1. (a) Prove that the series

$$\sum_{n=2}^{\infty} \frac{1}{(\log_2 n)^{p(\log_2 n)}}$$

is convergent for all $p > 1$. Here $\log_2 x$ denotes the logarithm base 2 of x. You may assume that $\log_2 n$ is increasing in n.

Proof. We attempt to satisfy the criterion in Rudin Theorem 3.27. Rewrite the series; and let $c_n : \mathbb{N} \to \mathbb{R}$:

$$c_n = \begin{cases} 0 & n = 1 \\ \frac{1}{(\log_2 n)^{p(\log_2 n)}} & n \geq 2 \end{cases}.$$

Then our summation becomes

$$\begin{aligned} \sum_{n=2}^{\infty} \frac{1}{(\log_2 n)^{p(\log_2 n)}} &= 0 + \sum_{n=2}^{\infty} \frac{1}{(\log_2 n)^{p(\log_2 n)}} \\ &= \sum_{n=1}^{\infty} c_n. \end{aligned}$$

Next we want to show that the general term is decreasing so we can use Theorem 3.27. Let $x < y$ be natural numbers.

$$\begin{aligned} x &< y \\ \log_2 x &< \log_2 y \\ (\log_2 x)^p &< (\log_2 y)^p \\ (\log_2 x)^{p \log_2 x} &< (\log_2 y)^{p \log_2 y} \\ \frac{1}{(\log_2 x)^{p \log_2 x}} &> \frac{1}{(\log_2 y)^{p \log_2 y}}. \end{aligned}$$

Now that our sum is indexed from 1 and we have shown that the general term is decreasing, we can apply Rudin Theorem 3.27. Our series of c_n converges if and only if the following series converges.

$$\begin{aligned} \sum_{k=0}^{\infty} 2^k c_k &= 0 \cdot 2^0 + \sum_{k=1}^{\infty} \frac{2^k}{(\log_2 2^k)^{p \log_2 2^k}} \\ &= \sum_{k=1}^{\infty} \left(\frac{2}{k^p}\right)^k. \end{aligned}$$

(b) For $a > 0$ find the sum of the series

$$\sum_{k=2}^{\infty} \left(\frac{a}{a+1}\right)^k \quad \text{(show your work)}$$

Solution: We notice a geometric series; since $a > 0$,we can say $a < a + 1$ and $\frac{a}{a+1} < 1$. Then the sum is given by:

$$\begin{aligned} \left(\frac{a}{a+1}\right)^2 \frac{1}{1 - \frac{a}{a+1}} &= \left(\frac{a}{a+1}\right)^2 \frac{1}{\frac{a+1}{a+1} - \frac{a}{a+1}} \\ &= \left(\frac{a}{a+1}\right)^2 \frac{1}{\frac{1}{a+1}} \\ &= \left(\frac{a}{a+1}\right)^2 (a+1) \\ &= \frac{a^2}{a+1}. \end{aligned}$$

2. (a) Prove that $f(x) = \sin(x^2)$ is not uniformly continuous in $[0, \infty)$.

Proof. Choose $\varepsilon = 1$, and let $\delta > 0$. Then let $k \in \mathbb{Z}$, and $k > \frac{1}{\delta^2}$ which we can do by the Archimedian Property.

We attempt to choose x, y so that the function's value on one is 0, and on the other is ± 1. Then let $x^2 = k\pi$ for some $k \in \mathbb{N}$, and $y^2 = k\pi + \frac{\pi}{2}$, and our final choice is

$$x = \sqrt{k\pi - \frac{\pi}{2}}, \quad y = \sqrt{k\pi}.$$

Then regardless of our choice of k,

$$|f(x) - f(y)| = \left|\sin\left(\left(\sqrt{k\pi}\right)^2\right) - \sin\left(\left(\sqrt{k\pi - \frac{\pi}{2}}\right)^2\right)\right| = \left|\sin(k\pi) - \sin\left(k\pi - \frac{\pi}{2}\right)\right|.$$

If n is odd, then $|\sin(k\pi) - \sin\left(k\pi - \frac{\pi}{2}\right)| = |\pm 1 - 0| = 1$, and if k is even, $|\sin(k\pi) - \sin\left(k\pi - \frac{\pi}{2}\right)| = |0 - \pm 1| = 1$.

We now have guaranteed that $|f(x) - f(y)| = 1$ for any k. It aids us to note that thanks to our choice of k, we can say that $\frac{1}{k} < \delta^2$ and $\frac{1}{\sqrt{k}} < \delta$. So then we proceed on $|y - x|$.

$$\begin{aligned}
|y - x| &= y - x && \text{Since } y > x \\
&= \sqrt{\pi k} - \sqrt{\pi k - \frac{\pi}{2}} \\
&= \frac{\left(\sqrt{\pi k} - \sqrt{\pi k - \frac{\pi}{2}}\right)\left(\sqrt{\pi k} + \sqrt{\pi k - \frac{\pi}{2}}\right)}{\left(\sqrt{\pi k} + \sqrt{\pi k - \frac{\pi}{2}}\right)} \\
&= \frac{\pi k - \left(\pi k - \frac{\pi}{2}\right)}{\left(\sqrt{\pi k} + \sqrt{\pi k - \frac{\pi}{2}}\right)} \\
&= \frac{\frac{\pi}{2}}{\left(\sqrt{\pi k} + \sqrt{\pi k - \frac{\pi}{2}}\right)} \\
&= \frac{\pi}{2\left(\sqrt{\pi k} + \sqrt{\pi k - \frac{\pi}{2}}\right)} \\
&= \frac{\pi}{2\sqrt{\pi}\left(\sqrt{k} + \sqrt{k - \frac{1}{2}}\right)} \\
&< \frac{\sqrt{\pi}}{2\left(\sqrt{k} + \sqrt{k}\right)} \\
&< \frac{\sqrt{\pi}}{4\sqrt{k}} && \text{Since } \frac{\sqrt{\pi}}{4} < 1 \\
&< \frac{1}{\sqrt{k}} \\
&< \delta.
\end{aligned}$$

(b) Show an example of a continuous function in $(0, 1)$ which is not uniformly continuous (no proof necessary).

Solution: $f(x) = \sin\left(\frac{1}{x^2}\right)$ is continuous in $(0, 1)$ however it is not uniformly continuous (as shown in class).