# Chapter 5

# Ruin Theory

## 5.1 The classical risk process

Short term risk models for a fixed time period have been studied in the previous sections. In this section, risk models that evolve over time will be presented. Suppose that an insurer

- begins with an initial capital $u$, called an initial surplus,

- collects premiums at a constant rate $c$ per unit time,
- and pays claims when losses occur.

The insurer is in ruin if the insurer's capital becomes negative at some point in time, i.e. the insurer's surplus falls to zero or below.

**Note** A surplus is an excess of income or assets over expenditure or liabilities in a given period, typically a financial year:

**Example 5.1.** *An insurer has initial surplus $u$ of 1 (in suitable units) and receives premium payments at a rate of 1 per year. Suppose claims from a portfolio of insurance over the first two years are as follows:*

| *Time (years)* | *0.4* | *0.9* | *1.5* |
|---|---|---|---|
| *Amount* | *0.8* | *0.7* | *1.2* |

*Plot a surplus process and determine whether ruin occurs within the first three years.*

**Example 5.2.** *As given in Example 5.1, suppose that the insurer has effected proportional reinsurance with retained proportion of 0.7. The reinsurance premium is 0.4 per year to be paid continuously. Plot a surplus process and determine whether ruin occurs within the first three years. Comment on the results.*

### 5.1.1 Classical risk process

The following assumptions are assumed for the study of the evolution of insurer's surplus over time.

1. The insurer's initial capital is $u$.
2. The premium rate per unit of time received continuously is $c$, i.e. the total amount of premiums received by time $t$ is $ct$.
3. The counting process $\{N(t)\}_{t\geq 0}$ for the number of claims occurred in the time interval $[0, t]$ is a Poisson process with parameter $\lambda$.
4. The claim sizes (or individual claim amounts) $X_1, X_2, \ldots$ are independent

and identically distributed random variables.

5. The claim sizes $X_1, X_2, \ldots$ are independent of the counting process $N(t)$.

The **surplus process** $\{U(t)\}_{t\geq 0}$ is then given by

$$U(t) = u + ct - S(t), \tag{5.1}$$

where the **aggregate claim amount up to time** $t$, $S(t)$ is

$$S(t) = \sum_{i=1}^{N(t)} X_i. \tag{5.2}$$

## Notes

1. The evolution of insurer's surplus defined in (5.1) is also known as the **classical risk process**.

2. The only random and uncertain quantity in (5.1) is the aggregate claims $S(t)$.

### 5.1.2 Poisson processes

A **Poisson process** is a special type of counting process. It can be represented by a continuous time stochastic process $\{N(t)\}_{t\geq 0}$ which takes values in the non-negative integers. It can be used to model the occurrence or arrival of events over a continuous time interval. The state space is discrete but the time set is continuous. Here $N(t)$ represents the number of events in the interval $(0, t]$.

The following examples can also be modelled by a Poisson process:

1. Claims arrivals at an insurance company,
2. Accidents occurring on the highway, and
3. Telephone calls to a call centre.

#### Counting Process

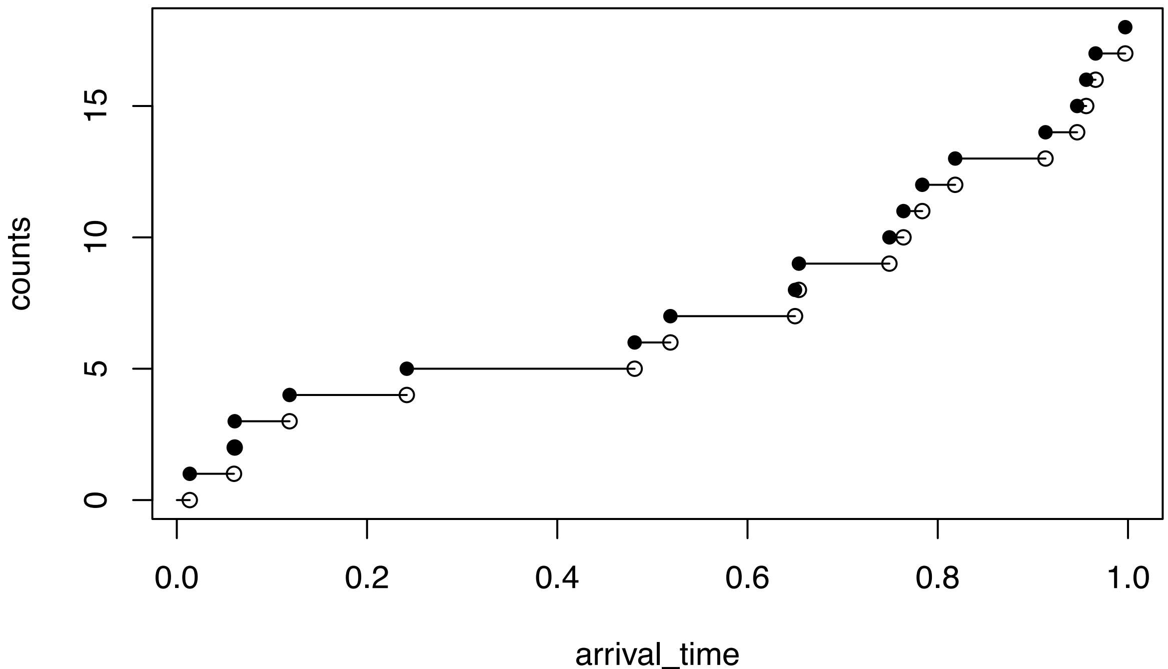
A counting process $\{N_t\}_{t\geq 0}$ is a collection of non-negative, integer-valued random variables such that if $0 \leq s \leq t$, then $N(s) \leq N(t)$.

