Exercise 1

Assume that μ is a measure on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$, which satisfies that

$$\mu((-\infty, x]) = \mu([x, \infty)) < \infty \quad \text{for all } x \in \mathbb{R}.$$

Show then that $\mu(B) = \mu(-B)$ for any Borel set B.

Solution: Suppose μ is a measure as above. Define a new measure ν by $\nu(B) = \mu(-B)$.

It is not too difficult to see ν is a measure; since $\nu(\emptyset) = \mu(-\emptyset) = \mu(\emptyset) = 0$ And, if $(A_n)_{n\in\mathbb{N}}$ is a sequence of subsets of $\mathbb{R}$, we have:

$$\nu\left(\bigcup_{n\in\mathbb{N}} A_n\right) = \mu\left(-\bigcup_{n\in\mathbb{N}} A_n\right) = \mu\left(\bigcup_{n\in\mathbb{N}} -A_n\right) = \sum_{n\in\mathbb{N}} \mu(-A_n) = \sum_{n\in\mathbb{N}} \nu(A_n).$$

So ν is a measure. We wish to apply Theorem 2.2.2 on the system $\mathcal{S}$:

$$\mathcal{S} = \{(-\infty, x] : x \in \mathbb{R}\}.$$

It's discussed in the textbook that the system is $\cap$-stable, and that the system generates $\mathcal{B}(\mathbb{R})$.

From the assumption we have:

$$\mu((-\infty, x]) = \mu([x, \infty)) = \mu(-(-\infty, x]) = \nu((-\infty, x]).$$

So μ, ν agree on $\mathcal{S}$ (and are finite). They also agree on the sequence $A_n = (-\infty, n]$ in $\mathcal{S}$, which has $\bigcup_{n\in\mathbb{N}} A_n = \mathbb{R}$.

Since all the conditions in 2.2.2 are satisfied, we have:

$$\mu(B) = \nu(B) = \mu(-B).$$