} \pi_j = 1,$$

and this distribution is independent of the initial state. Such probability distribution π is called the **limiting probability distribution**. In addition, the limiting distribution $\pi = (\pi_j)_{j\in S}$ is the stationary probability distribution of the Markov chain, i.e. it also satisfies $\pi P = \pi$.

Example 4.32. *Recall the Markov chain as given in ExampleÂ ExampleÂ 22. The Markov chain is finite, irreducible and aperiodic. We have also shown that*

$$\pi = \left(\frac{b}{a+b}, \frac{a}{a+b}\right)$$

is the limiting distribution, i.e.

$$\lim_{n\to\infty} p_{ij}^{(n)} = \pi_j > 0, \text{ for } j \in \{1, 2\},$$

independent of i. This limiting distribution is also the unique stationary distribution of the Markov chain, which can be verified by

$$\left(\frac{b}{a+b}, \frac{a}{a+b}\right) \begin{bmatrix} 1-a & a \\ b & 1-b \end{bmatrix} = \left(\frac{b}{a+b}, \frac{a}{a+b}\right),$$

where $0 < a, b < 1$.

Example 4.33. *The above result can be applied to the NCD system because it is finite, irreducible and aperiodic. Indeed, there is the unique limiting probability distribution*

$$\pi = \left(\frac{(p-1)^2}{p^2-p+1}, \frac{p-p^2}{p^2-p+1}, \frac{p^2}{p^2-p+1} \right),$$

which is the stationary distribution of the chain. This gives the long-term behaviour of the Markov chain, i.e. the probability of finding the Markov chain in state j is approximately π_j independent of the initial distribution.

For example, let $p = 0.8$ in the transition probability matrix P. We compute several powers of P as follows:

$$P = \begin{bmatrix} 0.2 & 0.8 & 0 \\ 0.2 & 0 & 0.8 \\ 0 & 0.2 & 0.8 \end{bmatrix}, \qquad P^2 = \begin{bmatrix} 0.2 & 0.16 & 0.64 \\ 0.04 & 0.32 & 0.64 \\ 0.04 & 0.16 & 0.8 \end{bmatrix},$$

$$P^4 = \begin{bmatrix} 0.072 & 0.1856 & 0.7424 \\ 0.0464 & 0.2112 & 0.7424 \\ 0.0464 & 0.1856 & 0.768 \end{bmatrix}, \qquad P^8 = \begin{bmatrix} 0.0482432 & 0.1903514 & 0.7614054 \\ 0.04758784 & 0.1910067 & 0.7614054 \\ 0.04758784 & 0.1903514 & 0.7620608 \end{bmatrix}$$

$$P^{16} = \begin{bmatrix} 0.04761946 & 0.1904761 & 0.7619044 \\ 0.04761903 & 0.1904765 & 0.7619044 \\ 0.04761903 & 0.1904761 & 0.7619049 \end{bmatrix}, \quad P^{32} = \begin{bmatrix} 0.04761905 & 0.1904762 & 0.7619048 \\ 0.04761905 & 0.1904762 & 0.7619048 \\ 0.04761905 & 0.1904762 & 0.7619048 \end{bmatrix}.$$

The limiting probability distribution is

$$\lim_{n\to\infty} P^n = \begin{bmatrix} 0.04761905 & 0.1904762 & 0.7619048 \\ 0.04761905 & 0.1904762 & 0.7619048 \\ 0.04761905 & 0.1904762 & 0.7619048 \end{bmatrix}.$$

Applications of Markov chains to NCD systems

Example 4.34. *A no-claims discount system for motor insurance has three levels of discount:*

Level	*1*	*2*	*3*
Discount	*0%*	*30%*	*50%*

The rules for moving between these levels are given as follows:

- *Following a claim-free year, move to the next higher level, or remain at level 3.*
- *Following a year with one claim, move to the next lower level, or remain at level 1.*
- *Following a year with two or more claims, move to level 1, or remain at level 1.*

A portfolio consists of 10000 policyholders, of which

- *5000 policyholders are classified as good drivers. The number of claims per year in this group is Poisson*(0.1).
- *5000 policyholders are classified as bad drivers. The number of claims per year in this group is Poisson*(0.2).

1. *Calculate* $\Pr[N = 0]$, $\Pr[N = 1]$, *and* $\Pr[N \geq 2]$ *for each group.*
2. *Write down the transition probability matrix of this no-claims discount system for each group.*

3. *Calculate the expected number of policyholders at each level for each group once stability has been achieved.*

4. *Calculate the expected premium income per driver from each group once stability has been achieved.*

5. *Calculate the ratio of the expected premium income per driver from the group of good drivers to that from the group of bad drivers once stability has been achieved.*

6. *Comments on the results obtained. Does this NCD system encourage good driving?*

Chapter 5

Poisson processes

5.1 Introduction

In this chapter, we will consider a class of stochastic processes, namely Poisson processes, that can be used to model the occurrence or arrival of events over a continuous time interval. Time domains of such processes are continuous and the state space is the set of whole numbers.

For instance, consider the number of claims that occur up to time t (denoted by $N(t) = N_t$) from a portfolio of health insurance policies (or other types of insurance products). Suppose that the average rate of occurrence of claims per time unit (e.g. day or week) is given by λ.

Here are some questions of interest:

1. Suppose that on average 20 claims arrive every day (i.e. $\lambda = 20$). What is the probability that more than 100 claims arrive within a week?
2. What is the expected time until the next claim?

The model used to model the insurance claims is an example of **Poisson processes**. The following examples can also be modelled by a Poisson process:

- Claims arrivals at an insurance company,
- Telephone calls to a call center,
- Accidents occurring on the highway.

5.2 Poisson process

A **Poisson process** is a special type of counting process. It can be represented by a continuous time stochastic process $\{N(t)\}_{t\geq 0}$ which takes values in the non-negative integers. The state space is discrete but the time set is continuous. Here $N(t)$ represents the number of events in the interval $(0, t]$.

5.2.1 Counting Process

A counting process $\{N(t)\}_{t\geq 0}$ is a collection of non-negative, integer-valued random variables such that if $0 \leq s \leq t$, then $N(s) \leq N(t)$.

Figure 5.1 illustrates a trajectory of the Poisson process. An R code to simulate the trajectory is also given below. The sample path of a Poisson process is a right-continuous step function. There are jumps occurring at time $t_1, t_2, t_3, \ldots$.

```
lambda <- 17
# the length of time horizon for the simulation T_length <- 31
last_arrival <- 0
arrival_time <- c()
inter_arrival <- rexp(1, rate = lambda)
T_length <- 1
while (inter_arrival + last_arrival < T_length) {
  last_arrival <- inter_arrival + last_arrival
  arrival_time <- c(arrival_time,last_arrival)
  inter_arrival <- rexp(1, rate = lambda)
}


n <- length(arrival_time)
counts <- 1:n

plot(arrival_time, counts, pch=16, ylim=c(0, n))
points(arrival_time, c(0, counts[-n]))
segments(
  x0 = c(0, arrival_time[-n]),
  y0 = c(0, counts[-n]),
  x1 = arrival_time,
  y1 = c(0, counts[-n])
)
```

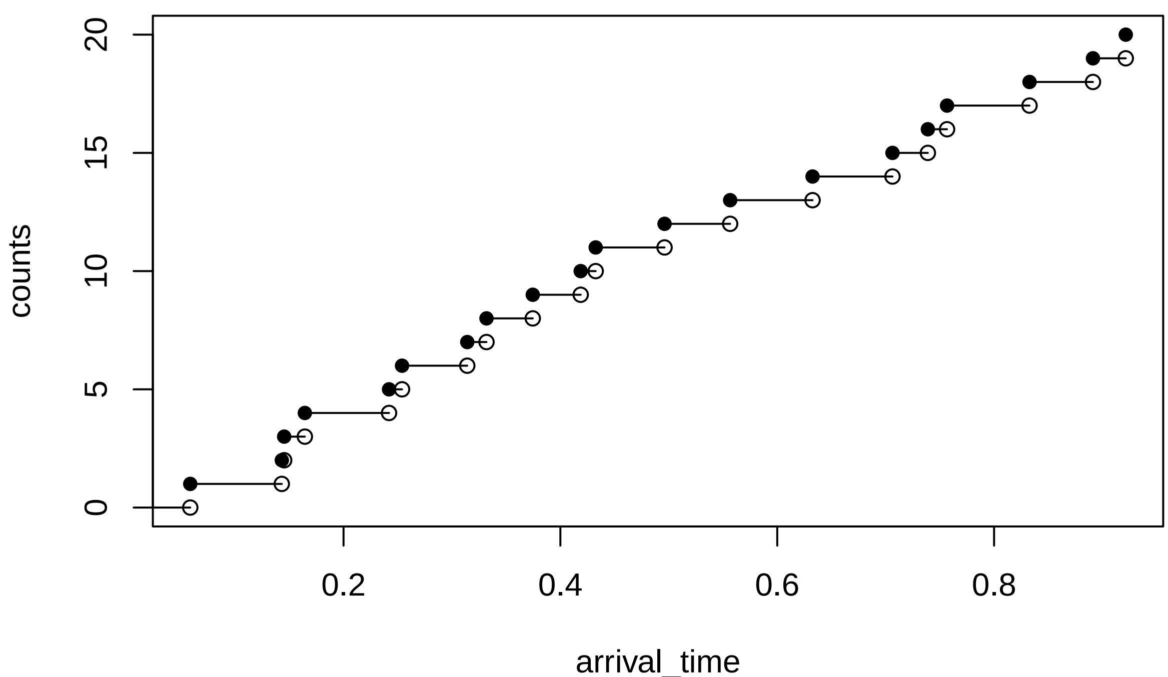


Figure 5.1: A trajectory of the Poisson process

We will see that there are several ways to describe a Poisson process. One can define it by the number of claims that occur up to time t or the times between those claims when they occur. Main properties of

Poisson processes will be discussed.

For $0 \le s < t$, the number of events in the interval $(s, t]$ is given by

$$N(s,t) = N(t) - N(s).$$

For any interval $I = (s, t]$,

$$N(I) = N(t) - N(s).$$

Therefore, $N(t) = N(0, t)$.

Formally, an integer-valued process $\{N(t)\}_{t\ge 0}$ is a **Poisson process** with rate λ (or intensity) if it satisfies the following two conditions:

1. For any t and $h > 0$ (h small),

$$\begin{aligned}
&\Pr(N(t+h) - N(t) = 1) = \lambda h + o(h) \\
&\Pr(N(t+h) - N(t) = 0) = 1 - \lambda h + o(h) \\
&\Pr(N(t+h) - N(t) \ge 2) = o(h)
\end{aligned}$$

2. For disjoint intervals $I_1, I_2, \dots, I_k$, the number of events $N(I_1), \dots, N(I_k)$ are independent random variables.

Notes The statement that $f(h) = o(h)$ as $h \to 0$ means $\lim\limits_{h\to 0} \dfrac{f(h)}{h} = 0$. Examples are h^2, h^3, etc.

1. The probability that an event occurs during the short time interval from time t to time $t + h$ is approximately equal to $\lambda\, h$ for small h, i.e.

$$\Pr[N(t+h) - N(t) = 1] \approx \lambda\, h.$$

   The parameter λ represents the average rate of occurrence of events (e.g. 20 claims (or events) per day).

2. The properties essentially require that, in a very small interval of length h, we have either a single point (or an occurrence) with probability λh, or no point with probability $1 - \lambda h$.

3. Any two of the three statements necessarily imply the third (since the sum of the probability of all possible outcomes is 1).

5.3 Properties of Poisson processes

In the following example, we show that $N(t)$ is a Poisson random variable with mean λt . In addition, $N(t+s) - N(s)$ is a Poisson random variable with mean $\lambda\, t$, independent of anything that has occurred before time s.

Example 5.1. Show that the process $N(t)$ is a Poisson random variable with mean λt, i.e. $N(t) \sim \text{Poisson}(\lambda\, t)$.

Solution:

To show that $N(t) \sim \text{Poisson}(\lambda t)$, we let

$$p_j(t) = \Pr(N(t) = j),$$

which is the probability that there have been exactly j events by time t.

We simply need to show that

$$p_j(t) = \frac{e^{-\lambda t}(\lambda t)^j}{j!}$$

Case 1: For any $j > 0$ and for small positive h, consider the following arguments

$$\begin{aligned}
p_j(t+h) &= \Pr(N(t+h) = j) \\
&= \Pr(N(t) = j \text{ and } N(t, t+h) = 0) \\
&+ \Pr(N(t) = j-1 \text{ and } N(t, t+h) = 1) \\
&+ \Pr(N(t) < j-1 \text{ and } N(t, t+h) \geq 2) \\
&= p_j(t)(1 - \lambda h) + p_{j-1}(t)(\lambda h) + o(h)
\end{aligned}$$

Rearranging the equation and dividing both sides of the equation by h yields

$$\frac{p_j(t+h) - p_j(t)}{h} = \lambda p_j(t) + \lambda p_{j-1}(t) + \frac{o(h)}{h}.$$

Letting $h \to 0$, we obtain

$$\frac{dp_j(t)}{dt} = -\lambda p_j(t) + \lambda p_{j-1}(t),$$

with initial condition $p_j(0) = 0 = \Pr(N(0) = j)$.

