1. Let $\{f_n\}$ be the sequence of functions defined by

$$f_n = \begin{cases} nx & \text{If } 0 \leq x \leq \frac{1}{n} \\ 1 & \text{If } \frac{1}{n} < x < 1 - \frac{1}{n} \\ n - nx & \text{If } 1 - \frac{1}{n} \leq x \leq 1 \end{cases}.$$

   (a) Find the pointwise limit f of the sequence.

   Solution: Proceed by cases. If $x = 0$, then the first case of the function will always be taken since $0 \leq x$. So $f_n(0) = n0 = 0$. Likewise if $x = 1$, then $f(1) = n - n1 = n - n = 0$.
   Now, if $x \in (0, 1)$, then we observe that $\frac{1}{n} \to 0$, and $1 - \frac{1}{n} \to 1$. Therefore the middle case of our piecewise function gives us $f(x) = 1$ for all x in this open interval.

   (b) Does $f_n \xrightarrow[{[0,1]}]{c.u} f$? Justify your answer.

2. Let $f_n(x) = \left(\cos\left(\frac{2x}{n}\right)\right)^{n^2}$

   (a) Compute the pointwise limit f of the sequence $\{f_n\}$.

   (b) Show that $f_n \xrightarrow[{[0,1]}]{c.u} f$.

3. Let $a \in \mathbb{R}_+$. Compute the limit

$$\lim_{n\to\infty} \int_a^{\pi} \frac{\sin(nx)}{nx}\, dx.$$

   What happens if $a = 0$?

4. Construct a sequence of functions defined in $[0, 1]$, each of which is discontinuous at every point of $[0, 1]$ and which converges uniformly to a function that is continuous at every point

   Solution: Take the series $\{f_n\}$ defined by:

$$f_n(x) = \begin{cases} \frac{1}{n} & \text{If } x \in \mathbb{Q} \\ 0 & \text{Otherwise} \end{cases}$$

   Then $\{f_n\}$ converges uniformly to the constant function 0, which we know to be continuous on the real line, as well as $[0, 1]$.

5. Consider the series of functions $\sum_{n\geq 1} \frac{x}{n(n+x)}$.

   (a) Show that the series converges uniformly in the interval $[0, b]$ for any $b > 0$.

   (b) Let $F(x) = \sum_{n\geq 1} \frac{x}{n(n+x)}$. Show that $F'(x) = \sum_{n\geq 1} \frac{1}{n(n+x)^2}$, $x \geq 0$.

6. Consider the series of functions $\sum_{n\geq 1} \frac{x}{1+n^2x^2}$. Show that the series doesn't converge uniformly in $\mathbb{R}_+$.

   Hint: You could start by showing that $\frac{x}{1+n^2x^2} \geq \int_n^{n+1} \frac{x}{1+t^2x^2}\, dt, \quad \forall x \in \mathbb{R}$.