Example 0.1

Show that $S^1 * S^1 = S^3$, and more generally, $S^m * S^n = S^{m+n+1}$.

Solution: We consider first the space $I \times S^m \times S^n$. This is compact and Hausdorff since all of its components are. Then we take the equivalence relation $\sim$ by $(x,0,y) \sim (x',0,y)$, and $(x,1,y) \sim (x,1,y')$

Define a function $f : I \times S^m \times S^n \to S^{m+n+1}$; for $x = (x_1, \dots, x_{m+1}) \in S^m \subseteq \mathbb{R}^{m+1}$ and $y = (y_1, \dots, y_{n+1}) \in S^n \subseteq \mathbb{R}^{n+1}$, let

$$f(t,x,y) = \sqrt{t}\begin{pmatrix} x_1 \\ \vdots \\ x_{m+1} \\ 0 \\ \vdots \\ 0 \end{pmatrix} + \sqrt{1-t}\begin{pmatrix} 0 \\ \vdots \\ 0 \\ y_1 \\ \vdots \\ y_{n+1} \end{pmatrix}.$$

First we write a diagram and state our goals.

$$\begin{tikzcd} I \times S^m \times S^n \arrow[rr, "f"] \arrow[dr, "\pi"'] & & S^{m+n+1} \\ & S^m * S^n \arrow[ur, "f/\sim"'] & \end{tikzcd}$$

We need to show that f respects equivalence classes, that it is continuous and that $f/\sim$ is a bijection. By showing all this we will know that $f/\sim$ is a homeomorphism and $S^m * S^n \cong S^{m+n+1}$

First take a moment to confirm that $|f(t,x,y)| = 1$:

$$\begin{aligned} |f(t,x,y)| &= \left| \sqrt{t}\begin{pmatrix} x_1 \\ \vdots \\ x_{m+1} \\ 0 \\ \vdots \\ 0 \end{pmatrix} + \sqrt{1-t}\begin{pmatrix} 0 \\ \vdots \\ 0 \\ y_1 \\ \vdots \\ y_{n+1} \end{pmatrix} \right| \\ &= \left(\sum_{i=1}^{m+1} (\sqrt{t} x_i)^2 + \sum_{i=1}^{n+1} (\sqrt{1-t} y_i)^2 \right)^{\frac{1}{2}} \\ &= \left(\sum_{i=1}^{m+1} t x_i^2 + \sum_{i=1}^{n+1} (1-t) y_i^2 \right)^{\frac{1}{2}} \\ &= (t|x| + (1-t)|y|)^{\frac{1}{2}} \\ &= 1. \end{aligned}$$

And check that f respects the equivalence classes, for $t = 1$:

$$f(1,x,y) = \sqrt{1}\begin{pmatrix} x_1 \\ \vdots \\ x_{m+1} \\ 0 \\ \vdots \\ 0 \end{pmatrix} + \sqrt{1-1}\begin{pmatrix} 0 \\ \vdots \\ 0 \\ y_1 \\ \vdots \\ y_{n+1} \end{pmatrix} = \begin{pmatrix} x_1 \\ \vdots \\ x_{m+1} \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

Clearly independent of y, so f respects the class $(1,x,y) \sim (1,x,y')$. The same holds for $(0,x,y) \sim (0,x',y)$

Then we wish to show continuity of f. Clearly its components are continuous and so f must be too. Since f is continuous, we know the induced map $f/\sim$ is continuous.

Then so long as $f/\sim$ is bijective, it is a homeomorphism and we will have our desired result.

We construct an inverse for $f/\sim$; let $g : S^{m+n+1} \to S^m * S^n$, where for $x \in S^{m+1}, y \in S^{n+1}$, we set:

$$t(x) = \sum_{i=1}^{m+1} x_i^2 = \|x\|^2.$$

Then define g by cases; if $0 < t < 1$;

$$g(x,y) = \begin{pmatrix} t(x) \\ \frac{x_1}{\sqrt{t(x)}} \\ \vdots \\ \frac{x_{m+1}}{\sqrt{t(x)}} \\ \frac{y_1}{\sqrt{1-t(x)}} \\ \vdots \\ \frac{y_{n+1}}{\sqrt{1-t(x)}} \end{pmatrix}.$$

If $t = 1$, we choose a basepoint $y_0 \in S^{n+1}$, and take:

$$g(x,y) = (t,x,y_0).$$

If $t = 0$, we choose a basepoint $x_0 \in S^{m+1}$, and take:

$$g(x,y) = (t,x_0,y).$$

It is not hard to see that f,g compose to identity both ways.