Case 2: For $j = 0$, we can also obtain

$$\frac{dp_0(t)}{dt} = -\lambda p_0(t)$$

with initial condition $p_0(0) = 1 = \Pr(N(0) = 0)$.

We can show that the solution to the initial value problem for both cases is

$$p_j(t) = \frac{e^{-\lambda t}(\lambda t)^j}{j!}.$$

Note This result explains why it is called the Poisson process, since number of events in an interval has a Poisson distribution.

Example 5.2. Consider a factory where machinery malfunctions happen as a Poisson process with rate once per 8 hours. The factory owner wants to estimate the probability of one or more failures in a given hour.

The following definition provides an alternative way to characterise the Poisson process.

5.4 Poisson process : Definition 2

A process $\{N(t)\}_{t \geq 0}$ that satisfies the following properties is called a **Poisson process** with rate (or intensity) $\lambda > 0$:

1. $N_0 = 0$.
2. **Poisson distribution** For all $t \geq 0$, $N(t)$ has as a Poisson process with parameter (mean) $\lambda\, t$.
3. **Independent increments** For $0 \leq q < r \leq s < t$, $N(t) - N(s)$ and $N(r) - N(q)$ are independent random variables.
4. **Stationary increments** For all $s, t > 0$, $N(t+s) - N(s)$ has the same distribution as N_t, i.e.

$$\Pr(N(t+s) - N(s) = n) = \Pr(N(t) = n) = \frac{e^{-\lambda t}(\lambda t)^n}{n!}, \quad \text{for } n = 0, 1, 2, \dots.$$

As may be seen from the definition, the increment $N(t+h) - N(t)$ of the Poisson process is independent of past values of the process and has a distribution which does not depend on t (only depends on the length of time interval h). It therefore follows that the Poisson process is a process with **stationary, independent increments** and, in addition, **satisfies the Markov property**.

Example 5.3. Starting at 9 a.m., customers arrive at a coffee shop according to Poisson process at the rate of 10 customers per hour.

1. Find the probability that more than 30 customers arrive between 11 a.m. and 1 p.m.

2. Find the probability that 30 customers arrive by noon and 50 customers by 2 p.m.

5.5 Inter arrival times (Inter event times or holding times)

The Poisson process can take only one-unit upward jumps so it can be characterised by the time between events. Let T_i denote the time between the i-th and the $i+1$-th events. The times T_i are referred to as **the time between events, interarrival times or holding times**

Notes 1. We choose the sample paths of N_t to be right continuous so that

```
-   $N(t)  = 0$ for $t \in [0, T_0)$

-   $N(t)  = 1$ for $t \in [T_0, T_0 + T_1)$

-   $N(t)  = 2$ for $t \in [T_0 + T_1, T_0 + T_1 + T_2)$, etc.
```

2. $N(t)$ is constant over intervals of the form $[a, b)$.

Important result

$\{T_i\}_{i\geq 0}$ is a sequence of independent exponential random variables, each with parameter λ.

Solution: For $t \geq 0$, we have

$$\Pr(T_0 > t) = \Pr(N(t) = 0) = e^{-\lambda t}.$$

This implies that T_0 has an exponential distribution with mean $1/\lambda$.

For $t \geq 0$ and $s \geq 0$, we have

$$\begin{aligned}
\Pr(T_0 > s, T_1 > t) &= \int_s^\infty \int_t^\infty f(u,v)\, dv\, du \\
&= \int_s^\infty \int_t^\infty f_{T_1|u}(v|u) f_{T_0}(u)\, dv\, du \\
&= \int_s^\infty \left[\int_t^\infty f_{T_1|u}(v|u)\, dv\right] f_{T_0}(u) du \\
&= \int_s^\infty \left[\int_t^\infty f_{T_1|u}(v|u)\, dv\right] \lambda e^{-\lambda u} du \\
&= \int_s^\infty \Pr(T_1 > t | T_0 = u)\, \lambda e^{-\lambda u} du \\
&= \int_s^\infty \Pr(\text{no events in the interval (u,u+t)})\, \lambda e^{-\lambda u} du \\
&= \int_s^\infty e^{-\lambda t}\, \lambda e^{-\lambda u} du \\
&= e^{-\lambda t}\, \lambda \int_s^\infty e^{-\lambda u} du \\
&= e^{-\lambda (t+s)}
\end{aligned}.$$

Putting $s = 0$, in the last expression, we obtain

$$\Pr(T_1 > t) = e^{-\lambda t}.$$

Hence, T_1 is exponentially distributed with mean $1/\lambda$.

Moreover,

$$\begin{aligned}
\Pr(T_0 > s, T_1 > t) &= e^{-\lambda(t+s)} \\
&= e^{-\lambda t} e^{-\lambda s} \\
&= \Pr(T_0 > s) \Pr(T_1 > t)
\end{aligned}.$$

This implies that T_1 and T_2 are independent.

Note A Poisson process is a counting process for which interarrival times are independent and identically distributed exponential random variables.

Example 5.4. In the previous example, the event of malfunctions can be modelled as a Poisson process with rate 1/8 failure per hour. Calculate the probability that a second failure will happen within one hour of the first failure in a day.

Example 5.5. Consider insurance claims arriving such that they follow a Poisson process with rate 5 per day.

1. Calculate the probability that there will be at least 2 claims reported on a given day.
2. Calculate the probability that another claim will be reported during the next hour.
3. Calculate the expected time until the next claim, if there haven't been any claims reported in the last two days.

Solution:

Following the properties of a Poisson process, the number of reported claims in an interval of t days has a Poisson distribution with parameter λt, Poisson(λt).

1. The probability that there will be at least 2 claims reported on a given day is

$$\begin{aligned}\Pr(N(t+1)-N(t)\geq 2)=\Pr(N(1)\geq 2) \quad &= 1-\frac{e^{-5}5^0}{0!}-\frac{e^{-5}5^1}{1!}\\ &=1-6e^{-5}\\ &=1-0.0404\\ &=0.9596\end{aligned}$$

2. The probability that another claim will be reported during the next hour is the same as the probability that there is at least one claim in the next hour. Therefore,

$$\begin{aligned}\Pr(N(t+1/24)-N(t)\geq 1)=1-\Pr(N(1/24)=0) \quad &= 1-\frac{e^{-5/24}5^0}{0!}\\ &=1-0.8119\\ &=0.1881\end{aligned}$$

Alternatively, the time between claims has an exponential distribution with parameter $\lambda = 5$. The required probability is

$$\begin{aligned}\Pr(T_i \leq 1/24) &= 1-e^{-5/24}\\ &=0.1881\end{aligned}$$

3. The waiting time has the lack of memory property, so the time before another claim comes in is independent of the time since the last one (i.e. independent of the fact that there have not been any claims reported in the last two days).

The expected time until the next claim is

$$\mathrm{E}[T_i]=\frac{1}{5}=0.2 \text{ days.}$$

5.6 Superposition and thinning properties

In this section, we consider two other important properties of the Poisson process.

Superposition property

Let $N_1(\cdot), N_2(\cdot), \dots, N_k(\cdot)$ be **independent** Poisson processes with rate parameters $\lambda_1, \lambda_2, \dots, \lambda_k$, respectively. Then the following statements hold:

1. The sum of the Poisson process $N(\cdot) = N_1(\cdot) + N_2(\cdot) + \cdots + N_k(\cdot)$ is also a Poisson process with rate $\lambda = \sum_{j=1}^{k} \lambda_j$.

2. Given the occurrence of a point of the process $N(\cdot)$ at time t, it belongs to any given original process $N_i(\cdot)$ with probability $p_i = \lambda_i/\lambda$, independent of all others.

The converse of the superposition property is the splitting property. It can be seen that the Poisson process behaves in an intuitive way when considering the problem of **splitting** or **sampling**.

Splitting (Thinning) property

Let $N(\cdot)$ be a Poisson process with rate λ. Suppose that each arrival of $N(\cdot)$, which is independent of other arrivals, is assigned or marked as a type i with probability p_i, where $p_1 + \dots + p_k = 1$. Let $N_i(\cdot)$ for $1 \le i \le k$ be the number of type i events in $[0, t]$. Then

1. The processes $N_1(\cdot), N_2(\cdot), \dots, N_k(\cdot)$ are **independent** Poisson processes with rates $p_1\lambda, \dots, p_k\lambda$.
2. Each processes is called a **thinned Poisson process**.

To illustrate the splitting property, let us assume (for explanation purpose) that $k = 2$. Let $N_1(t)$ denote the number of events of type 1 by time t and Let $N_2(t)$ denote the number of events of type 2. Therefore, $N(t) = N_1(t) + N_2(t)$. The joint probability mass function of $(N_1(t), N_2(t))$ is

$$\begin{aligned}
\Pr(N_1(t) = n_1, N_2(t) = n_2) &= \Pr(N_1(t) = n_1, N_2(t) = n_2, N(t) = n) \\
&= \Pr(N_1(t) = n_1, N_2(t) = n_2 | N(t) = n) \Pr(N(t) = n) \\
&= \Pr(N_1(t) = n_1 | N(t) = n) \Pr(N(t) = n) \\
&= \left(\binom{n}{n_1} p^{n_1} (1-p)^{n_2} \right) \left(\frac{e^{-\lambda t} (\lambda t)^n}{n!} \right) \\
&= \frac{p^{n_1} (1-p)^{n_2} e^{-\lambda t} (\lambda t)^n}{n_1! n_2!} \\
&= \frac{p^{n_1} (1-p)^{n_2} e^{-\lambda t} (\lambda t)^n}{n_1! n_2!} \\
&= \frac{p^{n_1} (1-p)^{n_2} e^{-\lambda t (p + (1-p))} (\lambda t)^{n_1 + n_2}}{n_1! n_2!} \\
&= \left(\frac{e^{-\lambda p t} (\lambda p t)^{n_1}}{n_1!} \right) \left(\frac{e^{-\lambda (1-p) t} (\lambda (1-p) t)^{n_2}}{n_2!} \right)
\end{aligned},$$

where $n = n_1 + n_2$, $p_1 = p$ and $p_2 = 1 - p_1 = 1 - p$. The above argument follows from the following fact. A type 1 event can be regarded as the result of a coin flip whose heads with probability is p. Assume that there are n events (of mixed types 1 and 2) by time t. Then the number of events of type 1 is the number of heads in n i.i.d. coin flips, which has a **binomial distribution** with parameters n and p.

Consequently, N_1 and N_2 are independent Poisson random variables with parameter $\lambda p t$ and $\lambda(1-p)t$, respectively. In addition, one can show that both processes stationary and independent increments from the original Poisson process.

Example 5.6. An insurance company has two types of policy, A and B. Reported claims under A follow a Poisson process with rate 4 per day. Reported claims independently under B follow a Poisson process with rate 6 per day. The probability that a claim from A is at least 5,000 THB is 2/5, while that from B is 1/3. Calculate the expected number of claims at least 5,000 THB in the next day.

Solution: For type A, $N_A(t) \sim \text{Poisson}(4t)$, a claim from type A is at least 5000 with probability of 2/5. By splitting property, claims under type A that are at least 5000 arrive as a Poisson process with rate $4 \times 2/5 = 8/5$ per day.

Similarly, for type B, $N_B(t) \sim \text{Poisson}(6t)$, a claim from type A is at least 5000 with probability of 1/3. By splitting property, claims under type B that are at least 5000 arrive as a Poisson process with rate $6 \times 1/3 = 2$ per day.

By superposition property, the overall claims that are at least 5000 has a Poisson distribution with parameter

$$8/5 + 2 = 18/5.$$

The expected number of claims at least 5,000 THB in the next day is 18/5 claims.

Example 5.7. Claims arriving from male happen as a Poisson process with rate 2 per day, wile claims arriving from female happen as a Poisson process with rate 6 per day.

1. Calculate the probability that in any given period of 1 week,
   1. no claims occur;
   2. at least 2 claims occur.
2. Calculate the probability that 4 claims from female have happened before 2 claims from male.

Solution:

By the superposition property, these two processes are equivalent to a single process

$$N(t) = N_m(t) + N_f(t)$$

of claims with rate \$ 2 + 6 = 8\$ per day, in which each claim has probability $2/8 = 0.25$ of being from male and $6/8 = 0.75$ of being from female.

