1. Let V and W be finite dimensional vector spaces with given bases $\mathcal{B} = \{\vec{b}_1, \ldots, \vec{b}_n\}$ and $\mathcal{D} = \{\vec{d}_1, \ldots, \vec{d}_m\}$, respectively.

   (a) For a given $\vec{x} \in V$, there are unique scalars so that $\vec{x} = a_1\vec{b}_1 + \cdots + a_n\vec{b}_n$. Define the vector $[\vec{x}]_{\mathcal{B}} := (a_1, \ldots, a_n)^T \in \mathbb{C}^n$. Show that the map $\vec{x} \mapsto [\vec{x}]_{\mathcal{B}}$ is a linear isomorphism from V into $\mathbb{C}^n$.

   Linearity: Let $\vec{x}, \vec{y} \in V$. Then write $\vec{x} = a_1\vec{b}_1 + \cdots + a_n\vec{b}_n$ and $\vec{y} = c_1\vec{b}_1 + \cdots + c_n\vec{b}_n$. Now:

$$[\vec{x}+\vec{y}]_{\mathcal{B}} = \begin{bmatrix} a_1 + c_1 \\ \vdots \\ a_n + c_n \end{bmatrix} = \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} + \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} = [\vec{x}]_{\mathcal{B}} + [\vec{y}]_{\mathcal{B}}$$

$$[\alpha\vec{x}]_{\mathcal{B}} = \begin{bmatrix} \alpha a_1 \\ \vdots \\ \alpha a_n \end{bmatrix} = \alpha \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} = \alpha[\vec{x}]_{\mathcal{B}}.$$

   So $[\cdot]_{\mathcal{B}}$ is linear.

   Isomorphism: Since $\dim V = \dim \mathbb{C}^n = n$, it will suffice to show that this mapping is injective. We do so by showing $\ker[\cdot]_{\mathcal{B}} = \{0\}$. Clearly 0 is in the kernel since $[0]_{\mathcal{B}} = [0\vec{x}]_{\mathcal{B}} = 0[\vec{x}]_{\mathcal{B}} = 0$. For inclusion the other way, let $\vec{x} \in \ker[\cdot]_{\mathcal{B}}$. Then $[\vec{x}]_{\mathcal{B}} = 0$; meaning the basis representation of $\vec{x}$ is through zero coefficients; and

$$\vec{x} = 0\vec{b}_1 + \ldots + 0\vec{b}_n = 0.$$

   So $\ker[\cdot]_{\mathcal{B}} = \{0\}$, and this map is injective. But since the spaces are of the same dimension it must also be surjective. So the map is a linear isomorphism from V to $\mathbb{C}^n$.

   (b) Let $T : V \to W$ be a linear map. In class, we defined the matrix representation of T with respect to $\mathcal{B}$ and $\mathcal{D}$ as the $m \times n$ matrix $[T]_{\mathcal{BD}} = [[T\vec{b}_1]_{\mathcal{D}}, \ldots, [T\vec{b}_n]_{\mathcal{D}}]$. In other words, the j-the column of $[T]_{\mathcal{BD}}$ is $[T\vec{b}_j]_{\mathcal{D}}$. Show that $[T]_{\mathcal{BD}}[\vec{x}]_{\mathcal{B}} = [T\vec{x}]_{\mathcal{D}}$ for any $\vec{x} \in V$.

   Solution: Let $T : V \to W$ be linear, then write $\vec{x} = a_1\vec{b}_1 + \ldots + a_n\vec{b}_n$.

$$\begin{aligned}
[T]_{\mathcal{BD}}[\vec{x}]_{\mathcal{B}} &= \big[[T\vec{b}_1]_{\mathcal{D}} \ldots [T\vec{b}_n]_{\mathcal{D}}\big] \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} \\
&= a_1[T\vec{b}_1]_{\mathcal{D}} + \ldots + a_n[T\vec{b}_n]_{\mathcal{D}} \\
&= [a_1T\vec{b}_1 + \ldots + a_nT\vec{b}_n]_{\mathcal{D}} && \text{By linearity of } [\cdot]_{\mathcal{D}} \\
&= \big[T(a_1\vec{b}_1 + \ldots + a_n\vec{b}_n)\big]_{\mathcal{D}} && \text{By linearity of } T \\
&= [T\vec{x}]_{\mathcal{D}}.
\end{aligned}$$

   (c) Show that $[T]_{\mathcal{BD}}$ is a linear isomorphism from $L(V, W)$ (the vector space of linear maps from V to W) to $M_{mn}(\mathbb{C})$ (vector space of $m \times n$ complex matrices).

