1. Let $\{f_n\}$ be the sequence of functions defined by

$$f_n = \begin{cases} nx & \text{If } 0 \le x \le \frac{1}{n} \\ 1 & \text{If } \frac{1}{n} < x < 1 - \frac{1}{n} \\ n - nx & \text{If } 1 - \frac{1}{n} \le x \le 1 \end{cases}.$$

(a) Find the pointwise limit f of the sequence.

Solution: Proceed by cases. If $x = 0$, then the first case of the function will always be taken since $0 \le x$. So $f_n(0) = n0 = 0$. Likewise if $x = 1$, then $f(1) = n - n1 = n - n = 0$.

Now, if $x \in (0, 1)$, then we observe that $\frac{1}{n} \to 0$, and $1 - \frac{1}{n} \to 1$. Therefore the middle case of our piecewise function gives us $f(x) = 1$ for all x in this open interval.

(b) Does $f_n \xrightarrow[{[0,1]}]{c.u} f$? Justify your answer.

Solution: This sequence is not uniformly convergent. Pick $\varepsilon = \frac{1}{3}$, and let $N \in \mathbb{N}$, and $n > N$. Pick $x = \frac{1}{2n}$ so that $0 \le x \le \frac{1}{n}$, and then $f_n(x) = nx = \frac{n}{2n} = \frac{1}{2}$. Then: $|f_n(x) - f(x)| = \left|\frac{1}{2} - 1\right| = \frac{1}{2} > \varepsilon$.

Therefore the sequence is not uniformly convergent.

2. Let $f_n(x) = \left(\cos\left(\frac{2x}{n}\right)\right)^{n^2}$

(a) Compute the pointwise limit f of the sequence $\{f_n\}$. **Hint:** Use the following double inequalities:

$$1 - \frac{1}{2}t^2 \le \cos t \le 1 - \frac{1}{2}t^2 + \frac{1}{24}t^4, \quad \forall t \in \mathbb{R}.$$

$$-t - t^2 \le \ln(1 - t) \le -t, \quad \forall t \in \left[0, \frac{1}{2}\right].$$

Solution: Begin with the first inequality.

$$1 - \frac{1}{2}t^2 \le \cos t$$
$$1 - \frac{1}{2}\frac{4x^2}{n^2} \le \cos\left(\frac{2x}{n}\right)$$
$$-\frac{2x^2}{n^2} \le \cos\left(\frac{2x}{n}\right) - 1$$
$$1 - \cos\left(\frac{2x}{n}\right) \le \frac{2x^2}{n^2}$$
.

And now using the second,

$$\ln\left(1 - \cos\left(\frac{2x}{n}\right)\right) \le \cos\left(\frac{2x}{n}\right) - 1 \le -\frac{2x^2}{n^2}.$$

Since ln is monotonically increasing,

$$\ln\left(-\cos\left(\frac{2x}{n}\right)\right) \le \ln\left(1-\cos\left(\frac{2x}{n}\right)\right) \le -\frac{2x^2}{n^2}$$

$$n^2 \ln\left(-\cos\left(\frac{2x}{n}\right)\right) \le -2x^2$$

$$\ln\left(-\cos^{n^2}\left(\frac{2x}{n}\right)\right) \le -2x^2$$

$$\exp\left[\ln\left(-\cos^{n^2}\left(\frac{2x}{n}\right)\right)\right] \le e^{-2x^2} \quad \text{exp is monotonically increasing}$$

$$-\cos^{n^2}\left(\frac{2x}{n}\right) \le e^{-2x^2}$$

$$\cos^{n^2}\left(\frac{2x}{n}\right) \le e^{-2x^2} \qquad \text{cos is an even function.}$$

(b) Show that $f_n \xrightarrow[{[0,1]}]{c.u} f$.

3. Let $a \in \mathbb{R}_+$. Compute the limit

$$\lim_{n\to\infty} \int_a^\pi \frac{\sin(nx)}{nx}\,dx.$$

What happens if $a = 0$?

We begin by considering our sequence of functions within the integral, each of which is a quotient of continuous functions, and is itself continuous (for all but $x = 0$). Call this $g_n(x) = \frac{\sin(nx)}{nx}$. Note that since $-1 \le \sin(nx) \le 1$, we can find (for nonzero x) that $-\frac{1}{nx} \le g_n(x) \le \frac{1}{nx}$. Both the sequences bounding g have a zero limit at infinity, so by the squeeze theorem on sequences of functions, we can say that for nonzero x, $g_n \to 0$. Now since we have already shown that our sequence g_n is bounded, and since each g_n is integrable, we can say:

$$\lim_{n\to\infty} \int_a^\pi \frac{\sin(nx)}{nx}\,dx = \int_a^\pi \lim_{n\to\infty} \frac{\sin(nx)}{nx}\,dx$$

$$= \int_a^\pi 0\,dx$$

$$= 0 - 0$$

$$= 0.$$

4. Construct a sequence of functions defined in $[0,1]$, each of which is discontinuous at every point of $[0,1]$ and which converges uniformly to a function that is continuous at every point

Solution: Take the series $\{f_n\}$ defined by:

$$f_n(x) = \begin{cases} \frac{1}{n} & \text{If } x \in \mathbb{Q} \\ 0 & \text{Otherwise} \end{cases}$$

Claim: $\{f_n\}$ converges uniformly to the constant function 0, which we know to be continuous on the real line, as well as $[0,1]$.