1. For any t, $N(t, t+7) = N(t+7) - N(t)$ has a Poisson distribution with mean \$ 7 ×8 = 56\$. In any given period of 1 week,
$$\Pr(\text{no claim}) = e^{-56} \approx 0,$$
and
$$\Pr(\text{at least two claims}) = 1 - e^{-56} - (56)e^{-56} = 1 - (57)e^{-56} \approx 1.$$

2. Each claim is independently "male" or "female". The required event happens if and only if, of the first 5 claims, at least 4 are claims from females. This follows because if the 4th claim from female occurs before or at the 5 claim, then it must have occurred before the 2nd claim from male Therefore, $N_f|5$ claims $= N_f|N = 5 \sim \mathcal{B}(5, 3/4)$ and
$$\begin{aligned}\Pr(N_f \geq 4|N=5) &= \Pr(N_f = 4|N=5) + \Pr(N_f = 5|N=5)\\ &= \binom{5}{4}\left(\frac{3}{4}\right)^4\left(\frac{1}{4}\right) + \binom{5}{5}\left(\frac{3}{4}\right)^5 = 0.6328.\end{aligned}$$

Note It should be noted that the probability of getting k successes before the rth failure can be calculated by taking the sum (over $j = 0, 1, \dots, r-1$) of the probability of $k-1$ successes and j failures followed by a success. This results in

$$\Pr(k \text{ successes before the } r\text{th failure}) = \sum_{j=0}^{r-1}\binom{k+j-1}{j}p^k(1-p)^j.$$

Applying the formula to the question above,

$$\Pr(4 \text{ claims from female before the 2 claims from male}) = \left(\frac{3}{4}\right)^4 + 4\left(\frac{3}{4}\right)^4\left(\frac{1}{4}\right) = 0.6328.$$

5.7 Memorylessness

The importance of the exponential distribution to the Poisson process lies in its unique memoryless property, a topic from probability that merits review. To illustrate the memoryless property, let us consider the following situation:

- Assume that John and Taylor each want to take a bus.
- Buses arrive at a bus stop according to a Poisson process with parameter $\lambda = 1/30$. That is, the times between buses have an exponential distribution, and buses arrive, on average, once every 30 minutes.

- Unlucky John gets to the bus stop just as a bus pulls out of the station. His waiting time for the next bus is about 30 minutes.
- Taylor arrives at the bus stop 10 minutes after John. Remarkably, the time that Taylor waits for a bus also has an exponential distribution with parameter $\lambda = 1/30$.

Memorylessness means that their waiting time distributions are the same, and they will both wait, on average, the same amount of time!

To prove it true, observe that Taylor waits more than t minutes if and only if John waits more than $t + 10$ minutes, given that a bus does not come in the first 10 minutes. Let A and B denote John and Taylor's waiting times, respectively. John's waiting time is exponentially distributed. Hence,

$$\begin{aligned}\Pr(B > t) = \Pr(A > t + 10|A > 10) &= \frac{\Pr(A > t + 10)}{\Pr(A > 10)} \\ &= \frac{e^{-(t+10)/30}}{e^{-10/30}} = e^{-t/30} \\ &= \Pr(A > t).\end{aligned}$$

from which it follows that A and Z have the same distribution.

Of course, there is nothing special about $t = 10$. Memorylessness means that regardless of how long you have waited, the distribution of the time you still have to wait is the same as the original waiting time.

The exponential distribution is the only continuous distribution that is memoryless. (The geometric distribution has the honors for the discrete case.) Here is the general statement of the property.

More precisely, a random variable X is **memoryless** if, for all

$$\Pr(X > s + t|X > s) = \Pr(X - s > t|X > s) = \Pr(X > t).$$

Notes 1. The exponential distribution is the only continuous distribution that exhibits this memoryless property, which is also called the Markov property.

2. Furthermore, it may be shown that, if X is a nonnegative continuous random variable having this memoryless property, then the distribution of X must be exponential.

3. For a discrete random variable, the geometric distribution is the only distribution with this property.

Example 5.8. Assume that the amount of time a patient spends in a dentist's office is exponentially distributed with mean equal to 40 minutes.

1. Calculate the probability that a patient spends more than 60 minutes in the dentist's office.
2. Calculate the probability that a patient will spend 60 minutes in the dentist's office given that she has already spent 40 minutes there.

Solution: Let T be the amount of time spending in the dentist office, which is exponentially distributed $\text{Exp}(\lambda)$ with $\lambda = 60$.

1. $\Pr(T > 60) = e^{-60\lambda} = e^{-60(1/40)} = e^{-1.5} = 0.2231$.

It should be noted that if $X \sim \text{Exp}(\lambda)$, then $\Pr(X > x) = e^{-\lambda x}$.

2. The required probability is

$$\begin{aligned}\Pr(T > 60|T > 40) &= \Pr(T > 40 + 20|T > 40) \\ &= \Pr(T - 40 > 20|T > 40) \\ &= \Pr(T > 20) = e^{-20\lambda} = e^{-0.5} = 0.6065.\end{aligned}$$

Chapter 6

Tutorials

6.1 Tutorial 1

1. Consider a random walk, S_n, with $S_0 = 0$ and where each step is normally distributed with mean 0 and variance 10.

    1. What is the distribution of S_{10}?
    2. Calculate $\Pr(S_{10} < -10)$.

    Solution:

    1. We have S_{10} is a sum of S_0 and the i.i.d random variables Z_i, each distributed normal distribution, $Z_i \sim N(0, 10)$. Moreover,

    $$\mathrm{E}[S_{10}] = \mathrm{E}[S_0 + \sum_{i=1}^{10} Z_i] = S_0 + \sum_{i=1}^{10} \mathrm{E}[Z_i] = 0$$

    and

    $$\mathrm{Var}[S_{10}] = \mathrm{Var}[S_0 + \sum_{i=1}^{10} Z_i] = \sum_{i=1}^{10} \mathrm{Var}[Z_i] = 100$$

    Therefore, the distribution of S_{10} is normally distributed, $S_{10} \sim N(0, 100)$.

    2. From the previous result,

    $$\Pr(S_{10} < -10) = \Pr(Z < -1) = 0.1587.$$

2. Suppose that the value of a commodity as a random walk, where each day the change in price has mean \$0.05 and variance \$0.1. Use the central limit theorem to estimate the probability that its value is more than \$6 after 100 days if the initial value is \$0.50.

    Solution: We have S_{100} is a sum of S_0 and the i.i.d random variables Z_i, each distributed normal distribution, $Z_i \sim N(0.05, 0.1)$. Moreover,

    $$\mathrm{E}[S_{100}] = \mathrm{E}[S_0 + \sum_{i=1}^{100} Z_i] = S_0 + \sum_{i=1}^{100} \mathrm{E}[Z_i] = 5.5$$

    and

    $$\mathrm{Var}[S_{100}] = \mathrm{Var}[S_0 + \sum_{i=1}^{100} Z_i] = \sum_{i=1}^{100} \mathrm{Var}[Z_i] = 10$$

Therefore, the distribution of S_{100} is normally distributed, $S_{100} \sim N(5.5, 10)$.

The Central Limit Theorem implies that S_{100} is approximately normally distributed, $S_{100} \sim N(5.5, 10)$.

$$\Pr(S_{100} > 6) = \Pr(Z > 0.1581) = 1 - \Pr(Z < 0.1581) = 1 - 0.5628 = 0.4372.$$

3. For the random walk process as described in the lecture note,

   1. Calculate $\Pr(X_8 = 96|X_0 = 100)$,
   2. Calculate $\Pr(X_1 = 99, X_8 = 96|X_0 = 100)$,
   3. Calculate $\Pr(X_1 = 99, X_4 = 98, X_8 = 96|X_0 = 100)$,
   4. Calculate $\Pr(X_4 = 98, X_8 = 96|X_1 = 99, X_0 = 100)$,
   5. Given $X_0 = 100$, calculate $\mathrm{E}[X_5]$.
   6. Write down the joint distribution of X_1 and X_3 given $X_0 = 100$. (Hint: consider all possible sample paths)

   Solution:

   1. Here the price must increase on any 2 day(s) and decrease on any 6 day(s), not necessarily in that order. There are $\binom{8}{2} = 28$ different possibilities and each of these has probability $p^2(1-p)^6$. Therefore, the required probability is

   $$\Pr(X_8 = 96|X_0 = 100) = 28p^2(1-p)^6.$$

   2. The problem can be divided into two periods:

      - The first period of 1 day(s): the price in this period must increase on any 0 day(s) and decrease on any 1 day(s), not necessarily in that order. There are $\binom{1}{0}$ different possibilities and each of these has probability $p^0(1-p)^1$.
      - The next period of 7 day(s): the price in this period must increase on any 2 day(s) and decrease on any 5 day(s), not necessarily in that order. There are $\binom{7}{2}$ different possibilities and each of these has probability $p^2(1-p)^5$.

      The required probability is

   $$\Pr(X_1 = 99, X_8 = 96|X_0 = 100) = 21p^2(1-p)^6.$$

   3. Similar to the previous problem, the required probability can be calculated as follows:

      The problem can be divided into three periods:

      - The first period of 1 day(s): the price in this period must increase on any 0 day(s) and decrease on any 1 day(s), not necessarily in that order. There are $\binom{1}{0}$ different possibilities and each of these has probability $p^0(1-p)^1$.
      - The next period of 3 day(s): the price in this period must increase on any 1 day(s) and decrease on any 2 day(s), not necessarily in that order. There are $\binom{3}{1}$ different possibilities and each of these has probability $p^1(1-p)^2$.
      - The last period of 4 day(s): the price in this period must increase on any 1 day(s) and decrease on any 3 day(s), not necessarily in that order. There are $\binom{4}{1}$ different possibilities and each of these has probability $p^1(1-p)^3$.

The required probability is

$$\Pr(X_1 = 99, X_4 = 98, X_8 = 96 | X_0 = 100) = 12p^2(1-p)^6.$$

4. By Markov property, we have

$$\Pr(X_4 = 98, X_8 = 96 | X_1 = 99, X_0 = 100), = \Pr(X_4 = 98, X_8 = 96 | X_1 = 99).$$

Again, the problem can be divided into two periods:

- The first period of 3 day(s): the price in this period must increase on any 1 day(s) and decrease on any 2 day(s), not necessarily in that order. There are $\binom{3}{1}$ different possibilities and each of these has probability $p^1(1-p)^2$.
- The next period of 4 day(s): the price in this period must increase on any 1 day(s) and decrease on any 3 day(s), not necessarily in that order. There are $\binom{4}{1}$ different possibilities and each of these has probability $p^1(1-p)^3$.

The required probability is

$$\Pr(X_4 = 98, X_8 = 96 | X_1 = 99, X_0 = 100) = \Pr(X_4 = 98, X_8 = 96 | X_1 = 99) = 12p^2(1-p)^5.$$

5. The variable X_5 can take the values from $95, 97, 99, 101, 103, 105$. In particular,

$$\begin{aligned}
\Pr(X_5 = 95) &= \binom{5}{0} p^0(1-p)^5 = 1p^0(1-p)^5 \\
\Pr(X_5 = 97) &= \binom{5}{1} p^1(1-p)^4 = 5p^1(1-p)^4 \\
\Pr(X_5 = 99) &= \binom{5}{2} p^2(1-p)^3 = 10p^2(1-p)^3 \\
&\vdots \\
\Pr(X_5 = 105) &= \binom{5}{5} p^5(1-p)^0 = 1p^5(1-p)^0
\end{aligned}$$

Therefore, $\mathrm{E}[X_5] = 95 \cdot (1p^0(1-p)^5) + 97 \cdot (5p^1(1-p)^4) + 99 \cdot (10p^2(1-p)^3) + \cdots + 105 \cdot (1p^5(1-p)^0) = 95 + 10 * x$. Alternatively,

$$\begin{aligned}
E[X_5] &= E\left[100 + \sum_{i=1}^{5} Z_i\right] \\
&= 100 + 5 \cdot E[Z_i] \\
&= 100 + 5(2p-1) \\
&= 95 + 10p
\end{aligned}$$

where $E[Z_i] = p - q = p - (1-p) = 2p - 1$.

4. For each event,

- Identify a stochastic process $\{X_t : t \in T\}$ and describe X_t in context.
- Describe the time domain and state space. State whether the time domain and state space are discrete or continuous.

1. Sociologists categorise the population of a country into upper-, middle- and lower-class groups. One of the government offices has monitored the movement of successive generations among these three groups.

2. The insurer's surplus (an excess of income or assets over expenditure or liabilities in a given period) at any future time which is defined as the initial surplus plus the premium income up to time t minus the aggregate claims up to time t.
3. In a working day, a coffee shop owner records customer arrival times.
4. The gambler starts with m and bets per game. The probability of winning is p and the probability of losing is q where $p + q = 1$. In addition, the gambler is ruined (or goes broke) if he reaches state 0, and also stops the game if he reaches state N.
5. In the board game Monopoly, there are 40 squares. A player is interested to know the successive board position.

