1. Let $\phi : \mathbb{R}^n \to \mathbb{R}$ be such that $\phi(x) = 0 \Leftrightarrow x = 0$ and $\phi(\lambda x) = |\lambda|\phi(x), \forall x \in \mathbb{R}^n, \forall \lambda \in \mathbb{R}$
   Show that if the set $B = \{x \in \mathbb{R}^n | \phi(x) \le 1\}$ is convex, then $\phi$ defines a norm on $\mathbb{R}^n$.

   **Solution:**    Non-degeneracy and scalar linearity are given from the definition of $\phi$.
   So all that is left to prove is the triangle inequality and non-negativity.

   Non-negativity: For the sake of contradiction, suppose that there exists some $x \in \mathbb{R}^n$
   so that $\phi(x) < 0$. Then let $n \in \mathbb{N}$. Take $\phi(nx) = n\phi(x) < 0 \le 1$. So if we take the set
   $\{nx : n \in \mathbb{N}\}$, which is clearly unbounded, we see that it is contained in $B$. However
   this is a contradiction since $B$ is bounded. So $\phi(x) > 0 \forall x \neq 0$.

   Triangle inequality: Let $x, y \in \mathbb{R}^n$, and take $\lambda = \frac{\phi(x)}{\phi(x)+\phi(y)} \le 1$. Note that $\phi\left(\frac{x}{\phi(x)}\right) = \frac{\phi(x)}{\phi(x)} = 1$, and likewise for $y$, so we may take

   $$\phi\left(\lambda \frac{x}{\phi(x)} + (1-\lambda)\frac{y}{\phi(y)}\right) = \phi\left(\frac{x\phi(x)}{\phi(x)(\phi(x)+\phi(y))} + \frac{y\phi(y)}{\phi(y)(\phi(x)+\phi(y))}\right)$$
   $$= \phi\left(\frac{x+y}{\phi(x)+\phi(y)}\right)$$
   $$= \frac{\phi(x+y)}{|\phi(x)+\phi(y)|} \le 1$$
   $$\phi(x+y) \le \phi(x) + \phi(y).$$

   And so the triangle inequality is satisfied.

2. Let E be a compact set in $\mathbb{R}^n$ and let F be a closed set in $\mathbb{R}^n$ such that $E \cap F = \emptyset$.

   (a) Show that there exists $d > 0$ such that $\|x - y\| > d, \forall x \in E$ and $\forall y \in F$.

      **Solution:**    Take $d = \inf_{x \in E, y \in F} \|x - y\|$. Clearly this is less than or equal to any
      $\|x - y\|$ for $x \in E$, $y \in F$, and it cannot be negative since the norm is positive. So
      then $d \ge 0$. For contradiction suppose $d = 0$.
      Since this is an inf, we can find a sequence $\{x_n - y_n\}_{n \ge 1}$, with $x_i \in E, y_i \in F$.
      Since $\{x_n\}_{n \ge 1}$ is a sequence in the bounded set $E$, we can find a convergent
      subsequence, say $x_{n_k} \to x$, with $x \in E$ by closure of $E$. But since $\|x_{n_k} - y_{n_k}\| \to 0$,
      we must have $y_{n_k} \to x$, meaning $x \in F$, giving us the contradiction we sought
      ($E \cap F = \emptyset$).

   (b) Does the result you proved in the previous question remain true if $E$ and $F$ are
      closed, but neither is compact? Justify your answer.

      **Solution:**    This does not remain true. Take the sequence $\{e_n\}_{n \ge 2}$ given by
      $e_n = n$, and $E$ as its image. Take $\{f_n\}_{n \ge 2} : f_n = e_n + \frac{1}{n}$ and $F$ as its image (These
      sets contain only isolated points, and are closed). Then for any $d > 0$, we can pick
      $N \in \mathbb{N} : \frac{1}{N} < d$; and the points $e_N$ and $f_N$ will have $|e_N - f_N| = \frac{1}{N} < d$, meaning we
      can have arbitrarily close points between the closed sets $E, F$, and no such $d$ can
      exist.

3. Let $E = \left\{(x, y) | y = \sin\left(\frac{1}{x}\right), x > 0\right\}$. Is E open? Is it closed? What are the accumulation points of E?

   **Solution:** This set is not open. Take an arbitrary ball of radius $r$ about the point $p = \left(\frac{1}{\pi}, 0\right) \in E$. Then the point $q = \left(\frac{1}{\pi}, \frac{r}{2}\right) \in B_r(p)$, but $q \notin E$ since sin is well-defined. So any ball about $p$ contains points not in $E$, and $E$ is not open.

   Clearly each point of $E$ is an accumulation point.

