0 Homotopy and CW Complexes

Definition 0.1 (*Homotopy of Paths*)

$f, g : X \to Y$ are homotopic if there exists some $F : X \times I \to Y$, so that $f(x) = F(x, 0)$ and $g(x) = F(x, 1)$. This induces an equivalence relation on the set of maps $X \to Y$.

Proof: *Consider the following maps:*

1. *$F : X \times I \to Y$, defined by*
$$F(x, t) = f(x).$$
2. *Suppose f, g are homotopic by some F, define $\bar{F} : X \times I \to Y$ by*
$$\bar{F}(x, t) = F(x, 1 - t).$$
3. *Suppose $f \sim^{F_1} g \sim^{F_2} h$.*

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Definition 0.2 (*Homotopy Equivalence*)

$f : X \to Y$ is a Homotopy equivalence if $\exists g : Y \to X$:

1. *fg is homotopic to $\mathbb{1}_Y$*
2. *gf is homotopic to $\mathbb{1}_Y$.*

Definition 0.3

$\iota : X \to Y$ is an inclusion of retraction if $\exists r : Y \to X$, so that $\iota r = id_X$

Definition 0.4

X is a Hausdorff space if $\forall p \neq q \in X$, there exist disjoint open U, V containing p, q

Definition 0.5

X is compact, if for every open cover $\bigcup_{\alpha \in A} G_\alpha$ of X, there exists some finite subcover $\bigcup_{i=1}^{n} G_{\alpha_i}$.

Theorem 0.1

Recall the following:

1. *If $A \subseteq X$ is a closed subset of a compact space, then A is compact.*
2. *If X is Hausdorff and $A \subseteq X$ is compact then A is closed*
3. *The image of a compact set under a continuous map is compact.*

Definition 0.6

Let X a space and $\sim$ and equivalence relation on X. Define a topology on the set of equivalence classes $X/\sim$, by considering π the projection map. Say $\pi^{-1}(U)$ open exactly when U is open.

For a subspace $A \subseteq X$ we may define X/A by using the relation $\sim_A$, by $a \sim b$ if $a = b$ or both $a, b \in A$.

Lemma 0.2

Suppose X has a n equivalence relation $\sim$, and $f : X \to Y$ a continuous map.

1. *The map $f/\sim : X/\sim \to Y$ is continuous*
2. *If X is compact and Hausdorff, and $f/\sim$ is bijective, then $f/\sim$ is a homeomorphism.*

Example 0.1

Take the compact Hausdorff space,space D^n, and identify the subspace S^{n-1}. Define $a \sim b$ if $a = b$ or $a, b \in S^{n-1}$.

Then define f :

Lemma 0.3

Let X be Hausdorff and $A \subseteq X$ compact. Then X/A is Hausdorff.

Lemma 0.4

Let X be Hausdorff, $f : S^{n-1} \to X$ be continuous. Define $Y = X \cup_f D^n = X \sqcup D^n/S^{n-1}$. Y is then Hausdorff.

Next we discuss a couple of methods of constructing spaces which are simple to work with.

Definition 0.7

A Finite CW complex (cell complex) is defined inductively: $X^0 \subseteq X^1 \subseteq \cdots \subseteq X^n$

1. X_0 *is a discrete set.*
2. $X^1 = X^0 \sqcup D^1/f_\alpha : S^0_\alpha = \partial D^1_\alpha \to X^0$

Theorem 0.5

Finite cell complexes are compact and Hausdorff.

Note: Many geomoetric spaces (compact manifolds) are homotopy equivalent to finite CW complexes.

Definition 0.8

Let X, p be a topological space with chosen basepoint p. We define a wedge product $(X, p) \vee (Y, q) = (X \sqcup Y)/p \sim q$

Definition 0.9

The smash product is defined:

$$(X, p) \wedge (Y, q) = (X \times Y)/X \vee Y.$$

Definition 0.10

We define the join $X * Y = X \times Y \times I/\sim$ where $\sim$ identifies:

1. $(X, 0, y) \sim (x', 0, y)$
2. $(x, 1, y) \sim (x, 1, y')$

Definition 0.11

If X is a cell complex, a subcomplex of X is a closed union of cells.

0.1 Projective Space

Definition 0.12

Real Projective Space $\mathbb{R}P^n = S^n/a \sim -a$

1 The Fundamental Group

2 Homology Groups

3 Homotopy Groups