1. Let $\phi : \mathbb{R}^n \to \mathbb{R}$ be such that $\phi(x) = 0 \Leftrightarrow x = 0$ and $\phi(\lambda x) = |\lambda|\phi(x), \forall x \in \mathbb{R}^n, \forall \lambda \in \mathbb{R}$ Show that if the set $B = \{x \in \mathbb{R}^n | \phi(x) \leq 1\}$ is convex, then ϕ defines a norm on $\mathbb{R}^n$.

   Solution: Non-degeneracy and scalar linearity are given from the definition of ϕ. So all that is left to prove is the triangle inequality and non-negativity.

   Non-negativity: For the sake of contradiction, suppose that there exists some $x \in \mathbb{R}^n$ so that $\phi(x) < 0$. Then let $n \in \mathbb{N}$. Take $\phi(nx) = n\phi(x) < 0 \leq 1$. So if we take the set $\{nx : n \in \mathbb{N}\}$, which is clearly unbounded, we see that it is contained in B. However this is a contradiction since B is bounded. So $\phi(x) > 0 \forall x \neq 0$.

   Triangle inequality: Let $x, y \in \mathbb{R}^n$, and take $\lambda = \frac{\phi(x)}{\phi(x)+\phi(y)} \leq 1$. Note that $\phi\left(\frac{x}{\phi(x)}\right) = \frac{\phi(x)}{\phi(x)} = 1$, and likewise for y, so we have $\frac{x}{\phi(x)}, \frac{y}{\phi(y)} \in B$ and may take:

$$\begin{aligned}
\phi\left(\lambda\frac{x}{\phi(x)} + (1-\lambda)\frac{y}{\phi(y)}\right) &= \phi\left(\frac{x\phi(x)}{\phi(x)(\phi(x)+\phi(y))} + \frac{y\phi(y)}{\phi(y)(\phi(x)+\phi(y))}\right) \\
&= \phi\left(\frac{x+y}{\phi(x)+\phi(y)}\right) \\
&= \frac{\phi(x+y)}{|\phi(x)+\phi(y)|} \leq 1 \\
\phi(x+y) \leq \phi(x) + \phi(y).
\end{aligned}$$

   And so the triangle inequality is satisfied.

2. Let E be a compact set in $\mathbb{R}^n$ and let F be a closed set in $\mathbb{R}^n$ such that $E \cap F = \emptyset$.

   (a) Show that there exists $d > 0$ such that $\|x - y\| > d$, $\forall x \in E$ and $\forall y \in F$.

   Solution: Take $d = \inf_{x\in E, y\in F} \|x - y\|$. Clearly this is less than or equal to any $\|x - y\|$ for $x \in E$, $y \in F$, and it cannot be negative since the norm is positive. So then $d \geq 0$.For contradiction suppose $d = 0$.
   Since this is an inf, we can find a sequence $\{x_n - y_n\}_{n\geq1}$, with $x_i \in E, y_i \in F$. Since $\{x_n\}_{n\geq1}$ is a sequence in the bounded set E, we can find a convergent subsequence, say $x_{n_k} \to x$, with $x \in E$ by closure of E. But since $\|x_{n_k} - y_{n_k}\| \to 0$, we must have $y_{n_k} \to x$, meaning $x \in F$, giving us the contradiction we sought ($E \cap F = \emptyset$). Therefore d is positive, and to ensure strict inequality, we simply take $d' = \frac{d}{2} < d \leq \|x - y\|$ for any $x \in E, y \in F$.

   (b) Does the result you proved in the previous question remain true if E and F are closed, but neither is compact? Justify your answer.

   Solution: This does not remain true. Take the sequence $\{e_n\}_{n\geq2}$ given by $e_n = n$, and E as its image. Take $\{f_n\}_{n\geq2} : f_n = e_n + \frac{1}{n}$ and F as its image (These sets contain only isolated points, and are closed). Then for any $d > 0$, we can pick $N \in \mathbb{N} : \frac{1}{N} < d$; and the points e_N and f_N will have $|e_N - f_N| = \frac{1}{N} < d$, meaning we can have arbitrarily close points between the closed sets E, F, and no such d can exist.

3. Let $E = \left\{(x, y) | y = \sin\left(\frac{1}{x}\right), x > 0\right\}$. Is E open? Is it closed? What are the accumulation points of E?

