Homework Set 2 - Thomas Boyko - 30191728

1. Define a parametric curve $\vec{r}(t) = (2t, \frac{t^2}{2}, 2\ln t)^T$ where $t > 0$.

   (a) Compute the arclength of the curve between the points $(2, 0.5, 0)$ and $(8, 8, 2\ln 4)$.
   First we begin by calculating $\vec{v}(t) = D\vec{r}(t) = (2, t, \frac{2}{t})^T$.
   We can normalize this to find $v(t) = \sqrt{4 + t^2 + \frac{4}{t^2}}$, which we can simplify to $\frac{t^2+2}{t}$ since $t > 0$.
   Notice that $\vec{r}(1) = (2, 0.5, 0)$ and $\vec{r}(4) = (8, 8, 2\ln 4)^T$. So the arc length between the two points is given by:
   $$\int_1^4 t + \frac{2}{t} dt = \left(\frac{t^2}{2} + 2\ln t\right)_1^4 = \frac{15}{2} + 2\ln 4.$$

   (b) Compute the velocity, speed, acceleration, tangent vector, normal vector, binormal vector, curvature, torsion, and the tangential and normal components of acceleration for the curve at an arbitrary $t > 0$.
   With the functions of t we already have.
   $$\vec{T}(t) = \frac{\vec{v}(t)}{v(t)} = \frac{t}{t^2+2}\begin{pmatrix} 2 \\ t \\ \frac{2}{t} \end{pmatrix}.$$
   Above we calculated $\vec{v}(t)$, and we will use this now to calculate $\vec{a}(t) = (0, 1, -\frac{2}{t^2})^T$.
   Now we find $\|\vec{a}(t)\| = \sqrt{1 + \frac{4}{t^4}} = \frac{t^4+4}{t^2}$.
   Another necessary calculation will be $\vec{v} \times \vec{a}$.
   $$\vec{v} \times \vec{a} = \det\begin{bmatrix} i & j & k \\ 2 & t & \frac{2}{t} \\ 0 & 1 & -\frac{2}{t^2} \end{bmatrix} = \begin{pmatrix} -\frac{4}{t} \\ \frac{4}{t^2} \\ 2 \end{pmatrix}.$$
   And normalizing this vector gives us:
   $$\|\vec{v} \times \vec{a}\| = \sqrt{\frac{16}{t^2} + \frac{16}{t^4} + 4} = \frac{2(t^2+2)}{t^2}.$$
   With these two quantities we can find:
   $$\vec{B} = \frac{\vec{v} \times \vec{a}}{\|\vec{v} \times \vec{a}\|} = \frac{1}{t^2+2}\begin{pmatrix} -2t \\ 2 \\ t^2 \end{pmatrix}.$$
   And we can use $\vec{B} \times \vec{T} = \vec{N}$.
   $$\vec{N} = \frac{t}{(t^2+2)^2}\det\begin{bmatrix} i & j & k \\ -2t & 2 & t^2 \\ 2 & t & \frac{2}{t} \end{bmatrix} = \frac{1}{t^2+2}\begin{pmatrix} 2-t^2 \\ 2t \\ -2t \end{pmatrix}.$$
   Now we find τ with $\tau = \frac{(\vec{v}\times\vec{a})\cdot\vec{a}'}{\|\vec{v}\times\vec{a}\|^2}$.
   $$\tau = \frac{t^4}{4(t^2+2)^2}\begin{pmatrix} -\frac{4}{t} \\ \frac{4}{t^2} \\ 2 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 0 \\ \frac{4}{t^3} \end{pmatrix} = \frac{2t}{(t^2+2)^2}.$$
   Using $\kappa = \frac{1}{v^3}\|\vec{v} \times \vec{a}\|$:
   $$\kappa = \left(\frac{t}{t^2+2}\right)^3 2\frac{t^2+2}{t^2} = \frac{2t}{(t^2+2)^2}.$$
   We can now also find a_T, a_N.
   $$a_T = v' = \frac{t^2-2}{t^2}$$
   $$a_N = \kappa v^2 = \frac{2t}{(t^2+2)^2}\left(\frac{t^2+2}{t}\right)^2 = \frac{2}{t}.$$

2. (a) Suppose $y = f(x)$ is a planar curve where f is twice differentiable. Show that f has zero curvature everywhere if and only if f is a line.

Proof. Let f be twice differentiable as above.

$\Longrightarrow$: Suppose f has zero curvature. That is,

$$\kappa = \frac{f''(x)}{(1+f'(x)^2)^{\frac{3}{2}}} = 0..$$

So $f''(x) = 0$. Then $f'(x) = c$ for some $c \in \mathbb{R}$, and $f(x) = cx + d$, for some $d \in \mathbb{R}$. This is the equation of a line.

