1. Consider the function $f : \mathbb{R}^3 \to \mathbb{R}^3$, defined by $f(x,y,z) = \begin{pmatrix} x^3 - y - z \\ 2x + y + z \\ x + y - z \end{pmatrix}$

   (a) Compute $Jf(x,y,z)$ and show that $\mathrm{d}f_{(x,y,z)}$ is invertible for any $(x,y,z) \in \mathbb{R}^3$.

   Solution: Compute:

$$Jf(x,y,z) = \begin{bmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} & \frac{\partial f_1}{\partial z} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} & \frac{\partial f_2}{\partial z} \\ \frac{\partial f_3}{\partial x} & \frac{\partial f_3}{\partial y} & \frac{\partial f_3}{\partial z} \end{bmatrix} = \begin{bmatrix} 3x^2 & -1 & -1 \\ 2 & 1 & 1 \\ 1 & 1 & -1 \end{bmatrix}.$$

   Then we take the determinant;

$$\det Jf(x,y,z) = 3x^2 \begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} - 2 \begin{vmatrix} -1 & -1 \\ 1 & -1 \end{vmatrix} + \begin{vmatrix} -1 & -1 \\ 1 & 1 \end{vmatrix} = -6x^2 - 4 + 0.$$

   And we can see that since this determinant has no real roots, it must be nonzero for any $(x,y,z) \in \mathbb{R}^3$, and so Jf, as well as df are both invertible in $\mathbb{R}^3$.

   (b) Find the largest open $U \subset \mathbb{R}^3$ where f has a continuously differentiable inverse function g.

2. Consider the system of equations: $(S) \begin{cases} x - y - u^2 + v^2 = 0 \\ x + y - 2uv = 0 \end{cases}$

   (a) Show that the system (S) can be solved for u, v in term of (x,y) near the point $(x,y,u,v) = (1,1,1,1)$.

   Solution: We solve for the Jacobian about $(1,1,1,1)$.

$$Jf(x,y,u,v) = \begin{bmatrix} 1 & -1 & -2u & 2v \\ 1 & 1 & -2v & -2u \end{bmatrix}$$

$$Jf(1,1,1,1) = \begin{bmatrix} 1 & -1 & -2 & 2 \\ 1 & 1 & -2 & -2 \end{bmatrix}.$$

   And if we break Jf into block matrices, we get the invertible right half of Jf as $\begin{bmatrix} -2 & 2 \\ -2 & -2 \end{bmatrix}$ which has nonzero determinant and must be invertible. So u, v can be implicitly defined about $(1,1,1,1)$by the Implicit Function theorem.

   (b) Compute $\partial_x u(1,1) + \partial_y v(1,1)$.

   Solution: Begin with the identity from the Implicit Function Theorem:

$$\begin{aligned} \begin{bmatrix} \partial_x u & \partial_y u \\ \partial_x v & \partial_y v \end{bmatrix} &= \begin{bmatrix} \partial_u f_1 & \partial_v f_1 \\ \partial_u f_2 & \partial_v f_2 \end{bmatrix}^{-1} \begin{bmatrix} \partial_x f_1 & \partial_y f_1 \\ \partial_x f_2 & \partial_y f_2 \end{bmatrix} \\ &= \left(\det \begin{bmatrix} -2u & 2v \\ -2v & -2u \end{bmatrix}\right)^{-1} \begin{bmatrix} -2u & -2v \\ 2v & -2u \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \\ &= \frac{1}{2(u^2+v^2)} \begin{bmatrix} -u & -v \\ v & -u \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \\ &= \frac{1}{2(u^2+v^2)} \begin{bmatrix} -u-v & u-v \\ v-u & -v-u \end{bmatrix}. \end{aligned}$$

And so if we want the sum $\partial_x u(1,1) + \partial_y v(1,1)$ we need only take the trace of this matrix and evaluate at $(1,1)$.

$$\begin{aligned}\partial_x u(1,1) + \partial_y v(1,1) &= \frac{1}{2(u^2+v^2)}\Big|_{(1,1)} \\ &= -2\frac{u+v}{2(u^2+v^2)}\Big|_{(1,1)} \\ &= -1.\end{aligned}$$

3. Let $f : \mathbb{R}^2 \to \mathbb{R} : (x,y) \to f(x,y)$. Show that if $f \in C^1(\mathbb{R}^2,\mathbb{R})$, then f can't be injective on $\mathbb{R}^2$. Hint: Use the implicit functions theorem.

4. Let $E = C([a,b],\mathbb{R})$ equipped with the norm of uniform convergence, let $u \in C(\mathbb{R},\mathbb{R})$, and consider the mapping $\phi : E \to E$, defined by $\phi(v) = u \circ v$. Is ϕ continuous? Make sure to justify your answer.

5. Find in $C([0,1],\mathbb{R})$ the distance from the function $u(t) = t$ to the subspace $\mathbb{P}_0$ of polynomials of degree 0. Make sure to justify your answer.

   Solution: Let $u(t) = t$, and take the distance:

$$\begin{aligned}d(u,\mathbb{P}_0) &= \inf_{p\in\mathbb{P}_0} d(u,p) \\ &= \inf_{c\in\mathbb{R}} \|u-c\| && p \text{ is simply a real constant} \\ &= \inf_{c\in\mathbb{R}} \sup_{t\in[0,1]} |u(t)-c| \\ &= \inf_{c\in\mathbb{R}} \sup_{t\in[0,1]} |t-c| \\ &.\end{aligned}$$

6. Let $f \in C([a,b],\mathbb{R})$ be such that $\int_a^b f(x)x^n \, dx = 0, \quad \forall n \in \mathbb{N}$ Show that f is identically zero. Hint: Use Weierstrass Theorem.