Solution: This set is not open. Take an arbitrary ball of radius r about the point $p = \left(\frac{1}{\pi}, 0\right) \in E$. Then the point $q = \left(\frac{1}{\pi}, \frac{r}{2}\right) \in B_r(p)$, but $q \notin E$ since sin is well-defined. So any ball about p contains points not in E, and E is not open.

Clearly each point of E is an accumulation point.

The accumulation points of E not contained in E are of the form $(0, a)$ for $a \in [-1, 1]$. Take one such point, and some $r > 0$, and consider the r-ball about $(0, a)$. Choose $k \in \mathbb{N}$ so that $\frac{1}{2\pi k} < r$, and let $x = \frac{1}{2\pi k + \arcsin a} \leq \frac{1}{2\pi k} < r$. Then:

$$\begin{aligned}
\frac{1}{x} &= 2\pi k + \arcsin a \\
\frac{1}{x} - 2\pi k &= \arcsin a \\
\sin\left(\frac{1}{x} - 2\pi k\right) &= a \\
\sin\left(\frac{1}{x}\right) &= a.
\end{aligned}$$

Then the point (x, a) is in E, and $\|(x, a) - (0, a)\| = \|(x, 0)\| = \sqrt{x^2} = x < r$, so x is in the arbitrary open ball we chose around $(0, a)$, and so every open ball around p contains a distinct point in E, and as such p is an accumulation point of E.

Clearly none of these accumulation points can be in E thanks to the condition $x > 0$, so E does not contain all its limit points and is not closed.

4. Let $f : \mathbb{R}^n \to \mathbb{R}$ be a function in $C^1(\mathbb{R}^n)$, i.e., $f, \partial_{x_1} f, \ldots, \partial_{x_n} f$ are continuous in $\mathbb{R}^n$. Suppose $f(tx) = tf(x), \forall x \in \mathbb{R}^n, \forall t \in \mathbb{R}$ Show that f is a linear function.

Solution: Take the partial derivative with respect to x_i for some $1 \leq i \leq n$.

$$\begin{aligned}
f(tx) &= tf(x) \\
\partial_i f(tx) &= \partial_i tf(x) \\
tf_{x_i}(tx) &= tf_{x_i}(x) \\
f_{x_i}(tx) &= f_{x_i}(x).
\end{aligned}$$

But this must mean any partial f_{x_i} is constant, and combined with $f(0) = f(0x) = 0f(x) = 0$, we can write that:

$$f(x_1, \ldots, x_n) = a_1 x_1 + \cdots + a_n x_n.$$

For real constants $a_1, \ldots, a_n$, so f is linear.

5. Given $u : \mathbb{R} \to \mathbb{R}$ a function in $C^2(\mathbb{R})$, define $f : \mathbb{R}^2 \to \mathbb{R}$ by $f(x,y) = \begin{cases} \frac{u(y)-u(x)}{y-x} & \text{if } y \neq x \\ u'(x) & \text{if } y = x \end{cases}$

   Show that f is differentiable at any point (a,a).

   Solution: Take the partial derivative with respect to y,

$$\begin{aligned} \frac{\partial f}{\partial y}(a,a) &= \lim_{h\to 0} \frac{f(a,a+h)-f(a,a)}{h} \\ &= \lim_{h\to 0} \frac{\frac{u(a+h)-u(a)}{a+h-a} - u'(a)}{h} \\ &= \lim_{h\to 0} \frac{\frac{u(a+h)-u(a)}{h} - u'(a)}{h}. \end{aligned}$$

   Before we get too far, let's do the same for x:

$$\begin{aligned} \frac{\partial f}{\partial x}(a,a) &= \lim_{h\to 0} \frac{f(a+h,a)-f(a,a)}{h} \\ &= \lim_{h\to 0} \frac{\frac{u(a)-u(a+h)}{a-a-h} - u'(a)}{h} \\ &= \lim_{h\to 0} \frac{\frac{u(a+h)-u(a)}{h} - u'(a)}{h}. \end{aligned}$$

   And so now we have $\frac{\partial f}{\partial x}(a,a) = \frac{\partial f}{\partial y}(a,a)$

   Now we apply l'Hopital in h, noticing that the first term on the numerator tends to $u'(a)$:

$$\lim_{h\to 0} \frac{\frac{u(a+h)-u(a)}{h} - u'(a)}{h} = \lim_{h\to 0} \frac{hu'(a+h) - u(a+h) + u(a)}{h^2}.$$

