Exercise 1

In the surface M_g of genus g, let C be a circle that separates G_g into two compact subsurfaces M'_h and M'_k obtained from the closed surfaces M_h and M_k by deleting an open disk from each. Show that M'_h does not retract onto its boundary circle C and hence M_g does not retract onto C. [Hint: abelianize π_1.] But show that M_g does retract onto the nonseparating circle C' in the figure (Hatcher).

Solution: Begin with a quick computation of the fundamental group of the punctured surface of genus g, M_g. Recall the construction of M_g consisted of a single 0-cell, $2g$ 1-cells, and a single 2-cell. Puncturing the 2-cell with a hole or by removing a single point allows us to retract the hole onto the boundary of the cell. After gluing the boundary to the 1-cells, we will just be left with the 1-cells, and we will have a retract onto $\bigvee^{2g} S^1$. And since we know the fundamental group of the wedge sum, we are left with:

$$\pi_1(S^1) = \overbrace{\pi_1(S^1) * \cdots * \pi_1(S^1)}^{2g \text{ times}} = \overbrace{\mathbb{Z} * \cdots * \mathbb{Z}}^{2g \text{ times}}.$$

Now suppose for the sake of contradiction that M'_h did retract onto the boundary circle $C \simeq S^1$, for some retraction r. Then the homomorphism induced by the inclusion $\iota_* : \pi_1(S_1) \to \pi_1(M'_h)$ would be injective, and the composition $r_*\iota_*$ would be identity. Then, applying π_1 and then ab,

$$\begin{tikzcd}
\pi_1(S_1) \arrow[r, two heads, "\iota_*"] \arrow[rr, bend right, "id_*"'] & \pi_1(M'_h) \arrow[r, "r_*"] & \pi_1(S^1)
\end{tikzcd}$$

$$\begin{tikzcd}
\mathbb{Z} \arrow[r, two heads, "\iota_*^{ab}"] \arrow[rr, bend right, "id_*^{ab}"'] & \mathbb{Z}^{2g} \arrow[r, "r_*^{ab}"] & \mathbb{Z}
\end{tikzcd}$$

However, since ι_* maps the generator of $\mathbb{Z}$ to some commutator $aba^{-1}b^{-1}$, and abelianization kills all the commutators through the quotient, we must have ι_*^{ab} identically zero (homomorphisms from a cyclic group are determined by their evaluation on generator). This contradicts $r_*^{ab}\iota_*^{ab} = id_*^{ab}$, and so M'_k cannot retract onto C.

Now for C'. Early in Hatcher we identified M_g as a $4g$ sided polygon, with sides

$$a_1 b_1 a_1^{-1} b_1^{-1} \dots a_g b_g a_g^{-1} b_g^{-1}.$$

(in that order). Take the quotient $M_g/\sim$, where is defined by $a_i \sim a_1$ and $b_i \sim b_1$ for all i. The quotient map $q : M_g \to M_g/\sim = M_1$ then gives a retract of M_g to M_1, the torus.

We further retract onto the circle C', by viewing M_1 as the typical square with sides $aba^{-1}b^{-1}$. Construct this retract r by retracting a point in the square to the closest point in a. This is clearly continuous except perhaps at the line a^{-1}. However this is not a problem since a^{-1} is identified with a in the construction of the torus.

By composing the retracts,

$$\begin{tikzcd}
M_g \arrow[r, "q"] \arrow[rr, bend right, "rq"'] & M_1 \arrow[r, "r"] & C'
\end{tikzcd}$$

We have our retract $M_g \to C'$