1. Let $M = (Q, \Sigma, T, \delta, q_0, q_{\text{accept}})$ be a Turing machine, where:

   $Q = \{q_0, q_1, q_2, q_3, q_{\text{accept}}\}$,

   $\Sigma = \{0, 1\}$ is the input alphabet

   $T = \{0, 1, \perp\}$ is the tape alphabet (with $\perp$ denoting the blank symbol).

   The transition function δ is defined by the following table:

δ	0	1	$\perp$
q_0	$(q_1, 0, R)$	$(q_1, 0, R)$	$(q_3, \perp, R)$
q_1	$(q_1, 0, R)$	$(q_2, 1, R)$	$(q_3, \perp, R)$
q_2	$(q_2, 0, R)$	$(q_0, 1, R)$	$(q_3, \perp, R)$
q_3	—	—	—

   (a) Simulate the behavior of the Turing machine M on the following inputs. For each case, provide the final tape content and the halting state:

      i. 1011

Solution: Begin with the input string and the state q_0.

	q_0				
$\perp$	1	0	1	1	$\perp$

		q_1			
$\perp$	0	0	1	1	$\perp$

			q_1		
$\perp$	0	0	1	1	$\perp$

				q_2	
$\perp$	0	0	1	1	$\perp$

					q_0
$\perp$	0	0	1	1	$\perp$

And at this point we transition to q_3 which is a halting state. We are left with the final tape content:

$\perp$	0	0	1	1	$\perp$

      ii. 111

Solution:

	q_0			
$\perp$	1	1	1	$\perp$

		q_1		
$\perp$	0	1	1	$\perp$

			q_2	
$\perp$	0	1	1	$\perp$

				q_0
$\perp$	0	1	1	$\perp$

And again, we will switch states to q_3 and halt, leaving us with the final tape:

$\perp$	0	1	1	$\perp$

iii. 010

Solution:

$$\begin{array}{|c|c|c|c|c|} \multicolumn{1}{c}{} & \multicolumn{1}{c}{\boxed{q_0}} & \multicolumn{1}{c}{} & \multicolumn{1}{c}{} & \multicolumn{1}{c}{} \\ \hline \perp & 0 & 1 & 0 & \perp \\ \hline \end{array}$$

$$\begin{array}{|c|c|c|c|c|} \multicolumn{1}{c}{} & \multicolumn{1}{c}{} & \multicolumn{1}{c}{\boxed{q_1}} & \multicolumn{1}{c}{} & \multicolumn{1}{c}{} \\ \hline \perp & 0 & 1 & 0 & \perp \\ \hline \end{array}$$

$$\begin{array}{|c|c|c|c|c|} \multicolumn{1}{c}{} & \multicolumn{1}{c}{} & \multicolumn{1}{c}{} & \multicolumn{1}{c}{\boxed{q_2}} & \multicolumn{1}{c}{} \\ \hline \perp & 0 & 1 & 0 & \perp \\ \hline \end{array}$$

$$\begin{array}{|c|c|c|c|c|} \multicolumn{1}{c}{} & \multicolumn{1}{c}{} & \multicolumn{1}{c}{} & \multicolumn{1}{c}{} & \multicolumn{1}{c}{\boxed{q_2}} \\ \hline \perp & 0 & 1 & 0 & \perp \\ \hline \end{array}$$

And again, we will switch states to q_3 and halt, leaving us with the final tape:

$$\begin{array}{|c|c|c|c|c|} \hline \perp & 0 & 1 & 0 & \perp \\ \hline \end{array}$$

(b) Describe the general behavior of M when the input is of the form 1^k for some $k \in \mathbb{N}$.

