1. Find three examples of functors not mentioned above.
2. Show that functors preserve isomorphism. That is, prove that if $F : \mathscr{A} \to \mathscr{B}$ is a functor and $A, A' \in \mathscr{A}$ with $A \cong A'$, then $F(A) \cong F(A')$.

   Proof: Suppose $F : \mathscr{A} \to \mathscr{B}$ is a functor, and $A \cong A'$ in $\mathscr{A}$. Then there exists a pair of morphisms $f : A \to A'$ and $g : A' \to A$ with $fg = 1_{A'}$ and $gf = 1_A$. And, the functor F gives another pair of morphisms Ff, Fg. Verify:
   $$(Ff)(Fg) = F(fg) = F1_{A'} = 1_{FA'}$$
   and likewise:
   $$(Fg)(Ff) = F(gf) = F1_A = 1_{FA}.$$
   And so we have $FA \cong FA'$. ■

3. Two categories $\mathscr{A}$ and $\mathscr{B}$ are isomorphic, written as $\mathscr{A} \cong \mathscr{B}$, if they are isomorphic as objects of $\mathbf{Cat}$.

   (a) Let G be a group, regarded as a one-object category all of whose maps are isomorphisms. Then its opposite G^{op} is also a one-object category all of whose maps are isomorphisms, and can therefore be regarded as a group too. What is G^{op}, in purely group-theoretic terms? Prove that G is isomorphic to G^{op}.

   Proof: Take the functors $F : G \to G^{op}$, and $F' : G^{op} \to G$. Define, for $g \in G$ and $h^{op} \in G^{op}$:
   $$F(g) = (g^{-1})^{op}, \qquad F'(h^{op}) = h^{-1}.$$
   We first check that these functors compose to identity:
   $$\begin{aligned} FF'(g^{op}) &= F(g^{-1}) \\ &= ((g^{-1})^{-1})^{op} \\ &= g^{op} \\ FF' &= 1_{G^{op}} \\ F'F(g) &= F'((g^{-1})^{op}) \\ &= (g^{-1})^{-1} \\ &= g \\ F'F &= 1_G. \end{aligned}$$
   And then we check that these mappings are indeed functors. Clearly F, F' map the single object in G to G^{op}, and vice versa. Then we check the morphism identities for F and F'. Let $g, h \in G$;
   $$\begin{aligned} F(gh) &= \left((gh)^{-1}\right)^{op} \\ &= \left(h^{-1}g^{-1}\right)^{op} \\ &= \left(g^{-1}\right)^{op}\left(h^{-1}\right)^{op} \\ &= F(g)F(h). \end{aligned}$$
   Then, if $g^{op}, h^{op} \in G^{op}$;
   $$\begin{aligned} F'(g^{op}h^{op}) &= F'((hg)^{op}) \\ &= (hg)^{-1} \\ &= g^{-1}h^{-1} \\ &= F(g^{op})F(h^{op}). \end{aligned}$$
   And all that is left to verify is that F, F' send identities to identities. Let $g \in G$, and $g^{op} \in G^{op}$. We wish to show that $F(1_G) = (1_G)^{op} = 1_{G^{op}}$, and that $F'(1_{G^{op}}) = 1_G$. Take $g^{op} \in G^{op}$, which we know to have a preimage g^{-1} under F.
   $$\begin{aligned} (1_G)^{op}g^{op} &= F(1_G)g^{op} \\ &= F(1_G)F(g^{-1}) \\ &= F(1_G g^{-1}) \\ &= F(g^{-1}) \\ &= g^{op}. \end{aligned}$$

And so $1_{G^{op}} = (1_G)^{op} = F(1_G)$ (Since identity of right composition follows from the same argument). Now for $g \in G$,

$$\begin{aligned} F'(1_{G^{op}}) &= F'((1_G)^{op}) \\ &= 1_G^{-1} \\ &= 1_G. \end{aligned}$$

So F and F' are functors which serve as inverses for one another, and $G \cong G^{op}$. ■

(b) Find a monoid which is not isomorphic to its opposite.

