1. An $m \times n$ matrix is said to be a queen if the restriction of A to the orthogonal complement of its kernel is an isometry.

   (a) Show that $A$ is a queen if and only if $A^*A$ is an orthogonal projection.

   **Solution:**   Suppose $A$ is a queen. Then $A$ is an isometry on $(\ker A)^\perp = (\ker A^*A)^\perp = \operatorname{ran} A^*A$. Take any $v \in \mathbb{C}^n$, which can be decomposed as $v = x + y$ with $x \in (\ker A^*A)^\perp$ and $y \in \ker A^*A$. Then:

   $$(A^*A)^2 v = (A^*A)^2(x + y)$$
   $$= (A^*A)^2 x + (A^*A)^2 y$$
   $$= (A^*A)^2 x$$
   .

   But since $A$ is an isometry on $(\ker A^*A)^\perp$, which contains $x$, we must have $A^*Ax = x$. Then

   $$(A^*A)^2 v = (A^*A)^2 x$$
   $$= A^*Ax$$
   $$= A^*Ax + 0$$
   $$= A^*Ax + A^*Ay$$
   $$= A^*Av.$$

   And therefore $(A^*A)^2 = A^*A$, and $A^*A$ is an orthogonal projection.
   Conversely, let $A^*A$ be an orthogonal projection, and $v \in (\ker A)^\perp = \operatorname{ran}(A^*A)$. But we know that $A^*A$ acts as identity on its range. So $A^*Av = v$, and

   $$\langle A^*Av, v \rangle = \langle v, v \rangle$$
   $$\langle Av, Av \rangle = \langle v, v \rangle$$
   $$\|Av\|^2 = \|v\|^2$$
   $$\|Av\| = \|v\|.$$

   And so $A$ is an isometry on the orthogonal complement of its kernel, and $A$ is a queen.

   (b) Show that $A$ is a queen if and only if $AA^*$ is an orthogonal projection.

   (c) Show that a queen A is an isometry if and only if $\ker A = 0$.

   **Solution:**   If $\ker A = \{0\}$, then $(\ker A)^\perp = V$, so the restriction of $A$ to the orthogonal complement of its kernel is $A$ restricted to all of $V$. Then $A$ is an isometry on any vector.
   Conversely, suppose $A$ is an isometry. Then:

   $$v \in \ker A \iff Av = 0$$
   $$\iff \|Av\| = \|v\| = 0$$
   $$\iff v = 0.$$

   Therefore $\ker A = \{0\}$.

   (d) Find an example of a $4 \times 2$ queen that has non-zero kernel. Be sure to prove it's a queen!

**Solution:**    Take the matrix:
$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

Then clearly $A \begin{bmatrix} t \\ 0 \end{bmatrix} = 0$ for any $t \in \mathbb{C}$, so $A$ has nonzero kernel, and all we must show is that $A$ is a queen.

Begin by observing that since $\ker A = \operatorname{span} \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\}$, we can find:

$(\ker A)^\perp = \operatorname{span} \left\{ \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$. Then let $v = \begin{bmatrix} 0 \\ t \end{bmatrix} \in (\ker A)^\perp$. Computing both $\|v\|, \|Av\|$, we see:

$$\|Av\| = \| \begin{bmatrix} 0 & 1 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ t \end{bmatrix} \| = \| \begin{bmatrix} t \\ 0 \\ 0 \\ 0 \end{bmatrix} \| = |t| = \| \begin{bmatrix} 0 \\ t \end{bmatrix} \| = \|v\|.$$

So the restriction of $A$ to the orthogonal complement of its kernel is an isometry, and $A$ is a queen.

2. (a) Given a singular value decomposition $A = W\Sigma V^*$ of a square matrix $A$, construct a polar decomposition of $A$ using $W, V, \Sigma$.

