1 The Spectrum and its Topology

Definition 1.1 (*Spectrum of a Ring*)

Let R be a ring. Define the spectrum of R,

$$Spec(R) = \{\mathfrak{p} \trianglelefteq R : \mathfrak{p} \text{ is prime}\}.$$

Definition 1.2

Upon $Spec(R)$ we define a topology by letting our closed sets be of the form:

$$V(I) = \{\mathfrak{p} \in Spec(R) : I \subseteq \mathfrak{p}\}.$$

For an ideal I.

Proposition 1.1

$V(I)$ induces a topology on $Spec(R)$.

1. *$\varnothing, Spec(R)$ are closed.*
2. *$\bigcup_{i=1}^{n} V(I_i)$ is closed for any $n \in \mathbb{N}$.*
3. *$\bigcap_{\alpha} V(I_i)$ for any $\alpha \in \Delta$, some indexing set.*

Proof:

1. Take $V(\{0\}) = \{\mathfrak{p} \trianglelefteq R \text{ prime} : \{0\} \subseteq \mathfrak{p}\}$. But naturally, the additive identity is contained within every prime ideal (which must be an abelian group w.r.t +), so we have $V(\{0\}) = Spec(R)$, and $Spec(R) \in \mathcal{T}$

   Now take $V(R) = \{\mathfrak{p} \trianglelefteq R \text{ prime} : R \subseteq \mathfrak{p}\}$. But by definition, no prime ideal can contain R, so $V(R) = \varnothing$ and $\varnothing \in \mathcal{T}$

2. It is sufficient to show that $V(I) \cup V(J) \in \mathcal{T}$ for any $V(I), V(J) \in \mathcal{T}$. Any finite union can be proven inductively using this result. We claim that $V(I) \cup V(J) = V(IJ)$.

   Suppose $\mathfrak{p} \in V(I)$ without loss of generality. Then $\mathfrak{p}$ is a prime ideal containing I. But we have $IJ \subseteq I \cap J \subseteq I \subseteq \mathfrak{p}$, so $\mathfrak{p} \in V(IJ)$.

   Conversely, suppose $\mathfrak{p} \in V(IJ)$. Then $\mathfrak{p}$ is a prime ideal containing IJ. If $J \subseteq \mathfrak{p}$, we are done, so suppose that $J \nsubseteq \mathfrak{p}$. Then take $i \in I, j \in J \setminus \mathfrak{p}$. We know that $ij \in IJ \subseteq \mathfrak{p}$, and then since $\mathfrak{p}$ is a prime ideal, either i or j must be in $\mathfrak{p}$. Since we supposed it was not j, we know it must be i. Therefore, $I \subseteq \mathfrak{p}$, and $\mathfrak{p} \in V(I)$.

3. We claim that $\bigcap_{\alpha} V(I_\alpha) = V\left(\sum_{\alpha} I_\alpha\right)$.

   If $\mathfrak{p} \in \bigcap_{\alpha} V(I_\alpha)$, then $P \supseteq I_\alpha$ for all our α. But $\sum_{\alpha} I_\alpha \supseteq I_\alpha$ for any fixed α, so $P \supseteq \sum_{\alpha} I_\alpha \supseteq I\alpha$, and $\mathfrak{p} \in V\left(\sum_{\alpha} I_\alpha\right)$.

   Conversely, if $\mathfrak{p} \in V\left(\sum_{\alpha} I_\alpha\right), \mathfrak{p} \supseteq \sum_{\alpha} I_\alpha$. Fix some β arbitrary in Δ, then we know already $\mathfrak{p} \supseteq \sum_{\alpha} I_\beta \supseteq I_\beta$. So $\mathfrak{p} \in V(I_\beta)$ for any β, and $\mathfrak{p} \in \bigcap_{\alpha} V(I_\alpha)$. ■

Proposition 1.2

If R is a ring, then the closed points of $Spec(R)$ correspond to $V(M) = \{M\}$, for the ideals of R.

Proof: Recall closed points in $\mathfrak{p} \in Spec(R)$ are those for which $\{\mathfrak{p}\}$ is closed.

Suppose M is some such point. Then for some $I \trianglelefteq R$, we have $V(I) = \{M\}$, and $I \subseteq M \subseteq R$. Suppose we have some $J \trianglelefteq R$ so that $M \subseteq J \subseteq R$. Then $I \subseteq J$, and we would have to have $J \in \{M\}$. Therefore M is maximal. ■

Proposition 1.3

For any $I \trianglelefteq R$, $V(\sqrt{I}) = V(I)$