Exercise 1

Let $K \in (0,\infty), \alpha \in \mathbb{R}$. Show that:

$$\int_0^K x^\alpha \lambda(dx) = \begin{cases} \infty & \text{if } \alpha \leq -1 \\ (\alpha+1)^{-1}K^{\alpha+1} & \text{if } \alpha > -1 \end{cases}.$$

Solution: Use the relationship between the Lebesgue and Riemann integral again.

$$I_\alpha = R\int_0^K x^\alpha \, dx.$$

Then proceed by cases. If $\alpha = -1$, we integrate as if it were high school.

$$\begin{aligned} I_{-1} &= \lim_{n\to\infty} R\int_{\frac{1}{n}}^K x^{-1}\,dx \\ &= \lim_{n\to\infty} (\ln x)_{\frac{1}{n}}^K \\ &= \lim_{n\to\infty} \ln K - \ln\frac{1}{n} \\ &= \infty. \end{aligned}$$

If $\alpha \neq -1$, use the usual power rule:

$$\begin{aligned} I_\alpha &= \lim_{n\to\infty} R\int_{\frac{1}{n}}^K x^\alpha\,dx \\ &= \lim_{n\to\infty} \left((\alpha+1)^{-1} x^{\alpha+1}\right)_{\frac{1}{n}}^K \\ &= (\alpha+1)^{-1} \lim_{n\to\infty} \left(x^{\alpha+1}\right)_{\frac{1}{n}}^K \\ &= \lim_{n\to\infty} (\alpha+1)^{-1}\left(K^{\alpha+1} - \frac{1}{n^{\alpha+1}}\right). \end{aligned}$$

Now we must again split into cases. If $\alpha < -1$, then $\alpha+1$ is negative and we will have the limit:

$$\lim_{n\to\infty} \frac{1}{n^{\alpha+1}} = \infty.$$

And $I_\alpha = \infty$.

If $\alpha > -1$, then $\alpha + 1$ is positive;

$$\lim_{n\to\infty} \frac{1}{n^{\alpha+1}} = 0.$$

And $I_\alpha = \frac{K^{\alpha+1}}{\alpha+1}$

Exercise 2

Let $K \in (0,\infty), \alpha \in \mathbb{R}$. Show that:

$$\int_K^\infty x^\alpha \, \lambda(dx) = \begin{cases} \infty & \textit{if } \alpha \leq -1 \\ -(\alpha+1)^{-1} & \textit{if } \alpha > -1 \end{cases}$$

Solution: Rewrite as a Riemann integral again:

$$\begin{aligned} I_\alpha &= \int_K^\infty x^\alpha \, d\lambda \\ &= \lim_{n\to\infty} R \int_K^n x^\alpha \, d\lambda. \end{aligned}$$

If $\alpha = -1$, again we have:

$$\begin{aligned} I_{-1} &= \lim_{n\to\infty} (\ln x)_K^n \\ &= \lim_{n\to\infty} \ln n - \ln K & = \infty. \end{aligned}$$

Now if $\alpha \neq -1$,

$$I_\alpha = \lim_{n\to\infty} \left(\frac{x^{\alpha+1}}{\alpha+1} \right)_K^n = (\alpha+1)^{-1} \lim_{n\to\infty} n^{\alpha+1} - K^{\alpha+1}.$$

If $\alpha < -1$, we have $\alpha + 1$ negative, and

$$\lim_{n\to\infty} n^{\alpha+1} = 0.$$

And in turn

$$I_\alpha = -\frac{K^{\alpha+1}}{\alpha+1}.$$

Now if $\alpha > -1$ $\alpha + 1$ is positive, and

$$\lim_{n\to\infty} n^{\alpha+1} = \infty.$$