1. (a) Prove that the series

$$\sum_{n=2}^{\infty} \frac{1}{(\log_2 n)^{p(\log_2 n)}}$$

is convergent for all $p > 1$. Here $\log_2 x$ denotes the logarithm base 2 of x. You may assume that $\log_2 n$ is increasing in n.

Proof. We compare with a p-series.

(b) For $a > 0$ find the sum of the series

$$\sum_{k=2}^{\infty} \left(\frac{a}{a+1}\right)^k \quad \text{(show your work)}$$

Solution: We notice a geometric series; since $a > 0$,we can say $a < a + 1$ and $\frac{a}{a+1} < 1$. Then the sum is given by:

$$\left(\frac{a}{a+1}\right)^2 \frac{1}{1-\frac{a}{a+1}} = \left(\frac{a}{a+1}\right)^2 \frac{1}{\frac{a+1}{a+1}-\frac{a}{a+1}} = \left(\frac{a}{a+1}\right)^2 \frac{1}{\frac{1}{a+1}} = \left(\frac{a}{a+1}\right)^2 (a+1) = \frac{a^2}{a+1}.$$

2. (a) Prove that $f(x) = \sin\left(x^2\right)$ is not uniformly continuous in $[0, \infty)$.

f is uniformly continuous on $E \subset X$ if and only if $\forall \varepsilon > 0$, $\exists \delta > 0$, $d(x, y) < \delta \implies d(f(x), f(y)) < \varepsilon$

f is NOT uniformly continuous on $E \subset X$ if and only if $\exists \varepsilon > 0$, $\forall \delta > 0$, we can choose x, y so that $d(x, y) < \delta$ and $d(f(x), f(y)) \geq \varepsilon = 1$

Proof. Choose $\varepsilon = 2$, and let $\delta > 0$. Then we must choose $|x - y| < \delta$ but $|\sin x^2 - \sin y^2| = 2$

We choose $x < y - \delta$, say $x^2 = n\pi + \frac{\pi}{2}$ for $n \in \mathbb{N}$ so $x = \sqrt{n\pi + \frac{\pi}{2}}$ (taking the positive root since we care only about the positive reals. Then we want $y^2 - x^2 = \pi$ so that the difference $\sin y^2 - \sin x^2 = 2$. So choose $y^2 = (n + 1)\pi + \frac{\pi}{2}$, and then $y = \sqrt{(n+1)\pi + \frac{\pi}{2}}$.

We now have guaranteed that $|f(x) - f(y)| = 2$, and we must choose n so that $|x - y| < \delta$ for any given δ. Rewrite with the assumptions $y > x$, and with our expressions for x, y above, so we may find an expression for n in terms of δ.

$$\begin{aligned}
y - x &< \delta \\
y^2 + x^2 - 2yx &< \delta^2 \\
(n\pi + \pi) + \frac{\pi}{2} - (n\pi + \frac{\pi}{2}) - 2yx &< \delta^2 \\
\pi - 2yx &< \delta^2 \\
2yx &> \pi - \delta^2 \\
2\sqrt{\left(n\pi + \frac{\pi}{2}\right)\left((n-1)\pi + \frac{\pi}{2}\right)} &> \pi - \delta^2
\end{aligned}$$

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(b) Show an example of a continuous function in $(0, 1)$ which is not uniformly continuous (no proof necessary).

Solution: $f(x) = \sin\left(\frac{1}{x^2}\right)$ is continuous in $(0, 1)$ however it is not uniformly continuous (as shown in class)