1. Let $V$ and $W$ be finite dimensional vector spaces with given bases $\mathcal{B} = \{\vec{b}_1, \ldots, \vec{b}_n\}$ and $\mathcal{D} = \{\vec{d}_1, \ldots, \vec{d}_m\}$, respectively.

   (a) For a given $\vec{x} \in V$, there are unique scalars so that $\vec{x} = a_1\vec{b}_1 + \cdots + a_n\vec{b}_n$. Define the vector $[\vec{x}]_\mathcal{B} := (a_1, \ldots, a_n)^T \in \mathbb{C}^n$. Show that the map $\vec{x} \mapsto [\vec{x}]_\mathcal{B}$ is a linear isomorphism from $V$ into $\mathbb{C}^n$.

   **Linearity:** Let $\vec{x}, \vec{y} \in V$. Then write $\vec{x} = a_1\vec{b}_1 + \cdots + a_n\vec{b}_n$ and $\vec{y} = c_1\vec{b}_1 + \cdots + c_n\vec{b}_n$. Now:

$$[\vec{x} + \vec{y}]_\mathcal{B} = \begin{bmatrix} a_1 + c_1 \\ \vdots \\ a_n + c_n \end{bmatrix} = \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} + \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} = [\vec{x}]_\mathcal{B} + [\vec{y}]_\mathcal{B}$$

$$[\alpha\vec{x}]_\mathcal{B} = \begin{bmatrix} \alpha a_1 \\ \vdots \\ \alpha a_n \end{bmatrix} = \alpha \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} = \alpha[\vec{x}]_\mathcal{B}.$$

   So $[\cdot]_\mathcal{B}$ is linear.

   **Isomorphism:** Since $\dim V = \dim \mathbb{C}^n = n$, it will suffice to show that this mapping is injective. We do so by showing $\ker[\cdot]_\mathcal{B} = \{0\}$. Clearly 0 is in the kernel since $[0]_\mathcal{B} = [0\vec{x}]_\mathcal{B} = 0[\vec{x}]_\mathcal{B} = 0$. For inclusion the other way, let $\vec{x} \in \ker[\cdot]_\mathcal{B}$. Then $[\vec{x}]_\mathcal{B} = 0$; meaning the basis representation of $\vec{x}$ is through zero coefficients; and

$$\vec{x} = 0\vec{b}_1 + \ldots + 0\vec{b}_n = 0.$$

   So $\ker[\cdot]_\mathcal{B} = \{0\}$, and this map is injective. But since the spaces are of the same dimension it must also be surjective. So the map is a linear isomorphism from $V$ to $\mathbb{C}^n$.

   (b) Let $T : V \to W$ be a linear map. In class, we defined the matrix representation of $T$ with respect to $\mathcal{B}$ and $\mathcal{D}$ as the $m \times n$ matrix $[T]_{\mathcal{B}\mathcal{D}} = [[T\vec{b}_1]_\mathcal{D}, \ldots, [T\vec{b}_n]_\mathcal{D}]$. In other words, the $j$-the column of $[T]_{\mathcal{B}\mathcal{D}}$ is $[T\vec{b}_j]_\mathcal{D}$. Show that $[T]_{\mathcal{B}\mathcal{D}}[\vec{x}]_\mathcal{B} = [T\vec{x}]_\mathcal{D}$ for any $\vec{x} \in V$.

   **Solution:** Let $T : V \to W$ be linear, then write $\vec{x} = a_1\vec{b}_1 + \ldots + a_n\vec{b}_n$.

$$\begin{aligned}
[T]_{\mathcal{B}\mathcal{D}}[\vec{x}]_\mathcal{B} &= \big[[T\vec{b}_1]_\mathcal{D} \ldots [T\vec{b}_n]_\mathcal{D}\big] \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} \\
&= a_1[T\vec{b}_1]_\mathcal{D} + \ldots + a_n[T\vec{b}_n]_\mathcal{D} \\
&= [a_1T\vec{b}_1 + \ldots + a_nT\vec{b}_n]_\mathcal{D} && \text{By linearity of } [\cdot]_\mathcal{D} \\
&= \big[T(a_1\vec{b}_1 + \ldots + a_n\vec{b}_n)\big]_\mathcal{D} && \text{By linearity of } T \\
&= [T\vec{x}]_\mathcal{D}.
\end{aligned}$$

   (c) Show that $[T]_{\mathcal{B}\mathcal{D}}$ is a linear isomorphism from $L(V, W)$ (the vector space of linear maps from $V$ to $W$) to $M_{mn}(\mathbb{C})$ (vector space of $m \times n$ complex matrices).