   And again,

$$\begin{aligned} \lim_{h\to 0} \frac{hu'(a+h) - u(a+h) + u(a)}{h^2} &= \lim_{h\to 0} \frac{hu''(a+h) + u'(a+h) - u'(a+h)}{2h} \\ &= \lim_{h\to 0} \frac{u''(a+h)}{2} \\ &= \frac{u''(a)}{2}. \end{aligned}$$

   Therefore:

$$\frac{\partial f}{\partial x}(a,a) = \frac{u''(a)}{2} = \frac{\partial f}{\partial y}(a,a).$$

   And thanks to $u \in C^2(\mathbb{R})$, both our partials exist and are continuous at any (a,a) and therefore f is differentiable at any (a,a).

6. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a function that is defined in an open set Ω in $\mathbb{R}^2$. Show that if $\partial_x f(x,y), \partial_y f(x,y)$ and $\partial_{xy} f(x,y)$ are continuous in Ω, then $\partial_{yx} f(x,y)$ exists in Ω and we have $\partial_{yx} f(x,y) = \partial_{xy} f(x,y), \forall (x,y) \in \Omega$ Hint: Consider the expression $\Delta(s,t) = f(a+s,b+t) - f(a+s,b) - f(a,b+t) + f(a,b)$.

Solution: Let $(x,y) \in \Omega$, with t,s real and small enough that $(x+s,y+t) \in \Omega$ and write $g(y) = f(x+s,y) - f(x,y)$. Apply the Mean Value Theorem in the interval $(y,y+t)$ to obtain some $\mu \in (0,1)$ so that:

$$\begin{aligned}
\Delta(s,t) &= (f(x+s,y+t) - f(x,y+t)) - (f(x,y) - f(x+s),y) \\
&= g(y+t) - g(y) \\
&= \frac{d}{dy} g(y+\mu t)(y+t-y) \\
&= t\frac{\partial}{\partial y}\left(f(x+s,y+\mu t) - f(x,y+\mu t)\right).
\end{aligned}$$

Now we take the function $h(x) = f_y(x,y+\mu t)$, and we use MVT again in conjunction with our above expression to find some $\tau \in (0,1)$:

$$\begin{aligned}
\Delta(s,t) &= t(f_y(x+s,y+\mu t) - f_y(x,y+\mu t)) \\
&= t(h(x+s) - h(x)) \\
&= t(x+s-x)\frac{d}{dx}(h(x+\tau s)) \\
&= ts\frac{d}{dx}\left(f_y(x+\tau s,y+\mu t)\right) \\
&= tsf_{yx}(x+\tau s,y+\mu t) \\
\frac{\Delta(s,t)}{st} &= f_{yx}(x+\tau s,y+\mu t).
\end{aligned}$$

If we perform the same process on x before y, we will obtain some λ, θ so that:

$$\frac{\Delta(s,t)}{st} = f_{xy}(x+\theta s,y+\lambda t).$$

Then we simply take the limit:

$$f_{yx}(x,y) = \lim_{t,s\to 0} f_{yx}(x+\tau s,y+\mu t) = \lim_{t,s\to 0}\frac{\Delta(s,t)}{st} = \lim_{t,s\to 0} f_{xy}(x+\theta s,y+\lambda t) = f_{xy}(x,y).$$

7. Compute the degree 3 Taylor polynomial $T_3(x, x_2)$ of the function $f : \mathbb{R}^2 \to \mathbb{R}$, defined by $f(x_1, x_2) = \frac{4x_1+6x_2-1}{2x_1+3x_2}$ at the point $(-1, 1)$.

Solution: Begin by computing all necessary partials, and evaluating at $(-1, 1)$:

