1. Let G be a non-abelian group of order 8. Prove that G has a subgroup of order 4.

   Solution: We already know every group of order 8. Particularly, we know that the non-abelian groups are Q_8, D_4. As well, we know that Q_8 has a subgroup $\{1, i, -1, -i\} = \langle i \rangle$, and that D_4 has a subgroup $\{e, r, r^2, r^3\} = \langle r \rangle$. Both of these are of order 4, and in fact both of these are normal, since they have index 2.

2. Exercises 9.1 #1(c). Find the length of D_4 and exhibit the composition factors.

   Solution: Write the composition series:

$$D_4 \trianglerighteq \langle r \rangle \trianglerighteq \left\langle r^2 \right\rangle \trianglerighteq \{e\}.$$

   Comparing the orders of each subgroup (8,4,2,1), we can see that each factor group has order 2, since the index $[G_{i+1} : G_i] = 2$. This not only validates normality of each subgroup, but tells us that each factor group must have $\frac{G_i}{G_{i+1}} \cong C_2$, a simple group (By Lagrange, the only subgroups it can have are of order 1 and 2). So this is a composition series of D_4, with each factor isomorphic to the cyclic group of order 2.

3. Exercises 9.1#10. For groups $G_1, G_2, \ldots, G_r$ show that $G_1 \times \cdots \times G_r$ has a composition series iff each G_i has a composition series. In this case, show the the length of $G_1 \times \cdots \times G_r$ is equal to the sum of the lengths of the G_i's.

   Proceed by induction. The base case of $r = 2$ is proved in Corollary 2 of theorem 2 in Nicholson. We use the same strategy.

   Start with a lemma; if $\theta : G \to H$ is a homomorphism, then G has a composition series if and only if both ker θ and $\theta(G)$ have composition series. In this case,

$$\text{length}\, G = \text{length} \ker \theta + \text{length}\, \theta(G).$$

   Proof of lemma: Let G have a composition series. We know that $\ker \theta$ is a normal subgroup, and from Theorem 2 it has a series. And from the first isomorphism theorem, $G/\ker \theta \cong \theta(G)$. And since $G/\ker \theta$ has a series, again by Theorem 2, so does $\theta(G)$.

   Finally, we use Theorem 2 one more time to see that

$$\text{length}\, G = \text{length} \ker \theta + \text{length}\, G/\ker \theta = \text{length} \ker \theta + \text{length}\, \theta(G).$$

   Now we begin on the original question, proceeding by induction on r.

   Base case: $r = 1$ is trivial, if G_1 has a series, then G_1 has a series. Take the case $r = 2$.

   Consider the homomorphism $\theta : G_1 \times G_2$ by $\theta(g_1, g_2) = g_1$. This is onto, since for any $g \in G_1$, we have $(g, e) \in G_1 \times G_2$, and $\theta(g_1, e) = g_1$. And, taking some $(a, b) \in \ker \theta$, we have $\theta(a, b) = a = e$. So $\ker \theta = G_1 \times \{e\} \cong G_1$. So using our previous result, we can see that $G_1 \times G_2$ has a series $\iff$ $G_1 \cong \ker \theta$ and $G_2 = \theta(G_1 \times G_2)$ both have series, and that $\text{length}\, G_1 \times G_2 = \text{length}\, G_1 + \text{length}\, G_2$.

   Inductive Hypothesis: Suppose that for some $k \geq 2$, $G_1 \times \ldots \times G_k$ has a composition series $\iff$ G_i has a series for $i = 1, 2, \ldots, k$, and that its length is equal to the sum of the lengths of each G_i.

   We want to show that this holds for $k + 1$.

   Inductive Step: Write $H = G_1 \times \ldots \times G_k$. We know this is a group and has a series $\iff$ $G_1, \ldots, G_k$ all have series by the inductive hypothesis. The problem then reduces to showing $H \times G_{k+1}$ has a series $\iff$ H, G_{k+1} have series, which was shown already in the base case. As well, from the inductive hypothesis, we can say that the length of H is the sum of the lengths of $G_1, \ldots, G_k$. Again from the base case, we simply add on the length of G_{k+1}.

   Therefore, by induction on r, $G_1 \times \cdots \times G_r$ has a composition series iff each G_i has a composition series, and the length of $G_1 \times \cdots \times G_r$ is equal to the sum of the lengths of the G_i's.