

1. Let  $V$  and  $W$  be finite dimensional vector spaces with given bases  $\mathcal{B} = \{\vec{b}_1, \dots, \vec{b}_n\}$  and  $\mathcal{D} = \{\vec{d}_1, \dots, \vec{d}_m\}$ , respectively.
- (a) For a given  $\vec{x} \in V$ , there are unique scalars so that  $\vec{x} = a_1\vec{b}_1 + \dots + a_n\vec{b}_n$ . Define the vector  $[\vec{x}]_{\mathcal{B}} := (a_1, \dots, a_n)^T \in \mathbb{C}^n$ . Show that the map  $\vec{x} \mapsto [\vec{x}]_{\mathcal{B}}$  is a linear isomorphism from  $V$  into  $\mathbb{C}^n$ .

**Linearity:** Let  $\vec{x}, \vec{y} \in V$ . Then write  $\vec{x} = a_1\vec{b}_1 + \dots + a_n\vec{b}_n$  and  $\vec{y} = c_1\vec{b}_1 + \dots + c_n\vec{b}_n$ . Now:

$$[\vec{x} + \vec{y}]_{\mathcal{B}} = \begin{bmatrix} a_1 + c_1 \\ \vdots \\ a_n + c_n \end{bmatrix} = \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} + \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} = [\vec{x}]_{\mathcal{B}} + [\vec{y}]_{\mathcal{B}}$$

$$[\alpha\vec{x}]_{\mathcal{B}} = \begin{bmatrix} \alpha a_1 \\ \vdots \\ \alpha a_n \end{bmatrix} = \alpha \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} = \alpha [\vec{x}]_{\mathcal{B}}.$$

So  $[\cdot]_{\mathcal{B}}$  is linear.

**Isomorphism:** Since  $\dim V = \dim \mathbb{C}^n = n$ , it will suffice to show that this mapping is injective. We do so by showing  $\ker[\cdot]_{\mathcal{B}} = \{0\}$ . Clearly  $0$  is in the kernel since  $[0]_{\mathcal{B}} = [0\vec{x}]_{\mathcal{B}} = 0[\vec{x}]_{\mathcal{B}} = 0$ . For inclusion the other way, let  $\vec{x} \in \ker[\cdot]_{\mathcal{B}}$ . Then  $[\vec{x}]_{\mathcal{B}} = 0$ ; meaning the basis representation of  $\vec{x}$  is through zero coefficients; and

$$\vec{x} = 0\vec{b}_1 + \dots + 0\vec{b}_n = 0.$$

So  $\ker[\cdot]_{\mathcal{B}} = \{0\}$ , and this map is injective. But since the spaces are of the same dimension it must also be surjective thanks to Rank-Nullity. So the map is a linear isomorphism from  $V$  to  $\mathbb{C}^n$ .

- (b) Let  $T : V \rightarrow W$  be a linear map. In class, we defined the matrix representation of  $T$  with respect to  $\mathcal{B}$  and  $\mathcal{D}$  as the  $m \times n$  matrix  $[T]_{\mathcal{B}\mathcal{D}} = [[T\vec{b}_1]_{\mathcal{D}}, \dots, [T\vec{b}_n]_{\mathcal{D}}]$ . In other words, the  $j$ -th column of  $[T]_{\mathcal{B}\mathcal{D}}$  is  $[T\vec{b}_j]_{\mathcal{D}}$ . Show that  $[T]_{\mathcal{B}\mathcal{D}}[\vec{x}]_{\mathcal{B}} = [T\vec{x}]_{\mathcal{D}}$  for any  $\vec{x} \in V$ .

**Solution:** Let  $T : V \rightarrow W$  be linear, then write  $\vec{x} = a_1\vec{b}_1 + \dots + a_n\vec{b}_n$ .

$$\begin{aligned} [T]_{\mathcal{B}\mathcal{D}}[\vec{x}]_{\mathcal{B}} &= [[T\vec{b}_1]_{\mathcal{D}} \dots [T\vec{b}_n]_{\mathcal{D}}] \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} \\ &= a_1[T\vec{b}_1]_{\mathcal{D}} + \dots + a_n[T\vec{b}_n]_{\mathcal{D}} \\ &= [a_1T\vec{b}_1 + \dots + a_nT\vec{b}_n]_{\mathcal{D}} && \text{By linearity of } [\cdot]_{\mathcal{D}} \\ &= [T(a_1\vec{b}_1 + \dots + a_n\vec{b}_n)]_{\mathcal{D}} && \text{By linearity of } T \\ &= [T\vec{x}]_{\mathcal{D}}. \end{aligned}$$

- (c) Show that  $[T]_{\mathcal{BD}}$  is a linear isomorphism from  $L(V, W)$  (the vector space of linear maps from  $V$  to  $W$ ) to  $M_{mn}(\mathbb{C})$  (vector space of  $m \times n$  complex matrices).

