1. An $m \times n$ matrix is said to be a queen if the restriction of A to the orthogonal complement of its kernel is an isometry.

   (a) Show that A is a queen if and only if A^*A is an orthogonal projection.

   Solution: Suppose A is a queen. Then A is an isometry on $(\ker A)^\perp = (\ker A^*A)^\perp = \operatorname{ran} A^*A$. Take any $v \in \mathbb{C}^n$, which can be decomposed as $v = x + y$ with $x \in (\ker A^*A)^\perp$ and $y \in \ker A^*A$. Then:

$$\begin{aligned}(A^*A)^2 v &= (A^*A)^2(x+y) \\ &= (A^*A)^2 x + (A^*A)^2 y \\ &= (A^*A)^2 x.\end{aligned}$$

   But since A is an isometry on $(\ker A^*A)^\perp$, which contains x, we must have $A^*Ax = x$. Then

$$\begin{aligned}(A^*A)^2 v &= (A^*A)^2 x \\ &= A^*Ax \\ &= A^*Ax + 0 \\ &= A^*Ax + A^*Ay \\ &= A^*A(x+y) \\ &= A^*Av.\end{aligned}$$

   And therefore $(A^*A)^2 = A^*A$, and A^*A is an orthogonal projection.
   Conversely, let A^*A be an orthogonal projection, and $v \in (\ker A)^\perp = \operatorname{ran}(A^*A)$. But we know that A^*A acts as identity on its range. So $A^*Av = v$, and

$$\begin{aligned}\langle A^*Av, v\rangle &= \langle v, v\rangle \\ \langle Av, Av\rangle &= \langle v, v\rangle \\ \|Av\|^2 &= \|v\|^2 \\ \|Av\| &= \|v\|.\end{aligned}$$

   And so A is an isometry on the orthogonal complement of its kernel, and A is a queen.

   (b) Show that A is a queen if and only if AA^* is an orthogonal projection.

   Solution: We already have that A is a queen $\iff$ A^*A is an orthogonal projection. Rather than repeat the previous argument with AA^*, we show that A^*A is an orthogonal projection $\iff$ AA^* is an orthogonal projection.
   $\implies$: Let A^*A be an orthogonal projection. Then take $v \in \mathbb{C}^n$, and since $\operatorname{ran} A^*A = (\ker A^*A)^\perp = (\ker A)^\perp = \operatorname{ran} A^*$, we know that $A^*v \in \operatorname{ran} A^*A$. Then since A^*A acts as identity on its range, we have:

$$(AA^*)^2 v = A(A^*A)(A^*v) = AA^*v.$$

   And so AA^* is an orthogonal projection.
   $\impliedby$: Let AA^* be an orthogonal projection. Then take $w \in \mathbb{C}^m$, and since $\operatorname{ran} AA^* = (\ker AA^*)^\perp = (\ker A^*)^\perp = \operatorname{ran} A$, we know that $Aw \in \operatorname{ran} A^*A$. Then since AA^* acts as identity on its range,

$$(A^*A)^2 w = A^*(AA^*)(Aw) = A^*Aw.$$

   And therefore A^*A is an orthogonal projection.
   Now we have: A is a queen $\iff$ A^*A is an orthogonal projection $\iff$ AA^* is an orthogonal projection.

(c) Show that a queen A is an isometry if and only if $\ker A = 0$.

Solution: If $\ker A = \{0\}$, then $(\ker A)^{\perp} = V$, so the restriction of A to the orthogonal complement of its kernel is A restricted to all of V. Then A is an isometry on any vector.
Conversely, suppose A is an isometry. Then:

$$\begin{aligned} v \in \ker A &\iff Av = 0 \\ &\iff \|Av\| = \|v\| = 0 \\ &\iff v = 0. \end{aligned}$$

Therefore $\ker A = \{0\}$.

(d) Find an example of a 4×2 queen that has non-zero kernel. Be sure to prove it's a queen!

Solution: Take the matrix:

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

Then clearly $A\begin{bmatrix} t \\ 0 \end{bmatrix} = 0$ for any $t \in \mathbb{C}$, so A has nonzero kernel, and all we must show is that A is a queen.
Begin by observing that since $\ker A = \operatorname{span}\left\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right\}$, we can find:
$(\ker A)^{\perp} = \operatorname{span}\left\{\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right\}$. Then let $v = \begin{bmatrix} 0 \\ t \end{bmatrix} \in (\ker A)^{\perp}$. Computing both $\|v\|, \|Av\|$, we see:

$$\|Av\| = \| \begin{bmatrix} 0 & 1 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ t \end{bmatrix} \| = \| \begin{bmatrix} t \\ 0 \\ 0 \\ 0 \end{bmatrix} \| = |t| = \| \begin{bmatrix} 0 \\ t \end{bmatrix} \| = \|v\|.$$

So the restriction of A to the orthogonal complement of its kernel is an isometry, and A is a queen.

