1. An $m \times n$ matrix is said to be a queen if the restriction of A to the orthogonal complement of its kernel is an isometry.

   (a) Show that A is a queen if and only if A^*A is an orthogonal projection.

   (b) Show that A is a queen if and only if $AA*$ is an orthogonal projection.

   (c) Show that a queen A is an isometry if and only if $\ker A = 0$.

   Solution: If $\ker A = \{0\}$, then $(\ker A)^{\perp} = V$, so the restriction of A to the orthogonal complement of its kernel is A restricted to its domain. Then A is an isometry on any vector.

   (d) Find an example of a 4×2 queen that has non-zero kernel. Be sure to prove it's a queen!

2.

   (a) Given a singular value decomposition $A = W\Sigma V^*$ of a square matrix A, construct a polar decomposition of A using W, V, Σ.

   (b) Using the method above, compute a polar decomposition for

$$A = \begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix}.$$

3. Find your favorite 4×2 matrix A of rank 2 and compute a singular value decomposition for A. All of the entries of A must be nonzero.

4. For an $m \times n$ matrix A, show that the set of nonzero eigenvalues for A^*A coincide with that of AA^*.

5. Suppose $A = W\Sigma V^*$ is a singular value decomposition for A. Show that the columns of W are eigenvectors for AA^*.