Example 0.1

Let $X \neq \emptyset$, and $B \subseteq X$ be given. Show that the set:

$$\mathcal{E}_B = \{A \subseteq X : B \subseteq A \text{ or } B \subseteq A^c\}.$$

Is a σ-algebra in X.

σ1**:** $X \in \mathcal{E}_B$ since $B \subseteq X$ by assumption.

σ2**:** If $A \in \mathcal{E}_B$, then either $B \subseteq A$ or $B \subseteq A^c$. If the first holds, then clearly $B \subseteq (A^c)^c = A$, and $A^c \in \mathcal{E}_B$. On the other hand, if $B \subseteq A^c$, then $A^c \in \mathcal{E}_B$. Therefore $\mathcal{E}_B$ is closed under complements.

σ3**:** Suppose $\{A_n\}_{n\in\mathbb{N}}$ is a sequence of elements in $\mathcal{E}_B$. Partition $\mathbb{N}$ into disjoint subsets I, J so that $B \subseteq A_i$ for $i \in I$, and $B \subseteq A_j^c$, for $j \in J$. The A_j require some consideration. By σ2, we know that each A_j must also be in $\mathcal{E}_B$. Then $B \subseteq A_i$, for all i, and $B \subseteq J$ for all j. Since $I \cup J = \mathbb{N}$, we have

$$B \subseteq \bigcup_{i\in I} A_i \cup \bigcup_{j\in J} A_j = \bigcup_{n\in\mathbb{N}} A_n.$$

So $\mathcal{E}_B$ is closed under countable union, and is therefore a σ-algebra.