   The accumulation points of $E$ not contained in $E$ are of the form $(0, a)$ for $a \in [-1, 1]$. Take one such point, and some $r > 0$, and consider the $r$-ball about $(0, a)$. Choose $k \in \mathbb{N}$ so that $\frac{1}{2\pi k} < r$, and let $x = \frac{1}{2\pi k + \arcsin a} \leq \frac{1}{2\pi k} < r$. Then:

   $$\frac{1}{x} = 2\pi k + \arcsin a$$
   $$\frac{1}{x} - 2\pi k = \arcsin a$$
   $$\sin\left(\frac{1}{x} - 2\pi k\right) = a$$
   $$\sin\left(\frac{1}{x}\right) = a.$$

   Then the point $(x, a)$ is in $E$, and $\|(x, a) - (0, a)\| = \|(x, 0)\| = \sqrt{x^2} = x < r$, so $x$ is in the arbitrary open ball we chose around $(0, a)$, and so every open ball around $p$ contains a distinct point in $E$, and as such $p$ is an accumulation point of $E$.

   Clearly none of these accumulation points can be in $E$ thanks to the condition $x > 0$, so $E$ does not contain all its limit points and is not closed.

4. Let $f : \mathbb{R}^n \to \mathbb{R}$ be a function in $C^1(\mathbb{R}^n)$, i.e., $f, \partial_{x_1} f, ..., \partial_{x_n} f$ are continuous in $\mathbb{R}^n$. Suppose $f(tx) = tf(x), \forall x \in \mathbb{R}^n, \forall t \in \mathbb{R}$ Show that f is a linear function.

   **Solution:** Take the partial derivative with respect to $x_i$ for some $1 \leq i \leq n$.

   $$f(tx) = tf(x)$$
   $$\partial_i f(tx) = \partial_i tf(x)$$
   $$tf_{x_i}(tx) = tf_{x_i}(x)$$
   $$f_{x_i}(tx) = f_{x_i}(x).$$

   But this must mean any partial $f_{x_i}$ is constant, and combined with $f(0) = f(0x) = 0f(x) = 0$, we can write that:

   $$f(x_1, \ldots, x_n) = a_1 x_1 + \cdots + a_n x_n.$$

   For real constants $a_1, \ldots, a_n$, so $f$ is linear.

5. Given $u : \mathbb{R} \to \mathbb{R}$ a function in $C^2(\mathbb{R})$, define $f : \mathbb{R}^2 \to \mathbb{R}$ by $f(x, y) = \begin{cases} \frac{u(y)-u(x)}{y-x} & \text{if } y \neq x \\ u'(x) & \text{if } y = x \end{cases}$

   Show that f is differentiable at any point $(a, a)$.

   **Solution:**   Take the partial derivative with respect to $y$ ($\partial x$ is done identically).

$$\frac{\partial f}{\partial y}(a, a) = \lim_{h \to 0} \frac{f(a, a + h) - f(a, a)}{h}$$
$$= \lim_{h \to 0} \frac{\frac{u(a+h)-u(a)}{a+h-a} - u'(a)}{h}$$
$$= \lim_{h \to 0} \frac{\frac{u(a+h)-u(a)}{h} - u'(a)}{h}$$
$$= \lim_{h \to 0} \frac{\frac{u(a+h)-u(a)}{h} - u'(a)}{h}.$$

Now we apply l'Hopital in $h$, noticing that the first term on the numerator tends to $u'(a)$:

$$\lim_{h \to 0} \frac{\frac{u(a+h)-u(a)}{h} - u'(a)}{h} = \lim_{h \to 0} \frac{h u'(a + h) - u(a + h) + u(a)}{h^2}.$$

And again,

$$\lim_{h \to 0} \frac{h u'(a + h) - u(a + h) + u(a)}{h^2} = \lim_{h \to 0} \frac{h u''(a + h) + u'(a + h) - u'(a + h)}{2h}$$
$$= \lim_{h \to 0} \frac{u''(a + h)}{2}$$
$$= \frac{u''(a)}{2}.$$

And by symmetry of $f$, we find:

$$\frac{\partial f}{\partial x}(a, a) = \frac{-u''(a)}{2}$$
$$\frac{\partial f}{\partial y}(a, a) = \frac{u''(a)}{2}.$$

And thanks to $u \in C^2(\mathbb{R})$, both our partials are continuous at any $(a, a)$ and therefore $f$ is differentiable at any $(a, a)$.

6. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a function that is defined in an open set $\Omega$ in $\mathbb{R}^2$. Show that if $\partial_x f(x, y), \partial_y f(x, y)$ and $\partial_{xy} f(x, y)$ are continuous in $\Omega$, then $\partial_{yx} f(x, y)$ exists in $\Omega$ and we have $\partial_{yx} f(x, y) = \partial_{xy} f(x, y), \forall (x, y) \in \Omega$ Hint: Consider the expression $\Delta(s, t) = f(a + s, b + t) - f(a + s, b) - f(a, b + t) + f(a, b)$.

7. Compute the degree 3 Taylor polynomial $T_3(x, x_2)$ of the function $f : \mathbb{R}^2 \to \mathbb{R}$, defined by $f(x_1, x_2) = \frac{4x_1 + 6x_2 - 1}{2x_1 + 3x_2}$ at the point $(-1, 1)$.

