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1. 1

   We must find the bias of each estimator. Begin with $\hat{\lambda}_1$.

$$\begin{aligned} B(\hat{\lambda}_1) &= E\left[\frac{Y_1 + Y_2}{2}\right] - \lambda \\ &= \frac{E[Y_1] + E[Y_2]}{2} - \lambda \\ &= \lambda - \lambda \\ &= 0. \end{aligned}$$

   So $\hat{\lambda}_1$ is unbiased, so its MSE is simply its variance. Now for $\hat{\lambda}_2$.

$$\begin{aligned} B(\hat{\lambda}_2) &= E[\bar{Y}] - \lambda \\ &= \frac{1}{25} E[\sum_{i=1}^{25} Y_i] - \lambda \\ &= \frac{1}{25} \sum_{i=1}^{25} E[Y_i] - \lambda \\ &= E[Y_i] - \lambda \\ &= \lambda - \lambda \\ &= 0. \end{aligned}$$

   And the efficiency of these two random variables is simply the ratio of their variances.

$$\begin{aligned} eff(\hat{\lambda}_1, \hat{\lambda}_2) &= \frac{V\left[\frac{Y_1+Y_2}{2}\right]}{V[\bar{Y}]} \\ &= \frac{\frac{1}{4}V\left[Y_1 + Y_2\right]}{\frac{1}{25^2}V[\sum_{i=1}^{25} Y_i]} \\ &= \frac{\frac{\lambda}{2}}{\frac{1}{25}V[\sum_{i=1}^{25} Y_i]} && \text{By independence} \\ &= \frac{\frac{\lambda}{2}}{\frac{1}{25}\lambda} \\ &= \frac{25}{2}. \end{aligned}$$

   So $\hat{\lambda}_2$ is more efficient than $\hat{\lambda}_1$.

2. 2

3. 3

$$\begin{aligned} \frac{\sum_{i=1}^{n}(X_i - \bar{X})^2 + \sum_{i=1}^{n}(Y_i - \bar{Y}^2}{2(n-1)} &= \frac{(n-1)S_X^2 + (n-1)S_Y^2}{2(n-1)} \\ &= \frac{S_X^2 + S_Y^2}{2}. \end{aligned}$$

   And since we know $S_X^2 \to^p \sigma^2$ and $S_Y^2 \to^p \sigma^2$, using summation and constant properties of consistency we know that $\frac{S_X^2+S_Y^2}{2} \to^p \sigma^2$, and our above expression converges in probability to σ^2.

4. 4

5. 5

   We start with a probability statement. Let $\epsilon > 0$.

$$\begin{aligned} P(|Y_{(1)} - \beta| \leq \epsilon) &= P(\beta - \epsilon \leq Y_{(1)} \leq \beta + \epsilon) \\ &= F_{(Y_1}(\beta + \epsilon) - F_{(Y_1)}(\beta - \epsilon). \end{aligned}$$

We can see that the second term of this must be zero since $\beta - \epsilon < \beta$, ϵ is positive. So our expression becomes:

$$P(|Y_{(1)} - \beta| \leq \epsilon) = 1 - \left(\frac{\beta}{\beta + \epsilon}\right)^{\alpha n}.$$

And we apply the limit to find

$$\lim_{n\to\infty} P(|Y_{(1)} - \beta| \leq \epsilon) = \lim_{A\to\infty} 1 - \left(\frac{\beta}{\beta + \epsilon}\right)^{\alpha n} = 1 - 0 = 1.$$

(The second last equality comes from $\beta + \epsilon > \beta$, so the fraction is less than 1, and its limit tends to 0. So $Y_{(1)}$ is consistent for β

6. 6
7. 7
   (a) We find an MoM estimator for β, by equating our first sample and theoretical moments.

   $$\bar{X} = E[X] = \int_{\beta}^{\infty} xe^{\beta - x}\,dx = \beta + 1.$$

   So we have $\bar{X} = \beta + 1$, and this gives $\hat{\beta}_{MoM} = \beta + 1$.

   (b) We begin with our likelihood function:

   $$\begin{aligned} L(\beta) &= \prod_{i=1}^{n} e^{b - x_i} \\ &= e^{\beta n} e^{\sum_{i=1}^{n} x_i} \\ &= e^{\beta n} e^{-n\bar{x}}. \end{aligned}$$

   And from this our log-likelihood function:

   $$\begin{aligned} l(\beta) &= \ln(e^{\beta n - n\bar{x}} \\ &= n(\beta - \bar{x}). \end{aligned}$$

   Setting this equal to zero,

   $$\begin{aligned} 0 &= n(\beta - \bar{x}) \\ \hat{\beta}_{MLE} &= \bar{x}. \end{aligned}$$

8. 8
9. 9

We begin with our likelihood function for θ.

$$\begin{aligned} L(\theta) &= \prod_{i=1}^{n} (\theta + 1) y^{\theta} \\ &= (\theta + 1)^n \prod_{i=1}^{\infty} y^{\theta} \\ &= (\theta + 1)^n \left(\prod_{i=1}^{\infty} y\right)^{\theta}. \end{aligned}$$

Taking ln for our log-likelihood,

$$\begin{aligned} l(\theta) &= \ln\left((\theta + 1)^n \prod_{i=1}^{n} y^{\theta}\right) \\ &= n\ln(\theta + 1) + \sum_{i=1}^{n} n \ln\theta \\ &= n\ln(\theta + 1) + \theta \sum_{i=1}^{n} \ln y. \end{aligned}$$

And now we differentiate w.r.t θ,

$$\begin{aligned}
\frac{\partial l}{\partial \theta} &= \frac{n}{\theta+1} + \sum_{i=1}^{n} \ln y \\
0 &= \frac{n}{\theta+1} + \sum_{i=1}^{n} \ln y \\
\frac{n}{\theta+1} &= -\sum_{i=1}^{n} \ln y \\
\frac{\theta+1}{n} &= -\frac{1}{\sum_{i=1}^{n} \ln y} \\
\theta+1 &= -\frac{n}{\sum_{i=1}^{n} \ln y} \\
\hat{\theta}_{MLE} &= -\frac{n}{\sum_{i=1}^{n} \ln y} - 1
\end{aligned}$$

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10. 10
11. 11

    Our null hypothesis H_0 is that the true population proportion of overweight children is $p \geq 0.15$, and our alternative H_a is that the true population proportion $p < 0.15$.

    We can do this with `prop.test` in R

    ```
    prop.test(13,100,.15,alternative="less",conf.level=.95,correct=F)
    ```

    Our test gives us a p-value of .02877, so we reject the null hypothesis and conclude the true value of p is less than 15%.
12. 12
13. 13

    (a) We can say by looking at the histogram, that the average internet usage of the Canadians surveyed appears normally distributed.

    (b) Our null hypothesis H_0 is that the mean personal time spent online by the average Canadian in a week is less than or equal to 12.7 hours, the amount observed in 2005. This means our alternative H_a is that the average Canadian spends strictly more than 12.7 hours a week online in their personal time.
    For this we use the R code:

    ```
    > t.test(data,alternative="less",mean=12.7)
    ```

    Which gives us the p-value of 0.9973, a test statistic of 2.8548, and since the p-value is much higher than our $\alpha = .05$, we fail to reject the null hypothesis and cannot say that the true mean is less than 12.7.

    (c) R gives us our confidence interval from `t.test`, -Inf 20.69673. Of course we cannot have any negative hours of time spent on the Internet, so our 95% CI for the true mean is:

    $$(0, 20.69673).$$
14. 14