Solution:

$$\begin{array}{|c|c|c|c|c|c|c|} \multicolumn{1}{c}{} & \multicolumn{1}{c}{\boxed{q_0}} & \multicolumn{1}{c}{} & \multicolumn{1}{c}{} & \multicolumn{1}{c}{} & \multicolumn{1}{c}{} & \multicolumn{1}{c}{} \\ \hline \perp & 1 & 1 & 1 & 1 & \cdots & \perp \\ \hline \end{array}$$

$$\begin{array}{|c|c|c|c|c|c|c|} \multicolumn{1}{c}{} & \multicolumn{1}{c}{} & \multicolumn{1}{c}{\boxed{q_1}} & \multicolumn{1}{c}{} & \multicolumn{1}{c}{} & \multicolumn{1}{c}{} & \multicolumn{1}{c}{} \\ \hline \perp & 0 & 1 & 1 & 1 & \cdots & \perp \\ \hline \end{array}$$

$$\begin{array}{|c|c|c|c|c|c|c|} \multicolumn{1}{c}{} & \multicolumn{1}{c}{} & \multicolumn{1}{c}{} & \multicolumn{1}{c}{\boxed{q_2}} & \multicolumn{1}{c}{} & \multicolumn{1}{c}{} & \multicolumn{1}{c}{} \\ \hline \perp & 0 & 1 & 1 & 1 & \cdots & \perp \\ \hline \end{array}$$

$$\begin{array}{|c|c|c|c|c|c|c|} \multicolumn{1}{c}{} & \multicolumn{1}{c}{} & \multicolumn{1}{c}{} & \multicolumn{1}{c}{} & \multicolumn{1}{c}{\boxed{q_0}} & \multicolumn{1}{c}{} & \multicolumn{1}{c}{} \\ \hline \perp & 0 & 1 & 1 & 1 & \cdots & \perp \\ \hline \end{array}$$

Now, we can see that we are in the same starting state, only now we are working with the string 1^{k-3}. So we repeat the above process on this substring, and continue until we find the character $\perp$, at which point we will be stuck in q_3, and halt. So we can say that the machine takes a string 1^k and converts every third 1 to a zero, starting with the first 1.

(c) Construct a Turing machine $M' = (Q', \Sigma, T, \delta', q'_0, q'_{\text{accept}})$, where $T = \{0, 1, \perp\}$, that satisfies each of the following:

i. Replaces the first occurrence of the substring 01 in the input with 10, and leaves the rest unchanged.

ii. Accepts if and only if the input contains the substring 010.

Specify only the state transitions relevant to this task (you may assume the rest lead to a rejecting state or halt).

2. Let $\Sigma = \{0, 1\}$. Define the language:

$$L' = \{0^n 1^n 0^n 1^n | n \in \mathbb{N}_0\}.$$

(a) Design a Turing machine that accepts the language L'.

δ	0	1	$\perp$
q_0	$(q_1, 0, R)$	$(q_1, 0, R)$	$(q_3, \perp, R)$
q_A	−	−	−

Solution: Let $M = (Q, \Sigma, T, \delta, q_0, q_{\text{accept}})$ be a Turing machine, where:

$Q = \{q_0, q_1, q_2, q_3, q_{\text{accept}}\}$,
$\Sigma = \{0, 1\}$ is the input alphabet
$T = \{0, 1, \perp\}$ is the tape alphabet (with $\perp$ denoting the blank symbol).
The transition function δ is defined by the following table:

Provided for readability is a summary of each state and its purpose:

i. Initial state. As well, this state scans right until it finds $\perp$, then moves to the left and switches to state 1.
ii. When this state is reached, the position on the tape should be on the furthest left 0 or 1. If it's a 1, we switch to state 2. If it's a 0 or $\perp$ we halt.
iii. This state scans left until it finds $\perp$, then moves to the right and switches to state 3.

(b) Prove that if the Turing machine accepts a string x, then $x \in L'$.

(c) Modify the Turing machine so that it replaces the input $x \in L'$ with the string xx (i.e., it duplicates the input).

(d) Prove that the modified machine correctly duplicates the input only if $x \in L'$.