   **Linearity:** Let $T, S$ be linear from $V$ to $W$. Then:

$$\begin{aligned}
[T + S]_{\mathcal{B}\mathcal{D}} &= \big[[(T + S)\vec{b}_1]_\mathcal{D} \ldots [(T + S)\vec{b}_n]_\mathcal{D}\big] \\
&= \big[[(T\vec{b}_1 + S\vec{b}_1)]_\mathcal{D} \ldots [(T\vec{b}_n + S\vec{b}_n)]_\mathcal{D}\big] \\
&= \big[[T\vec{b}_1]_\mathcal{D} \ldots [T\vec{b}_n]_\mathcal{D}\big] + \big[[S\vec{b}_1]_\mathcal{D} \ldots [S\vec{b}_n]_\mathcal{D}\big] && \text{By Linearity of } [\cdot]_\mathcal{D} \\
&= [S]_{\mathcal{B}\mathcal{D}} + [T]_{\mathcal{B}\mathcal{D}}.
\end{aligned}$$

And then letting $\alpha \in \mathbb{C}$,

$$\begin{aligned}\alpha[T]_{\mathcal{BD}} &= \alpha\big[[T\vec{b}_1]_{\mathcal{D}} \dots [T\vec{b}_n]_{\mathcal{D}}\big] \\ &= \big[\alpha[T\vec{b}_1]_{\mathcal{D}} \dots \alpha[T\vec{b}_n]_{\mathcal{D}}\big] && \text{Linearity of } [\cdot]_{\mathcal{D}} \\ &= \big[[\alpha T\vec{b}_1]_{\mathcal{D}} \dots [\alpha T\vec{b}_n]_{\mathcal{D}}\big] \\ &= [\alpha T]_{\mathcal{BD}}.\end{aligned}$$

**Injective:** Clearly $0 \in \ker[\cdot]_{\mathcal{BD}}$; take any transformation $T$ and $[0]_{\mathcal{BD}} = [T - T]_{\mathcal{BD}} = [T]_{\mathcal{BD}} - [T]_{\mathcal{BD}} = 0$ .
Ten let $\mathcal{B} = \{b_1, \dots, b_n\}$ and $T \in \ker[\cdot]_{\mathcal{BD}}$. Then:

$$\begin{aligned}[T]_{\mathcal{BD}} &= 0 \\ [[Tb_1]_{\mathcal{D}} \dots [Tb_n]_{\mathcal{D}}] &= [0 \dots 0].\end{aligned}$$

Then $[Tb_i]_{\mathcal{D}} = 0$ for any basis vector $b_i$. In particular this means that $TB_i = 0$, since $[\cdot]_{\mathcal{D}}$ is an isomorphism. Now for any arbitrary $v \in V$, write $v = a_1b_1 + \dots a_nb_n$. Then $Tv = T(a_1b_1 + \dots a_nb_n) = a_1Tb_1 + \dots + a_nTb_n = 0 + \dots + 0 = 0$ and $T = 0$.
Therefore $\ker[\cdot]_{\mathcal{BD}} = \{0\}$.

**Surjective:** The argument that $\dim L(V, W) = \dim M_{mn}(\mathbb{C})$ proves difficult, so instead we show directly that $[L(V, W)]_{BD} = M_{mn}(\mathbb{C})$

2. Let $V$, $W$ and $U$ be finite dimensional vector spaces with given bases:
$\mathcal{B} = \{\vec{b}_1, \dots, \vec{b}_n\}, \mathcal{D} = \{\vec{d}_1, \dots, \vec{d}_m\}$, and $\mathcal{F} = \{f_1, \dots, f_k\}$, respectively. Suppose $T : V \to W$ and $S : W \to U$ are linear. Prove or disprove the following statement for the composition linear map $ST : V \to U$:

$$[ST]_{\mathcal{BF}} = [S]_{\mathcal{DF}}[T]_{\mathcal{BD}}.$$

**Solution:** We make great use of the property shown in 1(b). Where it is used will be marked with $(*)$. Let $v \in V$ be arbitrary and recall that $[v]_{\mathcal{B}}$ is unique since $[\cdot]_{\mathcal{B}}$ is an isomorphism.

$$\begin{aligned}[ST]_{\mathcal{BF}}[v]_{\mathcal{B}} &= [STv]_{\mathcal{F}} && (*) \\ &= [S]_{\mathcal{DF}}[Tv]_{\mathcal{D}} && (*) \\ &= [S]_{\mathcal{DF}}[T]_{\mathcal{BD}}[v]_{\mathcal{B}} && (*)\end{aligned}$$

So we have shown that these matrices $[ST]_{\mathcal{BF}}$ and $[S]_{\mathcal{DF}}[T]_{\mathcal{BD}}$ agree upon all vectors in the image of $[\cdot]_{\mathcal{B}}$. However since this particular mapping is onto, we know this to be all of $\mathbb{C}^n$. This means the matrices agree upon all of $\mathbb{C}^n$ and therefore they must be equal.

