

1. Consider the function $f : \mathbb{R}^3 \rightarrow \mathbb{R}^3$, defined by $f(x, y, z) = \begin{pmatrix} x^3 - y - z \\ 2x + y + z \\ x + y - z \end{pmatrix}$

(a) Compute $Jf(x, y, z)$ and show that $df_{(x,y,z)}$ is invertible for any $(x, y, z) \in \mathbb{R}^3$.

Solution: Compute:

$$Jf(x, y, z) = \begin{bmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} & \frac{\partial f_1}{\partial z} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} & \frac{\partial f_2}{\partial z} \\ \frac{\partial f_3}{\partial x} & \frac{\partial f_3}{\partial y} & \frac{\partial f_3}{\partial z} \end{bmatrix} = \begin{bmatrix} 3x^2 & -1 & -1 \\ 2 & 1 & 1 \\ 1 & 1 & -1 \end{bmatrix}.$$

Then we take the determinant;

$$\det Jf(x, y, z) = 3x^2 \begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} - 2 \begin{vmatrix} -1 & -1 \\ 1 & -1 \end{vmatrix} + \begin{vmatrix} -1 & -1 \\ 1 & 1 \end{vmatrix} = -6x^2 - 4 + 0.$$

And we can see that since this determinant has no real roots, it must be nonzero for any $(x, y, z) \in \mathbb{R}^3$, and so Jf , as well as df are both invertible in \mathbb{R}^3 .

- (b) Find the largest open $U \subset \mathbb{R}^3$ where f has a continuously differentiable inverse function g .

Solution: Begin by showing that f is injective in \mathbb{R}^3 . Suppose:

$$x_1^3 - y_1 - z_1 = x_2^3 - y_2 - z_2 \quad (1)$$

$$2x_1 + y_1 + z_1 = 2x_2 + y_2 + z_2 \quad (2)$$

$$x_1 + y_1 - z_1 = x_2 + y_2 - z_2. \quad (3)$$

However, if we add (1) + (2), we get $x_1^3 + 2x_1 = x_1(x_1^2 + 2) = x_2(x_2^2 + 2) = x_2^3 + 2x_2$

2. Consider the system of equations: (S) $\begin{cases} x - y - u^2 + v^2 = 0 \\ x + y - 2uv = 0 \end{cases}$

(a) Show that the system (S) can be solved for u, v in term of (x, y) near the point $(x, y, u, v) = (1, 1, 1, 1)$.

Solution: We solve for the Jacobian about $(1, 1, 1, 1)$.

$$Jf(x, y, u, v) = \begin{bmatrix} 1 & -1 & -2u & 2v \\ 1 & 1 & -2v & -2u \end{bmatrix}$$

$$Jf(1, 1, 1, 1) = \begin{bmatrix} 1 & -1 & -2 & 2 \\ 1 & 1 & -2 & -2 \end{bmatrix}.$$

And if we break Jf into block matrices, we get the invertible right half of Jf as $\begin{bmatrix} -2 & 2 \\ -2 & -2 \end{bmatrix}$ which has nonzero determinant and must be invertible. So u, v can be implicitly defined about $(1, 1, 1, 1)$ by the Implicit Function theorem.

- (b) Compute $\partial_x u(1, 1) + \partial_y v(1, 1)$.

Solution: Begin with the identity from the Implicit Function Theorem:

$$\begin{aligned} \begin{bmatrix} \partial_x u & \partial_y u \\ \partial_x v & \partial_y v \end{bmatrix} &= \begin{bmatrix} \partial_u f_1 & \partial_v f_1 \\ \partial_u f_2 & \partial_v f_2 \end{bmatrix}^{-1} \begin{bmatrix} \partial_x f_1 & \partial_y f_1 \\ \partial_x f_2 & \partial_y f_2 \end{bmatrix} \\ &= \left(\det \begin{bmatrix} -2u & 2v \\ -2v & -2u \end{bmatrix} \right)^{-1} \begin{bmatrix} -2u & -2v \\ 2v & -2u \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \\ &= \frac{1}{2(u^2 + v^2)} \begin{bmatrix} -u & -v \\ v & -u \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \\ &= \frac{1}{2(u^2 + v^2)} \begin{bmatrix} -u-v & u-v \\ v-u & -v-u \end{bmatrix}. \end{aligned}$$

And so if we want the sum $\partial_x u(1, 1) + \partial_y v(1, 1)$ we need only take the trace of this matrix and evaluate at $(1, 1)$.

$$\begin{aligned} \partial_x u(1, 1) + \partial_y v(1, 1) &= \frac{1}{2(u^2 + v^2)} \Big|_{(1,1)} \\ &= -2 \frac{u+v}{2(u^2 + v^2)} \Big|_{(1,1)} \\ &= -1. \end{aligned}$$

3. Let $f : \mathbb{R}^2 \rightarrow \mathbb{R} : (x, y) \rightarrow f(x, y)$. Show that if $f \in C^1(\mathbb{R}^2, \mathbb{R})$, then f can't be injective on \mathbb{R}^2 . Hint: Use the implicit functions theorem.
4. Let $E = C([a, b], \mathbb{R})$ equipped with the norm of uniform convergence, let $u \in C(\mathbb{R}, \mathbb{R})$, and consider the mapping $\phi : E \rightarrow E$, defined by $\phi(v) = u \circ v$. Is ϕ continuous? Make sure to justify your answer.

