1. Find three examples of adjoint functors not mentioned above. Do the same for initial and terminal objects.

   (a) Let $\mathbb{E}$ be a field extension of another field $\mathbb{F}$. Let G be the Galois group $G = \mathrm{Gal}(\mathbb{E} : \mathbb{F})$ and take the category $\mathcal{E}$, with objects being intermediate field extensions between $\mathbb{E}$ and $\mathbb{F}$, and morphisms as inclusions. Take $\mathcal{G}$, with objects being subgroups, and morphisms $\mathrm{Hom}(H, H') = \{\leq |H \leq H'\}$.
   Then, for $\mathbb{k} \in \mathcal{E}$, we define the map:

   $$\phi : \mathcal{E} \to \mathcal{G}^{op}, \quad \phi(\mathbb{k}) = \mathrm{Gal}(\mathbb{E} : \mathbb{k}).$$

   And for $H \in \mathcal{G}$, define:

   $$\phi : \mathcal{G}^{op} \to \mathcal{E}, \quad \psi(H) = \{x \in \mathbb{E} : \alpha(x) = x \forall \alpha \in H\}.$$

   (Recalling that elements of H are automorphisms of $\mathbb{E}$). A study of Galois theory tells us that these maps produce a bijective correspondence of field extensions and subgroups, we see that they are functors which are both adjoint to each other.

   (b) In the category of fields of characteristic p, for some fixed p, the field $\mathbb{F}_p$ is initial.

   (c)

2. 2.1.14 (may be hard)

3. Let G be a group.

   (a) What interesting functors are there (in either direction) between Set and the category $[G, \mathrm{Set}]$ of left G-sets? Which of those functors are adjoint to which?

   Solution: Consider F, the forgetful functor from $[G, \mathrm{Set}] \to \mathrm{Set}$, which takes an action and maps it to the set it acts on.
   Then take $L : \mathrm{Set} \to [G, \mathrm{Set}]$, which takes a set X in Set and maps it to an action $G \times X$, defined by $h(g, x) = (hg, x)$. For a functor in Set, we map $L(f(x)) = (g, f(x))$
   No take $R : \mathrm{Set} \to [G, \mathrm{Set}]$ defined by $R(X) = \mathrm{Map}(G, X) = \{f : G \to X\}$. We equip $\mathrm{Map}(G, X)$ with an action, $g\phi(h) = \phi(gh)$. For a functor g in Set, we map $R(g) =$
   Finally we claim that these are adjoint:

   $$\begin{tikzcd}
   {[G, \mathrm{Set}]} \arrow[d, "F" description] \\
   \mathrm{Set} \arrow[u, shift left=6, "L"] \arrow[u, shift right=6, "R"']
   \end{tikzcd}$$

4. Let $f : A \to B$ and $g : B \to A$ be order-preserving operations between ordered sets. Prove TFAE:

   (a) For any $a \in A, b \in B$,
   $$f(a) \leq b \iff a \leq f(b).$$

   (b) $a \leq g(f(a))$ for any $a \in A$ and $f(g(b)) \leq b$ for any $b \in B$

5. Show that for any adjunction, the right adjoint is full and faithful if and only if the counit is an isomorphism