1. (a) Prove that the series

$$\sum_{n=2}^{\infty} \frac{1}{(\log_2 n)^{p(\log_2 n)}}$$

is convergent for all $p > 1$. Here $\log_2 x$ denotes the logarithm base 2 of x. You may assume that $\log_2 n$ is increasing in n.

Proof. We compare with a p-series. ∎

(b) For $a > 0$ find the sum of the series

$$\sum_{k=2}^{\infty} \left(\frac{a}{a+1}\right)^k \quad \text{(show your work)}$$

Solution: We notice a geometric series; since $a > 0$, $\frac{a}{a+1} < 1$. Then the sum is given by:

$$\left(\frac{a}{a+1}\right)^2 \frac{1}{1 - \frac{a}{a+1}} = \left(\frac{a}{a+1}\right)^2 \frac{1}{\frac{a+1}{a+1} - \frac{a}{a+1}} = \left(\frac{a}{a+1}\right)^2 \frac{1}{\frac{1}{a+1}} = \left(\frac{a}{a+1}\right)^2 (a+1) = \frac{a^2}{a+1}.$$

2. (a) Prove that $f(x) = \sin\left(x^2\right)$ is not uniformly continuous in $[0, \infty)$.

f is uniformly continuous on $E \subset X$ if and only if $\forall \varepsilon > 0$, $\exists \delta > 0$, $d(x, y) < \delta \implies d(f(x), f(y)) < \varepsilon$

f is NOT uniformly continuous on $E \subset X$ if and only if $\exists \varepsilon > 0$, $\forall \delta > 0$, we can choose x, y so that $d(x, y) < \delta$ and $d(f(x), f(y)) \geq \varepsilon = 1$

Proof. Choose $\varepsilon = 2$, and let $\delta > 0$. Then we must choose $|x - y| < \delta$ but $|\sin x^2 - \sin y^2| = 2$

WLOG choose $x < y$, in fact $x < y - \delta$

So we wish to choose $x^2 + \pi = y^2$ and $\sin x^2 = 1$ and $x - \delta = y$ and $|\sin x^2 - \sin y^2| = 2 \impliedby x^2 + \pi = y^2 \impliedby \pi = y^2 - x^2 = (x - y)(x + y)$

Can do dumb algebra lter to finish this ∎

(b) Show an example of a continuous function in $(0, 1)$ which is not uniformly continuous (no proof necessary).

Solution: $f(x) = \sin\left(\frac{1}{x^2}\right)$ is continuous in $(0, 1)$ however it is not uniformly continuous (as shown in class)