2. (a) Given a singular value decomposition $A = W\Sigma V^*$ of a square matrix A, construct a polar decomposition of A using W, V, Σ.

Solution: Suppose $A = W\Sigma V^*$ is given, we wish to find $|A|$ and some U unitary with $A = U|A|$.

$$|A| = \sqrt{A^*A} = \sqrt{V\Sigma^* W^* W \Sigma V^*} = \sqrt{V\Sigma^*\Sigma V^*}.$$

But recalling that Σ is a real diagonal matrix, we have $\Sigma = \Sigma^*$:

$$|A| = \sqrt{V\Sigma V^* V\Sigma V^*} = V\Sigma V^*.$$

Now we wish to right cancel V, and get back our W. So take $U = WV^*$ as the unitary (since it is the product of unitaries); and then:

$$U|A| = (WV^*)(V\Sigma V^*) = W\Sigma V^* = A.$$

(b) Using the method above, compute a polar decomposition for

$$A = \begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix}.$$

Solution: Compute A^*A;

$$A^*A = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix} = \begin{bmatrix} 5 & 15 \\ 15 & 45 \end{bmatrix}.$$

And find the characteristic polynomial:

$$C_{A^*A}(z) = \det(A - zI) = \begin{vmatrix} 5-z & 15 \\ 15 & 45-z \end{vmatrix} = z^2 - 50 = z(z-50).$$

Which gives the nonzero singular value $\sigma_1 = 5\sqrt{2}$, and our $\Sigma = \begin{bmatrix} 5\sqrt{2} & 0 \\ 0 & 0 \end{bmatrix}$. Then find an associated eigenvector for σ_1^2.

$$\begin{aligned} (50I - A^*A)v_1 = 0 &\implies \begin{bmatrix} -45 & 15 \\ 15 & -5 \end{bmatrix} v_1 = 0 \\ &\implies \begin{bmatrix} -3 & 1 \\ 0 & 0 \end{bmatrix} v_1 = 0 \\ &\implies v_1 = t \begin{bmatrix} 1 \\ -3 \end{bmatrix} \\ &\implies v_1 = \frac{1}{\sqrt{10}} \begin{bmatrix} 1 \\ -3 \end{bmatrix}. \end{aligned}$$

Now that we have v_1, we need only pick v_2 so that V is unitary, so by inspection take $v_2 = \frac{1}{\sqrt{10}} \begin{bmatrix} 3 \\ -1 \end{bmatrix}$, which already has norm 1, and is orthogonal to v_1. And so we have our matrix $V = \frac{1}{\sqrt{10}} \begin{bmatrix} 1 & 3 \\ 3 & -1 \end{bmatrix}$.

Now we find W. Begin by computing:

$$w_1 = \frac{1}{\sigma_1} A v_1 = \frac{1}{5\sqrt{2}} \frac{1}{\sqrt{10}} \begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \end{bmatrix} = \frac{1}{\sqrt{5}} \begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$

And again by inspection, $w_2 = \frac{1}{\sqrt{5}} \begin{bmatrix} -2 \\ 1 \end{bmatrix}$, and $W^* = \frac{1}{\sqrt{5}} \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}$. So then we have our SVD:

$$A = \frac{1}{\sqrt{10}} \begin{bmatrix} 1 & 3 \\ 3 & -1 \end{bmatrix} \begin{bmatrix} 5\sqrt{2} & 0 \\ 0 & 0 \end{bmatrix} \frac{1}{\sqrt{5}} \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}.$$

After a quick sanity check that all our matrix multiplication gives us back A, we just need to find $|A| = V\Sigma V^*$ and $U = WV^*$.