**Linearity:** Let  $T, S$  be linear from  $V$  to  $W$ . Then:

$$\begin{aligned} [T + S]_{\mathcal{BD}} &= [(T + S)\vec{b}_1]_{\mathcal{D}} \dots [(T + S)\vec{b}_n]_{\mathcal{D}} \\ &= [(T\vec{b}_1 + S\vec{b}_1)]_{\mathcal{D}} \dots [(T\vec{b}_n + S\vec{b}_n)]_{\mathcal{D}} \\ &= [T\vec{b}_1]_{\mathcal{D}} \dots [T\vec{b}_n]_{\mathcal{D}} + [S\vec{b}_1]_{\mathcal{D}} \dots [S\vec{b}_n]_{\mathcal{D}} \quad \text{By Linearity of } [\cdot]_{\mathcal{D}} \\ &= [S]_{\mathcal{BD}} + [T]_{\mathcal{BD}}. \end{aligned}$$

And then letting  $\alpha \in \mathbb{C}$ ,

$$\begin{aligned} \alpha[T]_{\mathcal{BD}} &= \alpha[T\vec{b}_1]_{\mathcal{D}} \dots [T\vec{b}_n]_{\mathcal{D}} \\ &= [\alpha T\vec{b}_1]_{\mathcal{D}} \dots [\alpha T\vec{b}_n]_{\mathcal{D}} \quad \text{Linearity of } [\cdot]_{\mathcal{D}} \\ &= [\alpha T\vec{b}_1]_{\mathcal{D}} \dots [\alpha T\vec{b}_n]_{\mathcal{D}} \\ &= [\alpha T]_{\mathcal{BD}}. \end{aligned}$$

**Injective:** Clearly  $0 \in \ker[\cdot]_{\mathcal{BD}}$ ; take any transformation  $T$  and  $[0]_{\mathcal{BD}} = [T - T]_{\mathcal{BD}} = [T]_{\mathcal{BD}} - [T]_{\mathcal{BD}} = 0$ .

Then let  $\mathcal{B} = \{b_1, \dots, b_n\}$  and  $T \in \ker[\cdot]_{\mathcal{BD}}$ . Then:

$$\begin{aligned} [T]_{\mathcal{BD}} &= 0 \\ [[Tb_1]_{\mathcal{D}} \dots [Tb_n]_{\mathcal{D}}] &= [0 \dots 0]. \end{aligned}$$

Then  $[Tb_i]_{\mathcal{D}} = 0$  for any basis vector  $b_i$ . In particular this means that  $Tb_i = 0$ , since  $[\cdot]_{\mathcal{D}}$  is an isomorphism. Now for any arbitrary  $v \in V$ , write  $v = a_1b_1 + \dots + a_nb_n$ . Then  $Tv = T(a_1b_1 + \dots + a_nb_n) = a_1Tb_1 + \dots + a_nTb_n = 0 + \dots + 0 = 0$  and  $T = 0$ .

Therefore  $\ker[\cdot]_{\mathcal{BD}} = \{0\}$ .

**Surjective:** The argument that  $\dim L(V, W) = \dim M_{mn}(\mathbb{C})$  proves difficult, so instead we show directly that  $[L(V, W)]_{\mathcal{BD}} = M_{mn}(\mathbb{C})$ .

