Homework 1 - Thomas Boyko - 30191728

1. (5 points) Define $f : \mathbb{R}^2 \to \mathbb{R}^2$

$$f(x,y) \begin{cases} 0 & (x,y) = (0,0) \\ \frac{xy^2}{x^2+y^2} & (x,y) \neq (0,0) \end{cases}.$$

(a) (1 point) Compute the partial derivatives for f at (0, 0).
Use the definition for directional derivatives:

$$\begin{aligned} f_x(x,y) = D_{e_1}f(0,0) &= \lim_{t\to 0} \frac{f(t,0) - f(0,0)}{t} \\ &= \lim_{t\to 0} \frac{\frac{0}{t^2} - 0}{t} \\ &= \lim_{t\to 0} \frac{0}{t} = 0 \end{aligned}$$

And for f_y:

$$\begin{aligned} f_y(x,y) = D_{e_2}f(0,0) &= \lim_{t\to 0} \frac{f(0,t) - f(0,0)}{t} \\ &= \lim_{t\to 0} \frac{\frac{0}{t^2} - 0}{t} \\ &= 0 \end{aligned}$$

So we see that $f_x(x,y) = f_y(x,y) = 0$.
Note for later that $Df(0,0) = \begin{bmatrix} 0 & 0 \end{bmatrix}$

(b) (2 point) Determine which directional derivatives exist for f at $(0,0)$ and compute their value.
Let $\vec{v} = \begin{bmatrix} v_1 & v_2 \end{bmatrix}, \vec{v} \in \mathbb{R}^2$ with $||\vec{v}|| = 1$. Note that $\sqrt{v_1^2 + v_2^2} = 1$ and $v_1^2 + v_2^2 = 1$.
Then the directional derivatives at $(0,0)$ will exist when the following limit exists.

$$\begin{aligned} D_{\vec{v}}f(x,y) &= \lim_{t\to 0} \frac{f(v_1t, v_2t) - f(0,0)}{t} \\ &= \lim_{t\to 0} \frac{\frac{v_1v_2^2t^3}{t^2(v_1^2+v_2^2)} - 0}{t} \\ &= \lim_{t\to 0} \frac{v_1v_2^2t}{t} \\ &= v_1v_2^2 \end{aligned}$$

So the directional derivative exists for any value of $\vec{v}$.

(c) (2 points) Is f differentiable at (0, 0)? Why or why not?
In order for f to be differentiable at $(0,0)$, the following limit must be 0, at any path to the origin.

$$\begin{aligned} \lim_{(x,y)\to(0,0)} \frac{f(x,y) - f(0,0) - \begin{bmatrix} 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}}{\sqrt{x^2 - y^2}} &= \lim_{(x,y)\to(0,0)} \frac{\frac{xy^2}{x^2+y^2}}{(x^2+y^2)^{\frac{1}{2}}} \\ &= \lim_{(x,y)\to(0,0)} \frac{xy^2}{(x^2+y^2)^{\frac{3}{2}}} \end{aligned}$$

Choose the path to the origin where $x = y$ and $x \to 0^+$.

$$\lim_{x\to 0^+}\frac{x^3}{(2x^2)^{\frac{3}{2}}} = \lim_{x\to 0^+}\frac{x^3}{2x^3}$$
$$= \frac{1}{2}$$

Since this limit does not equal 0 for our curve near the origin, the function f is not differentiable at $(0,0)$.

2. (4 points) Define $f(x,y,z) = e^{x^2+yz}$.

   (a) (1 point) Compute the gradient for f at the point $(1,1,-1)$.

$$\nabla f(x,y,z) = \begin{pmatrix} f_x \\ f_y \\ f_z \end{pmatrix} = \begin{pmatrix} 2xe^{x^2+yz} \\ ze^{x^2+yz} \\ ye^{x^2+yz} \end{pmatrix}$$

$$\nabla f(1,1,-1) = \begin{pmatrix} 2e^0 \\ -e^0 \\ 1e^0 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}$$

   (b) (1 point) Determine the maximal rate of ascent for f at $(1,1,-1)$. In which direction does this occur?
   The maximal rate of ascent will be given by $||\nabla f(1,1,-1)||$.

$$||\nabla f(1,1,-1)|| = \sqrt{2^2+1^2+(-1)^2} = \sqrt{6}$$

   The direction in which this occurs will be given by:

$$\frac{\nabla f}{||\nabla f||} = \frac{1}{\sqrt{6}}\begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}$$

