

# 1 The Spectrum and its Topology

## Definition 1.1 (Spectrum of a Ring)

Let  $R$  be a ring. Define the spectrum of  $R$ ,

$$\text{Spec}(R) = \{\mathfrak{p} \trianglelefteq R : \mathfrak{p} \text{ is prime}\}.$$

## Example 1.1

Describe the spectrum  $\text{Spec}(\mathbb{Z}_n)$ .

## Definition 1.2 (Zarski Topology)

Upon  $\text{Spec}(R)$  we define a topology by letting our closed sets be of the form:

$$V(I) = \{\mathfrak{p} \in \text{Spec}(R) : I \subseteq \mathfrak{p}\}.$$

For an ideal  $I$ .

## Proposition 1.1

$V(I)$  induces a topology on  $\text{Spec}(R)$ .

1.  $\emptyset, \text{Spec}(R)$  are closed.
2.  $\bigcup_{i=1}^n V(I_i)$  is closed for any  $n \in \mathbb{N}$ .
3.  $\bigcap_{\alpha} V(I_i)$  for any  $\alpha \in \Delta$ , some indexing set.

**Proof:** 1. Take  $V(\{0\}) = \{\mathfrak{p} \trianglelefteq R \text{ prime} : \{0\} \subseteq \mathfrak{p}\}$ . But naturally, the additive identity is contained within every prime ideal (which must be an abelian group w.r.t +), so we have  $V(\{0\}) = \text{Spec}(R)$ , and  $\text{Spec}(R) \in \mathcal{T}$

Now take  $V(R) = \{\mathfrak{p} \trianglelefteq R \text{ prime} : R \subseteq \mathfrak{p}\}$ . But by definition, no prime ideal can contain  $R$ , so  $V(R) = \emptyset$  and  $\emptyset \in \mathcal{T}$

2. It is sufficient to show that  $V(I) \cup V(J) \in \mathcal{T}$  for any  $V(I), V(J) \in \mathcal{T}$ . Any finite union can be proven inductively using this result. We claim that  $V(I) \cup V(J) = V(IJ)$ .

Suppose  $\mathfrak{p} \in V(I)$  without loss of generality. Then  $\mathfrak{p}$  is a prime ideal containing  $I$ . But we have  $IJ \subseteq I \cap J \subseteq I \subseteq \mathfrak{p}$ , so  $\mathfrak{p} \in V(IJ)$ .

Conversely, suppose  $\mathfrak{p} \in V(IJ)$ . Then  $\mathfrak{p}$  is a prime ideal containing  $IJ$ . If  $J \subseteq \mathfrak{p}$ , we are done, so suppose that  $J \not\subseteq \mathfrak{p}$ . Then take  $i \in I, j \in J \setminus \mathfrak{p}$ . We know that  $ij \in IJ \subseteq \mathfrak{p}$ , and then since  $\mathfrak{p}$  is a prime ideal, either  $i$  or  $j$  must be in  $\mathfrak{p}$ . Since we supposed it was not  $j$ , we know it must be  $i$ . Therefore,  $I \subseteq \mathfrak{p}$ , and  $\mathfrak{p} \in V(I)$ .

3. We claim that  $\bigcap_{\alpha} V(I_{\alpha}) = V\left(\sum_{\alpha} I_{\alpha}\right)$ .

If  $\mathfrak{p} \in \bigcap_{\alpha} V(I_{\alpha})$ , then  $\mathfrak{p} \supseteq I_{\alpha}$  for all our  $\alpha$ . But  $\sum_{\alpha} I_{\alpha} \supseteq I_{\alpha}$  for any fixed  $\alpha$ , so  $\mathfrak{p} \supseteq \sum_{\alpha} I_{\alpha} \supseteq I_{\alpha}$ , and  $\mathfrak{p} \in V\left(\sum_{\alpha} I_{\alpha}\right)$ .

Conversely, if  $\mathfrak{p} \in V\left(\sum_{\alpha} I_{\alpha}\right)$ ,  $\mathfrak{p} \supseteq \sum_{\alpha} I_{\alpha}$ . Fix some  $\beta$  arbitrary in  $\Delta$ , then we know already  $\mathfrak{p} \supseteq \sum_{\alpha} I_{\beta} \supseteq I_{\beta}$ . So  $\mathfrak{p} \in V(I_{\beta})$  for any  $\beta$ , and  $\mathfrak{p} \in \bigcap_{\alpha} V(I_{\alpha})$ . ■

## Example 1.2

Describe the Zarski Topology on  $\mathbb{Z}_n$ .

**Solution:** We know already that  $\text{Spec}(\mathbb{Z}_n) = \{\mathbb{Z}_d : d|n\}$

## Proposition 1.2

If  $R$  is a ring, then the closed points of  $\text{Spec}(R)$  correspond to  $V(M) = \{M\}$ , for the ideals of  $R$ .

**Proof:** Recall closed points in  $\mathfrak{p} \in \operatorname{Spec}(R)$  are those for which  $\{\mathfrak{p}\}$  is closed.

Suppose  $M$  is some such point. Then for some  $I \trianglelefteq R$ , we have  $V(I) = \{M\}$ , and  $I \subseteq M \subseteq R$ . Suppose we have some  $J \trianglelefteq R$  so that  $M \subseteq J \subseteq R$ . Then  $I \subseteq J$ , and we would have to have  $J \in \{M\}$ . Therefore  $M$  is maximal. ■

**Proposition 1.3**

For any  $I \trianglelefteq R$ ,  $V(\sqrt{I}) = V(I)$