**Solution:** Begin by computing all necessary partials, and evaluating at $(-1, 1)$:

$$f(x_1, x_2) = \frac{4x_1 + 6x_2 - 1}{2x_1 + 3x_2}, \qquad f(-1, 1) = 1$$

$$\frac{\partial f}{\partial x_1} = \frac{2}{(2x_1 + 3x_2)^2}, \qquad \frac{\partial f}{\partial x_1}(-1, 1) = 2$$

$$\frac{\partial f}{\partial x_2} = \frac{3}{(2x_1 + 3x_2)^2}, \qquad \frac{\partial f}{\partial x_2}(-1, 1) = 3$$

$$\frac{\partial^2 f}{\partial x_1 x_2} = \frac{-12}{(2x_1 + 3x_2)^3}, \qquad \frac{\partial^2 f}{\partial x_1 x_2}(-1, 1) = -12$$

$$\frac{\partial^2 f}{\partial x_1 x_1} = \frac{-8}{(2x_1 + 3x_2)^3}, \qquad \frac{\partial^2 f}{\partial x_1 x_1}(-1, 1) = -8$$

$$\frac{\partial^2 f}{\partial x_2 x_2} = \frac{-18}{(2x_1 + 3x_2)^3}, \qquad \frac{\partial^2 f}{\partial x_2 x_2}(-1, 1) = -18$$

$$\frac{\partial^3 f}{\partial x_1 x_1 x_1} = \frac{48}{(2x_1 + 3x_2)^4}, \qquad \frac{\partial^3 f}{\partial x_1 x_1 x_1}(-1, 1) = 48$$

$$\frac{\partial^3 f}{\partial x_2 x_2 x_2} = \frac{162}{(2x_1 + 3x_2)^4}, \qquad \frac{\partial^3 f}{\partial x_2 x_2 x_2}(-1, 1) = 162$$

$$\frac{\partial^3 f}{\partial x_1 x_2 x_2} = \frac{72}{(2x_1 + 3x_2)^4}, \qquad \frac{\partial^3 f}{\partial x_1 x_2 x_2}(-1, 1) = 72$$

$$\frac{\partial^3 f}{\partial x_1 x_1 x_2} = \frac{108}{(2x_1 + 3x_2)^4}, \qquad \frac{\partial^3 f}{\partial x_1 x_1 x_2}(-1, 1) = 108.$$

Then begin expanding the first three terms of the Taylor expansion

$$f((-1,1)+x) = f(-1,1) + \left( \sum_{k=1}^{3} \frac{1}{k!} \left( (x_1+1)\frac{\partial}{\partial x_1} + (x_2-1)\frac{\partial}{\partial x_2} \right)^k f(-1,1) \right)$$

$$= 1 + \frac{1}{1!} \left( (x_1+1)\frac{\partial}{\partial x_1} + (x_2-1)\frac{\partial}{\partial x_2} \right)^1 f(-1,1)$$

$$+ \frac{1}{2!} \left( (x_1+1)\frac{\partial}{\partial x_1} + (x_2-1)\frac{\partial}{\partial x_2} \right)^2 f(-1,1)$$

$$+ \frac{1}{3!} \left( (x_1+1)\frac{\partial}{\partial x_1} + (x_2-1)\frac{\partial}{\partial x_2} \right)^3 f(-1,1)$$

$$= 1 + (x_1+1)\frac{\partial f}{\partial x_1}(-1,1) + (x_2-1)\frac{\partial f}{\partial x_2}(-1,1)$$

$$+ \frac{1}{2}\Big( (x_1+1)^2\frac{\partial^2}{\partial x_1 x_1} + 2(x_1+1)(x_2-1)\frac{\partial}{\partial x_1 x_2}$$

$$+ (x_2-1)^2\frac{\partial^2}{\partial x_2 x_2} \Big) f(-1,1)$$

$$+ \frac{1}{6}\Big( (x_1+1)^3\frac{\partial^3}{\partial x_1 x_1 x_1} + 3(x_1+1)^2(x_2-1)\frac{\partial^3}{\partial x_1 x_1 x_2}$$

$$+ 3(x_1+1)(x_2-1)^2\frac{\partial^3}{\partial x_1 x_2 x_2} + (x_2-1)^3\frac{\partial^3}{\partial x_2 x_2 x_2} \Big) f(-1,1)$$

$$= 1 + 2(x_1+1) + 3(x_2-1)$$

$$+ \frac{1}{2}\left( -8(x_1+1)^2 - 24(x_1+1)(x_2-1) - 18(x_2-1)^2 \right)$$

$$+ \frac{1}{6}\Big( 48(x_1+1)^3 + 3\cdot 108(x_1+1)^2(x_2-1)$$

$$+ 3\cdot 72(x_1+1)(x_2-1)^2 + 162(x_2-1)^3 \Big)$$

$$= 2x_1 - 3x_2 - 4(x_1+1)^2 - 12(x_1+1)(x_2-1) - 9(x_2-1)^2$$

$$+ 8(x_1+1)^3 + 54(x_1+1)^2(x_2-1)$$

$$+ 36(x_1+1)(x_2-1)^2 + 27(x_2-1)^3.$$