Let $\varepsilon > 0$, and choose N such that $0 < \frac{1}{N} < \varepsilon$. Then any $n \geq N$ will have $0 < \frac{1}{n} < \frac{1}{N} < \varepsilon$. Now by cases, if $x \in \mathbb{Q}$, then we have

$$|f_n(x) - f(x)| = \left|\frac{1}{n} - 0\right| = \frac{1}{n} < \frac{1}{N} < \varepsilon.$$

And for $x \notin \mathbb{Q}$,

$$|f_n(x) - f(x)| = |0 - 0| = 0 < \varepsilon.$$

Therefore $\{f_n\}$ is a sequence of functions which are continuous nowhere, convergent to the zero function which is continuous everywhere.

5. Consider the series of functions $\sum_{n \geq 1} \frac{x}{n(n+x)}$.

   (a) Show that the series converges uniformly in the interval $[0, b]$ for any $b > 0$.

   Solution:

   $$\frac{x}{n(n+x)} = \frac{x}{n^2 + nx}$$
   $$\leq \frac{x}{n^2}$$
   $$\leq \frac{b}{n^2}.$$

   Define $u_n = \frac{b}{n^2}$, then by the Weierstrass Comparison test, since $\sum_{n \geq 1} u_n$ is convergent as a p-series with $p = 2$, $\frac{x}{n(n+x)} \leq u_n$, this series must converge.

   (b) Let $F(x) = \sum_{n \geq 1} \frac{x}{n(n+x)}$. Show that $F'(x) = \sum_{n \geq 1} \frac{1}{n(n+x)^2}$, $x \geq 0$.

   Solution:

   $$F'(x) = \frac{d}{dx}\left(\sum_{k=1}^{\infty} \frac{x}{k(k+x)}\right)$$
   $$= \frac{d}{dx}\left(\lim_{n \to \infty} \sum_{k=1}^{n} \frac{x}{k(k+x)}\right)$$
   $$= \lim_{n \to \infty}\left(\frac{d}{dx} \sum_{k=1}^{n} \frac{x}{k(k+x)}\right)$$
   $$= \lim_{n \to \infty} \sum_{k=1}^{n} \frac{d}{dx} \frac{x}{k(k+x)}$$
   $$= \lim_{n \to \infty} \sum_{k=1}^{n} \frac{k(x+k) - kx}{k^2(k+x)^2}$$
   $$= \lim_{n \to \infty} \sum_{k=1}^{n} \frac{k}{k^2(k+x)^2}$$
   $$= \lim_{n \to \infty} \sum_{k=1}^{n} \frac{1}{k(k+x)^2}$$
   $$= \sum_{k=1}^{\infty} \frac{1}{k(k+x)^2}.$$

6. Consider the series of functions $\sum_{n\geq 1} \frac{x}{1+n^2x^2}$. Show that the series doesn't converge uniformly in $\mathbb{R}_+$.

Hint: You could start by showing that $\frac{x}{1+n^2x^2} \geq \int_n^{n+1} \frac{x}{1+t^2x^2}\, dt, \quad \forall x \in \mathbb{R}$.

Solution: Begin with the hint. If we take $P_0 = \{n, n+1\}$, the trivial partition on $[n, n+1]$, then we will have the upper sum:

$$U\left(P_0, \frac{x}{1+t^2x^2}\right) = \sum_{k=1}^{1} \sup_{t\in[n,n+1]} \left(\frac{x}{1+t^2x^2}\right)((n+1)-n) = \frac{x}{1+n^2x^2}.$$

But from the definition of the Riemann integral, we have (For σ the set of all partitions of $[n, n+1]$):

$$\int_n^{n+1} \frac{x}{1+t^2x^2}\, dt = \inf_{P\in\sigma} U\left(P, \frac{x}{1+t^2x^2}\right)$$

$$\leq U\left(P_0, \frac{x}{1+t^2x^2}\right)$$

$$= \frac{x}{1+n^2x^2}.$$

Suppose by way of contradiction that the series does converge uniformly. Then there exists some m such that, for any $x \in \mathbb{R}$

$$\left|\sum_{n=1}^{\infty} \frac{x}{1+n^2x^2} - \sum_{n=1}^{m} \frac{x}{1+n^2x^2}\right| < \frac{1}{2}.$$

However,

$$\left|\sum_{n=1}^{\infty} \frac{x}{1+n^2x^2} - \sum_{n=1}^{m} \frac{1}{1+n^2x^2}\right| = \left|\sum_{n=m+1}^{\infty} \frac{x}{1+x^2n^2}\right|$$

$$= \sum_{n=m+1}^{\infty} \frac{x}{1+x^2n^2}$$

$$\geq \sum_{n=m+1}^{\infty} \int_n^{n+1} \frac{x}{1+t^2x^2}\, dt$$

$$= \int_{m+1}^{\infty} \frac{x}{1+t^2x^2}\, dt \qquad\qquad \text{Let } u = tx$$

$$= \int_{(m+1)x}^{\infty} \frac{1}{1+u^2}\, du \qquad\qquad du = x\, dt$$

$$= (\arctan u)_{u=(m+1)x}^{\infty}$$

$$= \frac{\pi}{2} - \arctan((m+1)x)$$

$$= \frac{\pi}{2} - \arctan(1) \qquad\qquad \text{Pick } x = \frac{1}{m+1}$$

$$= \frac{\pi}{2} - \frac{\pi}{4}$$

$$= \frac{\pi}{4}$$

$$> \frac{1}{2}.$$

A contradiction, so our series cannot converge uniformly.