

1. Let S be a subset (not necessarily a subspace) of a finite dimensional inner product space V . Show that $(S^\perp)^\perp = \text{span } S$, where

$$\text{span } S := \left\{ \sum_{j=1}^m \alpha_j s_j : \alpha_j \in \mathbb{C}, s_j \in S, m \in \mathbb{N} \right\}$$

is the smallest subspace of V containing S (think of this as the set of all possible linear combinations of vectors from S).

Lemma: To prove the following fact we use the following:

For any subset S of a finite dimensional inner product space V , $(\text{span } S)^\perp = S^\perp$.

Proof. \subseteq : Let $v \in (\text{span } S)^\perp$. Since v is perpendicular to any element of $\text{span } S$, it must be perpendicular to any element of S (since $S \subseteq \text{span } S$).

\supseteq : Let $v \in S^\perp$, and let $w = a_1 s_1 + \dots + a_k s_k$ for some spanning set $\{s_1, \dots, s_k\} \subseteq S$. Then take:

$$\langle w, v \rangle = \left\langle \sum_{j=1}^k a_j s_j, v \right\rangle = \sum_{j=1}^k a_j \langle s_j, v \rangle = \sum_{j=1}^k a_j 0 = 0.$$

So $v \perp w$ and therefore $v \in \text{span } S$. □

Solution: Let $B = \{b_1, \dots, b_n\}$ be an orthonormal basis for $\text{span } S$.

\subseteq : Let $v \in \text{span } S$. Then write $v = a_1 b_1 + \dots + a_n b_n$. Let $w \in S^\perp$ be given, and note that $w \perp b_j$ for any basis element in S . Take:

$$\langle v, w \rangle = \langle a_1 b_1 + \dots + a_n b_n, w \rangle = a_1 \langle b_1, w \rangle + \dots + a_n \langle b_n, w \rangle = 0 + \dots + 0 = 0.$$

And we see that $v \perp w$, so v is perpendicular to any element of S^\perp , and $v \in (S^\perp)^\perp$.

\supseteq : Recall the identity $V = (\text{span } S)^\perp \oplus \text{span } S$. Then for any $v \in (S^\perp)^\perp$, $v = x + y$ for some $x \in (\text{span } S)^\perp = S^\perp$ and $y \in \text{span } S$.

Now take the inner product $\langle v, x \rangle = 0$ since $v \perp x$. Expanding,

$$0 = \langle v, x \rangle = \langle x + y, x \rangle = \langle x, x \rangle + \langle y, x \rangle = \langle x, x \rangle + 0 = \|x\|^2.$$

Then since $\|x\| = 0$ we must have $x = 0$, meaning $v = y$ and $v \in \text{span } S$.

2. Let V and W be finite dimensional inner product spaces and suppose $\ker A = \{0\}$. Find a left inverse for A in terms of A and A^* .

Solution: Begin with the identity,

$$\{0\} = \ker A = \ker A^* A.$$

So the composition of transformations $A^* A : V \rightarrow V$ has zero kernel and is injective, and by rank-nullity it must too be surjective. Then this map is invertible, and if we take $(A^* A)^{-1} A^* A = I$, we see that $(A^* A)^{-1} A^*$ is a left inverse for A .

3. Let V be a finite dimensional inner product space.

- (a) We can think of any $x \in V$ as a linear map from $\mathbb{C} \rightarrow V$ by setting $x(\lambda) := \lambda x$. You do not have to prove that this is linear. Show that $x^* : V \rightarrow \mathbb{C}$ satisfies

$$x^* y = \langle y, x \rangle.$$

Use this to deduce that the map xy^* is given by $xy^* v = \langle v, y \rangle x$. HINT: The inner product on \mathbb{C} is assumed to be $\langle z, w \rangle = z \bar{w}$.

- (b) Show that if $T : V \rightarrow \mathbb{C}$ is any linear map, then there is a vector y so that $T = y^*$.

Solution:

- (a) Recall from the definition of an adjoint operator, that the adjoint $x^* : V \rightarrow \mathbb{C}$ is given by:

$$\begin{aligned} \langle x(\lambda), y \rangle_V &= \langle \lambda, x^*(y) \rangle_{\mathbb{C}} \\ \langle \lambda x, y \rangle_V &= \lambda \overline{x^*(y)} \\ \lambda \langle x, y \rangle_V &= \lambda \overline{x^*(y)} \\ \overline{\lambda \langle x, y \rangle_V} &= \overline{\lambda \overline{x^*(y)}} \\ \bar{\lambda} \langle y, x \rangle_V &= \bar{\lambda} x^*(y) \\ \langle y, x \rangle &= x^* y. \end{aligned}$$

Then for the map $xy^* : V \rightarrow V$,

$$x(y^*(v)) = x(\langle v, y \rangle_V) = \langle v, y \rangle x.$$

- (b) Choose $y = T^*(1)$. Then, for any $v \in V$,

$$y^*(v) = \langle v, y \rangle_V = \langle v, T^*(1) \rangle_V = \langle Tv, 1 \rangle_{\mathbb{C}} = Tv.$$

And $y^* = T$.

4. Let V and W be finite dimensional vector spaces. You may find problem 3 useful here.

- (a) Suppose $T : V \rightarrow W$ satisfies $\text{rank } T = 1$. Show that there are vectors $x \in W$ and $y \in V$ so that $T = xy^*$.
- (b) Suppose $T : V \rightarrow W$ satisfies $\text{rank } T = k$. Show that T is the sum of k rank one operators. Hint: $PT = T$ where P is the orthogonal projection onto $\text{ran } T$.

Solution:

- (a) Since the dimension of the image of T has dimension 1, we must have $\text{ran } T = \text{span } \{b\}$ for some $b \in W$. Let $v \in V$, then $Tv = \alpha b$ for some $\alpha \in \mathbb{C}$. Choose $x = b$, and $y^*(v) = \alpha$. Now we have

$$xy^*(v) = x(y^*(v)) = x(\alpha) = \alpha x = \alpha b = Tv.$$

- (b) Let T be linear from V to W of rank k . Then let $\{b_1, \dots, b_k\}$ be an orthogonal basis for $\text{ran } T$. Then for $1 \leq j \leq k$ and some $v \in V$, define $T_j v = P_{b_j}(Tv)$. Now:

$$\sum_{j=1}^k T_j v = \sum_{j=1}^k P_{b_j}(Tv) = \sum_{j=1}^k \frac{\langle Tv, b_j \rangle}{\|b_j\|} b_j = Tv.$$

The final equality is simply the projection of Tv onto its range, which is Tv itself.

5. Suppose that A and B are unitarily equivalent $n \times n$ matrices. That is, there is a unitary matrix U so that $U^*AU = B$. Show that E is an invariant subspace for B if and only if UE is invariant for A . Recall that a subspace E of V is invariant for T if $Tv \in E$ for all $v \in E$.

Solution: Suppose that $U^*AU = B$, and recall that from unitary equivalence, we can see that $AU = UB$;

$$\begin{aligned} E \text{ is invariant under } B &\iff Bv \in E \quad \forall v \in E \\ &\iff Bv = w \in E \\ &\iff UBv = Uw \\ &\iff AUv = Uw \\ &\iff AUv \in UE \quad \forall Uv \in UE \\ &\iff UE \text{ is invariant under } A. \end{aligned}$$