Solution: Let $\varepsilon > 0$, and $v, w \in E$. Recall that the image of compact sets under continuous functions is compact, and the union of compact sets is compact. Then since continuous functions are uniformly continuous on compact sets, u must be uniformly continuous on $v([a, b]) \cup w([a, b])$. Let $x \in [a, b]$, and let δ be chosen so that $|w(x) - v(x)| < \delta \implies |u(w(x)) - u(v(x))| < \varepsilon$. Suppose

$$\|w - v\| = \sup_{x \in [a, b]} |w(x) - v(x)| < \delta. \quad (*)$$

Then we must have $|w(x) - v(x)| < \delta$ for any $x \in [a, b]$. But by continuity of u , we have

$$|\phi(w) - \phi(v)| = |u(w(x)) - u(v(x))| < \varepsilon.$$

for any $x \in [a, b]$. Then recall that since $u, v, w \in E$ are continuous, the composition, difference and absolute value $|u \circ w - u \circ v|$ is continuous. Therefore the supremum of this function is attained in the compact set $[a, b]$, and when we take the supremum $\sup_{x \in [a, b]} |u(w(x)) - u(v(x))|$, we can say that it is attained for some $x_0 \in [a, b]$. And from $(*)$, we have:

$$\begin{aligned} \|\phi(w) - \phi(v)\| &= \|u \circ w - u \circ v\| \\ &= \sup_{x \in [a, b]} |u(w(x)) - u(v(x))| \\ &= |u(w(x_0)) - u(v(x_0))| \\ &< \varepsilon. \end{aligned}$$

And ϕ is continuous as desired.

5. Find in $C([0, 1], \mathbb{R})$ the distance from the function $u(t) = t$ to the subspace \mathbb{P}_0 of polynomials of degree 0. Make sure to justify your answer.

Solution: Let $u(t) = t$, and take the distance:

$$\begin{aligned}
 d(u, \mathbb{P}_0) &= \inf_{p \in \mathbb{P}_0} d(u, p) \\
 &= \inf_{c \in \mathbb{R}} \|u - c\| && p \text{ is simply a real constant} \\
 &= \inf_{c \in \mathbb{R}} \sup_{t \in [0,1]} |u(t) - c| \\
 &= \inf_{c \in \mathbb{R}} \sup_{t \in [0,1]} |t - c| \\
 &= \frac{1}{2}.
 \end{aligned}$$

6. Let $f \in C([a, b], \mathbb{R})$ be such that $\int_a^b f(x)x^n dx = 0$, $\forall n \in \mathbb{N}$. Show that f is identically zero. Hint: Use Weierstrass Theorem.

Solution: First, we claim that if p is any real polynomial, then $\int_a^b f(x)p(x) dx = 0$. Write $p = \sum_{i=0}^n c_i x^i$. Then:

$$\begin{aligned}
 \int_a^b f(x)p(x) dx &= \int_a^b f(x) \sum_{i=0}^n c_i x^i dx \\
 &= \sum_{i=0}^n \int_a^b c_i x^i f(x) dx \\
 &= \sum_{i=0}^n c_i \int_a^b x^i f(x) dx \\
 &= \sum_{i=1}^n c_i \cdot 0 \\
 &= 0.
 \end{aligned}$$

By Weierstrass, there exists a sequence of real polynomials convergent to f . Let $\{p_n\}$ be such a sequence, and take:

$$\begin{aligned}
 \int_a^b f^2(x) dx &= \int_a^b \lim_{n \rightarrow \infty} p_n(x) f(x) dx \\
 &= \lim_{n \rightarrow \infty} \int_a^b f(x) p_n(x) dx && f p_n \in C([a, b]) \\
 &= \lim_{n \rightarrow \infty} 0 \\
 &= 0.
 \end{aligned}$$

Define $F(x)$ so that $\frac{d}{dx} F(x) = f^2(x)$ by the Fundamental Theorem of Calculus. Then $\int_a^b f^2(x) dx = F(a) - F(b) = 0$. Then $F(a) = F(b)$. But since $f^2(x) \geq 0$, F is increasing, and we must have $F(x) = c$ for some constant real $c \in \mathbb{R}$. Then since $f^2(x) = \frac{d}{dx} F(x) = \frac{d}{dx} c = 0$, we have f^2 is identically 0, and then f must also be identically 0.