$$\begin{aligned}
|A| &= V\Sigma V^* \\
&= \frac{1}{10}\begin{bmatrix}1 & 3\\ 3 & -1\end{bmatrix}\begin{bmatrix}5\sqrt{2} & 0\\ 0 & 0\end{bmatrix}\begin{bmatrix}1 & 3\\ 3 & -1\end{bmatrix} \\
&= \frac{1}{10}\begin{bmatrix}1 & 3\\ 3 & -1\end{bmatrix}\begin{bmatrix}5\sqrt{2} & 15\sqrt{2}\\ 0 & 0\end{bmatrix} \\
&= \frac{1}{10}\begin{bmatrix}5\sqrt{2} & 15\sqrt{2}\\ 15\sqrt{2} & 45\sqrt{2}\end{bmatrix} \\
&= \frac{1}{\sqrt{2}}\begin{bmatrix}1 & 3\\ 3 & 9\end{bmatrix} \\
U &= WV^* \\
&= \frac{1}{\sqrt{200}}\begin{bmatrix}2 & -4\\ 4 & 2\end{bmatrix}\begin{bmatrix}1 & 3\\ 3 & -1\end{bmatrix} \\
&= \frac{1}{10\sqrt{2}}\begin{bmatrix}-10 & 10\\ 10 & 10\end{bmatrix} \\
&= \frac{1}{\sqrt{2}}\begin{bmatrix}-1 & 1\\ 1 & 1\end{bmatrix}.
\end{aligned}$$

And so we have the polar decomposition:

$$A = \frac{1}{\sqrt{2}}\begin{bmatrix}-1 & 1\\ 1 & 1\end{bmatrix}\frac{1}{\sqrt{2}}\begin{bmatrix}1 & 3\\ 3 & 9\end{bmatrix} = \begin{bmatrix}1 & 3\\ 2 & 6\end{bmatrix}.$$

3. Find your favorite 4×2 matrix A of rank 2 and compute a singular value decomposition for A. All of the entries of A must be nonzero.

Solution: Take the matrix:

$$A = \begin{bmatrix}2 & 1\\ 1 & 2\\ 2 & 1\\ 1 & 2\end{bmatrix}; \quad A^*A = \begin{bmatrix}2 & 1 & 2 & 1\\ 1 & 2 & 1 & 2\end{bmatrix}\begin{bmatrix}2 & 1\\ 1 & 2\\ 2 & 1\\ 1 & 2\end{bmatrix} = \begin{bmatrix}10 & 8\\ 8 & 10\end{bmatrix}.$$

And compute $C_{A^*A}(z) = (z-10)^2 - 64 = z^2 - 20z - 36 = (z-18)(z-2)$. So now we have our Σ:

$$\Sigma = \begin{bmatrix}3\sqrt{2} & 0\\ 0 & \sqrt{2}\end{bmatrix}.$$

And singular values: $\sigma_1 = 3\sqrt{2}, \sigma_2 = \sqrt{2}$. Now compute eigenvectors for σ_1^2, σ_2^2 :

$$\begin{aligned}
(A^*A - 18I)v_1 &= \begin{bmatrix}-8 & 8\\ 8 & -8\end{bmatrix} \\
v_1 &= \frac{1}{\sqrt{2}}\begin{bmatrix}1\\ 1\end{bmatrix} \\
(A^*A - 2I)v_2 &= \begin{bmatrix}8 & 8\\ 8 & 8\end{bmatrix} \\
v_1 &= \frac{1}{\sqrt{2}}\begin{bmatrix}1\\ -1\end{bmatrix}.
\end{aligned}$$

So we have our V:

$$V = \frac{1}{\sqrt{2}}\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix}.$$

And we are free to compute w_1, w_2 from v_1, v_2:

$$\begin{aligned} w_1 &= \frac{1}{\sigma_1} A v_1 \\ &= \frac{1}{3\sqrt{2}} \\ &= \frac{1}{3\sqrt{2}} \begin{bmatrix} 2 & 1 \\ 1 & 2 \\ 2 & 1 \\ 1 & 2 \end{bmatrix} \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \\ &= \frac{1}{6} \begin{bmatrix} 3 \\ 3 \\ 3 \\ 3 \end{bmatrix} \\ &= \frac{1}{2} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \\ w_2 &= \frac{1}{\sigma_2} A v_2 \\ &= \frac{1}{\sqrt{2}} \\ &= \frac{1}{\sqrt{2}} \begin{bmatrix} 2 & 1 \\ 1 & 2 \\ 2 & 1 \\ 1 & 2 \end{bmatrix} \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix} \\ &= \frac{1}{2} \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix}. \end{aligned}$$