Solution: Take $\mathbb{N}$,

4. Of the functors appearing in this section, which are faithful and which are full?

5. Give an example of a functor that is full, faithful, both, and neither.

Solution:

(a) The forgetful functor $F : \text{CRing} \to \text{Ring}$ that forgets commutativity is faithful, for distinct commutative rings will necessarily map to distinct rings. However it is not full; there exist rings which are not commutative ($\mathbb{M}_2(\mathbb{R})$)

(b) For a full but not faithful functor, we can take the categorical representation of the trivial group, and a functor $F : \text{Set} \to \{e\}$, which maps every $X \in \text{Set}$ to the single object, and morphisms map to the identity.

(c) A functor which is neither full nor faithful, we take $F : \text{Set} \to \text{Set}$ defined by $F(X) = \varnothing$ for any $X \in \text{Set}$, and $F(f) = 1_\varnothing$ for any morphism in Set

(d) The functors in the group exercise is both full and faithful, being bijections between the set of morphisms in G and G^{op}.

6. Let A and B be sets, and denote B^A the set of functions from A to B. Write down:

(a) a canonical function $A \times B^A \to B$;

Solution: The God-Given function from $A \times B^A \to B$ is the function $f : A \times B^A \to B$, given by $f(a, g) = g(a)$ where $g : A \to B$.

(b) a canonical function $A \to B^{(B^A)}$.

Solution: The God-Given function from $A \to B^{(B^A)}$ is the function $h(a)$, which for any $a \in A$ corresponds to a function ev_a, which takes a function from $A \to B$ and outputs its value at a. That is, $h(a)$ gives the evaluation map on B^A at a.

7. In this exercise, you will prove Proposition 1.3.18. Let $F : \mathscr{A} \to \mathscr{B}$ be a functor.

(a) Suppose that F is an equivalence. Prove that F is full, faithful and essentially surjective on objects. (Hint: prove faithfulness before fullness.)

Proof: Suppose that F is an equivalence. Then there exists a functor $G : \mathscr{B} \to \mathscr{A}$, and natural isomorphisms $\eta : 1_{\mathscr{A}} \to GF$ and $\varepsilon : FG \to 1_{\mathscr{B}}$.

$$\begin{tikzcd} A \arrow[r, "1_{\mathscr{A}} f = f"] \arrow[d, "\eta_A"] & A' \arrow[d, "\eta'_A"] \\ GF(A) \arrow[r, "GF(f)"] & GF(A') \end{tikzcd} \qquad \begin{tikzcd} F(A) \arrow[r, "F(f)"] \arrow[d, "\alpha_A"'] & F(A') \arrow[d, "\alpha_{A'}"] \\ G(A) \arrow[r, "G(f)"'] & G(A') \end{tikzcd}.$$

■

Faithfulness:

(b) Now suppose instead that F is full, faithful and essentially surjective on objects. For each $B \in \mathscr{B}$, choose an object $G(B)$ of $\mathscr{A}$ and an isomorphism $\varepsilon_B : F(G(B)) \to B$. Prove that G extends to a functor in such a way that $(\varepsilon_B)_{B\in\mathscr{B}}$ is a natural isomorphism $FG \to 1_B$. Then construct a natural isomorphism $1_A \to GF$, thus proving that F is an equivalence.

8. Kristaps' favorite: If you understood the "groupoid with one object" example, determine what functors between two such groupoids correspond to in terms of groups. Then, determine what natural transformations correspond to.

Solution: Let G, H be groupoids with one object, and ϕ, ψ be functors $G \to H$. We know already for any elements g, g' of the group G (Morphisms in the sigle object category),

$$\phi(gg') = \phi(g)\phi(g').$$

Which we know already to be the identity required by a group homomorphism. Then let α be a natural transformation:

$$A^F_G\alpha \qquad B$$