Let  $A \in M_{mn}(\mathbb{C})$ , and write  $A = [\vec{a}_1 \dots \vec{a}_n]$ , where  $\vec{a}_j$  are column vectors in  $\mathbb{C}^m$ . Then take the inverse map for  $[\cdot]_{\mathcal{D}}$  (which was shown to exist in 1(a)), denote it  $[\cdot]_{\mathcal{D}}^{-1}$  and define  $T$  on the basis vectors in  $\mathcal{B}$  such that  $Tb_j = [\vec{a}_j]_{\mathcal{D}}^{-1}$ . Then:

$$\begin{aligned} [T]_{\mathcal{BD}} &= [[Tb_1]_{\mathcal{D}} \dots [Tb_n]_{\mathcal{D}}] \\ &= [[[\vec{a}_1]_{\mathcal{D}}^{-1}]_{\mathcal{D}} \dots [[\vec{a}_n]_{\mathcal{D}}^{-1}]_{\mathcal{D}}] \\ &= [\vec{a}_1 \dots \vec{a}_n] \\ &= A. \end{aligned}$$

So every arbitrary matrix has a preimage in the space of linear transformations, and therefore the mapping is onto. Since  $[\cdot]_{\mathcal{BD}}$  is bijective and linear, it must then be an isomorphism.

2. Let  $V$ ,  $W$  and  $U$  be finite dimensional vector spaces with given bases:

$\mathcal{B} = \{\vec{b}_1, \dots, \vec{b}_n\}$ ,  $\mathcal{D} = \{\vec{d}_1, \dots, \vec{d}_m\}$ , and  $\mathcal{F} = \{f_1, \dots, f_k\}$ , respectively. Suppose  $T : V \rightarrow W$  and  $S : W \rightarrow U$  are linear. Prove or disprove the following statement for the composition linear map  $ST : V \rightarrow U$ :

$$[ST]_{\mathcal{B}\mathcal{F}} = [S]_{\mathcal{D}\mathcal{F}}[T]_{\mathcal{B}\mathcal{D}}.$$

**Solution:** We make great use of the property shown in 1(b). Where it is used will be marked with (\*). Let  $v \in V$  be arbitrary and recall that  $[v]_{\mathcal{B}}$  is unique since  $[\cdot]_{\mathcal{B}}$  is an isomorphism.

$$\begin{aligned} [ST]_{\mathcal{B}\mathcal{F}}[v]_{\mathcal{B}} &= [STv]_{\mathcal{F}} & (*) \\ &= [S]_{\mathcal{D}\mathcal{F}}[Tv]_{\mathcal{D}} & (*) \\ &= [S]_{\mathcal{D}\mathcal{F}}[T]_{\mathcal{B}\mathcal{D}}[v]_{\mathcal{B}} & (*) \end{aligned}$$

So we have shown that these matrices  $[ST]_{\mathcal{B}\mathcal{F}}$  and  $[S]_{\mathcal{D}\mathcal{F}}[T]_{\mathcal{B}\mathcal{D}}$  agree upon all vectors in the image of  $[\cdot]_{\mathcal{B}}$ . However since this particular mapping is onto, we know this to be all of  $\mathbb{C}^n$ . This means the matrices agree upon all of  $\mathbb{C}^n$  and therefore they must be equal.

3. Let  $V$  be a finite dimensional vector space and  $T : V \rightarrow V$  be linear. Show that  $\sigma(T) = \sigma([T]_{\mathcal{B}})$  where  $\mathcal{B}$  is any basis for  $V$ .

**Linearity of inverse:** To show this we use the fact that  $[\cdot]_{\mathcal{B}}^{-1}$  is linear. Included is a brief demonstration of this fact. Let  $T, S \in L(V, V)$  and  $\alpha \in \mathbb{C}$ .

$$\begin{aligned} [T]_{\mathcal{B}}^{-1} + [S]_{\mathcal{B}}^{-1} &= [[T]_{\mathcal{B}}^{-1} + [S]_{\mathcal{B}}^{-1}]_{\mathcal{B}}^{-1} \\ &= [[T]_{\mathcal{B}}^{-1}]_{\mathcal{B}} + [[S]_{\mathcal{B}}^{-1}]_{\mathcal{B}}^{-1} \\ &= [T + S]_{\mathcal{B}}^{-1}. \end{aligned}$$

$$\begin{aligned} \alpha[T]_{\mathcal{B}}^{-1} &= [[\alpha[T]_{\mathcal{B}}^{-1}]_{\mathcal{B}}]_{\mathcal{B}}^{-1} \\ &= [\alpha[T]_{\mathcal{B}}^{-1}]_{\mathcal{B}}^{-1} \\ &= [\alpha T]_{\mathcal{B}}^{-1}. \end{aligned}$$

**Solution:**  $\subseteq$ : Let  $\lambda \in \sigma(T)$ , and let  $\vec{v}$  be an associated eigenvector. We show that  $[\vec{v}]_{\mathcal{B}}$  is an eigenvector for  $\lambda$  under  $[T]_{\mathcal{B}}$ .

