## Exercise 1

*Compute the Lebesgue integrals of the following functions:*

$$f_1(x) = x^2, \qquad f_2(x) = \frac{1}{1+x^2}, \qquad f_3(x) = e^{-|x|}.$$

As suggested, we use the property:

$$\int f(x)\mathbb{1}_{[a,b]}\,d\lambda = \int_a^b f(x)\,dx.$$

Along with the convergent increasing sequence of real functions:

$$\lim_{n\to\infty} \mathbb{1}_{[-n,n]} = 1.$$

Starting with $f_1$:

$$\begin{aligned}
\int f_1(x)\,d\lambda &= \int \lim_{n\to\infty} f_1(x)\mathbb{1}_{[-n,n]}\,d\lambda \\
&= \lim_{n\to\infty} \int x^2\mathbb{1}_{[-n,n]}\,d\lambda && \text{By Monotone convergence} \\
&= \lim_{n\to\infty} \mathcal{R}\int_{-n}^{n} x^2\,dx \\
&= \lim_{n\to\infty} \left(\frac{x^3}{3}\right)_{-n}^{n} \\
&= \lim_{n\to\infty} \left(\frac{n^3}{3} - \frac{-n^3}{3}\right) \\
&= \infty + \infty \\
&= \infty.
\end{aligned}$$

And for $f_2$:

$$\begin{aligned}
\int f_2(x)\,d\lambda &= \int \lim_{n\to\infty} \frac{1}{1+x^2}\mathbb{1}_{[-n,n]}\,d\lambda \\
&= \lim_{n\to\infty} \int \frac{1}{1+x^2}\mathbb{1}_{[-n,n]}\,d\lambda && \text{By Monotone Convergence} \\
&= \lim_{n\to\infty} \mathcal{R}\int_{-n}^{n} \frac{1}{1+x^2}\,dx \\
&= \lim_{n\to\infty} (\arctan x)_{-n}^{n} \\
&= lim\,(\arctan n - \arctan -n) \\
&= \frac{\pi}{2} - \left(-\frac{\pi}{2}\right) \\
&= \pi.
\end{aligned}$$

Finally for $f_3$

$$\begin{aligned}
\int f_3(x)\,d\lambda &= \int \lim_{n\to\infty} e^{-|x|}\mathbb{1}_{[-n,n]}\,d\lambda \\
&= \lim_{n\to\infty} \int e^{-|x|}\mathbb{1}_{[-n,n]}\,d\lambda && \text{By Monotone Convergence} \\
&= \lim_{n\to\infty} \mathcal{R}\int_{-n}^{n} e^{-|x|}\,dx \\
&= \lim_{n\to\infty} \left(\mathcal{R}\int_{-n}^{0} e^{x}\,dx + \mathcal{R}\int_{0}^{n} e^{-x}\,dx\right) \\
&= \lim_{n\to\infty} (e^x)_{-n}^{0} + \left(-e^{-x}\right)_0^n \\
&= \lim_{n\to\infty} e^0 - e^{-n} - e^{-n} + e^0 \\
&= 1 - 0 - 0 + 1 \\
&= 2.
\end{aligned}$$