1. Let S be a subset (not necessarily a subspace) of a finite dimensional inner product space V. Show that $(S^\perp)^\perp = \text{span } S$, where

$$\text{span } S := \left\{ \sum_{j=1}^{m} \alpha_j s_j : \alpha_j \in \mathbb{C}, s_j \in S, m \in \mathbb{N} \right\}$$

is the smallest subspace of V containing S (think of this as the set of all possible linear combinations of vectors from S).

Lemma: To prove the following fact we use the following:
For any subset S of a finite dimensional inner product space V, $(\text{span } S)^\perp = S^\perp$.

Proof. $\subseteq$: Let $v \in (\text{span } S)^\perp$. Since v is perpendicular to any element of span S, it must be perpendicular to any element of S (since $S \subseteq \text{span } S$).
$\supseteq$: Let $v \in S^\perp$, and let $w = a_1 s_1 + \cdots + a_k s_k$ for some spanning set $\{s_1, \ldots, s_k\} \subseteq S$. Then take:

$$\langle w, v \rangle = \langle \sum_{j=1}^{k} a_j s_j, v \rangle = \sum_{j=1}^{k} a_j \langle s_j, v \rangle = \sum_{j=1}^{k} a_j 0 = 0.$$

So $v \perp w$ and therefore $v \in \text{span } S$. □

Solution: Let $B = \{b_1, \ldots, b_n\}$ be an orthonormal basis for span S.

$\subseteq$: Let $v \in \text{span } S$. Then write $v = a_1 b_1 + \cdots + a_n b_n$. Let $w \in S^\perp$ be given, and note that $w \perp b_j$ for any basis element in S. Take:

$$\langle v, w \rangle = \langle a_1 b_1 + \cdots + a_n b_n, w \rangle = a_1 \langle b_1, w \rangle + \cdots + a_n \langle b_n, w \rangle = 0 + \cdots + 0 = 0.$$

And we see that $v \perp w$, so v is perpendicular to any element of $S^\perp$, and $v \in S^{\perp\perp}$.

$\supseteq$: Recall the identity $V = (\text{span } S)^\perp \oplus \text{span } S$. Then for any $v \in (S^\perp)^\perp$, $v = x + y$ for some $x \in (\text{span } S)^\perp = S^\perp$ and $y \in \text{span } S$.
Now take the inner product $\langle v, x \rangle = 0$ since $v \perp x$. Expanding,

$$0 = \langle v, x \rangle = \langle x + y, x \rangle = \langle x, x \rangle + \langle y, x \rangle = \langle x, x \rangle + 0 = \|x\|^2.$$

Then since $\|x\| = 0$ we must have $x = 0$, meaning $v = y$ and $v \in \text{span } S$.

2. Let V and W be finite dimensional inner product spaces and suppose $\ker A = \{0\}$. Find a left inverse for A in terms of A and A^*.

Solution: Begin with the identity,

$$\{0\} = \ker A = \ker A^* A.$$

So the composition of transformations $A^* A : V \to V$ has zero kernel and is injective, and by rank-nullity it must too surjective. Then this map is invertible, and if we take $(A^* A)^{-1} A^* A = I$, we see that $(A^* A)^{-1} A^*$ is a left inverse for A.

3. Let V be a finite dimensional inner product space.

   (a) We can think of any $x \in V$ as a linear map from $\mathbb{C} \to V$ by setting $x(\lambda) := \lambda x$. You do not have to prove that this is linear. Show that $x^* : V \to \mathbb{C}$ satisfies

   $$x^* y = \langle y, x \rangle.$$

   Use this to deduce that the map xy^* is given by $xy^* v = \langle v, y \rangle x$. HINT: The inner product on $\mathbb{C}$ is assumed to be $\langle z, w \rangle = z\overline{w}$.

   (b) Show that if $T : V \to \mathbb{C}$ is any linear map, then there is a vector y so that $T = y^*$.

   Solution:

   (a) Recall from the definition of an adjoint operator, that the adjoint $x^* : V \to \mathbb{C}$ is given by:

   $$\begin{aligned}
   \langle x(\lambda), y \rangle_V &= \langle \lambda, x^*(y) \rangle_{\mathbb{C}} \\
   \langle \lambda x, y \rangle_V &= \lambda \overline{x^*(y)} \\
   \lambda \langle x, y \rangle_V &= \lambda \overline{x^*(y)} \\
   \overline{\lambda \langle x, y \rangle_V} &= \overline{\lambda \overline{x^*(y)}} \\
   \overline{\lambda} \langle y, x \rangle_V &= \overline{\lambda} x^*(y) \\
   \langle y, x \rangle &= x^* y.
   \end{aligned}$$

   Then for the map $xy^* : V \to V$,

   $$x(y^*(v)) = x(\langle v, y \rangle_V) = \langle v, y \rangle x.$$

   (b) Choose $y = T^*(1)$. Then, for any $v \in V$,

   $$y^*(v) = \langle v, y \rangle_V = \langle v, T^*(1) \rangle_V = \langle Tv, 1 \rangle_{\mathbb{C}} = Tv.$$

   And therefore T is induced by $y = T^*(1)$

4. Let V and W be finite dimensional vector spaces. You may find problem 3 useful here.

   (a) Suppose $T : V \to W$ satisfies rank $T = 1$. Show that there are vectors $x \in W$ and $y \in V$ so that $T = xy^*$.

   (b) Suppose $T : V \to W$ satisfies rank $T = k$. Show that T is the sum of k rank one operators. Hint: $PT = T$ where P is the orthogonal projection onto ran T.

   Solution:

   (a) Since the dimension of the image of T has dimension 1, we must have ran $T =$ span $\{b\}$ for some $b \in W$. Let $v \in V$, then $Tv = \alpha b$ for some $\alpha \in \mathbb{C}$. Choose $x = b$, and $y^*(v) = \alpha$. Now we have

   $$xy^*(v) = x(y^*(v)) = x(\alpha) = \alpha x = \alpha b = Tv.$$

   (b) Let T be linear from V to W of rank k. Then let $\{b_1, \ldots, b_k\}$ be an orthogonal basis for ran T. Then for $1 \le j \le k$ and some $v \in V$, define $T_j v = P_{b_j}(Tv)$. Now:

   $$\sum_{j=1}^{k} T_k v = \sum_{j=1}^{k} P_{b_j}(Tv) = \sum_{j=1}^{k} \frac{\langle Tv, b_j \rangle}{\|b_j\|} b_j = Tv.$$

   Note the last equality comes from the orthogonal expansion of a vector discussed in class.

5. Suppose that A and B are unitarily equivalent $n \times n$ matrices. That is, there is a unitary matrix U so that $U^*AU = B$. Show that E is an invariant subspace for B if and only if UE is invariant for A. Recall that a subspace E of V is invariant for T if $Tv \in E$ for all $v \in E$.

   Solution: Suppose that $U^*AU = B$, and recall that from unitary equivalence, we can see that $AU = UB$;

$$\begin{aligned} E \text{ is invariant under } B &\iff Bv \in E \quad \forall v \in E \\ &\iff Bv = w \in E \\ &\iff UBv = Uw \\ &\iff AUv = Uw \\ &\iff AUv \in UE \quad \forall Uv \in UE \\ &\iff UE \text{ is invariant under } A. \end{aligned}$$