Problem Set 3 - Thomas Boyko - 30191728

1. Consider the following subset of S_4:

$$K = \{e, (12)(34), (13)(24), (14)(23)\}.$$

   (a) Show that $K \trianglelefteq S_4$.

   We write the Cayley table for K, and we observe closure which is sufficient to show that K is a subgroup thanks to the fact that S_4 and K are finite groups.

$\cdot$	e	$(1\,2)(3\,4)$	$(1\,3)(2\,4)$	$(1\,4)(2\,3)$
e	e	$(1\,2)(3\,4)$	$(1\,3)(2\,4)$	$(1\,4)(2\,3)$
$(1\,2)(3\,4)$	$(1\,2)(3\,4)$	e	$(1\,4)(2\,3)$	$(1\,3)(2\,4)$
$(1\,3)(2\,4)$	$(1\,3)(2\,4)$	$(1\,4)(2\,3)$	e	$(1\,2)(3\,4)$
$(1\,4)(2\,3)$	$(1\,4)(2\,3)$	$(1\,3)(2\,4)$	$(1\,2)(3\,4)$	e

   Take some $\tau \in K$, and any $\sigma \in S_4$. Since conjugation in S_n preserves cycle structure, and K contains all the products of disjoint $2-$cycles in S_4, $\sigma\tau\sigma^{-1}$ is a product of disjoint $2-$cycles and $\sigma\tau\sigma^{-1} \in K$.

   (b) Write down the set of distinct left cosets of K, i.e. list the elements of $\frac{S_4}{K}$.

$$\begin{aligned}
eK &= K \\
(1\,2)K &= \{(1\,2), (3\,4), (1\,3\,2\,4), (1\,4\,2\,3)\} \\
(2\,3)K &= \{(2\,3), (1\,3\,4\,2), (1\,2\,4\,3), (1\,4)\} \\
(1\,3)K &= \{(1\,3), (1\,2\,3\,4), (1\,4), (4\,2)\} \\
(1\,2\,3)K &= \{(1\,2\,3), (1\,3\,4), (2\,4\,3), (1\,4\,2)\} \\
(1\,3\,2)K &= \{(1\,3\,2), (1\,4\,3), (3\,4\,2), (2\,4\,1)\}
\end{aligned}$$

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   (c) Show that $S_4/K \cong S_3$.

   Above we have written each coset of K using a cycle free of 4. Knowing $S_3 = \{e, (1\,2), (1\,3), (1, 2), (1\,2\,3), (1\,3\,2)\}$ we can see each element of S_3 has one coset representation in K, and since cosets partition S_4 we know that each element of S_3 will appear in exactly one coset.

   So take the mapping $f : S_4/K \to S_3$, $f(\sigma K) = \sigma$. Thanks to how we wrote our above cosets we can see that this is both onto and one-to-one. As well since each coset has only one representation in terms of a cycle in S_3, our mapping is well-defined. All that remains is to check homomorphism.

   Take $\sigma K, \tau K \in S_4/K$. Then:

$$f(\sigma K \tau K) = f(\sigma\tau K) = \sigma\tau = f(\sigma K \tau K).$$

   So f is a homomorphism, and since f is a bijection, it is an isomorphism, and therefore $S_4/K \cong S_3$.

2. Let G be a group and H be a subgroup of G.

   (a) Show that $H \trianglelefteq G$ if and only if H is a union of conjugacy classes.

   $\implies$**:** Suppose $H \trianglelefteq G$ and let $C = \bigcup_{h \in H} Cl(h)$.

   We will show that $H = C$. Clearly every $h \in H$ is in C, since the conjugacy class for h will contain $h = ehe^{-1}$. So $H \subseteq C$

   Now let some $c \in C$, where $c \in Cl(h_i)$ for some $h_i \in H$. Then for some $g \in G$, $c = gh_ig^{-1}$ which means that $c \in H$ since $H \trianglelefteq G$. Therefore $C \subseteq H$, $C = H$ and H is a union of conjugacy classes.

$\Longleftarrow$ **:** Suppose H is a union of n conjugacy classes. So for any $h \in H$, $h \in Cl(h_i)$, where $i \in \{1, 2, \ldots, n\}$. This means that for any $g \in G$, we can write $h = gh_ig^{-1}$, and $g^{-1}hg = h_i$. Since $h_i \in H$, for any $h \in H$ and any $g \in G$, $g^{-1}hg \in H$, and $g^{-1}Hg \subseteq H$ and $H \trianglelefteq G$.

(b) Suppose $|G| = 20$, and the class equation of G is given by $20 = 1 + 4 + 5 + 5 + 5$ Does G have a subgroup of order 4 ? what about order 5 ? Can G have a normal subgroup of order 4? what about order 5 ? Justify.

By Cauchy's Theorem, G must have an element of order 5, and a subgroup generated by that element of order 5. G may not necessarily have a subgroup of order 4 since 4 is not prime.

By what we proved above, we can make a subgroup of order 5 from the singleton $\{e\}$ and the class of order 4. So G has a normal subgroup of order 5. However there is no way to make a subgroup of order 4, since the only way to make 4 is with the class containing 4 elements, which would not contain identity. So G does not have a normal subgroup of order 4.

3. (a) Deduce with proper justification, the class equation of the dihedral group D_4.
Write out the center and conjugacy classes for D_4.

$$\begin{aligned} Z(D_4) &= \{e, r^2\} \\ Cl(r) &= \{r, r^3\} \\ Cl(s) &= \{s, sr^2\} \\ Cl(sr) &= \{sr, sr^3\}. \end{aligned}$$

And so our class equation is $|D_4| = 1 + 1 + 2 + 2 + 2$.

