1. (a) Prove that the series

$$\sum_{n=2}^{\infty} \frac{1}{(\log_2 n)^{p(\log_2 n)}}$$

is convergent for all $p > 1$. Here $\log_2 x$ denotes the logarithm base 2 of x. You may assume that $\log_2 n$ is increasing in n.

Proof. We attempt to satisfy the criterion in Rudin Theorem 3.27. Rewrite the series;

$$\begin{aligned}\sum_{k=1}^{\infty} \frac{2^k}{(\log_2 2^k)^{p \log_2 2^k}} &= \sum_{k=1}^{\infty} \frac{2^k}{k^{pk}} \\ &= \sum_{k=1}^{\infty} \left(\frac{2}{k^p}\right)^k .\end{aligned}$$

(b) For $a > 0$ find the sum of the series

$$\sum_{k=2}^{\infty} \left(\frac{a}{a+1}\right)^k \quad \text{(show your work)}$$

Solution: We notice a geometric series; since $a > 0$,we can say $a < a + 1$ and $\frac{a}{a+1} < 1$. Then the sum is given by:

$$\begin{aligned}\left(\frac{a}{a+1}\right)^2 \frac{1}{1 - \frac{a}{a+1}} &= \left(\frac{a}{a+1}\right)^2 \frac{1}{\frac{a+1}{a+1} - \frac{a}{a+1}} \\ &= \left(\frac{a}{a+1}\right)^2 \frac{1}{\frac{1}{a+1}} \\ &= \left(\frac{a}{a+1}\right)^2 (a+1) \\ &= \frac{a^2}{a+1}.\end{aligned}$$

2. (a) Prove that $f(x) = \sin\left(x^2\right)$ is not uniformly continuous in $[0, \infty)$. f is uniformly continuous on $E \subset X$ if and only if $\forall \varepsilon > 0, \exists \delta > 0, d(x, y) < \delta \implies d(f(x), f(y)) < \varepsilon$

f is NOT uniformly continuous on $E \subset X$ if and only if $\exists \varepsilon > 0, \forall \delta > 0$, we can choose x, y so that $d(x, y) < \delta$ and $d(f(x), f(y)) \geq \varepsilon = 1$

Proof. Choose $\varepsilon = 1$, and let $\delta > 0$. Then we must choose $|x - y| < \delta$ but $|\sin x^2 - \sin y^2| \geq 1$ We attempt to choose x, y so that the function's value on one is 0, and on the other is ± 1. Then let $x^2 = n\pi$ for some $n \in \mathbb{N}$, and $y^2 = n\pi + \frac{\pi}{2}$, and our final choice is

$$x = \sqrt{n\pi}, \quad y = \sqrt{n\pi + \frac{\pi}{2}}.$$

Then regardless of our choice of n, $|f(x) - f(y)| = |\sin(n\pi) - \sin\left(n\pi - \frac{\pi}{2}\right)|$. If n is odd, then $|\sin(n\pi) - \sin\left(n\pi - \frac{\pi}{2}\right)| = |\pm 1 - 0| = 1$, and if n is even, $|\sin(n\pi) - \sin\left(n\pi - \frac{\pi}{2}\right)| = |0 - \pm 1| = 1$ We now have guaranteed that $|f(x) - f(y)| = 1$ for any n, so now we must choose n so that $|x - y| < \delta$, for any given δ.

$$\begin{aligned}
|y - x| &= y - x && \text{Since } y > x \\
&= \sqrt{n\pi + \frac{\pi}{2}} - \sqrt{n\pi} \\
&= \frac{n\pi + \frac{\pi}{2} - n\pi}{\sqrt{n\pi + \frac{\pi}{2}} + \sqrt{n\pi}} \\
&= \frac{\frac{\pi}{2}}{\sqrt{n\pi + \frac{\pi}{2}} + \sqrt{n\pi}} \\
&< \frac{2}{\sqrt{n\pi + \frac{\pi}{2}} + \sqrt{n\pi}} && \text{Since } \frac{\pi}{2} < 2 \\
&< \frac{2}{\sqrt{n}\left(\sqrt{\pi + \frac{\pi}{2\sqrt{n}}} + \sqrt{\pi}\right)} \\
&< \frac{2}{\sqrt{n}\left(2\sqrt{\pi}\right)} \\
&< \frac{1}{\sqrt{n}}.
\end{aligned}$$

So then choose $n > \frac{1}{\delta^2}$ (Which we can do by the Archimedian Property),

$$\begin{aligned}
|y - x| &= y - x \\
&= \sqrt{n\pi + \frac{\pi}{2}} - \sqrt{n\pi} \\
&< \sqrt{\frac{\pi}{\delta^2} + \frac{\pi}{2}} - \sqrt{\frac{\pi}{\delta^2}}
\end{aligned}$$

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(b) Show an example of a continuous function in $(0, 1)$ which is not uniformly continuous (no proof necessary).

Solution: $f(x) = \sin\left(\frac{1}{x^2}\right)$ is continuous in $(0, 1)$ however it is not uniformly continuous (as shown in class)