$$
\begin{aligned}
f(x_1, x_2) &= \frac{4x_1 + 6x_2 - 1}{2x_1 + 3x_2}, & f(-1, 1) &= 1 \\
\frac{\partial f}{\partial x_1} &= \frac{2}{(2x_1 + 3x_2)^2}, & \frac{\partial f}{\partial x_1}(-1, 1) &= 2 \\
\frac{\partial f}{\partial x_2} &= \frac{3}{(2x_1 + 3x_2)^2}, & \frac{\partial f}{\partial x_2}(-1, 1) &= 3 \\
\frac{\partial^2 f}{\partial x_1 x_2} &= \frac{-12}{(2x_1 + 3x_2)^3}, & \frac{\partial^2 f}{\partial x_1 x_2}(-1, 1) &= -12 \\
\frac{\partial^2 f}{\partial x_1 x_1} &= \frac{-8}{(2x_1 + 3x_2)^3}, & \frac{\partial^2 f}{\partial x_1 x_1}(-1, 1) &= -8 \\
\frac{\partial^2 f}{\partial x_2 x_2} &= \frac{-18}{(2x_1 + 3x_2)^3}, & \frac{\partial^2 f}{\partial x_2 x_2}(-1, 1) &= -18 \\
\frac{\partial^3 f}{\partial x_1 x_1 x_1} &= \frac{48}{(2x_1 + 3x_2)^4}, & \frac{\partial^3 f}{\partial x_1 x_1 x_1}(-1, 1) &= 48 \\
\frac{\partial^3 f}{\partial x_2 x_2 x_2} &= \frac{162}{(2x_1 + 3x_2)^4}, & \frac{\partial^3 f}{\partial x_2 x_2 x_2}(-1, 1) &= 162 \\
\frac{\partial^3 f}{\partial x_1 x_2 x_2} &= \frac{72}{(2x_1 + 3x_2)^4}, & \frac{\partial^3 f}{\partial x_1 x_2 x_2}(-1, 1) &= 72 \\
\frac{\partial^3 f}{\partial x_1 x_1 x_2} &= \frac{108}{(2x_1 + 3x_2)^4}, & \frac{\partial^3 f}{\partial x_1 x_1 x_2}(-1, 1) &= 108.
\end{aligned}
$$

Then begin expanding the first three terms of the Taylor expansion

$$
\begin{aligned}
f((-1,1)+x) = f(-1,1) &+ \left(\sum_{k=1}^{3}\frac{1}{k!}\left((x_1+1)\frac{\partial}{\partial x_1}+(x_2-1)\frac{\partial}{\partial x_2}\right)^k f(-1,1)\right)\\
&= 1+\frac{1}{1!}\left((x_1+1)\frac{\partial}{\partial x_1}+(x_2-1)\frac{\partial}{\partial x_2}\right)^1 f(-1,1)\\
&\quad+\frac{1}{2!}\left((x_1+1)\frac{\partial}{\partial x_1}+(x_2-1)\frac{\partial}{\partial x_2}\right)^2 f(-1,1)\\
&\quad+\frac{1}{3!}\left((x_1+1)\frac{\partial}{\partial x_1}+(x_2-1)\frac{\partial}{\partial x_2}\right)^3 f(-1,1)\\
&= 1+(x_1+1)\frac{\partial f}{\partial x_1}(-1,1)+(x_2-1)\frac{\partial f}{\partial x_2}(-1,1)\\
&\quad+\frac{1}{2}\Big((x_1+1)^2\frac{\partial^2}{\partial x_1 x_1}+2(x_1+1)(x_2-1)\frac{\partial}{\partial x_1 x_2}\\
&\quad+(x_2-1)^2\frac{\partial^2}{\partial x_2 x_2}\Big)f(-1,1)\\
&\quad+\frac{1}{6}\Big((x_1+1)^3\frac{\partial^3}{\partial x_1 x_1 x_1}+3(x_1+1)^2(x_2-1)\frac{\partial^3}{\partial x_1 x_1 x_2}\\
&\quad+3(x_1+1)(x_2-1)^2\frac{\partial^3}{\partial x_1 x_2 x_2}+(x_2-1)^3\frac{\partial^3}{\partial x_2 x_2 x_2}\Big)f(-1,1)\\
&= 1+2(x_1+1)+3(x_2-1)\\
&\quad+\frac{1}{2}\left(-8(x_1+1)^2-24(x_1+1)(x_2-1)-18(x_2-1)^2\right)\\
&\quad+\frac{1}{6}\Big(48(x_1+1)^3+3\cdot 108(x_1+1)^2(x_2-1)\\
&\quad+3\cdot 72(x_1+1)(x_2-1)^2+162(x_2-1)^3\Big)\\
&= 2x_1-3x_2-4(x_1+1)^2-12(x_1+1)(x_2-1)-9(x_2-1)^2\\
&\quad+8(x_1+1)^3+54(x_1+1)^2(x_2-1)\\
&\quad+36(x_1+1)(x_2-1)^2+27(x_2-1)^3.
\end{aligned}
$$