Solution:

1. Let X_n be the class of n^{th} generation of a family. The state space, $S = \{\text{Upper, Middle, Lower}\}$, is discrete. The index set, $I = \{0, 1, 2, ...\}$, is also discrete.
2. Let $S(t)$ be the insurer's surplus at time t. $S = \mathbb{R}$ and $I = [0, \infty)$.
3. Let X_n be the amount of money of the gambler after game n.

   $S = \{0, 1, 2 \ldots, N\}$ and $I = \{0, 1, 2, ...\}$.
4. Suppose that the opening hours of the coffee shop are from 7:00 am to 6:00 pm (i.e. 11 hours). Let X_n be the arrival time of customer n. State space (continuous)

   $S = [0, 11 \times 60] = [0, 660]$ (in minutes) and $I = \{1, 2, ...\}$.
5. Let x_n be a player's board position after n plays.

   $S = \{1, 2, 3, \ldots, 40\}$ and $I = \{0, 1, 2, ...\}$.

5. The simple weather pattern can be classified into three types including rainy (R), cloudy (C) and sunny (S). The weather is observed daily. The following information is provided.

   - On any given rainy day, the probability that it will rain the next day is 0.7; the probability that it will be cloudy the next day 0.2.
   - On any given cloudy day, the probability that it will rain the next day is 0.75; the probability that it will be sunny the next day 0.1.
   - On any given sunny day, the probability that it will rain the next day is 0.2; the probability that it will be sunny the next day 0.4.

   Explain how this may be modelled by a Markov chain.

   Solution: The three weather conditions describe the three state of the Markov chain. Let $X_n$be the weather condition on day n.

   - State 1 (R) rainy day
   - State 2 (C) cloudy day
   - State 3 (S) sunny day

   The transition probability matrix P for this Markov chain is

$$P = \begin{bmatrix} 0.7 & 0.2 & 0.1 \\ 0.75 & 0.15 & 0.1 \\ 0.2 & 0.4 & 0.4 \end{bmatrix}$$

   This stochastic process has the Morkov property because the weather condition on the next day depends only on the condition today.

6. Explain whether an independent and identically distributed sequence of random variables has a Markov property.

   Solution: Assume that this Markov chain $X_0, X_1, X_2 \ldots$, takes values in $\{1, 2, \ldots, k\}$ with

   $$P(X_n = j) = p_j \quad \text{for } j = 1, 2, \ldots, k \quad \text{and } n \geq 0.$$

   Note that this equality holds for all n because $\{X_n\}_{n\geq 0}$ have the same distribution.

   By independence,

   $$P(x_n = j \mid x_{n-1} = i) = P(x_n = j) = p_j,$$

   This proves our claim that the i.i.d. sequence of random variables has a Markov property. Note also that the transition matrix is

   $$P = \begin{bmatrix} p_1 & p_2 & \cdots & p_k \\ p_1 & p_2 & \cdots & p_k \\ \vdots & \vdots & & \vdots \\ p_1 & p_2 & \cdots & p_k \end{bmatrix}.$$

7. The random variables $Z_1, Z_2, \ldots$ are independent and with the common probability mass function

   $$Z_i = \begin{cases} 1, & \text{with probability } 0.2 \\ 2, & \text{with probability } 0.3 \\ 3, & \text{with probability } 0.4 \\ 4, & \text{with probability } 0.1 \end{cases}$$

   Let $X_0 = 1$ and $X_n = \max\{Z_1, Z_2, \ldots, Z_n\}$ be the largest Z observed to date. Explain how this may be modelled by a Markov chain. **Solution:**

   Given $X_0 = 1$ and $X_n = \max\{Z_1, Z_2, \ldots, Z_n\}$ where $\{Z_i\}$ are i.i.d. random variables with

i	1	2	3	4
$P(Z_i = i)$	0.2	0.3	0.4	0.1

   We note that

   $$\begin{aligned} X_{n+1} &= \max\{Z_1, Z_2, \ldots, Z_{n+1}\} \\ &= \max\{X_n, Z_{n+1}\} \end{aligned}$$

   Consider the transition probabilities

   $$\begin{aligned} P(X_{n+1} = j \mid X_n = i) &= P(\max\{X_n, Z_{n+1}\} = j \mid X_n = i) \\ &= P(\max\{i, Z_{n+1}\} = j \mid X_n = i) \end{aligned}$$

   Case 1: If $i = 1$, then

   $$\max\{1, Z_{n+1}\} = \begin{cases} 1 & \text{w.p. } 0.2 \\ 2 & \text{w.p. } 0.3 \\ 3 & \text{w.p. } 0.4 \\ 4 & \text{w.p. } 0.1 \end{cases}$$

Case 2: If $i = 2$, then

$$\max\{2, Z_{n+1}\} = \begin{cases} 1 & \text{w.p. } 0 \\ 2 & \text{w.p. } 0.5 \quad (\text{i.e. } z_{n+1} = 1 \text{ or } 2) \\ 3 & \text{w.p. } 0.4 \\ 4 & \text{w.p. } 0.1 \end{cases}$$

Case 3: If $i = 3$, then

$$\max\{3, Z_{n+1}\} = \begin{cases} 1 & \text{w.p. } 0 \\ 2 & \text{w.p. } 0 \\ 3 & \text{w.p. } 0.9 \\ 4 & \text{w.p. } 0.1 \end{cases}$$

Case 4: If $i = 4$, then

$$\max\{4, Z_{n+1}\} = \begin{cases} 1 & \text{w.p. } 0 \\ 2 & \text{w.p. } 0 \\ 3 & \text{w.p. } 0 \\ 4 & \text{w.p. } 1 \end{cases}$$

The transition probability matrix is then

$$P = \begin{bmatrix} 0.2 & 0.3 & 0.4 & 0.1 \\ 0 & 0.5 & 0.4 & 0.1 \\ 0 & 0 & 0.9 & 0.1 \\ 0 & 0 & 0 & 0.1 \end{bmatrix}.$$

Clearly the sequence $X_0, X_1, X_2, ...$ can be modelled by the Markov chain with the transition probability matrix P. Moreover, given the most recent value X_n, its future value X_{n+1} is independent of the past history $X_0, X_1, ..., X_{n-1}$.

6.2 Tutorial 2

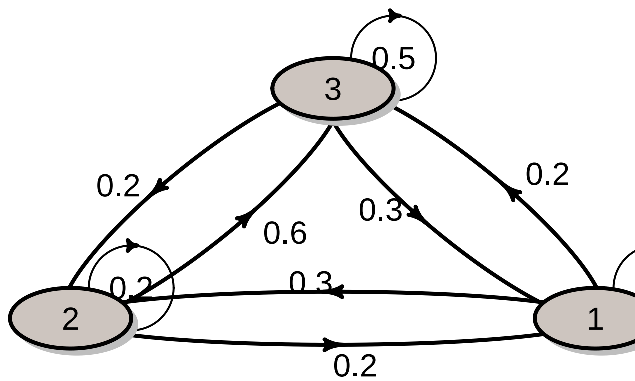
1. A Markov chain $X_0, X_1, ...$ on states $1, 2, 3$ has the following transition matrix

$$P = \begin{bmatrix} 0.5 & 0.3 & 0.2 \\ 0.2 & 0.2 & 0.6 \\ 0.3 & 0.2 & 0.5 \end{bmatrix}.$$

The distribution of the initial random variable X_0 is $\mu = (0.3, 0.3, 0.4)$.

1. Draw a transition diagram for the chain.
2. Determine $\Pr(X_0 = 1, X_1 = 2, X_2 = 3)$.
3. Determine $\Pr(X_1 = 2, X_2 = 3|X_0 = 1)$.
4. Determine $\Pr(X_{11} = 2, X_{12} = 3|X_{10} = 1)$.
5. Determine $\Pr(X_2 = 3|X_0 = 1)$.
6. Determine $\Pr(X_3 = 3|X_1 = 1)$.
7. Determine $\Pr(X_2 = 3)$.
8. Determine $\mathrm{E}[X_2]$

Solutions:



1. The transition diagram for the chain is shown in the figure below:
2. $\Pr(X_0 = 1, X_1 = 2, X_2 = 3) = \mu_1 p_{12} p_{23} = (0.3)(0.3)(0.6) = 0.054$.
3. $\Pr(X_1 = 2, X_2 = 3 | X_0 = 1) = p_{12} p_{23} = (0.3)(0.6) = 0.18$.
4. From the time homogeneous assumption, it follows that

$$\Pr(X_{11} = 2, X_{12} = 3 | X_{10} = 1) = \Pr(X_1 = 2, X_2 = 3 | X_0 = 1) = 0.18.$$

5. $\Pr(X_2 = 3 | X_0 = 1) = (P^2)_{13} = 0.38$.
6. $\Pr(X_3 = 3 | X_1 = 1) = \Pr(X_2 = 3 | X_0 = 1) = (P^2)_{13} = 0.38$.
7. $\Pr(X_2 = 3) = (\mu P^2)_3 = 0.424$.
8. $\mathrm{E}[X_2] = \sum_{k=1}^{3} k \Pr(X_2 = k) = (1,2,3) \cdot (0.343, 0.233, 0.424) = 2.081$.

Note that

```
## P^2
##  A  3 - dimensional discrete Markov Chain defined by the following states:
##  1, 2, 3
##  The transition matrix  (by rows)  is defined as follows:
##      1    2    3
## 1 0.37 0.25 0.38
## 2 0.32 0.22 0.46
## 3 0.34 0.23 0.43
```

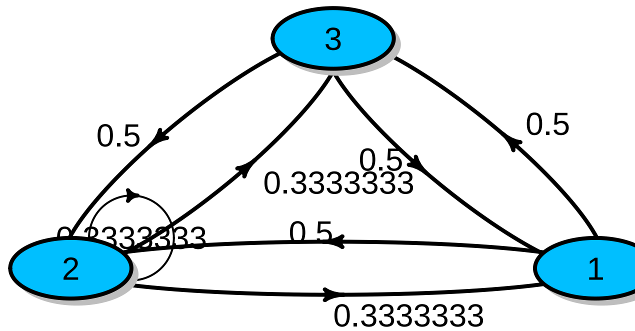
2. A Markov chain $X_0, X_1, \ldots$ on states 1,2,3 has the following transition matrix

$$P = \begin{bmatrix} 0 & 1/2 & 1/2 \\ 1/3 & 1/3 & 1/3 \\ 1/2 & 1/2 & 0 \end{bmatrix}.$$

The process starts in states $X_0 = 1$.

1. Draw a transition diagram for the chain.
2. Determine $\Pr(X_0 = 1, X_1 = 3, X_2 = 2)$.
3. Determine $\Pr(X_1 = 3, X_2 = 2 | X_0 = 1)$.
4. Determine $\Pr(X_2 = 2 | X_0 = 1)$.
5. Determine $\Pr(X_3 = 2 | X_1 = 1)$.
6. Determine $\Pr(X_2 = 2)$.

Solution:



1. The transition diagram for the chain is shown in the figure below:
2. $\Pr(X_0 = 1, X_1 = 3, X_2 = 2) = \mu_1 p_{13} p_{32} = (1)(1/2)(1/2) = 1/4$.
3. $\Pr(X_1 = 3, X_2 = 2|X_0 = 1) = p_{13} p_{32} = (1/2)(1/2) = 1/4$.
4. $\Pr(X_2 = 2|X_0 = 1) = (P^2)_{12} = 5/12$.
5. $\Pr(X_3 = 2|X_1 = 1) = \Pr(X_2 = 2|X_0 = 1) = (P^2)_{12} = 5/12$.
6. $\Pr(X_2 = 2) = (\mu P^2)_2 = 5/12$.

Note that

```
## P^2
##  A  3 - dimensional discrete Markov Chain defined by the following states:
##  1, 2, 3
##  The transition matrix  (by rows)  is defined as follows:
##           1         2         3
## 1 0.4166667 0.4166667 0.1666667
## 2 0.2777778 0.4444444 0.2777778
## 3 0.1666667 0.4166667 0.4166667
```

3. A Markov chain $X_0, X_1, \ldots$ on sates 1,2 has the following transition matrix

$$P = \begin{bmatrix} 1-a & a \\ b & 1-b \end{bmatrix},$$

where $0 < a, b < 1$.

   1. Draw a transition diagram for the chain.
   2. the distribution of X_1.
   3. Show that
$$P^n = \frac{1}{a+b} \begin{bmatrix} b & a \\ b & a \end{bmatrix} + \frac{(1-a-b)^n}{a+b} \begin{bmatrix} a & -a \\ -b & b \end{bmatrix}.$$
   4. Given that $X_0 = 1$, what is the probability that in the long run the system will be in state 1? (Hint: consider $\lim_{n\to\infty} \mu P^n$)
   5. Given that $X_0 = 1$, what is the probability that in the long run the system will be in state 2?
   6. Given that $X_0 = 2$, what is the probability that in the long run the system will be in state 1?
   7. Given that $X_0 = 2$, what is the probability that in the long run the system will be in state 2?