   Linearity: Let T, S be linear from V to W. Then:

$$\begin{aligned}
[T+S]_{\mathcal{BD}} &= \big[[(T+S)\vec{b}_1] \ldots [(T+S)\vec{b}_n]\big]_{\mathcal{D}} \\
&= \big[[(T\vec{b}_1 + S\vec{b}_1)] \ldots [(T\vec{b}_n + S\vec{b}_n)]\big]_{\mathcal{D}} \\
&= \big[[T\vec{b}_1] \ldots [T\vec{b}_n]\big]_{\mathcal{D}} + \big[[S\vec{b}_1] \ldots [S\vec{b}_n]\big]_{\mathcal{D}} && \text{By Linearity of } [\cdot]_{\mathcal{D}} \\
&= [S]_{\mathcal{BD}} + [T]_{\mathcal{BD}}.
\end{aligned}$$

And then letting $\alpha \in \mathbb{C}$,

$$\begin{aligned}
\alpha[T]_{\mathcal{BD}} &= \alpha\big[[T\vec{b}_1]\dots[T\vec{b}_n]\big]_{\mathcal{D}} \\
&= \big[\alpha[T\vec{b}_1]\dots\alpha[T\vec{b}_n]\big]_{\mathcal{D}} && \text{Linearity of } [\cdot]_{\mathcal{D}} \\
&= \big[[\alpha T\vec{b}_1]\dots[\alpha T\vec{b}_n]\big]_{\mathcal{D}} \\
&= [\alpha T]_{\mathcal{BD}}.
\end{aligned}$$

2. Let V, W and U be finite dimensional vector spaces with given bases: $\mathcal{B} = \{\vec{b}_1,\dots,\vec{b}_n\}, \mathcal{D} = \{\vec{d}_1,\dots,\vec{d}_m\}$, and $\mathcal{F} = \{f_1,\dots,f_k\}$, respectively. Suppose $T : V \to W$ and $S : W \to U$ are linear. Prove or disprove the following statement for the composition linear map $ST : V \to U$:

$$[ST]_{\mathcal{BF}} = [S]_{\mathcal{DF}}[T]_{\mathcal{BD}}..$$

3. Let V be a finite dimensional vector space and $T : V \to V$ be linear. Show that $\sigma(T) = \sigma([T]_{\mathcal{B}})$ where $\mathcal{B}$ is any basis for V.

   Solution: $\subseteq$: Let $\lambda \in \sigma(T)$, and let $\vec{v}$ be an associated eigenvector. We show that $[\vec{v}]_{\mathcal{B}}$ is an eigenvector for λ under $[T]_{\mathcal{B}}$.

$$\begin{aligned}
[T]_{\mathcal{B}}[\vec{v}]_{\mathcal{B}} &= [T\vec{v}]_{\mathcal{B}} && \text{By 1(b)} \\
&= [\lambda\vec{v}]_{\mathcal{B}} \\
&= \lambda[\vec{v}]_{\mathcal{B}} && [\cdot]_{\mathcal{B}} \text{ is linear .}
\end{aligned}$$

   $\supseteq$: Let τ be an eigenvalue of $[T]_B$ with associated eigenvector $\vec{y}$. Since $[\cdot]_{\mathcal{B}}$ is an isomorphism, $\vec{y}$ has a unique preimage under the mapping, say $\vec{x}$ so that $[\vec{x}]_{\mathcal{B}} = \vec{y}$. Recall that $[\cdot]_{\mathcal{B}}$ also must have an inverse. Denote this $[\cdot]_{\mathcal{B}}^{-1}$ for lack of better notation.

$$\begin{aligned}
T\vec{x} &= [[T\vec{x}]_{\mathcal{B}}]_{\mathcal{B}}^{-1} \\
&= [[T]_{\mathcal{B}}[\vec{x}]_{\mathcal{B}}]_{\mathcal{B}}^{-1} && \text{Again by 1(b)} \\
&= [[T]_{\mathcal{B}}\vec{y}]_{\mathcal{B}}^{-1} \\
&= [\tau\vec{y}]_{\mathcal{B}}^{-1} \\
&= \tau[\vec{y}]_{\mathcal{B}}^{-1} && [\cdot]_{\mathcal{B}} \text{ is linear} \\
&= \tau\vec{x}.
\end{aligned}$$

   Therefore $\sigma(T) = \sigma([T]_{\mathcal{B}})$.

4. Let A be an $n \times n$ complex matrix with $\sigma(A) = \{1\}$. Show that A is diagonalizable if and only if A is the identity matrix.

   $\implies$: Let A be a diagonalizable matrix and $\sigma(A) = \{1\}$. Then there exists some invertible S so that $S^{-1}AS = D = \mathrm{diag}\{1,\dots,1\} = I$. Multiply both sides:

$$\begin{aligned}
S^{-1}AS &= I \\
SS^{-1}ASS^{-1} &= SIS^{-1} \\
A &= SS^{-1} \\
A &= I.
\end{aligned}$$

   $\impliedby$: Conversely, if $A = I$, then take the invertible matrix I, so that $IAI^{-1} = A = I$, and since I is diagonal, A is diagonalizable.

5. Determine whether or not the derivative map $D : P_n \to P_n$ given by $Dp(z) = p'(z)$ is diagonalizable.

Claim: The derivative map defined above is nilpotent; the $k+1$-th derivative of Proceed by induction on the degree of p. If p has degree 0, then p is constant and has zero derivative, and as such, any subsequent derivative will be zero.

Solution: The derivative map is nilpotent on polynomial spaces; hence it is not diagonalizable.