Thomas Boyko - Math 271 Assignment 4 - 30191728

1. Let $A = \{1,2,3,4,5,6,7,8,9\}$. Let $\mathcal{P} = \mathcal{P}(A)$ be the power set of A

   Let R be the relation on A defined by:

   $$\text{For all } X, Y \in \mathcal{P}, X \; R \; Y \iff |X \cup Y| \leq |X \cup \{1,2\}|.$$

   (a) Is R:

   (i) Reflexive? R is reflexive. We must prove that for all $X \in \mathcal{P}, X \; R \; X$.

   Proof: Suppose $X \in \mathcal{P}$ and that X has x elements.
   So $|X \cup X| = |X| = x$.
   Note that $|X \cup \{1,2\}| = x + 2 - |X \cap \{1,2\}|$. We can see this through summing the size of both sets, and then subtracting their intersection since it will be counted twice.
   For an element to be in $|X \cap \{1,2\}|$, it must be 1 or 2. So the set $X \cap \{1,2\}$ has between 0 and 2 elements.
   Since $0 \leq |X \cap \{1,2\}| \leq 2$, we can say that $|X \cup \{1,2\}| = x + n$ where $n \in \mathbb{Z}, 0 \leq n \leq 2$.
   $|X \cup X| = x \leq x + n = |X \cup \{1,2\}|$
   Therefore, for all $X \in \mathcal{P}, X \; R \; X$ ■

   (ii) Symmetric?
   R is not symmetric. Or, there exist $X, Y \in \mathcal{P}$ so that $X \; R \; Y$ but $Y \not R \; X$.
   Proof: Choose the following sets $X, Y \in \mathcal{P}$.

   $$\begin{aligned} X &= \{1,2,3\} \\ Y &= \varnothing \end{aligned}$$

   Note the following:

   $$\begin{aligned} X \cup Y &= \{1,2,3\} \\ X \cup \{1,2\} &= \{1,2,3\} \\ Y \cup \{1,2\} &= \{1,2\} \\ |X \cup Y| = 3 &\leq 3 = |X \cup \{1,2\}| \end{aligned}$$

   So, $X \; R \; Y$. However,

   $$|X \cup Y| = 3 > 2 = |Y \cup \{1,2\}|$$

   So, $Y \not R \; X$. Therefore, there exist $X, Y \in \mathcal{P}$ so that $X \; R \; Y$ but $Y \not R \; X$, and R is not symmetric. ■

   (iii) Antisymmetric?
   R is not antisymmetric. We must prove that there exist $X, Y \in \mathcal{P}$ so that $X \; R \; Y$ and $Y \; R \; X$ but $Y \neq X$.
   Proof: Choose the following sets $X, Y \in \mathcal{P}$:

   $$\begin{aligned} X &= \{1\} \\ Y &= \varnothing \end{aligned}$$

   So,

   $$\begin{aligned} X \cup Y &= \{1\} \\ X \cup \{1,2\} &= \{1,2\} \\ Y \cup \{1,2\} &= \{1,2\} \\ |X \cup Y| = 1 &\leq 2 = X \cup \{1,2\} \\ |X \cup Y| = 1 &\leq 2 = Y \cup \{1,2\} \end{aligned}$$

   So $Y \; R \; X$ and $X \; R \; Y$.
   But since $1 \in X$ and $1 \notin Y$, $Y \neq X$.
   Therefore, there exist $X, Y \in \mathcal{P}$ so that $X \; R \; Y$ and $Y \; R \; X$ but $Y \neq X$, and R is not antisymmetric. ■

(iv) Transitive?

R is not transitive. So we must find some $X, Y, Z \in \mathcal{P}$ so that $X \; R \; Y$ and $Y \; R \; Z$, but $X \not R \; Z$.

Proof: Choose the following sets $X, Y, Z \in \mathcal{P}$.

$$\begin{aligned} X &= \varnothing \\ Y &= \{3, 4\} \\ Z &= \{3, 4, 5\} \end{aligned}$$

Therefore we can see that:

$$\begin{aligned} X \cup Y &= \{3, 4\} \\ Y \cup Z &= \{3, 4, 5\} \\ X \cup Z &= \{3, 4, 5\} \\ X \cup \{1, 2\} &= \{1, 2\} \\ Y \cup \{1, 2\} &= \{1, 2, 3, 4\} \\ |X \cup Y| = 2 \leq 2 &= |X \cup \{1, 2\}| \\ |Y \cup Z| = 3 \leq 4 &= |Y \cup \{1, 2\}| \end{aligned}$$

So $X \; R \; Y$ and $Y \; R \; Z$.

$$|X \cup Z| = 3 > 2 = |X \cup \{1, 2\}|$$

So we know that $X \not R \; Z$.

Therefore there exist sets $X, Y, Z \in \mathcal{P}$ so that $X \; R \; Y$ and $Y \; R \; Z$, but $X \not R \; Z$. ■

(b) How many $S \in \mathcal{P}$ are there so $\{3\}RS$?