**Solution:**    Suppose $A = W\Sigma V^*$ is given, we wish to find $|A|$ and some $U$ unitary with $A = U|A|$.
$$|A| = \sqrt{A^*A} = \sqrt{V\Sigma^* W^* W\Sigma V^*} = \sqrt{V\Sigma^* \Sigma V^*}.$$

But recalling that $\Sigma$ is a real diagonal matrix, we have $\Sigma = \Sigma^*$:

$$|A| = \sqrt{V\Sigma V^* V\Sigma V^*} = V\Sigma V^*.$$

Now we wish to right cancel $V$, and get back our $W$. So take $U = WV^*$ as the unitary (since it is the product of unitaries); and then:

$$U|A| = (WV^*)(V\Sigma V^*) = W\Sigma V^* = A.$$

(b) Using the method above, compute a polar decomposition for
$$A = \begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix}.$$

**Solution:**    Compute $A^*A$;

$$A^*A = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix} = \begin{bmatrix} 5 & 15 \\ 15 & 45 \end{bmatrix}.$$

And find the characteristic polynomial:

$$C_{A^*A}(z) = \det(A - zI) = \begin{vmatrix} 5-z & 15 \\ 15 & 45-z \end{vmatrix} = z^2 - 50 = z(z-50).$$

Which gives the nonzero singular value $\sigma_1 = 5\sqrt{2}$, and our $\Sigma = \begin{bmatrix} 5\sqrt{2} & 0 \\ 0 & 0 \end{bmatrix}$. Then find an associated eigenvector for $\sigma_1^2$.

$$(50I - A^*A)v_1 = 0 \implies \begin{bmatrix} -45 & 15 \\ 15 & -5 \end{bmatrix} v_1 = 0$$

$$\implies \begin{bmatrix} -3 & 1 \\ 0 & 0 \end{bmatrix} v_1 = 0$$

$$\implies v_1 = t \begin{bmatrix} 1 \\ -3 \end{bmatrix}$$

$$\implies v_1 = \frac{1}{\sqrt{10}} \begin{bmatrix} 1 \\ -3 \end{bmatrix}.$$

Now that we have $v_1$, we need only pick $v_2$ so that $V$ is unitary, so by inspection take $v_2 = \frac{1}{\sqrt{10}} \begin{bmatrix} 3 \\ -1 \end{bmatrix}$, which is normal, and orthogonal to $v_1$. And so we have our matrix $V = \frac{1}{\sqrt{10}} \begin{bmatrix} 1 & 3 \\ 3 & -1 \end{bmatrix}$.

Now we find $W$. Begin by computing:

$$w_1 = \frac{1}{\sigma_1} A v_1 = \frac{1}{5\sqrt{2}} \frac{1}{\sqrt{10}} \begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \end{bmatrix} = \frac{1}{\sqrt{20}} \begin{bmatrix} 2 \\ 4 \end{bmatrix}.$$

And again by inspection, $w_2 = \frac{1}{\sqrt{20}} \begin{bmatrix} -4 \\ 2 \end{bmatrix}$, and $W^* = \frac{1}{\sqrt{20}} \begin{bmatrix} 2 & 4 \\ -4 & 2 \end{bmatrix}$. So then we have our SVD:

$$A = \frac{1}{\sqrt{10}} \begin{bmatrix} 1 & 3 \\ 3 & -1 \end{bmatrix} \begin{bmatrix} 5\sqrt{2} & 0 \\ 0 & 0 \end{bmatrix} \frac{1}{\sqrt{20}} \begin{bmatrix} 2 & 4 \\ -4 & 2 \end{bmatrix}.$$

After a quick sanity check that all our matrix multiplication gives us back $A$, we just need to find $|A| = V\Sigma V^*$ and $U = WV^*$.

$$|A| = V\Sigma V^*$$

$$= \frac{1}{10} \begin{bmatrix} 1 & 3 \\ 3 & -1 \end{bmatrix} \begin{bmatrix} 5\sqrt{2} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 3 & -1 \end{bmatrix}$$

$$= \frac{1}{10} \begin{bmatrix} 1 & 3 \\ 3 & -1 \end{bmatrix} \begin{bmatrix} 5\sqrt{2} & 15\sqrt{2} \\ 0 & 0 \end{bmatrix}$$

$$= \frac{1}{10} \begin{bmatrix} 5\sqrt{2} & 15\sqrt{2} \\ 15\sqrt{2} & 45\sqrt{2} \end{bmatrix}$$

$$= \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 3 \\ 3 & 9 \end{bmatrix}$$

$$U = WV^*$$

$$= \frac{1}{\sqrt{200}} \begin{bmatrix} 2 & -4 \\ 4 & 2 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 3 & -1 \end{bmatrix}$$

$$= \frac{1}{10\sqrt{2}} \begin{bmatrix} -10 & 10 \\ 10 & 10 \end{bmatrix}$$

$$= \frac{1}{\sqrt{2}} \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}.$$

