Assignment 2 - Thomas Boyko - 30191728

1. Let $a_n = \sum\limits_{i=1}^{n} \frac{2i-1}{2^i}$ where n is a positive integer.

   (a) Find a_1, a_2, a_3, and a_4.

$$\begin{aligned} a_1 &= \frac{1}{2} \\ a_2 &= \frac{5}{4} \\ a_3 &= \frac{15}{8} \\ a_4 &= \frac{37}{16} \end{aligned}$$

   (b) Guess a simple formula for a_n for all integers $n \geq 1$.
   Guess:

$$a_n = 3 - \frac{2n+3}{2^n}$$

   (c) Prove by induction on n that your guess in part (b) is correct.

   Proof: We will prove by induction. Suppose $n \in \mathbb{Z}$ and $n \geq 1$.
   Base case: $n = 1$.

$$\sum_{i=1}^{1} \frac{2i-1}{2^i} = \frac{2(1)-1}{2^1} = \frac{1}{2} = 3 - \frac{5}{2} = 3 - \frac{2(1)+3}{2^1}$$

   Induction Hypothesis:
   Suppose $k \in \mathbb{Z}$ and $k \geq 1$. Further suppose:

$$\sum_{i=1}^{k} \frac{2i-1}{2^i} = 3 - \frac{2k+3}{2^k}$$

   We want to prove:

$$\sum_{i=1}^{k+1} \frac{2i-1}{2^i} = 3 - \frac{2k+5}{2^{k+1}}$$

   Let's begin.

$$\begin{aligned} \sum_{i=1}^{k+1} \frac{2i-1}{2^i} &= \frac{2(k+1)-1}{2^{k+1}} + \sum_{i=1}^{k} \frac{2i-1}{2^i} \\ &= \frac{2(k+1)-1}{2^{k+1}} + 3 - \frac{2k+3}{2^k} \text{ (By the Induction Hypothesis)} \\ &= 3 + \frac{2k+1}{2^{k+1}} - \frac{2(2k+3)}{2^{k+1}} \\ &= 3 + \frac{2k+1-4k-6}{2^{k+1}} \\ &= 3 - \frac{2k+5}{2^{k+1}} \end{aligned}$$

   Therefore, by induction on n, $\sum\limits_{i=1}^{n} \frac{2i-1}{2^i} = 3 - \frac{2n+3}{2^n}$ for all positive integers n. ■

2. The Fibonacci numbers $f_1, f_2, f_3, \ldots$ are defined by setting $f_1 = f_2 = 1$ and $f_n = f_{n-1} + f_{n-2}$ for all integers $n \geq 1$.

A fact that will become helpful in these proofs is that $\gcd(a+b, b) = \gcd(a, b)$.
Here is a proof, so that we can use it in the following questions.

Proof: Suppose $\gcd(a, b) = d$ where $d \in \mathbb{Z}$. d is the greatest integer that divides both a and b.
So $d|b$ and $d|a$ which means $a = dk$ and $b = dl$ for some $k, l \in \mathbb{Z}$.
$a + b = kd + ld = d(l + k)$ where $l + k \in \mathbb{Z}$, which means $d|a + b$.

Now we will let $e = \gcd(a + b, b)$. e is the greatest integer that divides both $a + b$ and b.
$em = a + b$ and $en = b$ for some $m, n \in \mathbb{Z}$.
$em = a + en$
$e(m - n) = a$ where $m - n \in \mathbb{Z}$ so e divides a.
Since d divides both $a + b$ and b, $d \leq e$.
And since e divides both a and b, $e \leq d$.
So $e \leq d \leq e$ which means $d = \gcd(a, b) = \gcd(a + b, b) = e$. ■

(a) Prove that for all integers $n \geq 3$, $\gcd(f_n, f_{n-1}) = \gcd(f_{n-1}, f_{n-2})$

Proof: Suppose $n \in \mathbb{Z}$, and $n \geq 3$.
Because $n \geq 3$, $\gcd(f_n, f_{n-1}) = \gcd(f_{n-1} + f_{n-2}, f_{n-2})$.
As proved above, this is equal to $\gcd(f_{n-1}, f_{n-2})$.
Therefore, for all integers $n \geq 3$, $\gcd(f_n, f_{n-1}) = \gcd(f_{n-1}, f_{n-2})$ ■

(b) Prove by induction on n that $\gcd(f_n, f_{n-1}) = 1$ for all integers $n \geq 2$.

Proof: Suppose $n \in \mathbb{Z}, n \geq 2$.
Base cases: (For this we need to define $f_3 = f_2 + f_1 = 1 + 1 = 2$)
$n = 2$: $\gcd(f_2, f_1) = \gcd(1, 1) = 1$.
$n = 3$: $\gcd(f_3, f_2) = \gcd(2, 1) = 1$.
Induction Hypothesis:
Suppose $k \in \mathbb{Z}, k > 3$
Further suppose that for all $m \in \mathbb{Z}$ where $2 \leq m < k$:

$$\gcd(f_m, f_{m-1}) = 1$$

We want to prove: $\gcd(f_k, f_{k-1}) = 1$.
$\gcd(f_k, f_{k-1}) = \gcd(f_{k-1}, f_{k-2})$ as proved in part (a).
By the Induction Hypothesis, since $2 \leq k - 1 < k \leq 3$, $1 = \gcd(f_k, f_{k-1})$
So, by induction on n, $\gcd(f_n, f_{n-1}) = 1$ for all $n \geq 2$. ■

(c) Prove by induction on n that $\sum_{i=1}^{n} f_i^2 = f_{n+1} f_n$ for all integers $n \geq 1$.

