Assignment 3 - Thomas Boyko - 30191728

1. For each of the following statements: If the statement is true, then give a proof; if the statement is false, then write out the negation and prove that.
   Notation: Let F be the set of all functions from $\mathbb{Z}$ to $\mathbb{Z}$.

   (a) For all $f, g, h \in F$, if $f \circ g = f \circ h$, then $g = h$.
   The statement is false. The negation is: "There exist functions $f, g, h \in F$ so that $f \circ g = f \circ h$ but $g \neq h$."

   Proof: Choose the functions $f, g, h \in F$ where $f(x) = x^2$, $g(x) = x$ and $h(x) = -x$ for all $x \in \mathbb{Z}$.
   Note that
   $$f \circ g = f(g(x)) = f(x) = x^2 = (-x)^2 = f(-x) = f(h(x)) = f \circ h$$
   So for all $x \in \mathbb{Z}$, $f(g(x)) = f(h(x))$
   However,
   $$g(1) = 1 \neq -1 = h(1)$$
   So $g \neq h$.
   Therefore, there exist functions $f, g, h \in F$ so that $f \circ g = f \circ h$ but $g \neq h$. ■

   (b) For all $f, g, h \in F$, if $g \circ f = h \circ f$, then $g = h$. The statement is false. The negation is: "There exist $f, g, h \in F$ so that $g \circ f = h \circ f$ but $g \neq h$."

   Proof: Choose the following functions $f, g, h \in F$.
   $$f(x) = x^2$$
   $$g(x) = \begin{cases} y & \exists y \in \mathbb{Z} \text{ so that } y^2 = x \\ 0 & \forall y \in \mathbb{Z}, y^2 \neq x \end{cases}$$
   $$h(x) = \begin{cases} y & \exists y \in \mathbb{Z} \text{ so that } y^2 = x \\ 1 & \forall y \in \mathbb{Z}, y^2 \neq x \end{cases}$$
   Note that the entire codomain of f will take the first case for g, h since every element in the codomain is a perfect square.
   So:
   $$h(f(x)) = x = g(f(x))$$
   But, $h(2) = 1 \neq 0 = g(2)$, so $h \neq g$.
   Therefore, there exist $f, g, h \in F$ so that $g \circ f = h \circ f$ but $g \neq h$. ■

   (c) For all $f, g, h \in F$, if $f \circ g = f \circ h$ and f is one-to-one, then $g = h$.

   Proof: Suppose $f, g, h \in F$. Further suppose $f \circ g(x) = f \circ h(x)$ for all $x \in \mathbb{Z}$. Finally, suppose f is one-to-one.
   Let $a = h(x)$ and $b = g(x)$ where $a, b \in \mathbb{Z}$.
   Then f(a)=f(b).
   Then by the definition of one-to-one, since $f(a) = f(b)$, $a = b$.
   Therefore, for all $f, g, h \in F$, if $f \circ g = f \circ h$ and f is one-to-one, then $g = h$. ■

   (d) For all $f, g, h \in F$, if $g \circ f = h \circ f$ and f is onto, then $g = h$.

   Proof: Suppose $f, g, h \in F$, and that $f \circ g = f \circ h$.
   Since f is onto, for all $b \in \mathbb{Z}$, there exists an $a \in \mathbb{Z}$ so that $f(a) = b$. So, the codomain of f is $\mathbb{Z}$.
   This means that for all $c \in \mathbb{Z}$, $g(c) = h(c)$.
   Therefore, $g = f$. ■

2. Let $f : \mathbb{Z} \to \mathbb{Z}$ be the function defined by $f(x) = 3x^2 + x$ for every $x \in \mathbb{Z}$.

(a) Is f one-to-one? Prove your answer.
f is one-to-one.

Proof: Let $f : \mathbb{Z} \to \mathbb{Z}$, $f(x) = 3x^2 + x$, where $x \in \mathbb{Z}$. Suppose $a, b \in \mathbb{Z}$ and that $f(a) = f(b)$. From this we will attempt to prove that $a = b$.
So we have:

$$\begin{aligned} 3a^2 + a &= 3b^2 + b \\ 3a^2 + a - 3b^2 - b &= 0 \\ 3(a^2 - b^2) + (a - b) &= \\ 3(a-b)(a+b) + (a-b) &= \\ (a-b)(3(a+b)+1) &= \end{aligned}$$

So, either $(a - b) = 0$ or $3(a + b) + 1 = 0$. However, if $3(a + b) + 1 = 0$, then $a + b = -\frac{1}{3}$ which means a, b cannot both be integers.
Since $3(a + b) + 1 \neq 0$, $a - b = 0$ and $a = b$.
Therefore, for all $a, b \in \mathbb{Z}$, if $f(a) = f(b)$, then $a = b$. So f is one-to-one.

■

(b) Is f onto? Prove your answer.
f is not onto. So, we must prove: "There exists some $b \in \mathbb{Z}$ so that for all $a \in \mathbb{Z}$, $f(a) \neq b$."