And so we have the polar decomposition:

$$A = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix} \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 3 \\ 3 & 9 \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix}.$$

3. Find your favorite $4 \times 2$ matrix A of rank 2 and compute a singular value decomposition for A. All of the entries of A must be nonzero.

4. For an $m \times n$ matrix $A$, show that the set of nonzero eigenvalues for $A^*A$ coincide with that of $AA^*$.

**Solution:** Let $0 \neq \lambda \in \sigma(A^*A)$, with an associated eigenvector $v$.

Then $A^*Av = \lambda v$. Applying $A$ on both sides, we have $AA^*Av = A\lambda v = \lambda Av$, and so $Av$ is an eigenvector for $AA^*$ associated with $\lambda$.

Now let $0 = \lambda \in \sigma(AA^*)$

Then suppose $AA^*v = \lambda v$. Applying $A^*$ on both sides, we have $A^*AA^*v = A^*\lambda v = \lambda A^*v$, and so $A^*v$ is an eigenvector for $A^*A$ associated with $\lambda$.

5. Suppose $A = W\Sigma V^*$ is a singular value decomposition for $A$. Show that the columns of $W$ are eigenvectors for $AA^*$.

**Solution:** Let $1 \leq i \leq n$, and take:

$$W = \begin{bmatrix} w_1 & \dots & w_n \end{bmatrix}, \qquad \Sigma = \begin{bmatrix} \sigma_1 & & & 0 \\ & \ddots & & \vdots \\ & & \sigma_r & 0 \\ 0 & \dots & 0 & 0 \end{bmatrix}.$$

Then begin the computation:

$$
\begin{aligned}
AA^*w_i &= W\Sigma V^*V\Sigma^*W^*w_i \\
&= W\Sigma^2 W^*w_i && \text{$V$ unitary, $\Sigma$ real, symmetric} \\
&= W\Sigma^2 \begin{bmatrix} w_1^* \\ \vdots \\ w_n^* \end{bmatrix} w_i \\
&= W\Sigma^2 \begin{bmatrix} w_1^* w_i \\ \vdots \\ w_n^* w_i \end{bmatrix} \\
&= W\Sigma^2 \begin{bmatrix} \langle w_i, w_1 \rangle \\ \vdots \\ \langle w_i, w_n \rangle \end{bmatrix} \\
&= W\Sigma^2 \begin{bmatrix} 0 \\ \vdots \\ \langle w_i, w_i \rangle \\ \vdots \\ 0 \end{bmatrix} && \text{Since $w_i$ form an o.n.b.} \\
&= W\Sigma^2 \begin{bmatrix} 0 \\ \vdots \\ \|w_i\|^2 \\ \vdots \\ 0 \end{bmatrix}.
\end{aligned}
$$

Now we split by cases. If $i > r$, then the $i$-th column of $\Sigma$ will be exactly zero, and we will have $AA^*w_i = W0 = 0$, and $w_i \in \ker AA^*$

But if $i \leq r$, then the $i$-th column of $\Sigma^2$ will be of the form $\Sigma^2 = \begin{bmatrix} 0 & \dots & \sigma_i^2 & \dots & 0 \end{bmatrix}^T$

Then our equation becomes

$$AA^* w_i = W\sigma_i^2 \begin{bmatrix} 0 \\ \vdots \\ \|w_i\|^2 \\ \vdots \\ 0 \end{bmatrix}$$

$$= \sigma_i^2 \begin{bmatrix} w_1 & \cdots & w_n \end{bmatrix} \begin{bmatrix} 0 \\ \vdots \\ \|w_i\|^2 \\ \vdots \\ 0 \end{bmatrix}$$

$$= (\sigma_i \|w_i\|)^2 w_i.$$

And as we wanted to show, $w_i$ is an eigenvector for $AA^*$.