1. Let $\phi : \mathbb{R}^n \to \mathbb{R}$ be such that $\phi(x) = 0 \Leftrightarrow x = 0$ and $\phi(\lambda x) = |\lambda|\phi(x), \forall x \in \mathbb{R}^n, \forall \lambda \in \mathbb{R}$ Show that if the set $B = \{x \in \mathbb{R}^n | \phi(x) \leq 1\}$ is convex, then ϕ defines a norm on $\mathbb{R}^n$.

   Solution: Non-degeneracy and scalar linearity are given from the definition of ϕ. So all that is left to prove is the triangle inequality and non-negativity.

   Non-negativity: Suppose $x \in \mathbb{R}^n$ is nonzero, and

   Let $x, y \in \mathbb{R}^n$, and take $r = \max\{\phi(x), \phi(y)\}$. Then $\frac{x}{r}, \frac{y}{r} \in B$

2. Let E be a compact set in $\mathbb{R}^n$ and let F be a closed set in $\mathbb{R}^n$ such that $E \cap F = \emptyset$.

   (a) Show that there exists d > 0 such that $\|x - y\| > d$, $\forall x \in E$ and $\forall y \in F$.

   Solution: Take $d = \inf_{x \in E, y \in F} \|x - y\|$. Clearly this is less than any $\|x - y\|$ for $x \in E$, $y \in F$, and it cannot be negative since the norm is positive. So then $d \geq 0$.For contradiction suppose $d = 0$.

   (b) Does the result you proved in the previous question remain true if E and F are closed, but neither is compact? Justify your answer.

3. Let $E = \left\{(x, y) | y = \sin\left(\frac{1}{x}\right), x > 0\right\}$. Is E open? Is it closed? What are the accumulation points of E?

   Solution: This set is not open. Take an arbitrary ball of radius r about the point $p = \left(\frac{1}{\pi}, 0\right) \in E$. Then the point $q = \left(\frac{1}{\pi}, \frac{r}{2}\right) \in B_r(p)$, but $q \notin E$ since sin is well-defined. So any ball about p contains points not in E, and E is not open.

   By continuity of sin and $\frac{1}{x}$, all points of E are accumulation points.

   The accumulation points of E not contained in E are of the form $(0, a)$ for $a \in [-1, 1]$. Take one such point, and some $r > 0$, and consider the r-ball about $(0, a)$. Choose $k \in \mathbb{N}$ so that $\frac{1}{2\pi k} < r$, and let $x = \frac{1}{2\pi k + \arcsin a} \leq \frac{1}{2\pi k} < r$. Then:

$$\frac{1}{x} = 2\pi k + \arcsin a$$

$$\frac{1}{x} - 2\pi k = \arcsin a$$

$$\sin\left(\frac{1}{x} - 2\pi k\right) = a$$

$$\sin\left(\frac{1}{x}\right) = a.$$

   Then the point (x, a) is in E, and $\|(x, a) - (0, a)\| = \|(x, 0)\| = \sqrt{x^2} = x < r$, so x is in the arbitrary open ball we chose around $(0, a)$, and so every open ball around p contains a distinct point in E, and as such p is an accumulation point of E.

   Clearly none of these accumulation points can be in E thanks to the condition $x > 0$, so E does not contain all its limit points and is not closed.

4. Let $f : \mathbb{R}^n \to \mathbb{R}$ be a function in $C^1(\mathbb{R}^n)$, i.e., $f, \partial_{x_1} f, \ldots, \partial_{x_n} f$ are continuous in $\mathbb{R}^n$. Suppose $f(tx) = tf(x), \forall x \in \mathbb{R}^n, \forall t \in \mathbb{R}$ Show that f is a linear function.

5. Given $u : \mathbb{R} \to \mathbb{R}$ a function in $C^2(\mathbb{R})$, define $f : \mathbb{R}^2 \to \mathbb{R}$ by $f(x, y) = \begin{cases} u(y) - u(x) & \text{if } y \neq x \\ u'(x) & \text{if } y = x \end{cases}$

   Show that f is differentiable at any point (a, a).

6. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a function that is defined in an open set Ω in $\mathbb{R}^2$. Show that if $\partial_x f(x,y), \partial_y f(x,y)$ and $\partial_{xy} f(x,y)$ are continuous in Ω, then $\partial_{yx} f(x,y)$ exists in Ω and we have $\partial_{yx} f(x,y) = \partial_{xy} f(x,y), \forall (x,y) \in \Omega$ Hint: Consider the expression $\Delta(s,t) = f(a+s,b+t) - f(a+s,b) - f(a,b+t) + f(a,b)$.

7. Compute the degree 3 Taylor polynomial $T_3(x,x_2)$ of the function $f : \mathbb{R}^2 \to \mathbb{R}$, defined by $f(x_1,x_2) = \frac{4x_1+6x_2-1}{2x_1+3x_2}$ at the point $(-1,1)$.