Contents

1 σ-algebras and Measures

We begin by discussing the concept of a measure, or how to 'measure' space on the real line, and in $\mathbb{R}^n$. Then we extend the concept, to allow measuring of non-Euclidean space.

Definition 1.1 (*Area in* $\mathbb{R}^2$)

Consider a set function $\lambda_2 : \mathcal{P}(\mathbb{R}^2) \to [0, \infty]$ satisfying:

1. $\lambda_2(\emptyset) = 0$
2. $\lambda_2\left(\bigcup_{i\in I} A_i\right) = \sum_{i\in I} \lambda_2(A_i)$ *for disjoint A_i and countable I.*
3. $\lambda_2((a_1, b_1) \times (a_2, b_2)) = (b_1 - a_1)(b_2 - a_2)$
4. *Measure is invariant of translation and rotation.*

Definition 1.2 (*σ-algebra in* $\mathbb{R}^2$)

A σ-algebra is a system $\mathcal{E}$ of subsets of $\mathbb{R}^2$ satisfying:

1. $\mathbb{R}^2 \in \mathcal{E}$
2. *If $A \in \mathcal{E}$, then $A^c \in \mathcal{E}$.*
3. *if $\{A_n\}_{n\in\mathbb{N}}$ is a sequence of sets in from $\mathcal{E}$, then $\bigcup_{n\in\mathbb{N}} A_n \in \mathcal{E}$*

The common choice for a σ—algebra in $\mathbb{R}^2$ is the Borel Algebra $\mathcal{B}(\mathbb{R}^2)$, which is the smallest σ-algebra containing all open rectangles $(a_1, b_1) \times (a_2, b_2) \subseteq \mathbb{R}^2$.

Definition 1.3 (*σ-algebra*)

A σ-algebra in a set X is a nonempty collenction $\mathcal{E}$ of subsets of X which contains X, and is closed under complement and countable union; if $A_1, A_2, \dots \in \mathcal{E}$; then

1. $X, \emptyset \in \mathcal{E}$
2. $A^c \in \mathcal{E}$
3. $\bigcup_{i=1}^{\infty} A_i \in \mathcal{E}$

Elements of $\mathcal{E}$ are called measureable sets, or just measurable. X is a measurable space.

We may consider only finite unions, in which case we call $\mathcal{E}$ just an Algebra (This is equivalent to only requiring $A \cup B \in \mathcal{E}$ when $A, B \in \mathcal{E}$).

Definition 1.4 (*Measure*)

Let $\mathcal{E}$ be a σ-algebra in a set X. A measure on $\mathcal{E}$ is a set function $\mu : \mathcal{E} \to [0, \infty]$, satisfying the following:

1. $\mu(\emptyset) = 0$
2. $\mu\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} \mu(A_n)$

A measure satisfying $\mu(X) = 1$ may be called a probability measure.

Proposition 1.1

Let $\mathcal{E}$ be an algebra on X. Then:

1. $\emptyset \in \mathcal{E}$.
2. *If $A, B \in \mathcal{E}$, then $A \cup B \in \mathcal{E}$.*
3. *If $A, B \in \mathcal{E}$ then $A \setminus B \in \mathcal{E}$.*
4. *If $\mathcal{E}$ is a σ-algebra, and $A_1, A_2, \dots \in \mathcal{E}$, then $\bigcap_{i=1}^{\infty} A_n \in \mathcal{E}$.*

Example 1.1

Below are some examples of σ-algebras, for some set X

1. $\mathcal{E}_{min} = \{\emptyset, X\}$
2. $\mathcal{E}_{max} = \mathcal{P}(X)$
3. $\mathcal{E} = \{\emptyset, A, A^c, X\}$
4. *If* $A_1, \ldots, A_n$ *are disjoint but cover* X, $\mathcal{E} = \left\{\bigcup_{j\in I} A_j : I \subseteq \{1, \ldots, n\}\right\}$

Theorem 1.2 (*Generated σ-algebras*)

Let $\{\mathcal{E}_i\}_{i\in I}$ *be a family of σ-algebras in X. Then the system:*

$$\bigcap_{i\in I} A_i = \{A \subseteq X : A \in \mathcal{E}_i \forall i \in I\}.$$

Is a σ-algebra in X. In fact this is the smallest σ-algebra in X. Members of this σ-algebra are sometimes called Borel sets.