Now we must extend w_1, w_2 to an orthonormal basis of $\mathbb{C}^4$. We could do this with Gram-Schmidt, or we could be brave and move negative signs around until all our inner products turn out to be zero. Opt for the second option, and find:

$$w_3 = \frac{1}{2} \begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \end{bmatrix}, \quad w_4 = \frac{1}{2} \begin{bmatrix} -1 \\ 1 \\ 1 \\ -1 \end{bmatrix}.$$

Then we take the transpose of all our w_i's, and get our W:

$$W^* = \frac{1}{2} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & -1 & 1 \\ 1 & 1 & -1 & -1 \\ -1 & 1 & 1 & -1 \end{bmatrix}.$$

And therefore we have our SVD:

$$A = \frac{1}{2} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & -1 & 1 \\ 1 & 1 & -1 & -1 \\ -1 & 1 & 1 & -1 \end{bmatrix} \begin{bmatrix} 3\sqrt{2} & 0 \\ 0 & \sqrt{2} \end{bmatrix} \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \\ 2 & 1 \\ 1 & 2 \end{bmatrix}.$$

4. For an $m \times n$ matrix A, show that the set of nonzero eigenvalues for A^*A coincide with that of AA^*.

Solution: Let $0 \neq \lambda \in \sigma(A^*A)$, with an associated eigenvector v.

Then $A^*Av = \lambda v$. Applying A on both sides, we have $AA^*Av = A\lambda v = \lambda Av$, and so Av is an eigenvector for AA^* associated with λ.

Now let $0 \ = \lambda \in \sigma(AA^*)$

Then suppose $AA^*v = \lambda v$. Applying A^* on both sides, we have $A^*AA^*v = A^*\lambda v = \lambda A^*v$, and so A^*v is an eigenvector for A^*A associated with λ.

5. Suppose $A = W\Sigma V^*$ is a singular value decomposition for A. Show that the columns of W are eigenvectors for AA^*.

Solution: Let $1 \leq i \leq n$, and take:

$$W = \begin{bmatrix} w_1 & \dots & w_n \end{bmatrix}, \qquad \Sigma = \begin{bmatrix} \sigma_1 & & & 0 \\ & \ddots & & \vdots \\ & & \sigma_r & 0 \\ 0 & \dots & 0 & 0 \end{bmatrix}.$$

Then begin the computation:

$$\begin{aligned}
AA^*w_i &= W\Sigma V^*V\Sigma^*W^*w_i \\
&= W\Sigma^2W^*w_i && V \text{ unitary, } \Sigma \text{ real, symmetric} \\
&= W\Sigma^2 \begin{bmatrix} w_1^* \\ \vdots \\ w_n^* \end{bmatrix} w_i \\
&= W\Sigma^2 \begin{bmatrix} w_1^*w_i \\ \vdots \\ w_n^*w_i \end{bmatrix} \\
&= W\Sigma^2 \begin{bmatrix} \langle w_i, w_1 \rangle \\ \vdots \\ \langle w_i, w_n \rangle \end{bmatrix} \\
&= W\Sigma^2 \begin{bmatrix} 0 \\ \vdots \\ \langle w_i, w_i \rangle \\ \vdots \\ 0 \end{bmatrix} && \text{Since } w_i \text{ form an o.n.b.} \\
&= W\Sigma^2 \begin{bmatrix} 0 \\ \vdots \\ \|w_i\|^2 \\ \vdots \\ 0 \end{bmatrix}.
\end{aligned}$$

Now we split by cases. If $i > r$, then the i-th column of Σ will be exactly zero, and we will have $AA^*w_i = W0 = 0 = 0w_i$, and w_i is an eigenvector associated with 0.

But if $i \leq r$, then the i-th column of Σ^2 will be of the form $\Sigma^2 = \begin{bmatrix} 0 & \dots & \sigma_i^2 & \dots & 0 \end{bmatrix}^T$

Then our equation becomes

$$
\begin{aligned}
AA^* w_i &= W\sigma_i^2 \begin{bmatrix} 0 \\ \vdots \\ \|w_i\|^2 \\ \vdots \\ 0 \end{bmatrix} \\
&= \sigma_i^2 \begin{bmatrix} w_1 & \dots & w_n \end{bmatrix} \begin{bmatrix} 0 \\ \vdots \\ \|w_i\|^2 \\ \vdots \\ 0 \end{bmatrix} \\
&= (\sigma_i \|w_i\|)^2 w_i.
\end{aligned}
$$

And as we wanted to show, w_i is an eigenvector for AA^*.