The following figure illustrates a trajectory of the Poisson process. The sample path of a Poisson process is a right-continuous step function. There are jumps occurring at time $t_1, t_2, t_3, \ldots$.

```
lambda <- 17
# the length of time horizon for the simulation T_length <- 31
last_arrival <- 0
arrival_time <- c()
inter_arrival <- rexp(1, rate = lambda)
T_length <- 1
while (inter_arrival + last_arrival < T_length) {
  last_arrival <- inter_arrival + last_arrival
  arrival_time <- c(arrival_time,last_arrival)
  inter_arrival <- rexp(1, rate = lambda)
}


n <- length(arrival_time)
counts <- 1:n
```

```r
plot(arrival_time, counts, pch=16, ylim=c(0, n))
points(arrival_time, c(0, counts[-n]))
segments(
  x0 = c(0, arrival_time[-n]),
  y0 = c(0, counts[-n]),
  x1 = arrival_time,
  y1 = c(0, counts[-n])
)
```




Recall that a stochastic process $\{N(t)\}_{t\geq 0}$ is a Poisson process with parameter $\lambda$ if the process satisfies the three properties:

1. $N(0) = 0$.

2. **Independent increments** For $0 < s < t \leq u < v$, the increment $N(t) - N(s)$ is independent of the increment $N(v) - N(u)$, i.e. the number of events in $(s, t]$ is independent of the number of events in $(u, v]$.

3. **Stationary increments** For $0 < s < t$, the distribution $N(t) - N(s)$ depends only on $t - s$ and not on the values $s$ and $t$, i.e. the increments of the process over time has a distribution that only depend on the time difference $t - s$, the lenght of the time interval.

4. **Poisson distribution** For $t \geq 0$, the random variable $N(t)$ has a Poisson distribution with mean $\lambda t$.

It follows from conditions the **Stationary Increments** and **Poisson Distribution** properties that

$$\Pr(N(t)-N(s) = n) = \Pr(N(t-s)-N(0) = n) = \frac{(\lambda(t-s))^n e^{-\lambda(t-s)}}{n!}, \qquad s < t,\, n =$$

**Notes**

1. The sample paths of $\{N(t)\}_{t\geq 0}$ are non-decreasing step functions, or the process is referred to be as a counting process.

2. A process with stationary and independent increments can be thought of as **starting over** at any point in time in a probabilistic sense. The 'starting over' property follows from the fact that

- the exponential distribution has the memoryless property, and
- the times between successive events (or interarrival times) are independent and identically distributed exponential random variables with mean $1/\lambda$. This follows from the following result:

3. For more details about Poisson processes, please refer to the contents of the course "SCMA 469 Actuarial Statistics"

### 5.1.3 Compound Poisson processes

The aggregate claims process $S(t)$ defined in (5.2) of the classical risk process is said to be a **compound Poisson process** with Poisson parameter $\lambda$. The compound Poisson process has the following important properties:

1. For each $t$, the random variable $S(t)$ has a compound Poisson distribution with parameter $\lambda t$, i.e.

$$S(t) \sim \mathcal{CP}(\lambda t, F_X(x)).$$

Thus, the mean and variance of the compound Poisson distribution are

$$\mathrm{E}[S(t)] = \lambda t \mathrm{E}[X], \quad \mathrm{Var}[S(t)] = \lambda t \mathrm{E}[X^2].$$

The moment generating function of $S(t)$ is

$$M_{S(t)}(r) = \exp(\lambda t(M_X(r) - 1)).$$

2. It has stationary and independent increments, i.e. for disjoint time intervals $0 < s < t \leq u < v$, the random variables $N(t) - N(s)$ and $N(v) - N(u)$ are independent and have $\mathcal{CP}(\lambda(t-s), F_X(x))$ and $\mathcal{CP}(\lambda(v-u), F_X(x))$ distributions, respectively.

