Q1

Let S be a subset (not necessarily a subspace) of a finite dimensional inner product space V. Show that $(S^{\perp})^{\perp} = \text{span } S$, where

$$\text{span } S := \left\{ \sum_{j=1}^{m} \alpha_j s_j : \alpha_j \in \mathbb{C}, s_j \in S, m \in \mathbb{N} \right\}$$

is the smallest subspace of V containing S (think of this as the set of all possible linear combinations of vectors from S).

Solution: Let $B = \{b_1, \ldots, b_n\}$ be an orthonormal basis for span S.

$\Longrightarrow$: Let $v \in \text{span } S$. Then write $v = a_1 b_1 + \cdots + a_n b_n$. Let $w \in S^{\perp}$ be given, and note that $w \perp b_j$ for any basis element in S. Take:

$$\langle v, w \rangle = \langle a_1 b_1 + \cdots + a_n b_n, w \rangle = a_1 \langle b_1, w \rangle + \cdots + a_n \langle b_n, w \rangle = 0 + \cdots + 0 = 0.$$

And we see that $v \perp w$, so v is perpendicular to any element of $S^{\perp}$, and $v \in S^{\perp\perp}$.

$\Longleftarrow$: Now let $v \in S^{\perp\perp}$. Then $v \perp w$ for any $w \in S^{\perp}$
Recall the identity $V = \text{span } S \oplus (\text{span } S)^{\perp}$

Q2

Let V and W be finite dimensional inner product spaces and suppose $\ker A = \{0\}$. Find a left inverse for A in terms of A and A^*.

Solution: Begin with the identity,

$$\{0\} = \ker A = \ker A^* A.$$

So the composition of transformations $A^*A : V \to V$ has zero kernel and is injective, and by rank-nullity it must too surjective. Then this map is invertible, and if we take $(A^*A)^{-1}A^*A = I$, we see that $(A^*A)^{-1}A^*$ is a left inverse for A.

Q3

Let V be a finite dimensional inner product space.

(a) We can think of any $x \in V$ as a linear map from $\mathbb{C} \to V$ by setting $x(\lambda) := \lambda x$. You do not have to prove that this is linear. Show that $x^* : V \to \mathbb{C}$ satisfies

$$x^* y = \langle y, x \rangle.$$

Use this to deduce that the map xy^* is given by $xy^*v = \langle v, y \rangle x$. HINT: The inner product on $\mathbb{C}$ is assumed to be $\langle z, w \rangle = z\overline{w}$.

(b) Show that if $T : V \to \mathbb{C}$ is any linear map, then there is a vector y so that $T = y^*$.

(a) Recall from the definition of an adjoint operator, that the adjoint $x^* : V \to \mathbb{C}$ is given by:

$$\begin{aligned}\langle x(\lambda), y\rangle_V &= \langle \lambda, x^*(y)\rangle_{\mathbb{C}}\\ \langle \lambda x, y\rangle_V &= \lambda\overline{x^*(y)}\\ \lambda\langle x, y\rangle_V &= \lambda\overline{x^*(y)}\\ \overline{\lambda\langle x, y\rangle_V} &= \overline{\lambda\overline{x^*(y)}}\\ \overline{\lambda}\langle y, x\rangle_V &= \overline{\lambda}x^*(y)\\ \langle y, x\rangle &= x^*y.\end{aligned}$$

Then for the map $xy^* : V \to V$,

$$x(y^*(v)) = x(\langle v, y\rangle_V) = \langle v, y\rangle x.$$

(b) Choose $y = T^*(1)$. Then, for any $v \in V$,

$$y^*(v) = \langle v, y\rangle_V = \langle v, T^*(1)\rangle_V = \langle Tv, 1\rangle_{\mathbb{C}} = Tv.$$

And therefore T is induced by $y = T^*(1)$

Q4

Let V and W be finite dimensional vector spaces. You may find problem 3 useful here.

(a) Suppose $T : V \to W$ satisfies rank $T = 1$. Show that there are vectors $x \in W$ and $y \in V$ so that $T = xy^*$.

(b) Suppose $T : V \to W$ satisfies rank $T = k$. Show that T is the sum of k rank one operators. Hint: $PT = T$ where P is the orthogonal projection onto ran T.

(a) Since the dimension of the image of T has dimension 1, we must have ran T = span $\{b\}$ for some $b \in W$. Let $v \in V$, then $Tv = \alpha b$ for some $\alpha \in \mathbb{C}$. Choose $x = b$, and $y^*(v) = \alpha$. Now we have

$$xy^*(v) = x(y^*(v)) = x(\alpha) = \alpha x = \alpha b = Tv.$$

(b) Let T be linear from V to W of rank k. Then let $\{b_1, \dots, b_k\}$ be an orthogonal basis for ran T. Then for $1 \le j \le k$ and some $v \in V$, define $T_j v = P_{b_j}(Tv)$. Now:

$$\sum_{j=1}^{k} T_k v = \sum_{j=1}^{k} P_{b_j}(Tv) = \sum_{j=1}^{k} \frac{\langle Tv, b_j\rangle}{\|b_j\|} b_j = Tv.$$

Note the last equality comes from the orthogonal expansion of a vector discussed in class.

Q5

Suppose that A and B are unitarily equivalent $n \times n$ matrices. That is, there is a unitary matrix U so that $U^*AU = B$. Show that E is an invariant subspace for B if and only if UE is invariant for A. Recall that a subspace E of V is invariant for T if $Tv \in E$ for all $v \in E$.

Solution: Suppose that $U^*AU = B$, and recall that from unitary equivalence, we can see that $AU = UB$;

$$
\begin{aligned}
E \text{ is invariant under } B &\iff Bv \in E \quad \forall v \in E \\
&\iff Bv = w \in E \\
&\iff UBv = Uw \\
&\iff AUv = Uw \\
&\iff AUv \in UE \quad \forall Uv \in UE \\
&\iff UE \text{ is invariant under } A.
\end{aligned}
$$