3. Let $V$ be a finite dimensional vector space and $T : V \to V$ be linear. Show that $\sigma(T) = \sigma([T]_{\mathcal{B}})$ where $\mathcal{B}$ is any basis for $V$.

**Solution:** $\subseteq$: Let $\lambda \in \sigma(T)$, and let $\vec{v}$ be an associated eigenvector. We show that $[\vec{v}]_{\mathcal{B}}$ is an eigenvector for $\lambda$ under $[T]_{\mathcal{B}}$.

$$\begin{aligned}[T]_{\mathcal{B}}[\vec{v}]_{\mathcal{B}} &= [T\vec{v}]_{\mathcal{B}} && \text{By 1(b)} \\ &= [\lambda\vec{v}]_{\mathcal{B}} \\ &= \lambda[\vec{v}]_{\mathcal{B}} && [\cdot]_{\mathcal{B}} \text{ is linear .}\end{aligned}$$

$\supseteq$: Let $\tau$ be an eigenvalue of $[T]_{\mathcal{B}}$ with associated eigenvector $\vec{y}$. Since $[\cdot]_{\mathcal{B}}$ is an isomorphism, $\vec{y}$ has a unique preimage under the mapping, say $\vec{x}$ so that $[\vec{x}]_{\mathcal{B}} = \vec{y}$. Recall that $[\cdot]_{\mathcal{B}}$ also must have an inverse. Denote this $[\cdot]_{\mathcal{B}}^{-1}$.

$$\begin{aligned}
T\vec{x} &= [[T\vec{x}]_{\mathcal{B}}]_{\mathcal{B}}^{-1} \\
&= [[T]_{\mathcal{B}}[\vec{x}]_{\mathcal{B}}]_{\mathcal{B}}^{-1} && \text{Again by 1(b)} \\
&= [[T]_{\mathcal{B}}\vec{y}]_{\mathcal{B}}^{-1} \\
&= [\tau\vec{y}]_{\mathcal{B}}^{-1} \\
&= \tau[\vec{y}]_{\mathcal{B}}^{-1} && [\cdot]_{\mathcal{B}} \text{ is linear} \\
&= \tau\vec{x}.
\end{aligned}$$

Therefore $\sigma(T) = \sigma([T]_{\mathcal{B}})$.

4. Let $A$ be an $n \times n$ complex matrix with $\sigma(A) = \{1\}$. Show that $A$ is diagonalizable if and only if $A$ is the identity matrix.

$\implies$ : Let $A$ be a diagonalizable matrix and $\sigma(A) = \{1\}$. Then there exists some invertible $S$ so that $S^{-1}AS = D = \operatorname{diag}\{1, \dots, 1\} = I$. Multiply both sides:

$$\begin{aligned}
S^{-1}AS &= I \\
SS^{-1}ASS^{-1} &= SIS^{-1} \\
A &= SS^{-1} \\
A &= I.
\end{aligned}$$

$\Longleftarrow$**:** Conversely, if $A = I$, then take the invertible matrix $I$, so that $IAI^{-1} = A = I$, and since $I$ is diagonal, $A$ is diagonalizable.

5. Determine whether or not the derivative map $D : P_n \to P_n$ given by $Dp(z) = p'(z)$ is diagonalizable.

**Claim:** The derivative map defined above is nilpotent; the $k+1$-th derivative of a polynomial of degree $k \in C[x]$ is identically zero.

*Proof of claim:* Proceed by induction on the degree of $p$.

**Base case:** If $p$ has degree 0, then $p$ is constant and has zero derivative, and as such, any subsequent derivative will be zero.

**Inductive hypothesis:** Suppose that the $k$-th derivative of any $p \in \mathbb{C}[x]$ with $\deg p = k - 1$ is identically zero.

**Inductive step:** Let $f(x) \in \mathbb{C}[x]$ be of degree $k$. Write $f(x) = a_0 + a_1x + \dots + a_kx^k$ for complex $a_i$ . Then $Df(x) = a_1 + 2a_2 + \dots + ka_nx^{k-1}$. And since this polynomial $Df$ is of degree $k-1$, by the inductive hypothesis the $k$-th derivative of this must be zero, and $D^{k+1}f = D(D^kf) = 0$. Therefore by induction on $\deg f$, the derivative operator is nilpotent on $\mathbb{C}[x]$ $\square$

**Solution:** The derivative map is nilpotent on $\mathbb{C}[x]$, and since nilpotent operators are not diagonalizable, the derivative operator is not diagonalizable