So if f has zero curvature, f must be a line!

$\Longleftarrow$: Suppose f is a line. That is $f(x) = ax + b$ for some $a, b \in \mathbb{R}$. Then $f'(x) = a$ and $f''(x) = 0$. Using our formula:

$$\kappa = \frac{f''(x)}{(1+f'(x)^2)^{\frac{3}{2}}} = \frac{0}{(1+a^2)^{\frac{3}{2}}} = 0.$$

We can see that if f is a line, then $\kappa = 0$.

Therefore, curvature is zero $\iff$ f is a line.

(b) Suppose C is a parametric curve with parametrization $\vec{r}(t)$. If C has zero curvature at all points, is it the case that $\vec{r}$ is a line? Provide a proof or a counterexample

It is not the case that if C has zero curvature that $\vec{r}$ is a line. Consider the example $\vec{r} = \begin{pmatrix} e^t & e^t & 0 \end{pmatrix}$. Note that $\vec{r}$ is not a line.

We can find that $\vec{r} = \vec{v} = \vec{a}$. So, calculating $\vec{v} \times \vec{a}$, we see:

$$\vec{v} \times \vec{a} = \det \begin{bmatrix} \vec{i} & \vec{j} & \vec{k} \\ e^t & e^t & 0 \\ e^t & e^t & 0 \end{bmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

And so $\|\vec{v} \times \vec{a}\| = 0$. Since $\kappa = \frac{1}{v^3}\|\vec{v} \times \vec{a}\|$, $\kappa = \frac{0}{2e^{2t}} = 0$. So C has zero curvature but $\vec{r}$ is not a line.

(c) Suppose C is a parametric curve with parametrization $\vec{r}(t)$. If C has zero torsion at all points, is it the case that $\vec{r}$ is a line? Provide proof or a counterexample.

Choose $\vec{r} = \begin{pmatrix} t^2 & t & 3t^2 \end{pmatrix}^T$.

Then we have:

$$\begin{aligned} \vec{v}(t) &= \begin{pmatrix} 2t & 1 & 6t \end{pmatrix}^T \\ \vec{a}(t) &= \begin{pmatrix} 2 & 0 & 6 \end{pmatrix}^T \\ \vec{a'}(t) &= 0. \end{aligned}$$

Now we find $\vec{v} \times \vec{a}$.

$$\vec{v} \times \vec{a} = \det \begin{bmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2t & 1 & 6t \\ 2 & 0 & 6 \end{bmatrix} = \begin{pmatrix} 6 & 0 & 2 \end{pmatrix}^T.$$

And from this we find $\|\vec{v} \times \vec{a}\| = 2\sqrt{10}$. From our formula for τ:

$$\tau = \frac{(\vec{v} \times \vec{a}) \cdot \vec{a}'}{\|\vec{v} \times \vec{a}\|^2} = \frac{0}{2\sqrt{10}} = 0.$$

So C has zero torsion but r is not a line.

3. (a) Suppose C is a curve parametrized at by $r(t)$. Suppose we know that $\vec{a} = (4,-2,6)^T$, $\vec{T} = \frac{1}{\sqrt{3}}(1,-1,1)^T$ and $\vec{N} = \frac{1}{\sqrt{5}}(1,0,2)^T$ at some point on the curve. Determine a_T and a_N there.

   We can find a_T by taking $\vec{a} \cdot \vec{T}$ at our point. So:

$$a_T = \frac{4}{\sqrt{3}} + \frac{2}{\sqrt{3}} + \frac{6}{\sqrt{3}} = \frac{12}{\sqrt{3}} = 4\sqrt{3}.$$

   Likewise, we can find a_N by taking $\vec{a} \cdot \vec{N}$ at our point:

$$a_N = \frac{4}{\sqrt{5}} + \frac{12}{\sqrt{5}} = \frac{16}{\sqrt{5}}.$$

   (b) Suppose C is a curve parametrized at by $\vec{r}(t)$. Is it possible for $a_T = 0$ at all points? If so, what kind of curve is C?

   a_T can equal 0 for all t. If so, we can see $v'(t) = 0$ which implies that $v(t) = c$ for some $c \in \mathbb{R}$. That is, the speed is constant (we have zero acceleration).

   (c) Suppose C is a curve parametrized at by $\vec{r}(t)$. Is it possible for $a_N = 0$ at all points? If so, what kind of curve is C?

   If $a_N = 0$, $v^2\kappa = 0$. This means that either $v = 0$ or $\kappa = 0$.

   Intuitively, we know that normal acceleration represents the change in direction of velocity. So in the case where we have zero normal acceleration we have no change in the direction of velocity.