Note that $|\{3\} \cup \{1, 2\}| = 3$, so:

$$\{3\} \; R \; S \iff |\{3\} \cup S| \leq 3$$

To create a set S so that $\{3\} \; R \; S$, we first can select whether or not $3 \in S$. Whether this is the case or not will not change if $\{3\} \; R \; S$ There are 2 ways to do this.

Then, we choose 2 elements from A excluding 3, without replacement. There are $\binom{9}{2}$ ways to do this. (We cannot choose 3, but we can choose to not add anything to S).

So there are $2\binom{9}{2} = 72$ ways to choose a set $S \in \mathcal{P}$ so that $\{3\} \; R \; S$.

(c) How many $S \in \mathcal{P}$ are there so $S \; R \; \{3\}$?

To count the number of sets S so that $S \; R \; \{3\}$, first we must separate into two cases.

Case 1: $3 \in S$.

If 3 is in S, then we can choose any elements from A to put in S and it will still be related to $\{3\}$.

So, if $3 \in S$, there are 2^8 ways to choose the rest of S so that $S \; R \; \{3\}$.

Case 2: $3 \notin S$.

If 3 is not in S, then at most one of $\{1, 2\}$ can be in S.

So first we choose whether $1 \in S$, $2 \in S$, or neither. There are 3 ways to do this.

Then for each remaining element of A, we can choose if it is in or out. There are 2^6 ways to do this.

Therefore, there are $3(2^6)$ ways to choose a set S so that $S \; R \; \{3\}$ when $3 \notin S$.

So the total number of ways to choose a set $S \in \mathcal{P}$ so that $S \; R \; \{3\}$ is $2^8 + 3(2^6) = 448$.

2. Let $A = \{1, 2, 3, 4, 5\}$. Let $\mathcal{F}$ be the set of all functions from A to A. Define a relation R on $\mathcal{F}$ as follows:

$$\forall f, g \in \mathcal{F}, fRg \iff \forall i \in A, f(i) \leq g(i)$$

(a) Is R:

(i) Reflexive?
R is reflexive.

Proof: Suppose $f \in \mathcal{F}$.
Notice that $\forall i \in A,\ f(i) = f(i)$ since f is a function.
So $f(i) \leq f(i)$.
So $f\ R\ f$ and R is reflexive. ■

(ii) Symmetric?
R is not symmetric. So, there exist $f, g \in \mathcal{F}$ so that $f\ R\ g$ but $g\ \not R\ f$.

Proof: Choose $f = \{(x, 1) : x \in A\}$ and $g = \{(x, 2) : x \in A\}$
So, $\forall i \in A, f(i) = 1$ and $g(i) = 2$.

$$f(i) = 1 \leq 2 = g(i)$$

So $f\ R\ g$
But, $\forall i \in A$,

$$g(i) = 2 > 1 = f(i)$$

So $f\ \not R\ g$.
Therefore, there exist $f, g \in \mathcal{F}$ so that $f\ R\ g$ but $g\ \not R\ f$, and R is not symmetric. ■

(iii) Antisymmetric?
R is antisymmetric. So, $\forall f, g \in \mathcal{F}$ and $i \in A$, if $f\ R\ g$ and $g\ R\ f$, then $g = f$.

Proof: Suppose that $g, f \in \mathcal{F}$ and that $f\ R\ g$ and $g\ R\ f$.
So, for all $i \in A,\ f(i) \leq g(i),\ g(i) \leq f(i)$
So we have:

$$f(i) \leq g(i) \leq f(i)$$

So $f(i) = g(i)$ and $f = g$.
Therefore, R is antisymmetric. ■

(iv) Transitive?
R is transitive. So, for all $f, g, h \in \mathcal{F}$, if $f\ R\ g$ and $g\ R\ h$, then $f\ R\ h$.

Proof: Suppose that $f, g, h \in \mathcal{F}$.
Further suppose that $f\ R\ g$ and $g\ R\ h$.
So for all $i \in A,\ f(i) \leq g(i)$ and $g(i) \leq h(i)$

$$f(i) \leq g(i) \leq h(i)$$
$$f(i) \leq h(i)$$

So $f\ R\ h$.
Therefore, for all $f, g, h \in \mathcal{F}$, if $f\ R\ g$ and $g\ R\ h$, $f\ R\ h$ and R is transitive. ■

(b) Prove or disprove: For all $f \in \mathcal{F}$, there exists $g \in \mathcal{F}$ so that $f\ R\ g$

Proof: Suppose $f \in \mathcal{F}$.
Choose $g = f$.
So, for all $i \in A,\ f(i) = g(i)$.
This satisfies the condition for the relation, so $f\ R\ g$. ■

(c) Prove or disprove: There exists $g \in \mathcal{F}$ so that for all $f \in \mathcal{F}$, $f\ R\ g$.

Proof: Choose $g = \{(a, 5) : a \in A\}$
Suppose $f \in \mathcal{F}$.
So, for all $i \in A$, $g(i) = 5$.
Note that since all values in A are between 0 and 5, $0 \leq f(i) \leq 5$.
Since $g(i) = 5$, $f(i) \leq g(i)$.
So, $f\ R\ g$.
Therefore, there exists $g \in \mathcal{F}$ so that for all $f \in \mathcal{F}$, $f\ R\ g$. ∎

Let $F \in \mathcal{F}$ be the function $f = \{(1, 3), (2, 3), (3, 3), (4, 1), (5, 5)\}$.