Solution:

1. Leave it to the reader.
2. Denote $\mu = (\mu_1, \mu_2)$ the initial probability distribution. Then the distribution of X_1 is $\mu^{(1)} = \mu P = (\mu_1(1-a) + \mu_2 b, \mu_1 a + \mu_2(1-b))$

3. We apply eigendecomposition of a matrix. For more details, please follow this link from Wikipedia link.

The eigenvalues of P are $\lambda_1 = 1$ and $\lambda_2 = 1 - a - b$ and the corresponding eigenvectors are

$$v_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad v_2 = \begin{bmatrix} -a/b \\ 1 \end{bmatrix}.$$

Then the transition matrix can be factorised as

$$P = \begin{bmatrix} 1 & -a/b \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1-a-b \end{bmatrix} \begin{bmatrix} 1 & -a/b \\ 1 & 1 \end{bmatrix}^{-1}.$$

Hence

$$\begin{aligned} P^n &= \left(\begin{bmatrix} 1 & -a/b \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & (1-a-b)^n \end{bmatrix} \right) \begin{bmatrix} 1 & -a/b \\ 1 & 1 \end{bmatrix}^{-1} \\ &= \begin{bmatrix} 1 & -\frac{a}{b}(1-a-b)^n \\ 1 & (1-a-b)^n \end{bmatrix} \left(\frac{1}{1+a/b} \begin{bmatrix} 1 & a/b \\ -1 & 1 \end{bmatrix} \right) \\ &= \frac{1}{a+b} \begin{bmatrix} b + a(1-a-b)^n & a - a(1-a-b)^n \\ b - b(1-a-b)^n & a + b(1-a-b)^n \end{bmatrix} \\ &= \frac{1}{a+b} \begin{bmatrix} b & a \\ b & a \end{bmatrix} + \frac{(1-a-b)^n}{a+b} \begin{bmatrix} a & -a \\ -b & b \end{bmatrix} \end{aligned}.$$

Note that

$$\lim_{n\to\infty} P^n = \frac{1}{a+b} \begin{bmatrix} b & a \\ b & a \end{bmatrix},$$

which follows from the facts that $-1 < 1 - a - b < 1$ and $(1-a-b)^n \to 0$ as $n \to \infty$.

4. It follows from the above results that in the long run

$$\lim_{n\to\infty} \Pr(X_n = 1 | X_0 = 1) = (\lim_{n\to\infty} P^n)_{11} = \frac{b}{a+b}.$$

5. In the long run, we have

$$\lim_{n\to\infty} \Pr(X_n = 2 | X_0 = 1) = (\lim_{n\to\infty} P^n)_{12} = \frac{a}{a+b}.$$

6. In the long run, we have

$$\lim_{n\to\infty} \Pr(X_n = 1 | X_0 = 2) = (\lim_{n\to\infty} P^n)_{21} = \frac{b}{a+b}.$$

7. In the long run, we have

$$\lim_{n\to\infty} \Pr(X_n = 2 | X_0 = 2) = (\lim_{n\to\infty} P^n)_{22} = \frac{a}{a+b}.$$

Furthermore, for any initial distribution μ, the limiting distribution with this initial distribution is

$$\lim_{n\to\infty} \mu P^n = (\frac{b}{a+b}, \frac{a}{a+b}).$$

This gives the long term proportion of the Markov chain, i.e. the probability of finding the process in state 1 is $\frac{b}{a+b}$ and in state 2 is $\frac{a}{a+b}$, irrespective of the stating state.

4. Let a be a constant and $\xi_1, \xi_2, ...$ be a sequence of independent and identically distributed (i.i.d.) random variables. The stochastic process $\{X_n\}$ is defined by

$$X_0 = a, \quad X_n = X_{n-1} + \xi_n,\ n > 1.$$

This process is known as a random walk.

1. Express the state X_n in terms of X_0 and the random variables $\xi_i, i = 1, 2 ....$
2. Find $\mathrm{E}[X_n]$ and $\mathrm{Var}[X_n]$.
3. Does the process have the Markov property? Explain.
4. Is the process stationary? Explain.

Solution:

1. From the definition, it follows that

$$\begin{aligned}
X_0 &= a \\
X_1 &= X_0 + \xi_1 = a + \xi_1 \\
X_2 &= X_1 + \xi_2 = a + \xi_1 + \xi_2 \\
&\vdots \\
X_n &= X_{n-1} + \xi_n = a + \sum_{i=1}^{n} \xi_i, \quad n \geq 1.
\end{aligned}$$

2. Let $\mu = \mathrm{E}[\xi_i]$ and $\sigma^2 = \mathrm{Var}[\xi_i]$ denote the mean and variance of the increments ξ_i. Then

$$\begin{aligned}
\mathrm{E}[X_n] &= \mathrm{E}\left[a + \sum_{i=1}^{n} \xi_i\right] = a + \sum_{i=1}^{n} \mathrm{E}[\xi_i] = a + n\mu, \\
\mathrm{Var}[X_n] &= \mathrm{Var}\left[a + \sum_{i=1}^{n} \xi_i\right] = \sum_{i=1}^{n} \mathrm{Var}[\xi_i] = n\sigma^2.
\end{aligned}$$

The last equality follows from the assumption that $\xi_1, \xi_2, ...$ are independent.

3. The process $\{X_n\}_{n\geq 0}$ has independent increments and , hence, has the Markov property. More details can be found from the lecture note link.

4. The process is **not** stationary because $\mathrm{E}[X_n]$ is not constant and $\mathrm{Var}[X_n]$ also depends on n.

5. Consider a homogeneous discrete-time Markov chain that describes the daily weather pattern. The weather patterns are classified into 3 conditions: R(rainy), C (cloudy) and S(sunny). Based on the daily observations, the following information are given:

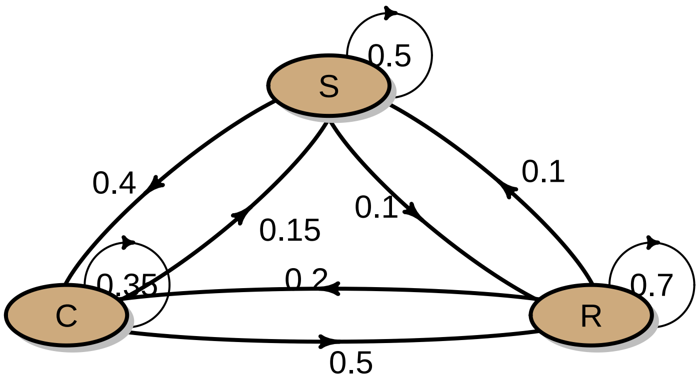
- On any rainy day, the probability that it will rain the next day is 0.7; the probability that tomorrow will be cloudy is 0.2 and the probability that tomorrow will be sunny is 0.1.
- On any cloudy day, the probability that it will rain the next day is 0.5; the probability that tomorrow will be cloudy is 0.35 and the probability that tomorrow will be sunny is 0.15.
- On any sunny day, the probability that it will rain the next day is 0.1; the probability that tomorrow will be cloudy is 0.4 and the probability that tomorrow will be sunny is 0.5.

1. Draw a transition diagram for the chain and write down a transition matrix.
2. Find the probability that tomorrow is cloudy and the day after is rainy, given that it is sunny today.
3. Given that today is rainy, find the probability that it will be sunny in two days time.

Solution:

1. The transition matrix P and the transition diagram are given in the results below :

```
## P
##  A  3 - dimensional discrete Markov Chain defined by the following states:
##  R, C, S
##  The transition matrix  (by rows)  is defined as follows:
##     R    C    S
## R 0.7 0.20 0.10
## C 0.5 0.35 0.15
## S 0.1 0.40 0.50
```



2. The probability that tomorrow is cloudy and the day after is rainy, given that it is sunny today is

$$\Pr(X_1 = C, X_2 = R | X_0 = S) = (0.4)(0.5) = 0.2.$$

3. Given that today is rainy, the probability that it will be sunny in two days time is

$$\Pr(X_2 = S | X_0 = R) = (P^2)_{13} = 0.15.$$

6.3 Tutorial 3

1. A Markov chain $X_0, X_1, ...$ on states 1, 2, 3 with initial distribution $\mu = (1/4, 1/4, 1/2)$. It has the following transition matrix

$$P = \begin{bmatrix} 1/2 & 1/4 & 1/4 \\ 1/3 & 1/3 & 1/3 \\ 1/5 & 2/5 & 2/5 \end{bmatrix}.$$

Compute the following probabilities:

1. $\Pr(X_{11} = 1, X_{12} = 2, X_{13} = 3 | X_{10} = 1)$.
2. $\Pr(X_0 = 3, X_1 = 2, X_2 = 1)$.
3. $\Pr(X_1 = 3, X_2 = 2, X_3 = 1)$.
4. $\Pr(X_1 = 2, X_3 = 2, X_5 = 2)$.

Solution:

```
1.  We have
```

$\Pr(X_{11} = 1, X_{12} = 2, X_{13} = 3 | X_{10} = 1) = p_{11} p_{12} p_{23} = (1/2)(1/4)(1/3) = 1/24.$

```
2.  We have
```

$\Pr(X_0 = 3, X_1 = 2, X_2 = 1) = \mu_3 p_{32} p_{21} = (0.5)(2/5)(1/3) = 1/15.$

3. We have

$$\begin{aligned}
\Pr(X_1 = 3, X_2 = 2, X_3 = 1) &= \sum_{i=1}^{3} \Pr(X_0 = i) \Pr(X_1 = 3, X_2 = 2, X_3 = 1 | X_0 = i) \\
&= \sum_{i=1}^{3} \mu_i p_{i3} p_{32} p_{21} \\
&= \left(\sum_{i=1}^{3} \mu_i p_{i3} \right) p_{32} p_{21} \\
&= (\mu P)_3 p_{32} p_{21} \\
&= (83/240)(2/5)(1/3) \\
&= 83/1800 = 0.0461111.
\end{aligned}$$

4. We have

$$\begin{aligned}
\Pr(X_1 = 2, X_3 = 2, X_5 = 2) &= \sum_{i=1}^{3} \Pr(X_0 = i) \Pr(X_1 = 2, X_3 = 2, X_5 = 2 | X_0 = i) \\
&= \sum_{i=1}^{3} \mu_i p_{i2} p_{22}^{(2)} p_{22}^{(2)} \\
&= \left(\sum_{i=1}^{3} \mu_i p_{i2} \right) p_{22}^{(2)} p_{22}^{(2)} \\
&= (\mu P)_2 p_{22}^{(2)} p_{22}^{(2)} \\
&= (83/240)(59/180)(59/180) \\
&= 9356/251805 = 0.0371557.
\end{aligned}$$

2. A Markov chain with state space $S = \{1, 2, 3, 4, 5, 6\}$ has the following transition matrix:

$$P = \begin{bmatrix} 1/4 & 0 & 3/4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1/2 & 0 & 0 & 1/2 \\ 1/5 & 1/5 & 1/5 & 0 & 1/5 & 1/5 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 1/3 & 0 & 0 & 0 & 0 & 2/3 \end{bmatrix}.$$

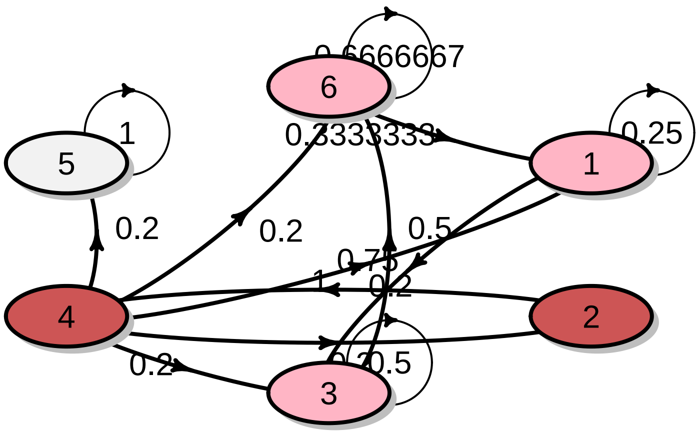
1. Draw a transition diagram.

2. Identify the communication classes and classify them as closed or non-closed.

3. Is the Markov chain irreducible?

Solution:

1. The transition diagram is shown in the figure below:



`2. There are two closed classes`

$C^1 = \{1, 3, 6\}$ and $C^2 = \{5\}$ because

- $p_{55} = 1$ and
- $1 \to 3 \to 6 \to 1$, and hence $1, 3, 6$ are in the same communication class. In addition, for each $i \in C^1$, $\sum_{j \in C^1} p_{ij} = 1$, which implies that escaping from C^1 is impossible. Therefore C^1 is a closed class.

