# Homework 4 - Thomas Boyko - 30191728

1. Consider the function $z^n$, where $n \in \mathbb{Z}$.

   (a) Calculate the integral
   $$\oint_\gamma z^n \, dz.$$
   $$\mathfrak{R}.$$
   Where $\gamma$ is the boundary of the pentagon whose vertices are the 5th roots of 1, traversed once clockwise.
   For $n \geq 0$, we know that $z^n$ is holomorphic in its domain $\mathbb{C}$, so $\gamma$ will be homotopic to a constant contour and the integral will be zero.
   For $n < -1$, we notice that our curve $\gamma$ is homotopic to the unit circle traveled clockwise, the integral over which will be the negative of the typical unit circle parameterization, call it $\delta$.
   $$\begin{aligned}\oint_\gamma z^n \, dz &= -\oint_\delta z^n \, dz \\ &= -\int_0^{2\pi} e^{nti} ie^{it} \, dt \\ &= -i \int_0^{2\pi} e^{(n+1)it)} \, dt \\ &= -i \left( \frac{e^{it(n+1)}}{n+1} \right)_0^{2\pi} \\ &= -i \left( \frac{e^{i2\pi(n+1)}}{n+1} - \frac{e^0}{n+1} \right) \\ &= -i \left( \frac{1}{n+1} - \frac{1}{n+1} \right) \\ &= 0.\end{aligned}$$
   We can see in our calculations that the primitive we found in this example is undefined for $n = -1$, which motivates a seperate case.
   For $n = -1$, we have the same homotopy and we can use the same parameterization; so we calculate
   $$\begin{aligned}\oint_\delta \frac{1}{z} \, dz &= -i \int_0^{2\pi} \frac{e^{it}}{e^{it}} \, dz \\ &= -i \int_0^{2\pi} 1 \, dz \\ &= -2\pi i.\end{aligned}$$
   So for $n = -1$, the integral is $-2\pi i$, and for any other $n \in \mathbb{Z}$, the integral is 0.

   (b) Calculate the integral of $z^n$ , where $n \in \mathbb{Z}$, on the first side of the same pentagon, originating from 1 and moving clockwise.
   We start by finding the necessary 5th roots of unity, which are given by $z_0 = 1$ and $z_1 = e^{\frac{8}{5}\pi i}$.
   First let's take the case $n \neq -1$, where our integral is not path-dependent. So we only depend on the endpoints, 1 and $e^{i\frac{5\pi}{8}}$, and a primitive for $z^n$ which is $\frac{z^{n+1}}{n+1}$ for $n \neq -1$.
   $$\int_\gamma z^n \, dz = F(e^{\frac{8\pi}{5}i}) - F(1) = \frac{1}{n+1} \left( e^{\frac{8\pi}{5}(n+1)} - 1 \right).$$
   For $n = -1$, we notice that though $\frac{1}{z}$ does not have a primitive in its domain, it does have a primitive in the domain of ln, $\mathbb{C} \setminus \mathbb{R}_{\leq 0}$ since this domain is simply connected. This primitive is given by $\ln z$; and so we can evaluate the integral:
   $$\int_\gamma \frac{1}{z} \, dz = \ln e^{\frac{8\pi}{5}i} - \ln 1 = \frac{8\pi}{5} i.$$

2. Calculate the integral

$$\oint_\gamma \frac{1}{z^2 - 4z + 3}\, dz.$$

where $\gamma$ is the circle centred on 0 and of radius 2, traversed once counterclockwise.

We can see singularities of this function at $z = 1, 3$. We also note that $\gamma$ is a positively oriented Jordan curve. Now if we let $f(z) = \frac{1}{z-3}$, we can modify our integral:

$$\begin{aligned}
\oint_\gamma \frac{1}{z^2 - 4z + 3}\, dz &= \oint_\gamma \frac{f(z)}{z-1}\, dx \\
&= \frac{2\pi i}{2\pi i} \oint_\gamma \frac{f(z)}{z-1}\, dz \\
&= 2\pi i f(1) \\
&= \frac{2\pi i}{1-3} \\
&= -\pi i.
\end{aligned}$$

So;

$$\oint_\gamma \frac{1}{z^2 - 4z + 3}\, dz = -\pi i.$$