

- Find three examples of adjoint functors not mentioned above. Do the same for initial and terminal objects.

- Let \mathbb{E} be a field extension of another field \mathbb{F} . Let G be the Galois group $G = \text{Gal}(\mathbb{E} : \mathbb{F})$ and take the category \mathcal{E} , with objects being intermediate field extensions between \mathbb{E} and \mathbb{F} , and morphisms as inclusions. Take \mathcal{G} , with objects being subgroups, and morphisms $\text{Hom}(H, H') = \{\leq | H \leq H'\}$.

Then, for $\mathbb{k} \in \mathcal{E}$, we define the map:

$$\phi : \mathcal{E} \rightarrow \mathcal{G}^{op}, \quad \phi(\mathbb{k}) = \text{Gal}(\mathbb{E} : \mathbb{k}).$$

And for $H \in \mathcal{G}$, define:

$$\psi : \mathcal{G}^{op} \rightarrow \mathcal{E}, \quad \psi(H) = \{x \in \mathbb{E} : \alpha(x) = x \forall \alpha \in H\}.$$

(Recalling that elements of H are automorphisms of \mathbb{E}). A study of Galois theory tells us that these maps produce a bijective correspondence of field extensions and subgroups, we see that they are functors which are both adjoint to each other.

(b)

(c)

- 2.1.14 (may be hard)

- Let G be a group.

- What interesting functors are there (in either direction) between Set and the category $[G, \text{Set}]$ of left G -sets? Which of those functors are adjoint to which?

Solution: Consider F , the forgetful functor from $[G, \text{Set}] \rightarrow \text{Set}$, which takes an action and maps it to the set it acts on.

Then take $L : \text{Set} \rightarrow [G, \text{Set}]$, which takes a set X in Set and maps it to an action $G \times X$, defined by $h(g, x) = (hg, x)$.

No take $R : \text{Set} \rightarrow [G, \text{Set}]$ defined by $R(X) = \text{Map}(G, X) = \{f : G \rightarrow X\}$. We equip $\text{Map}(G, X)$ with an action, $g\phi(h) = \phi(gh)$.

Finally we claim that these are adjoint:

$$\begin{array}{c} [G, \text{Set}] \\ \begin{array}{c} \uparrow L \quad \downarrow F \\ \downarrow \dashv \quad \uparrow \dashv \\ \text{Set} \end{array} \end{array}$$

- Similarly, what interesting functors are there between Vect_k and the category $[G, \text{Vect}_k]$ of k -linear representations of G , and what adjunctions are there between those functors?

- Let $f : A \rightarrow B$ and $g : B \rightarrow A$ be order-preserving operations between ordered sets. Prove TFAE:

- For any $a \in A, b \in B$,

$$f(a) \leq b \iff a \leq f(b).$$

- $a \leq g(f(a))$ for any $a \in A$ and $f(g(b)) \leq b$ for any $b \in B$

- Show that for any adjunction, the right adjoint is full and faithful if and only if the counit is an isomorphism