Proof: Suppose $n \in \mathbb{Z}$, $n \geq 1$.
Base case: $n = 1$.

$$\sum_{i=1}^{1} f_i^2 = 1^2 = 1 = 1 \times 1 = f_2 \times f_1$$

Induction Hypothesis:
Suppose $k \in \mathbb{Z}$, $k \geq 1$.
Further suppose that $\sum_{i=0}^{k} f_i^2 = f_{k+1} f_k$.
We want to prove that: $\sum_{i=0}^{k+1} f_i^2 = f_{k+2} f_{k+1}$.
Let's begin.

$$\begin{aligned}
\sum_{i=0}^{k+1} f_i^2 &= {f_{k+1}}^2 \sum_{i=0}^{k} f_i^2 \\
&= {f_{k+1}}^2 + f_{k+1} f_k \text{ By the Induction Hypothesis} \\
&= f_{k+1}(f_k + f_{k+1}) \\
&= f_{k+1}(f_{k+2})
\end{aligned}$$

Therefore, by induction on n, $\sum_{i=1}^{n} f_i^2 = f_{n+1} f_n$ for all integers $n \geq 1$. ■

3. For each of the following statements, if the statement is true , then give a proof, and if the statement is false, then write out its negation and prove that.

   (a) For all sets A, B, and C, $A \cup (B \cap C)$ is a subset of $(A \cup B) \cap (A \cup C)$.

   Proof: Suppose A, B and C are sets. Let $x \in A \cup (B \cap C)$.
   So, x is in A, or x is in both B and C.

   Case 1: $x \in A$
   If $x \in A$, x is in both $A \cup B$ and $A \cup C$.
   So, $x \in (A \cup B) \cap (A \cup C)$ if $x \in A$.

   Case 2: $x \notin A$
   If $x \notin A$, $x \in B \cap C$.
   This means that $x \in C$ and $x \in B$.
   So x must be in both $A \cup B$ and $A \cup C$.
   Therefore, if $x \notin A$, $x \in (A \cup B) \cap (A \cup C)$

   Since we know that all x must be in A or (B and C),
   And we know that all x in $A \cup (B \cap C)$ must be in $(A \cup B) \cap (A \cup C)$,
   $A \cup (B \cap C)$ is a subset of $(A \cup B) \cap (A \cup C)$. ■

   (b) For all sets A, B, and C, if $A \subset B$, then $(C - A) \subset (C - B)$.
   The statement is false, so we will negate it and prove that.
   The negation is: "There exist sets A, B, and C so that $A \subset B$ but $(C - A) \not\subset (C - B)$."

   Proof: Choose the following sets.
   $A = \varnothing$
   $B = \{1\}$
   $C = \{1\}$
   Note that $A \subset B$, because there are no elements in A.
   $C - A = \{1\}$
   $C - B = \varnothing$
   So, $A \subset B$ but $(C - A) \not\subset (C - B)$ because the element 1 is in $C - A$ but is not in $C - B$. Therefore, there exist sets A, B, and C so that $A \subset B$ but $(C - A) \not\subset (C - B)$. ■

   (c) For all sets A, B, and C, $(A \cup B) - C = (A - C) \cup (B - C)$

   Proof: Suppose A, B, and C are sets.
   Suppose $x \in (A \cup B) - C cases$.
   So x is in A or B, but x is not in C.
   We will consider two cases.
   Case: $x \in A$.
   Since $x \in A$, and $x \notin C$, $x \in (A - C)$
   So, when $x \in A$, $x \in (A - C) \cup (B - C)$.
   Case: $x \notin A$.
   Since $x \notin A$, x must be in B.
   Therefore $x \in (B - C)$ because x is in B but not C.
   So $x \in (A - C) \cup (B - C)$.
   All elements of the left side of the equation must be in the right side, so $(A \cup B) - C \subset (A - C) \cup (B - C)$
   Let $y \in (A - C) \cup (B - C)$.
   So y must be in either $A - C$ or $B - C$. This means y must be in A or B.
   Either way, $y \notin C$
   Since y must be in A or B, it will always be in $A \cup B$.
   And since $y \notin C$, y must always be in $(A \cup B) - C$.
   Therefore, $(A - C) \cup (B - C) \subset (A \cup B) - C$
   Since the sets $(A - C) \cup (B - C)$ and $(A \cup B) - C$ are subsets of each other, they are equal. ■

(d) For all sets A and B, $A \subset B \iff A - B = \varnothing$.

Proof: Suppose A, B are sets.
To prove that $A \subset B \iff A - B = \varnothing$, we will first prove $A \subset B \implies A - B = \varnothing$
We will prove this by contradition. Suppose $A \subset B$ and $\exists x \in (A - B)$.
So, $x \in A$ but $x \notin B$.
But all elements of A must be in B since $A \subset B$. A contradiction!
So, $A \subset B \implies A - B = \varnothing$.
Next we will prove that $A - B = \varnothing \implies A \subset B$.
Since there are no elements in $A - B$, $\forall y \in A, y \in B$
This is the same as saying "A is a subset of B"
So, $A - B = \varnothing \implies A \subset B$
$A \subset B \implies A - B = \varnothing$ and $A - B = \varnothing \implies A \subset B$
Which means that $A \subset B \iff A - B = \varnothing$. ■