Proof: Choose $b = -1$. We must prove that for all $a \in \mathbb{Z}$, $f(a) > -1$.
Note that since $a \in \mathbb{Z}$, we know that $(a + \frac{1}{6})^2 \geq 0$.
We can follow this to show that f is greater than a certain value.

$$\begin{aligned} (a + \frac{1}{6})^2 &\geq 0 \\ 3(a + \frac{1}{6})^2 &\geq 0 \\ 3(a^2 + \frac{a}{3} + \frac{1}{36}) &\geq 0 \\ (3a^2 + a + \frac{1}{12}) &\geq 0 \\ 3a^2 + a &\geq -\frac{1}{12} > -1 \end{aligned}$$

So the function $f(a) \neq -1$ for all $a \in \mathbb{Z}$.
Since there exists some $b \in \mathbb{Z}$ so that for all $a \in \mathbb{Z}$, $f(a) \neq b$, f is not onto. ■

(c) Is there a function $g : \mathbb{Z} \to \mathbb{Z}$ so that $g \circ f = I_{\mathbb{Z}}$ where $I_{\mathbb{Z}}$ is the identity function from $\mathbb{Z}$ to $\mathbb{Z}$? Prove your answer.

Proof: Choose the following function $g : \mathbb{Z} \to \mathbb{Z}$.

$$g(x) = \begin{cases} y & \text{If } 3y^2 + y = x \text{ for some } y \in \mathbb{Z} \\ 0 & \text{If } \forall y \in \mathbb{Z},\ 3y^2 + y \neq x \end{cases}$$

Note that $g \circ f = g(f(x)) = g(3x^2 + x)$.
Looking back at the definition for $g(x)$ we can see that every input for $g \circ f$ will take the form $3y^2 + y$ for some integer y, where y will always equal x.
So, for all integers x, $g(f(x)) = x$ which is the identity function $I_{\mathbb{Z}}$.
Therefore, there exists a function $g : \mathbb{Z} \to \mathbb{Z}$ so that $g \circ f = I_{\mathbb{Z}}$ ■

3. Let $S = \{1, 2, 3, 4, 5\}$.
Let $f : S \to S$ be the function defined by $f = \{(1, 1), (2, 1), (3, 3), (4, 3), (5, 5)\}$.

   (a) How many functions $g : S \to S$ are there so that $g \circ f(2) = 1$? Explain.

   Note that $f(2) = 1$ so $g(f(2)) = g(1) = 1$. So $(1, 1) \in g$.

   Now we must map the remaining 4 elements of S to something in S. Since there are 5 elements in S, there are 5^4 ways to do this.

   So the total number of functions $g : S \to S$ so that $g \circ f(2) = 1$ is $5^4 = 625$

   (b) How many functions $g : S \to S$ are there so that $f \circ g(2) = 1$? Explain.

   Note that for $f(i)$ to equal 1, i must equal 1 or 2. So, either $g(2) = 1$ or $g(2) = 2$.

   So, to count the number of functions, we will first choose whether $g(2)$ equals 1 or 2. (there are two ways to do this). Then, for the remaining elements of S, we can choose it to map to any other element of S under g. There are 5^4 ways to do this.

   So, the total number of functions $g : S \to S$ so that $f \circ g(2) = 1$ is $2 \times 5^4 = 1250$

   (c) How many functions $g : S \to S$ are there so that $g \circ f(i) = 1$ for some $i \in S$? Explain.

   Note that the codomain of f is $\{1, 3, 5\}$. So the number of functions g so that $g(f(i)) = 1$ equals the total number of functions $g : S \to S$ minus those where $1, 3, 5$ are all not mapped to 1.

   We can build the former section with 5^5, since we are choosing an element from S for each element of S.

   The latter can be built by choosing for 1,3 and 5, to map it to any element that is not 1. There are 4^3 ways to do this. Then we can map 2 and 4 to any element, and there are 5^2 ways to do this. So there are $5^2 \times 4^3$ ways to build a function so that 1, 3, and 5 are all not mapped to 1.

   Subtracting these, we find there are $5^5 - 5^2 \times 4^3 = 3125 - 1600 = 1525$ ways to build a function $g : S \to S$ so that $g \circ f(i) = 1$

   (d) How many functions $g : S \to S$ are there so that $f \circ g(i) = 1$ for some $i \in S$? Explain.

   Note that $f(n)$ equals 1 if $n = 1$ or $n = 2$. Therefore, $g(i)$ must equal 1 or 2.

   We can count these functions by taking the total number of functions $g : S \to S$ and subtracting the number of functions where no element is mapped to 1 or 2.

   The total number of functions is 5^5.

   We can count the functions where no element is mapped to 1 or 2 by mapping each element in S to $3, 4$ or 5. There are 3^5 ways to choose this.

   So the total number of ways to choose a function $g : S \to S$ so that $f \circ g(i) = 1$ for some $i \in S$ is $5^5 - 3^5 = 3125 - 243 = 2882$.