1. Let $\{f_n\}$ be the sequence of functions defined by

$$f_n = \begin{cases} nx & \text{If } 0 \le x \le \frac{1}{n} \\ 1 & \text{If } \frac{1}{n} < x < 1 - \frac{1}{n} \\ n - nx & \text{If } 1 - \frac{1}{n} \le x \le 1 \end{cases}.$$

   (a) Find the pointwise limit f of the sequence.

   Solution: Proceed by cases. If $x = 0$, then the first case of the function will always be taken since $0 \le x$. So $f_n(0) = n0 = 0$. Likewise if $x = 1$, then $f(1) = n - n1 = n - n = 0$.
   Now, if $x \in (0,1)$, then we observe that $\frac{1}{n} \to 0$, and $1 - \frac{1}{n} \to 1$. Therefore the middle case of our piecewise function gives us $f(x) = 1$ for all x in this open interval.

   (b) Does $f_n \xrightarrow[{[0,1]}]{c.u} f$? Justify your answer.

   Solution: This sequence is not uniformly continuous. Pick $\varepsilon = \frac{1}{3}$, and let $N \in \mathbb{N}$, and $n > N$. Pick $x = \frac{1}{2n}$ so that $0 \le x \le \frac{1}{n}$, and then $f_n(x) = nx = \frac{n}{2n} = \frac{1}{2}$. Then: $|f_n(x) - f(x)| = \left|\frac{1}{2} - 1\right| = \frac{1}{2} > \varepsilon$.
   Therefore the sequence is not uniformly continuous.

2. Let $f_n(x) = \left(\cos\left(\frac{2x}{n}\right)\right)^{n^2}$

   (a) Compute the pointwise limit f of the sequence $\{f_n\}$.

   (b) Show that $f_n \xrightarrow[{[0,1]}]{c.u} f$.

3. Let $a \in \mathbb{R}_+$. Compute the limit

$$\lim_{n\to\infty} \int_a^\pi \frac{\sin(nx)}{nx}\, dx.$$

   What happens if $a = 0$?

   We begin by considering our sequence of functions within the integral, each of which is a quotient of continuous functions, and is itself continuous (for all but $x = 0$). Call this $g_n(x) = \frac{\sin(nx)}{nx}$. Note that since $-1 \le \sin(nx) \le 1$, we can find (for nonzero x) that $-\frac{1}{nx} \le g_n(x) \le \frac{1}{nx}$. Both the sequences bounding g have a zero limit at infinity, so by the squeeze theorem on sequences of functions, we can say that for nonzero x, $g_n \to 0$. Now since we have already shown that our sequence g_n is bounded, and since each g_n is integrable, we can say:

$$\begin{aligned} \lim_{n\to\infty} \int_a^\pi \frac{\sin(nx)}{nx}\, dx &= \int_a^\pi \lim_{n\to\infty} \frac{\sin(nx)}{nx}\, dx \\ &= \int_a^\pi 0\, dx \\ &= 0 - 0 \\ &= 0. \end{aligned}$$

4. Construct a sequence of functions defined in $[0,1]$, each of which is discontinuous at every point of $[0,1]$ and which converges uniformly to a function that is continuous at every point

Solution: Take the series $\{f_n\}$ defined by:

$$f_n(x) = \begin{cases} \frac{1}{n} & \text{If } x \in \mathbb{Q} \\ 0 & \text{Otherwise} \end{cases}$$

Claim: $\{f_n\}$ converges uniformly to the constant function 0, which we know to be continuous on the real line, as well as $[0,1]$.

Let $\varepsilon > 0$, and choose (By Archimedian Principle), N such that $0 < \frac{1}{N} < \varepsilon$. Then any $n \geq N$ will have $0 < \frac{1}{n} < \frac{1}{N} < \varepsilon$. Now by cases, if $x \in \mathbb{Q}$, then we have

$$|f_n(x) - f(x)| = \left|\frac{1}{n} - 0\right| = \frac{1}{n} < \frac{1}{N} < \varepsilon.$$

And for $x \notin \mathbb{Q}$,

$$|f_n(x) - f(x)| = |0 - 0| = 0 < \varepsilon.$$

Therefore $\{f_n\}$ is a sequence of functions which are continuous nowhere, convergent to the zero function which is continuous everywhere.

5. Consider the series of functions $\sum_{n\geq 1} \frac{x}{n(n+x)}$.

   (a) Show that the series converges uniformly in the interval $[0, b]$ for any $b > 0$.

   Solution:

$$\begin{aligned} \frac{x}{n(n+x)} &= \frac{x}{n^2 + nx} \\ &\leq \frac{x}{n^2} \\ &\leq \frac{b}{n^2}. \end{aligned}$$

   Define $u_n = \frac{b}{n^2}$, then by the Weierstrass Comparison test, since $\sum_{n\geq 1} u_n$ is convergent as a p-series with $p = 2$, $\frac{x}{n(n+x)} \leq u_n$, this series must converge.

   (b) Let $F(x) = \sum_{n\geq 1} \frac{x}{n(n+x)}$. Show that $F'(x) = \sum_{n\geq 1} \frac{1}{n(n+x)^2}$, $x \geq 0$.

6. Consider the series of functions $\sum_{n\geq 1} \frac{x}{1+n^2x^2}$. Show that the series doesn't converge uniformly in $\mathbb{R}_+$.

   Hint: You could start by showing that $\frac{x}{1+n^2x^2} \geq \int_n^{n+1} \frac{x}{1+t^2x^2}\, dt, \quad \forall x \in \mathbb{R}$.