**Notes** Various properties of the aggregate claims process $S(t)$ can be summarised as follows:

1. $S(1) \sim \mathcal{CP}(\lambda, F_X(x))$ is the aggregate claims in the first year.
2. $S(n) - S(n-1) \sim \mathcal{CP}(\lambda, F_X(x))$ is the aggregate claims in the $n$th year, for $n = 1, 2, \ldots$.

3. The process $\{S(n) - S(n-1)\}_{n=1}^{\infty}$ is a sequence of **independent and identically distributed** random variables representing the aggregate claims in successive years.

### 5.1.4 The relative safety loading

Let $\mu_X$ and $\sigma_X^2$ denote the mean and the variance of claim sizes $X_i$.

**Example 5.3.** Consider the following questions.

1. Calculate the expected surplus and the variance surplus at time $t$.
2. Calculate the expected profit per unit time in $(0, t]$.

Based on the expected value premium principle, the premium rate $c$ per unit time can be defined by

$$c = (1 + \theta)\mathrm{E}[S(1)] = (1 + \theta)\lambda\mu_X.$$

Hence the **relative safety loading** (or **premium loading factor** or **relative security loading**) $\theta$ is given by

$$\theta = \frac{c - \lambda\mu_X}{\lambda\mu_X}.$$

## Notes

1. The insurer can make a profit provided that $c > \lambda\mu_X$ or the relative safety loading $\theta$ is positive. In this case, the surplus will drift to $\infty$, but ruin could still occur. The rate at which premium income comes in is greather than the rate at which claims are paid out.

2. On the other hand, if $c < \lambda\mu_X$, then the surplus will drift to $-\infty$, but ruin is certain.

### 5.1.5 Ruin probabilities

Various definitions of ruin probabilities are given.

1. The **probability of ruin in infinite time** (or the **ultimate ruin probability**) is defined by

$$\psi(u) = \Pr(U(t) < 0 \quad \text{for some } t > 0).$$

2. The **finite-time ruin probability** (or the **probability of ruin by time** $t$) is defined by

$$\psi(u,t) = \Pr(U(s) < 0 \quad \text{for some } s \in (0,t]).$$

3. The **discrete time ultimate ruin probability** is defined by

$$\psi_h(u) = \Pr(U(t) < 0 \quad \text{for some } t \in \{h, 2h, 3h, \ldots\}).$$

4. The **discrete time ruin probability in finite time** is defined by

$$\psi_h(u,t) = \Pr(U(s) < 0 \quad \text{for some } s \in \{h, 2h, 3h, \dots, t\}).$$

**Notes**

1. For $0 \le u_1 \le u_2$,
$$\psi(u_1) \ge \psi(u_2),$$
i.e. the ultimate ruin probability is non-increasing in $u$.

2. If ruin occurs under the discrete time, it must occur under the continuous time, i.e.
$$\psi_h(u) < \psi(u).$$
Similarly,
$$\psi_h(u,t) < \psi(u,t).$$

3. The discrete time ultimate ruin probability $\psi_h(u)$ could be used as an approximation of $\psi(u)$ provided $h$ is sufficiently small.

4. The discrete time ruin probability in finite time $\psi_h(u,t)$ could be used as an approximation of $\psi(u,t)$ provided $h$ is sufficiently small.

**Example 5.4.** *Suppose the annual aggregate claims for a portfolio of policies is approximately normal.*

- *The insurer's initial surplus is 1000 (in suitable units) and the premium rate is 1500 per year.*
- *The number of claims per year has a Poisson distribution with parameter 50.*
- *The distribution of claim sizes is lognormal with parameters $\mu = 3$ and $\sigma^2 = 0.9$.*

*Calculate the probability that the insurer's surplus at time 2 will be negative.*

## 5.2 Simulation of ruin probabilities

**Example 5.5.** *Let* $F(x)$ *be a continuous cumulative density function. Let* $Y$ *be a random variable with a* $U(0,1)$ *distribution. Define the random variable* $X$ *by*

$$X = F^{-1}(Y).$$

*Show that the cumulative density function of* $X$, $F_X(x)$ *is* $F(x)$.

**Example 5.6.** *The aggregate claims process for a risk is compound Poisson with Poisson parameter* $\lambda = 100$ *per year. Individual claim amounts have* $Pa(4, 3)$*. The premium income per year is* $c = 110$ *(in suitable units), received continuously.*

*Using either Excel or R to simulate 1000 values of aggregate claims* $S$*, assuming that* $S$ *is approximated by a translated gamma approximation,*

1. *Estimate* $\hat{\psi}_1(50, 5)$*, an estimate of* $\psi_1(50, 5)$*.*
2. *Estimate the standard error of* $\hat{\psi}_1(50, 5)$*.*
3. *Calculate a 95% confidence interval for your estimate in 1.*
4. *Estimate* $\psi_{0.5}(50, 5)$*.*