   (c) (1 point) Compute the tangent plane for f at $(1,1,-1)$.

$$\vec{L}(1,1,-1) = f(1,1,-1) + \begin{pmatrix} 2 & -1 & 1 \end{pmatrix}\begin{pmatrix} x-1 \\ y-1 \\ z+1 \end{pmatrix}$$
$$= 1 + (2x-2) + (1-y) + z + 1$$
$$\vec{L}(1,1,-1) = 2x - y + z + 1$$

   (d) (1 point) Estimate the value $f(0.9, 1.1, -0.8)$ using part (c)

$$f(0.9,1.1,-0.8) \approx 2(0.9) - 1.1 - 0.8 + 1 = 0.9$$

3. (6 points)Define the function $\vec{f}(x,y) = \left(xy, x^2+y^2\right)^T on\mathbb{R}^2$.

   (a) (3 point) Determine on which set of points in $\mathbb{R}^2$ the hypotheses of the inverse function theorem is satisfied.
   Begin by finding $D\vec{f}(x,y)$.

$$D\vec{f}(x,y) = \begin{bmatrix} y & x \\ 2x & 2y \end{bmatrix}$$

   This matrix is invertible whenever $\det(D\vec{f}(x,y)) \neq 0$.

$$\det(D\vec{f}(x,y)) = 2y^2 - 2x^2$$

   So $2y^2 - 2x^2 \neq 0$.
   So $\vec{f}$ has a local inverse wherever $y \neq |x|$

(b) Apply the inverse function theorem to f at the point (2, 1) and use it to find $D\vec{g}(2,5)$ where g is a local inverse for f at the point $(2,1)$.

From above, $D\vec{f}(2,1) = \begin{bmatrix} 1 & 2 \\ 4 & 2 \end{bmatrix}$, and $\det(D\vec{f}(2,1)) = 2 - 8 = -6$.

So $D\vec{f}^{-1}(2,1) = D\vec{g}(2,5) = -\frac{1}{6}\begin{bmatrix} 2 & 2 \\ -4 & 1 \end{bmatrix}$

4. Consider the equations

$$\begin{aligned} x^2 + y + 3z + u + v + 4 &= 0 \\ xy^2 - z + u - v + 3 &= 0 \end{aligned}$$

around the point $(x,y,z,u,v) = (1,0,-1,-3,1)$. Let $\vec{F}(x,y,z,u,v) = (x^2+y+3z+u+v+4, xy^2 - z + u - v + 3)^T$. Use the implicit function theorem to show that there is a function $\vec{g} : U \to \mathbb{R}^2$ where U is an open set containing $(x,z,v) = (1,-1,1)$, so that $g_1(x,z,v) = y, g_2(x,z,v) = u$ and $\vec{F}(x, g_1(x,z,v), z, g_2(x,z,v), v) = (3,2)^T$. Find $D\vec{g}(1,2,1)$

First we must find $D\vec{F}_{(1,0,-1,-3,1)} = \begin{pmatrix} 2x & 1 & 3 & 1 & 1 \\ y^2 & 2xy & -1 & 1 & -1 \end{pmatrix}_{(1,0,-1,-3,1)}$

Evaluating at our chosen point we get $D\vec{F}_{(1,0,-1,-3,1)} = \begin{pmatrix} 2 & 1 & 3 & 1 & 1 \\ 0 & 0 & -1 & 1 & -1 \end{pmatrix}$.

If an implicit function exists then the matrix $B = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ will be invertible. Clearly this matrix is invertible since it is upper triangular and $\det B = 1$ which is the product of the main diagonal. And the inverse of B is the matrix $B^{-1} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$.

So now we can take the remaining columns of our large matrix and create the matrix $A = \begin{pmatrix} 2 & 3 & 1 \\ 0 & -1 & -1 \end{pmatrix}$. Finally we can compute the product

$$-B^{-1}A = -\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 2 & 3 & 1 \\ 0 & -1 & -1 \end{pmatrix} = \begin{pmatrix} -2 & -4 & -2 \\ 0 & 1 & 1 \end{pmatrix}$$

And by the implicit function theorem, $D\vec{g}(1,2,1) = \begin{pmatrix} -2 & -4 & -2 \\ 0 & 1 & 1 \end{pmatrix}$.