(b) Deduce with proper justification, the class equation of S_4.
Write out the center and conjugacy classes for S_4, this is made easier by the fact that cycle structure is maintained by conjugation.

$$\begin{aligned} Z(S_4) &= \{e\} \\ Cl((1\,2)) &= \{(1\,2), (1\,3), (1\,4), (2\,3), (2\,4), (3\,4)\} \\ Cl((1\,2\,3)) &= \{(1\,2\,3), (1\,3\,2), (1\,2\,4), (1\,4\,2), (1\,3\,4), (1\,4\,3), (2\,3\,4), (2\,4\,3)\} \\ Cl((1\,2\,3\,4)) &= \{(1\,2\,3\,4), (1\,2\,4\,3), (1\,3\,2\,4), (1\,3\,4\,2), (1\,4\,2\,3), (1\,4\,3\,2)\} \\ Cl((1\,2)(3\,4)) &= \{(1\,2)(3\,4), (1\,3)(2\,4), (1\,4)(2\,3)\}. \end{aligned}$$

So our class equation is given by:

$$|S_4| = 1 + 6 + 8 + 6 + 3.$$

4. This question is all about finding an appropriate homomorphism and directly applying the first isomorphism theorem. Show that

(a) $S_n/A_n \cong \mathbb{Z}_2$.
Consider the function:

$$f : S_n \to \mathbb{Z}_2, \quad f(\tau) = \begin{cases} [0], & \text{If } \sigma \text{ is an even permutation} \\ [1], & \text{If } \sigma \text{ is an odd permutation} \end{cases}.$$

We begin by checking well-definedness. Suppose $\sigma_1 = \sigma_2$, and $\sigma_1, \sigma_2 \in S_n$. Since the two are equal, they will be both even or both odd. So they will both be mapped to [1] or both mapped to [0].

Now for homomorphism: Suppose $\sigma_1, \sigma_2 \in S_n$. Take $f(\sigma_1, \sigma_2)$. If both σ_1, σ_2 are even or odd cycles, their product will be even. So in this case $f(\sigma_1\sigma_2) = [0] = f(\sigma_1)f(\sigma_2)$. And if exactly one of σ_1, σ_2 is odd, then their product will be odd. So $f(\sigma_1\sigma_2) = [0] = f(\sigma_1)f(\sigma_2)$, and since this covers every case, f is a homomorphism.

What is the kernel of this transformation? All elements mapped to [0] will be even permutations, so $\ker f = A_n$ since A_n is the group of only even permutations.

And the image of this homomorphism is $\mathbb{Z}_2$, since we will be able to find both an even and an odd cycle in any S_n, when $n \geq 2$.

So by the first isomorphism theorem, $S_n/A_n \cong \mathbb{Z}_2$.

(b) $GL_n(\mathbb{Q})/SL_n(\mathbb{Q}) \cong (\mathbb{Q}, \cdot)$.

Consider:

$$f : GL_n(\mathbb{Q}) \to \mathbb{Q}, \quad f(A) = \det A.$$

First we check well-definedness of f. Let $A = B$ where $A, B \in GL_n(\mathbb{Q})$. Then $f(A) = \det A = \det B = f(B)$ so f is well-defined.

Now we check homomorphism. Let $A, B \in GL_n(\mathbb{Q})$. Then $f(AB) = \det AB = \det A \det B = f(A)f(B)$

The kernel of this homomorphism is given by all A so that $\det A = 1$, which is the definition of $SL_n(\mathbb{Q})$, so $\ker f = SL_n(\mathbb{Q})$.

And the image of this homomorphism is $\mathbb{Q}$, since all the entries of a matrix in $GL_n(\mathbb{Q})$ are rational, so their products and sums will be in $\mathbb{Q}$ since $\mathbb{Q}$ is closed under multipliation and addition.

So by the first isomorphism theorem, $GL_n(\mathbb{Q})/SL_n(\mathbb{Q}) \cong (\mathbb{Q}, \cdot)$.

(c) $\mathbb{R}/\mathbb{Z} \cong C^0$.

Consider the mapping:

$$f : \mathbb{R} \to \mathbb{C}^0, \quad f(\theta)) = e^{2\pi i\theta}, \quad k \in \mathbb{Z}.$$

We check well-definedness, let $\theta = \varphi \in \mathbb{R}$. Then

$$f(\theta) = e^{2\pi i\theta} = e^{2\pi i\varphi} = f(\varphi).$$

So f is well-defined.

Next we check homomorphism. Let $\theta, \varphi \in \mathbb{R}$.

$$f(\theta + \varphi) = e^{2\pi i(\theta+\phi)} = e^{2\pi i\theta+2\pi i\varphi} = e^{2\pi i\theta}e^{2\pi i\varphi} = f(\theta)f(\varphi)$$

So f is a homomorphism.

Next observe the image of f. Let $z = f(\theta)$ for some $\theta \in \mathbb{R}$. Then $z = e^{i2\pi\theta}$, and by taking the modulus of both sides we see that $|z| = 1$. So the image of f is all elements in C with modulus 1, or $\operatorname{Im} f = \mathbb{C}^0$.

Now we check the kernel of f. Suppose $f(\theta) = 1$ for some $\theta \in \mathbb{R}$. Then $e^{2\pi i\theta} = 1$, and $\cos 2\pi\theta + i \sin 2\pi\theta = 1$, which is true only for $\theta \in \mathbb{Z}$, meaning $\ker f = \mathbb{Z}$.

So by the first isomorphism theorem, $\mathbb{R}/\mathbb{Z} \cong \mathbb{C}^0$.