  There is one non-closed class $O = \{2, 4\}$. This is because $2 \leftrightarrow 4$ and $p_{43} > 0$.

  3. The Markov chain is reducible because it contains more than one communication classes.

3. For each of the Markov chains whose transition matrix is given below, identify the closed classes and the vector of absorption probabilities associated with each of these closed classes. Assume that the states are labelled $1, 2, 3 \ldots$.

   1. $$\begin{bmatrix} 1/6 & 0 & 1/3 & 1/2 \\ 0 & 1/3 & 2/3 & 0 \\ 1/2 & 1/2 & 0 & 0 \\ 0 & 0 & 1/4 & 3/4 \end{bmatrix}.$$

   2. $$\begin{bmatrix} 0 & 1/4 & 3/4 & 0 \\ 0 & 1/3 & 0 & 2/3 \\ 1/3 & 0 & 1/3 & 1/3 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

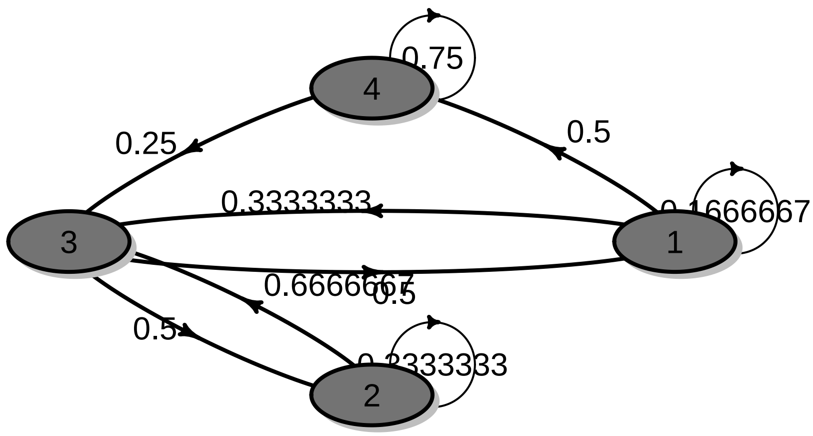
   3. $$\begin{bmatrix} 1/4 & 1/4 & 1/4 & 1/4 \\ 0 & 3/4 & 1/4 & 0 \\ 0 & 3/4 & 1/4 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

   4. $$\begin{bmatrix} 0 & 0 & 0 & 1/2 & 0 & 0 & 1/2 \\ 1/6 & 0 & 1/6 & 0 & 0 & 1/6 & 1/2 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1/2 & 0 & 0 & 1/2 & 0 & 0 & 0 \\ 1/4 & 1/4 & 0 & 0 & 0 & 0 & 1/2 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1/2 & 0 & 0 & 0 & 0 & 0 & 1/2 \end{bmatrix}.$$

Solution:

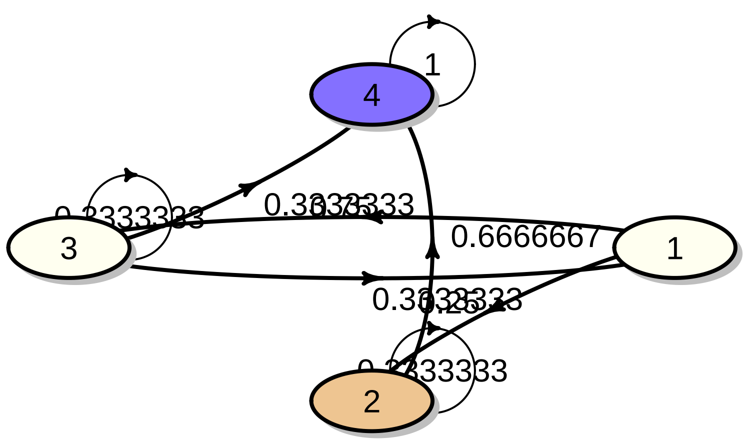
`1. Every two states communicate, so`

$\{1, 2, 3, 4\}$ is a single closed class (since $1 \to 4 \to 3 \to 2 \to 3 \to 1$). The absorption probabilities are 1, since each state is in this closed class. The transition diagram is shown in the figure below:



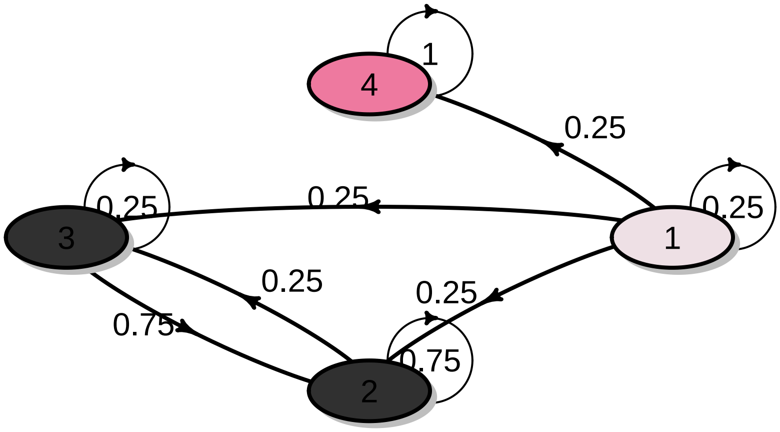
`2. There are two non-closed classes`

$O^1 = \{1, 3\}$ and $O^2 = \{2\}$ and a closed class $C^1 = \{4\}$. Since we have a single closed class, all absorption probabilities to this closed class $C^1 = \{4\}$ are equal to 1. The transition diagram is shown in the figure below:



`3. There are two closed classes`

$C^1 = \{2, 3\}$ and $C^2 = \{4\}$ and a non-closed class $O^1 = \{1\}$. The transition diagram is shown in the figure below:



Let $\mathbf{u} = \mathbf{u}^{C_2}$ be the vector of absorption probabilities in the closed class $C^2 = \{4\}$. Write $\mathbf{u} = (u_1, u_2, u_3, u_4)^T$ and $u_4 = 1$ and $u_2 = u_3 = 0$,

From $\mathbf{u} = P \cdot \mathbf{u}$,

$$\begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{pmatrix} = \begin{bmatrix} 1/4 & 1/4 & 1/4 & 1/4 \\ 0 & 3/4 & 1/4 & 0 \\ 0 & 3/4 & 1/4 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{pmatrix} \text{ gives}$$

$$u_1 = \frac{1}{4}u_1 + \frac{1}{4}$$

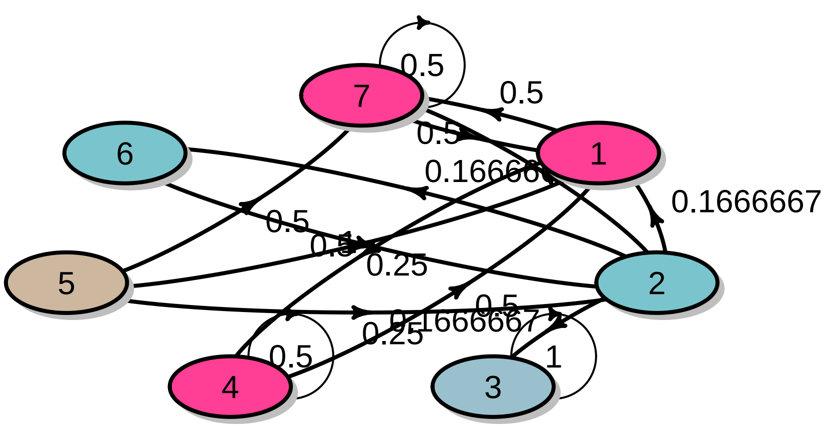
Solving the linear system for u_1 yields $u_1 = 1/3$. Hence, the absorption probabilities in the closed class C_2 is

$$\mathbf{u} = (1/3, 0, 0, 1)^T.$$

In addition, since there are two closed classes, $\mathbf{u}^{C_1} = \mathbf{1} - \mathbf{u}^{C_2} = (2/3, 1, 1, 0)^T$.

`4. There are two closed classes`

$C^1 = \{1, 4, 7\}$ and $C^2 = \{3\}$ and two non-closed classes $O^1 = \{2, 6\}$ and $O^2 = \{5\}$. The transition diagram is shown in the figure below:



Let $\mathbf{u} = \mathbf{u}^{C_2}$ be the vector of absorption probabilities in the closed class $C^2 = \{3\}$. Write $\mathbf{u} = (u_1, u_2, u_3, \ldots, u_7)^T$ and $u_3 = 1$ and $u_1 = u_4 = u_7 = 0$,

$$\begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_7 \end{pmatrix} = P \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_7 \end{pmatrix} \text{ gives}$$

$$u_2 = \frac{1}{6} + \frac{1}{6}u_6$$
$$u_5 = \frac{1}{4}u_2$$
$$u_6 = u_2.$$

Solving the linear system for u_2, u_5 and u_6 yields $u_2 = 1/5, u_5 = 1/20$ and $u_6 = 1/5$. Hence, the absorption probabilities in the closed class C_2 is

$$\mathbf{u} = (0, 1/5, 1, 0, 1/20, 1/5, 0)^T.$$

In addition, since there are two closed classes, $\mathbf{u}^{C_1} = \mathbf{1} - \mathbf{u}^{C_2} = (1, 4/5, 0, 1, 19/20, 4/5, 1)^T$.

4. If the Markov chain defined in Question 3 is irreducible, i.e. it has a unique stationary distribution, then find the stationary distribution of the chain.

5. Assume that a Markov chain has more than one closed classes (say r closed classes). The Markov chain can **have many stationary distributions**. Assume further that within each of these r closed classes, the associated Markov chain is aperiodic. The followings hold:

   - Within a closed class C_1, let P_1 be a reduction of a matrix P which is formed by deleting all rows and columns corresponding to states from other classes. Then there exists a unique stationary distribution, denoted by $\{\pi_j^{(1)}\}_{j \in C_1}$.
   - Similarly, let $\{\pi_j^{(2)}\}_{j \in C_2}, \dots, \{\pi_j^{(r)}\}_{j \in C_r}$ be stationary distributions within other classes.

   1. Show that for any numbers $\gamma_1, \gamma_2, \dots, \gamma_r$ such that $\sum_{m=1}^{r} \gamma_m = 1$, the following distribution $\{\pi_j\}$ is stationary, where

   $$\pi_j = \begin{cases} \pi_j^{(k)}\gamma_k & \text{for } j \in C_k,\ k = 1, \dots, r \\ 0 & \text{if } j \text{ is in a nonclosed class.} \end{cases} \tag{6.1}$$

   (In particular, any stationary distribution of the Markov chain is of this form.)

   2. Write down the general form of stationary distributions of the Markov chain in Questions 3.3 and 3.4.

   3. Now we will focus on limiting distributions. Consider the three following possible cases.

      1. If $X_0 = i$ and $i \in C_k$ for some closed class C_k, then verify that the limiting distribution is defined as in Eqn.(6.1) where $\gamma_k = 1$ and $\gamma_m = 0$, for $m \neq k$.

      2. If $X_0 = i$ and i is in a nonclosed class, then verify that the limiting distribution is defined as in Eqn.(6.1) where $\gamma_k = \alpha_i^{(k)}$ for $k = 1, 2, \dots r$ where $\alpha_i^{(k)}$ is the probability of absorption in class C_k. More precisely,

      $$\pi_j = \begin{cases} \pi_j^{(k)}\alpha_i^{(k)} & \text{for } j \in C_k,\ k = 1, \dots, r \\ 0 & \text{if } j \text{ is in a nonclosed class.} \end{cases}$$

      3. If X_0 is random, then this will leave as extra exercise. (Hint: you may need to apply first step anslysis)

6. A no-claims discount system for motor insurance has four levels of discount:

Level	1	2	3	4
Discount	0%	10%	30%	50%

The rules for moving between these levels are given as follows:

- Following a claim-free year, move to the next higher level, or remain at level 4.
- Following a year with one claim, move to the next lower level, or remain at level 1.
- Following a year with two or more claims, move down two levels, or move to level 1 (from level 2), or remain at level 1.

A portfolio consists of 10,000 policyholders. Suppose also that the number of claims per year is $\mathcal{Poisson}(0.1)$.

1. Calculate $\Pr[N = 0]$, $\Pr[N = 1]$, and $\Pr[N \geq 2]$ for each group.
2. Write down the transition probability matrix of this no-claims discount system.
3. Find the probability that a policyholder who has the 30% discount has no discount after 2 years.
4. Calculate the expected number of policyholders at each level at times 1 and 2, assuming no exits.
5. Calculate the expected number of policyholders at each level once stability has been achieved, assuming no exits.