(d) How many functions $g \in \mathcal{F}$ are there so that fRg? Explain.

We can count the number of functions g so that $f\ R\ g$ by counting the number of functions where $g(i)$ is greater than or equal $f(i)$ for each $i \in A$.

First, $f(1) = 3 \leq g(1)$. So g must equal 3, 4, or 5. (3 ways to choose g(1)).

The same applies for $2, 3$ since $3 = f(3) = f(2)$ so there are 3 ways each to choose $g(2)$, $g(3)$.

There are 5 ways to choose $g(4)$ since $f(4) = 1 \leq g(4)$ so $g(4)$ can be any element of A.

Finally, there is only one way to choose $g(5)$ since it must equal 5 in order for it to be greater than or equal to $f(5) = 5$.

So there are $3^3 \times 4 = 108$ ways to choose a function g so that $f\ R\ g$.

(e) How many functions $g \in \mathcal{F}$ are there so that gRf? Explain.

We can count the number of functions g so that $g\ R\ f$ by counting the number of functions where $g(i) \leq f(i)$ for each $i \in A$.

First we choose values for $g(1), g(2)$, and $g(3)$. (Like in the last case, these all have the same number of options since $f(1) = f(2) = f(3) = 3$). There are 3 ways to choose all of these since each can equal $1, 2$, or 3.

There is only one way to choose $g(4)$ since it must equal 1 in order to be less than or equal to $f(4) = 1$.

Finally, there are 5 ways to choose $g(5)$ since all 5 elements of A are less than or equal to $5 = f(5)$.

So there are $3^3 \times 4 = 108$ ways to choose a function g so that $f\ R\ g$.

3. Prove or disprove the following statements by using the definitions of "congruence modulo n" and "divides."

   (a) For all positive integers a, x and y, if $(a+x) \equiv (a+y) \pmod{12}$, then $x \equiv y \pmod{12}$

   Proof: Suppose $x, y, a \in \mathbb{Z}$, and that $a + x \equiv a + y \pmod{12}$.
   Then $12k = (a+x) - (a+y)$ for some $k \in \mathbb{Z}$.
   So $12k = a - y$
   Therefore, $x \equiv y \pmod{12}$. $\blacksquare$

   (b) For all positive integers a, x and y, if $ax \equiv ay \pmod{12}$, then $x \equiv y \pmod{12}$

   The statement is false. The negation is: "There exist positive integers a, x and y so that $ax \equiv ay \pmod{12}$, but $x \not\equiv y \pmod{12}$"

   Proof: Choose $a = 12$, $x = 2$, $y = 1$.
   Note that $12(2) \equiv 12(1) \pmod{12}$ since $12|24 - 12$ because $12 = (1)(12)$.
   However, $1 \not\equiv 2 \pmod{12}$ since $12 \nmid 2 - 1$, because $12 > 1$.
   Therefore, there exist positive integers a, x and y so that $ax \equiv ay \pmod{12}$, but $x \not\equiv y \pmod{12}$ $\blacksquare$

   (c) There exists a positive integer $a > 1$ so that for all $x, y \in \mathbb{Z}$, if $ax \equiv ay \pmod{12}$, then $x \equiv y \pmod{12}$

   Proof: Choose $a = 7$, and suppose that $7x \equiv 7y \pmod{12}$.
   So $12|7x - 7y$.
   It will prove helpful to check that $\gcd(12, 7) = 1$.

$$
\begin{aligned}
\gcd(12,7) &= \gcd(7,5) \\
&= \gcd(5,2) \\
&= \gcd(2,1) \\
&= \gcd(1,0) \\
&= 1
\end{aligned}
$$

   By Euclid's Lemma, since $\gcd(12, 7) = 1$ and $12|7(x-y)$, we know that $12|x-y$ and $x \equiv y \pmod{12}$.
   Therefore, there exists a positive integer $a > 1$ so that for all $x, y \in \mathbb{Z}$, if $ax \equiv ay \pmod{12}$, then $x \equiv y \pmod{12}$. $\blacksquare$

   (d) For all positive integers a and b, if $a^2 \equiv b^2 \pmod{12}$, then $a \equiv b \pmod{12}$.

   The statement is false. The negation is: "There exist positive integers a and b so that $a^2 \equiv b^2 \pmod{12}$ but $a \not\equiv b \pmod{12}$."

   Proof: Choose $a = 12$, $b = 6$.
   Then $144 \equiv 36 \pmod{12}$ since $12|144 - 36$, because $12(9) = 108$.
   However, 12 is not congruent to 6 modulo 12 because $12 \nmid 12 - 6$ since $12 > 6$.
   Therefore, there exist positive integers a and b so that
   $a^2 \equiv b^2 \pmod{12}$ but $a \not\equiv b \pmod{12}$. $\blacksquare$