$$\begin{aligned} [T]_{\mathcal{B}}[\vec{v}]_{\mathcal{B}} &= [T\vec{v}]_{\mathcal{B}} && \text{By 1(b)} \\ &= [\lambda\vec{v}]_{\mathcal{B}} \\ &= \lambda[\vec{v}]_{\mathcal{B}} && [\cdot]_{\mathcal{B}} \text{ is linear.} \end{aligned}$$

$\supseteq$ : Let  $\tau$  be an eigenvalue of  $[T]_{\mathcal{B}}$  with associated eigenvector  $\vec{y}$ . Since  $[\cdot]_{\mathcal{B}}$  is an isomorphism,  $\vec{y}$  has a unique preimage under the mapping, say  $\vec{x}$  so that  $[\vec{x}]_{\mathcal{B}} = \vec{y}$ . Recall that  $[\cdot]_{\mathcal{B}}$  also must have an inverse. Denote this  $[\cdot]_{\mathcal{B}}^{-1} : M_{mn}(\mathbb{C}) \rightarrow L(V, W)$ .

$$\begin{aligned} T\vec{x} &= [[T\vec{x}]_{\mathcal{B}}]_{\mathcal{B}}^{-1} \\ &= [[T]_{\mathcal{B}}[\vec{x}]_{\mathcal{B}}]_{\mathcal{B}}^{-1} && \text{Again by 1(b)} \\ &= [[T]_{\mathcal{B}}\vec{y}]_{\mathcal{B}}^{-1} \\ &= [\tau\vec{y}]_{\mathcal{B}}^{-1} \\ &= \tau[\vec{y}]_{\mathcal{B}}^{-1} && [\cdot]_{\mathcal{B}} \text{ is linear} \\ &= \tau\vec{x}. \end{aligned}$$

Therefore  $\sigma(T) = \sigma([T]_{\mathcal{B}})$ .

4. Let  $A$  be an  $n \times n$  complex matrix with  $\sigma(A) = \{1\}$ . Show that  $A$  is diagonalizable if and only if  $A$  is the identity matrix.

$\Rightarrow$  : Let  $A$  be a diagonalizable matrix and  $\sigma(A) = \{1\}$ . Then there exists some invertible  $S$  so that  $S^{-1}AS = D = \text{diag}\{1, \dots, 1\} = I$ . Multiply both sides:

$$\begin{aligned} S^{-1}AS &= I \\ SS^{-1}ASS^{-1} &= SIS^{-1} \\ A &= SS^{-1} \\ A &= I. \end{aligned}$$

$\Leftarrow$  : Conversely, if  $A = I$ , then take the invertible matrix  $I$ , so that  $IAI^{-1} = A = I$ , and since  $I$  is diagonal,  $A$  is diagonalizable.

5. Determine whether or not the derivative map  $D : P_n \rightarrow P_n$  given by  $Dp(z) = p'(z)$  is diagonalizable.

**Claim:** The derivative map defined above is nilpotent; the  $k + 1$ -th derivative of a polynomial of degree  $k \in \mathbb{C}[x]$  is identically zero.

*Proof of claim:* Proceed by induction on the degree of  $p$ .

**Base case:** If  $p$  has degree 0, then  $p$  is constant and has zero derivative, and as such, any subsequent derivative will be zero.

**Inductive hypothesis:** Suppose that the  $k$ -th derivative of any  $p \in \mathbb{C}[x]$  with  $\deg p = k - 1$  is identically zero.

**Inductive step:** Let  $f(x) \in \mathbb{C}[x]$  be of degree  $k$ . Write  $f(x) = a_0 + a_1x + \dots + a_kx^k$  for complex  $a_i$ . Then  $Df(x) = a_1 + 2a_2x + \dots + ka_kx^{k-1}$ . And since this polynomial  $Df$  is of degree  $k - 1$ , by the inductive hypothesis the  $k$ -th derivative of this must be zero, and  $D^{k+1}f = D(D^kf) = 0$ . Therefore by induction on  $\deg f$ , the derivative operator is nilpotent on  $\mathbb{C}[x]$   $\square$

**Solution:** The derivative map is nilpotent on  $\mathbb{C}[x]$ , and since nilpotent operators are not diagonalizable, the derivative operator is not diagonalizable.