7. A no-claims discount system for motor insurance has four levels of discount:

Level	1	2	3	4
Discount	0%	20%	30%	50%

The rules for moving between these levels are given as follows:

- For a claim-free year, a policyholder moves to the next higher level, or remains at level 4.
- For every claim in a year, the policyholder moves down a discount level or remains at level 1, for example if the policyholder is in level 4 and has one accident, he/she moves to level 3, and 2 accidents, he/she moves to level 2, and 2 or more accidents to level 1.

For a given policyholder, the number of claims each year, N, has a negative binomial distribution with parameters $k = 2$ and $p = 0.5$.

Note that a random variable N has a negative distribution with parameters k and p, denoted by $N \sim \mathcal{NB}(k, p)$ if its probability mass function is given by

$$f_N(n) = \Pr(N = n) = \frac{\Gamma(k+n)}{\Gamma(n+1)\Gamma(k)} p^k (1-p)^n \quad n = 0, 1, 2, \ldots .$$

1. Draw a transition diagram for the chain.
2. Write down the transition matrix of this no-claims discount system.
3. Find the probability that a policyholder who has the maximum discount level will have 20% discount after two years.

6.4 Tutorial 4

1. Customers arrive in a shop according to a Poisson process of rate $\lambda = 2$. Let $N(t)$ be the number of customers that have arrived up to time t. Determine the following probabilities, conditional probabilities

and expectations.

1. $\Pr(N(1) = 2)$.
2. $\Pr(N(1) = 2 \text{ and } N(3) = 6)$.
3. $\Pr(N(1) = 2 | N(3) = 6)$.
4. $\Pr(N(3) = 6 | N(1) = 2)$.
5. $\Pr(N(1) \leq 2)$.
6. $\Pr(N(1) = 1 \text{ and } N(2) = 3)$.
7. $\Pr(N(1) \geq 2 | N(1) \geq 1)$.
8. $\mathrm{E}[N(2)]$
9. $\mathrm{E}[N(1)^2]$
10. $\mathrm{E}[N(1)N(2)]$

2. Customers arrive in a shop according to a Poisson process of rate $\lambda = 4$ per hour. The shop opens at 9 am. Calculate the probability that exactly one customer has arrived by 9.30 am and a total of five customers have arrived by 11.30.

3. Defects occur along a cable according to a Poisson process of rate $\lambda = 0.1$ per kilometre.

   1. Calculate the probability that no defects appear in the first two kilometres of cable.
   2. Given that there are no defects in the first two kilometres of cable, calculate the probability of no defects between two and three kilometres of cable.

4. Customers arrive at a department store according to a Poisson process with rate $\lambda = 2$ per minute.

   1. Calculate the probability that in a given 5 minute period there will be no customers arriving?
   2. Calculate the probability that the 10th customer after 11 am will arrive before 11:05 am?
   3. If a third of customers are men, calculate the probability that in a 5 minute period more than 3 men arrive given more than 4 women arrive.
   4. If every 5th customer receives a discount voucher, calculate the distribution of the times between these vouchers being given out? What is the probability that a time longer than 5 minutes will pass between one voucher being given out and the next?

5. You have a bird table in your garden which attracts tailorbirds and pigeons. Tailorbirds arrive according to a Poisson process with rate λ_1 and the pigeons arrive according to a Poisson process with rate λ_2.

   1. How long does it take for the first bird to arrive after a fixed point in time?
   2. Calculate the probability this bird is a tailorbird?
   3. What is the distribution of the number of birds in the time interval $[t_1, t_2)$?
   4. Calculate the probability that exactly 3 tailorbirds arrive before the first pigeon after a fixed point in time?

6. Let X and Y be independent Poisson distributed random variables with parameters λ_X and λ_Y, respectively. Determine the conditional distribution of X, given that $N = X + Y = n$.

7. Accidents occur on an highway according to a Poisson process at the rate of 20 accidents per week. One out of four accidents involve speeding.

   1. What is the probability that ten accidents involving speeding will occur next week?
   2. What is the probability that at least one accident occurs tomorrow?

3. If sixty accidents occur in four weeks, what is the probability that less than half of them involve speeding?

8. Severe floods hit a southern of Thailand according to a Poisson process with $\lambda = 4$. The number of insurance claims filed after any sever flood has a Poisson distribution with mean 60. The number of server floods is independent of the number of insurance claims. Find the expectation and standard deviation of the total number of claims filed by time t.

9. Assume that births occur at a hospital at the average rate of 3 births per hour. Assume that the probability that any birth is a boy is 0.52.

   1. On an 8-hour shift, what is the expectation and standard deviation of the number of male births?
   2. Assume that ten babies were born yesterday. Find the probability that six are boys.
   3. Find the probability that only boys were born between 6 and 10 a.m.

Solution:

1. Customers arrive in a shop according to a Poisson process of rate $\lambda = 2$. Let $N(t)$ be the number of customers that have arrived up to time t. It follows that $N(t)$ has a Poisson distribution with parameter λt. In addition, $N(t+s) - N(s)$ is a Poisson random variable with mean λt , independent of anything that has occurred before time s.

   1. $\Pr(N(1) = 2) = \frac{\exp(-(2)(1))((2)(1))^2}{(2!)} = 0.2706706$.
   2. $$\begin{aligned}\Pr(N(1) = 2 \text{ and } N(3) = 6) &= \Pr(N(1) = 2 \text{ and } N(3) - N(1) = 4)\\ &= \Pr(N(1) = 2)\Pr(N(3) - N(1) = 4)\\ &= \frac{\exp(-(2)(1))((2)(1))^2}{(2!)}\frac{\exp(-(2)(2))((2)(2))^4}{(4!)} = (0.2706706)(0.1953668)\\ &= 0.05288.\end{aligned}$$
   Here we have used the fact that $N(3) - N(1)$ and $N(1)$ are independent.
   3. $$\Pr(N(1) = 2|N(3) = 6) = \frac{\Pr(N(1) = 2 \text{ and } N(3) = 6)}{\Pr(N(3) = 6)}$$ $$\frac{0.05288}{\frac{\exp(-(2)(3))((2)(3))^6}{(6!)}}$$ $$= 0.3292181.$$
   4. $$\begin{aligned}\Pr(N(3) = 6|N(1) = 2) &= \Pr(N(3) - N(1) = 4|N(1) = 2)\\ &= \Pr(N(3) - N(1) = 4)\frac{\exp(-(2)(2))((2)(2))^4}{(4!)} = 0.1953668.\end{aligned}$$
   Here we have used the fact that $N(3) - N(1)$ and $N(1)$ are independent.
   5. $$\Pr(N(1) \le 2) = 0.6766764.$$
   6. $$\begin{aligned}\Pr(N(1) = 1 \text{ and } N(2) = 3) &= \Pr(N(1) = 1 \text{ and } N(2) - N(1) = 2)\\ &= \Pr(N(1) = 1)\Pr(N(2) - N(1) = 2)\\ &= \frac{\exp(-(2)(1))((2)(1))^1}{(1!)}\frac{\exp(-(2)(1))((2)(1))^2}{(2!)} = (0.2706706)(0.2706706)\\ &= 0.0732626.\end{aligned}$$

Here we have used the fact that $N(2) - N(1)$ and $N(1)$ are independent.

7.
$$\begin{aligned}\Pr(N(1) \geq 2 | N(1) \geq 1) &= \frac{\Pr(N(1) \geq 2 \text{ and } N(1) \geq 1)}{\Pr(N(1) \geq 1)} \\ &= \frac{\Pr(N(1) \geq 2)}{\Pr(N(1) \geq 1)} \\ &= \frac{1 - \Pr(N(1) \leq 1)}{1 - \Pr(N(1) = 0)} \\ &= \frac{0.5939942}{0.8646647} \\ &= 0.5136058.\end{aligned}$$

8.
$$\mathrm{E}[N(2)] = 4.$$

9.
$$\mathrm{E}[N(1)^2] = \mathrm{Var}[N(1)] + (\mathrm{E}[N(1)])^2 = 6.$$

10.
$$\begin{aligned}\mathrm{E}[N(1)N(2)] &= \mathrm{E}[N(1) \cdot (N(1) + (N(2) - N(1))] \\ &= \mathrm{E}[N(1)^2] + \mathrm{E}[(N(1) \cdot (N(2) - N(1))] \\ &= \mathrm{E}[N(1)^2] + \mathrm{E}[N(1)] \cdot \mathrm{E}[N(2) - N(1)] \\ &= 6 + (2)(2) = 10.\end{aligned}$$

Here we have used the fact that $N(2) - N(1)$ and $N(1)$ are independent.

2. Let $N(t)$ be the number of customers that have arrived up to time t.

$$\begin{aligned}\Pr(N(1/2) = 1 \text{ and } N(5/2) = 5) &= \Pr(N(1/2) = 1 \text{ and } N(5/2) - N(1/2) = 4) \\ &= \Pr(N(1/2) = 1)\Pr(N(5/2) - N(1/2) = 4) \\ &= \frac{\exp(-(4)(0.5))((4)(0.5))^1}{(1!)} \frac{\exp(-(4)(2))((4)(2))^4}{(4!)} = (0.2706706)(0.0572523) \\ &= 0.0154965.\end{aligned}$$

Here we have used the fact that $N(5/2) - N(1/2)$ and $N(1/2)$ are independent.

3. Let $X(t)$ be the number of defects of cable of length t. We have $X(t) \sim \text{Poisson}(\lambda t)$.

   1. We have

$$\Pr(X(2) = 0) = \frac{\exp(-(0.1)(2))((0.1)(2))^0}{(0!)} = 0.8187308.$$

2. Using the independent of $X(2)$ and $X(3) - X(2)$, this implies that the conditional probability is the same as the unconditional probability. Therefore, we have

$$\begin{aligned}\Pr(X(3) - X(2) = 0 | X(2) = 0) &= \Pr(X(3) - X(2) = 0) \\ &= \frac{\exp(-(0.1)(1))((0.1)(1))^0}{(0!)} \\ &= 0.9048374.\end{aligned}$$

4. Again, let $N(t)$ be the number of customers that have arrived up to time t. We have $N(t) \sim \text{Poisson}(\lambda t)$ with $\lambda = 2$.

```
1.  $\Pr(N(5) = 0) = \frac{\exp(-(2)(5))((2)(5))^{0}  }{(0!)} = \ensuremath{4.539993\times 10^{-5}}$
```

```
2. The 10th customer will arrive after 11:00 am but before 11:05 am is the same as saying that at l
```

$$\begin{aligned}\Pr(N(t, t+5) \geq 10) &= 1 - \Pr(N(t, t+5) \leq 9)\\ &= 1 - 0.4579297\\ &= 0.5420703.\end{aligned}$$

```
3.  Given that a third of customers are men, the probability that a customer is male is 1/3, which 
```

$$\lambda_m = (0.3333333)(2) = 0.6666667, \quad \lambda_f = (0.6666667)(2) = 1.3333333.$$

It follows that the required probability can be founded as follows:

$$\begin{aligned}\Pr(N_m(5) \geq 4 | N_f(5) \geq 5) &= \Pr(N_m(5) \geq 4)\\ &= 1 - \Pr(N_m(5) \leq 3)\\ &= 0.427014.\end{aligned}$$

4. Each interarrival time has exponential distribution Exp(2). The sum of 5 independent exponentially distribution random variables, denoted by Z, has gamma distribution $\mathcal{G}(5, 2)$ The probability that a time longer than 5 minutes will pass between one voucher being given out and the next is

$$\Pr(Z > 5) = \Pr(N(5) \leq 4) = 0.0292527.$$

Here we use $N(5) \sim \text{Poisson}(5 \times 2)$. Alternatively, the following command in R produces the required result `pgamma(5, shape = 5, rate = 2, lower.tail = FALSE)`.

5. The solutions are given as follows:

```
1. The sum of two independent Poisson processes is also a Poisson process with rate
```

$\lambda_1 + \lambda_2$. Then the first event of this process (i.e. the time that the first bird arrives) occurs at random time having Exp$(\lambda_1 + \lambda_2)$ distribution.

```
2. According to the superposition theorem, the probability that this bird is a tailorbird is
```

$$\frac{\lambda_1}{\lambda_1 + \lambda_2}.$$

```
3. The process is homogeneous in time, i.e. only the lenght of the  interval matters. So the distri
```

$[t_1, t_2)$ is a Poisson distribution with parameter $(\lambda_1 + \lambda_2)(t_2 - t_1)$.

```
4. The required probability is
```

$$\Pr(\text{TB}, \text{TB}, \text{TB}, \text{P}) = p^3(1 - p),$$

where $p = \frac{\lambda_1}{\lambda_1 + \lambda_2}$, which is the probability that a bird is a tailorbird.

