1. Construct an explicit deformation retraction of the torus with one point deleted onto a graph consisting of two circles intersecting in a point, namely, longitude and meridian circles of the torus.

2. Construct an explicit deformation retraction of $\mathbb{R}^n - \{0\}$ onto S^{n-1}.

   Solution: Let $n \in \mathbb{N}$, and take the map:

   $$f_t(x) = \frac{tx}{|x|} + (1-t)x.$$

   Then clearly $f_0(x) = x \forall x \in \mathbb{R}^n$, and $f_1(x) = \frac{x}{|x|} \in S^{n-1}$ (justified since $\frac{x}{|x|}$ has norm 1).

3. (a) Show that the composition of homotopy equivalences $X \to Y$ and $Y \to Z$ is a homotopy equivalence $X \to Z$. Deduce that homotopy equivalence is an equivalence relation.

   Solution: If

   For reflexivity, we need only consider the identity on X, which is clearly continuously invertible function.

   (b) Show that the relation of homotopy among maps $X \to Y$ is an equivalence relation.

   (c) Show that a map homotopic to a homotopy equivalence is a homotopy equivalence.

4. Show that a space X is contractible iff every map $f : X \to Y$, for arbitrary Y, is nullhomotopic. Similarly, show X is contractible iff every map $f : X \to Y$ is nullhomotopic.

   $\implies$: Suppose that X is contractible. Then there exists some family $g_t : X \times I \to X$, so that:

   (a) $g_0(x) = x \forall x \in X$
   (b) $g_1(x) = c \forall x \in X$
   (c) The mapping $(x, t) \mapsto g_t(x)$ is continuous.

   Then, given some $f : X \to Y$ for some arbitrary topological space Y, we can construct the family of maps $f_t(x) : X \times I \to Y$;

   $$\begin{tikzcd} X \times I \arrow[r, "g_t"] \arrow[dr, dashed, "\exists f_t"'] & X \arrow[d, "f"] \\ & Y \end{tikzcd}$$

   So that the diagram commutes; $f_t(x) = f(g_t(x))$

   $\impliedby$: Suppose that, for any Y, every map $f : X \to Y$ is nullhomotopic. Then choose $Y = X$ and f the identity on X. Then there exist a family of maps $f_t : I \times X \to X$ so that, for some $c \in X$:

   (a) $f_0(x) = x \forall x \in X$
   (b) $f_1(x) = c \forall x \in X$
   (c) The map $(x, t) \mapsto f_t(x)$ is continuous.

   This satisfies the conditions for a contractible space (save for $f_t(c) = c$ for any c).

5. Show that $f : X \to Y$ is a homotopy equivalence if there exist maps $g, h : Y \to X$ such that $fg \simeq \mathbb{1}$ and $hf \simeq \mathbb{1}$. More generally, show that f is a homotopy equivalence if fg and hf are homotopy equivalences.