6. X and Y are independent Poisson distributed random variables with paramers λ_X and λ_Y. The conditional distribution of X given $N = X + Y = n$ is

$$\begin{aligned}
\Pr(X = k|N = n) &= \Pr(X = k|X + Y = n) \\
&= \Pr(X = k, Y = n - k|X + Y = n) \\
&= \frac{\left(\frac{e^{-\lambda_X}\lambda_X^k}{k!}\right) \times \left(\frac{e^{-\lambda_Y}\lambda_Y^{n-k}}{(n-k)!}\right)}{\left(\frac{e^{-(\lambda_X+\lambda_Y)}(\lambda_X+\lambda_Y)^n}{n!}\right)} \\
&= \frac{n!}{k!(n-k)!}\left(\frac{\lambda_X}{\lambda_X+\lambda_Y}\right)^k \left(\frac{\lambda_Y}{\lambda_X+\lambda_Y}\right)^{n-k}.
\end{aligned}$$

Hence this conditional distribution is a bionomial distribution with parameters n and $\frac{\lambda_X}{\lambda_X+\lambda_Y}$.

7. Let $N(t)$ be the number of accidents occurring on the highway and $N_S(t)$ be the process of speeding-related accidents.

   `1.`

$$\Pr(N_S(1) = 10) = \frac{\exp(-(20)(0.25))((20)(0.25))^{10}}{(10!)} = 0.0181328.$$

   `2. The probability that at least one accident occurs tomorrow is`

$$1 - e^{-20/7} = 0.9425674.$$

   3. There are $n = 60$ accidents occurring in four weeks ($t = 4$ weeks). The number of accidents involve speeding N_S in $[0, t]$ has a binomial distribution with parameters n and $p = 1/4$.

   `The required probability is`

$$\begin{aligned}
\Pr(N_S(4) < 30|N(4) = 60) &= \sum_{k=0}^{29}\binom{60}{k}\left(\frac{1}{4}\right)^k\left(\frac{3}{4}\right)^{60-k} \\
&= 0.9999733.
\end{aligned}$$

   The probability above can be calculated using the following command in R `pbinom(29,60,1/4)`.

8. Let $X_1, X_2, ...$$ be an i.i.d. sequence where X_i is the number of claims filed after the ith flood. Let T denote the total number of claims filed. Then

$$T = X_1 + X_2 + ... X_{N_t},$$

   where N_t is the number of severe floods that occur by time t. Using results for random sums of random variables,

$$\mathrm{E}(T) = \mathrm{E}(N_t)\mathrm{E}(X_1) = (4t)(60) = 240t,$$

   and

$$\begin{aligned}
\mathrm{Var}(T) &= \mathrm{Var}(X_1)\mathrm{E}(N_t) + [\mathrm{E}(X_1)]^2\mathrm{Var}(N_t) = (60)(4t) + (60)^2(4t) \\
&= 14640t,
\end{aligned}$$

   and, hence, $SD(T) = \sqrt{14640t}$.

Note that the distribution of T is said to have a compound Poisson distribution. For more details about how to compute $\mathrm{E}(T)$ and $\mathrm{Var}(T)$, please refer to the following website: https://pairote-sat.github.io/SCMA470/collective-risk-model.html#compound-poisson-distributions.

9. Let $(N_t)_{t\geq 0}, (M_t)_{t\geq 0}$ and $(F_t)_{t\geq 0}$ denote the overall birth, male, and female processes, respectively.

   `1. Male births form a Poisson process with parameter`

$$\lambda \cdot p = 3(0.52) = 1.56.$$

The number of male births on an 8-hour shift M_8 has a Poisson distribution with expectation

$$E(M_8) = \lambda \cdot p \cdot 8 = 12.48.$$

and standard deviation

$$SD(M_8) = \sqrt{12.48} = 3.5327043.$$

```
2. Conditional on there being five births in a given interval,the number of boys in that interval ha
```

$n = 10$ and $p = 0.52$. The desired probability is

$$\frac{10!}{6!4!}(0.52)^6(1-0.52)^4 = 0.2203963$$

```
3. The required probability is
```

$$\Pr(M_4 > 0, F_4 > 0).$$

By independence,

$$\begin{aligned}\Pr(M_4 > 0, F_4 = 0) &= \Pr(M_4 > 0)\Pr(F_4 = 0)\\ &= (1 - e^{-3(0.52)(4)})(e^{-3(1-0.52)(4)})\\ &= 0.003145.\end{aligned}$$

6.5 Tutorial 5

1. Suppose that we observe life i at exact age 74 years and 3 months. The observation will continue until the earlier of the life's 75th birthday or death. Assume that the force of mortality equal 0.08.
    1. Calculate the probability function of D_i, i.e. calculate $\Pr(D_i = 0)$ and $\Pr(D_i = 1)$.
    2. Calculate $\mathrm{E}[D_i]$.
    3. Calculate the probability density/mass function of V_i (Hint: consider two cases (i) when $v_i < 0.75$ and (ii) when $v_i = 0.75$).
    4. Calculate $\mathrm{E}[V_i]$.
2. For life i, recall that
    - $x + a_i$ is the age at which observation begins, $\quad 0 \leq a_i < 1$.
    - $x + b_i$ is the age at which observation ends, if life does not die, $\quad 0 \leq b_i < 1$.

    The terms a_i and b_i are known constants.
    1. Show that $\mathrm{E}[D_i] = \displaystyle\int_0^{b_i - a_i} e^{-\mu t}\mu \,\mathrm{d}t$.
    2. Show that $\mathrm{E}[V_i] = \displaystyle\int_0^{b_i - a_i} te^{-\mu t}\mu \,\mathrm{d}t + (b_i - a_i)e^{-\mu(b_i - a_i)}$.
3. In terms of the probability function of (D_i, V_i),

1. explain why the following expression holds:

$$\int_0^{b_i-a_i} e^{-\mu v_i}\mu\,\mathrm{d}v_i + e^{-\mu(b_i-a_i)} = 1.$$

2. Differentiating the above expression with respect to μ, show that

$$\mathrm{E}[D_i - \mu V_i] = 0.$$

3. Differentiating the above expression twice with respect to μ, show that

$$\mathrm{Var}[D_i - \mu V_i] = \mathrm{E}[D_i].$$

4. Show that

$$\mathrm{Var}[\widehat{\mu}] \to \left.\frac{\mu^2}{\mathrm{E}[D]}\right|_{\mu=\mu_0},$$

and hence

$$\mathrm{Var}[\widehat{\mu}] \to \left.\frac{\mu}{\mathrm{E}[V]}\right|_{\mu=\mu_0} = \frac{\mu_0}{E[V]}.$$

5. 1300 lives aged between 70 and 71 have been observed. We wish to calculate the force of mortality over this period.

   Suppose the true value of the force of mortality is 0.12 for lives aged between 70 and 71. Calculate the probability that the observed force of mortality is greater than 0.15.

   Hint: use the fact that the estimate $\widehat{\mu}$ is asymptotically normal

$$\widehat{\mu} \approx \mathcal{N}(\mu_0, \frac{\mu_0}{E[V]}).$$

6. Consider the following mortality data on ten lives all aged between 75 and 76.

Life	a_i	b_i	d_i	t_i
1	0	1	1	0.5
2	0	1	1	0.75
3	0	1	0	1
4	0	1	0	1
5	0	1	0	1
6	0.1	0.6	1	0.5
7	0.2	0.7	1	0.6
8	0.2	0.4	0	0.4
9	0.5	0.8	0	0.8
10	0.5	1	0	1

1. Using the Markov model, estimate the force of mortality μ_{75} assuming that it is constant from 75 to 76.
2. Estimate the variance of $\widehat{\mu}$.
3. Construct the 95% confidence interval for the force of mortality.
4. Estimate $\widehat{q}_{75} = {}_1\widehat{q}_{75}$, the probability of a life aged (75) dying within one year.
5. Use the $\Delta-$method to estimate the variance of $\widehat{q}_{75}$.

6.6 Tutorial 6

1. Consider the 3-state model of terminal illness for healthy, ill and dead states as shown below.

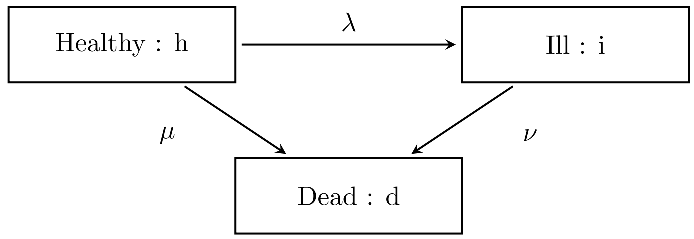


Figure 6.1: The 3-state model of terminal illness

`1. What can you say about`

$p_{hh}(t)$ and $p_{\overline{hh}}(t)$, and about $p_{ii}(t)$ and $p_{\overline{ii}}(t)$?

```
2. Write down the Kolmogorov forward differential equations (FDE)
   for
```

$\frac{d}{dt}p_{hh}(t)$, $\frac{d}{dt}p_{ii}(t)$ and solve these equations.

```
3. Write down the Kolmogorov forward differential equations (FDE)
   for
```

$\frac{d}{dt}p_{hi}(t)$, $\frac{d}{dt}p_{hd}(t)$ and explain how to solve these equations.

2. Consider a model of the mortality of two lives (husband and wife) consisting of for states :

- $b =$ both lives are alive,
- $u =$ husband is alive, but wife is dead,
- $v =$ wife is alive, but husband is dead, and
- $d =$ both are dead.

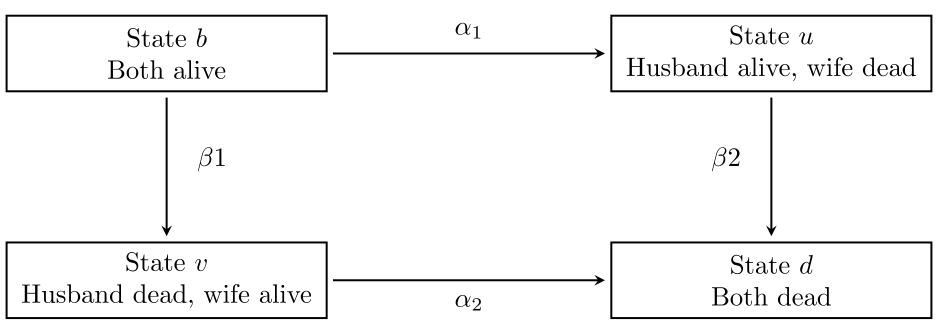


Figure 6.2: A model of the mortality of two lives

```
The model is also referred to as the joint life and last survivor
model. Write down the Kolmogorov equation for
```

$\frac{d}{dt}p_{bu}(t)$, $\frac{d}{dt}p_{bv}(t)$ and $\frac{d}{dt}p_{bd}(t)$.

3. A group of lives who hold health insurance policies can be classified into able (a), ill (i), dead (d) and withdrawn (w). The lives can move between state according to the following diagram:

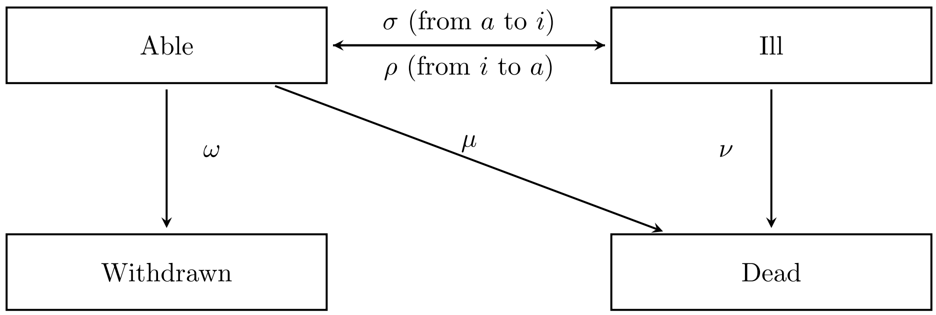


Figure 6.3: A Markov model for health insurance policies

```
Assuming that all transition rates are constant, write down the
Kolmogorov forward differential equations (FDE) for
```

$p_{aa}(t), p_{ai}(t)$ and $p_{aw}(t)$.

4. An actuary wishes to study a model in which states relate to marital status comprising of five states, single, married, divorced, widowed and dead. Draw the transition diagram that illustrate the possible transitions between these five states.ß

Chapter 7

Applications

Some *significant* applications are demonstrated in this chapter.

7.1 DataCamp Light

By default, `tutorial` will convert all R chunks.

eyJsYW5ndWFnZSI6InIiLCJzYW1wbGUiOiJhIDwtIDJcbmIgPC0gM1xuXG5hICsgYiJ9

Chapter 8

Final words

Some *significant* applications are demonstrated in this chapter.

8.1 DataCamp Light

By default, `tutorial` will convert all R chunks.

eyJsYW5ndWFnZSI6InIiLCJzYW1wbGUiOiJhIDwtIDJcbmIgPC0gM1xuYSArIGIifQ==