1. Let S be a set. The indiscrete topological space $I(S)$ is the space whose set of points is S and whose only open subsets are $\emptyset$ and S itself. Imitating Example 0.5, find a universal property satisfied by the space $I(S)$.

2. Find three examples of categories not mentioned above.

   (a) Mat_R is the category whose objects are positive integers, and where the set of morphisms from n to m is the set of $m \times n$ matrices with values in $\mathbb{R}$. Composition is by matrix multiplication, and identity for $n \in \mathbb{Z}_{>0}$ is the $n \times n$ identity matrix.

   (b) We can form a category out of regular languages, since strings form a monoid under concatenation.

   (c) Meas has measurable spaces as objects and measurable functions as morphisms.

3. Show that a map in a category can have at most one inverse. That is, given a map $f : A \to B$, show that there is at most one map $g : B \to A$ such that $gf = 1_A$ and $fg = 1_B$.

   Solution: Suppose there are two such mappings, $g, h : B \to A$ so that $fg = fh = 1_B$ and $gf = hf = 1_A$. Then left-compose with g:

   $$\begin{aligned} fg &= fh \\ gfg &= gfh \\ 1_A g &= 1_A h \\ g &= h. \end{aligned}$$

   So an inverse for f must be unique.

4. Let $\mathscr{A}, \mathscr{B}$ be categories. The construction of the product category:

   $$\mathrm{ob}(\mathscr{A} \times \mathscr{B}) = \mathrm{ob}(\mathscr{A}) \times \mathrm{ob}(\mathscr{B})$$

   $$\mathrm{Hom}(\mathscr{A} \times \mathscr{B}) = \mathrm{Hom}(\mathscr{A}) \times \mathrm{Hom}(\mathscr{B})$$

   has only one choice for compositions and identities. Give both.

   Solution: Let f, g, h be morphisms in $\mathscr{A} \times \mathscr{B}$. Write $f = (f_1, f_2)$, $g = (g_1, g_2)$, $h = (h_1, h_2)$. Then the sensible composition is $gf = (g_1 f_1, g_2 f_2)$. And associativity follows;

   $$h(gf) = h(g_1 f_1, g_2 f_2) = (h_1(g_1 f_1), h_2(g_2 f_2)) = ((h_1 g_1) f_1, (h_2 g_2) f_2) = (hg)(f).$$

   Then for an object $(a, b) \in \mathscr{A} \times \mathscr{B}$, the sensible identity is $1_{(a,b)} = (1_a, 1_b)$. Then for a morphism $f = (f_1, f_2)$ with domain (a, b), we have

   $$f 1_{(a,b)} = (f_1 1_a, f_2 1_b) = (f_1, f_2) = f.$$

   And likewise for some $g = (g_1, g_2)$ with codomain (a, b), we have:

   $$1_{(a,b)} g = (1_a g_1, 1_b g_2) = (g_1, g_2) = g.$$

5. Find three examples of functors not mentioned above.

6. Show that functors preserve isomorphism. That is, prove that if $F : \mathscr{A} \to \mathscr{B}$ is a functor and $A, A' \in \mathscr{A}$ with $A \cong A'$, then $F(A) \cong F(A')$.

   Proof: Suppose $F : \mathscr{A} \to \mathscr{B}$ is a functor, and $A \cong A'$ in $\mathscr{A}$. Then there exists a pair of morphisms $f : A \to A'$ and $g : A' \to A$ with $fg = 1_{A'}$ and $gf = 1_A$. And, the functor F gives another pair of morphisms Ff, Fg. Verify:

   $$(Ff)(Fg) = F(fg) = F1_{A'} = 1_{FA'}$$

   and likewise:

   $$(Fg)(Ff) = F(gf) = F1_A = 1_{FA}.$$

   And so we have $FA \cong FA'$. ∎

7. Two categories $\mathscr{A}$ and $\mathscr{B}$ are isomorphic, written as $\mathscr{A} \cong \mathscr{B}$, if they are isomorphic as objects of $\mathbf{Cat}$.

   (a) Let G be a group, regarded as a one-object category all of whose maps are isomorphisms. Then its opposite G^{op} is also a one-object category all of whose maps are isomorphisms, and can therefore be regarded as a group too. What is G^{op}, in purely group-theoretic terms? Prove that G is isomorphic to G^{op}.

   Proof: Take the functors $F : G \to G^{op}$, and $F' : G^{op} \to G$. Define, for $g \in G$ and $h^{op} \in G^{op}$:

$$F(g) = (g^{-1})^{op}, \qquad F'(h^{op}) = h^{-1}.$$

   We first check that these functors compose to identity:

$$\begin{aligned} FF'(g^{op}) &= F(g^{-1}) \\ &= ((g^{-1})^{-1})^{op} \\ &= g^{op} \\ FF' &= 1_{G^{op}} \\ F'F(g) &= F'((g^{-1})^{op}) \\ &= (g^{-1})^{-1} \\ &= g \\ F'F &= 1_G. \end{aligned}$$

   And then we check that these mappings are indeed functors. Clearly F, F' map the single object in G to G^{op}, and vice versa. Then we check the morphism identities for F and F'. Let $g, h \in G$;

$$\begin{aligned} F(gh) &= \left((gh)^{-1}\right)^{op} \\ &= \left(h^{-1}g^{-1}\right)^{op} \\ &= \left(g^{-1}\right)^{op}\left(h^{-1}\right)^{op} \\ &= F(g)F(h). \end{aligned}$$

   Then, if $g^{op}, h^{op} \in G^{op}$;

$$\begin{aligned} F'(g^{op}h^{op}) &= F'((hg)^{op}) \\ &= (hg)^{-1} \\ &= g^{-1}h^{-1} \\ &= F(g^{op})F(h^{op}). \end{aligned}$$

   And all that is left to verify is that F, F' send identities to identities. Let $g \in G$, and $g^{op} \in G^{op}$. We wish to show that $F(1_G) = (1_G)^{op} = 1_{G^{op}}$, and that $F'(1_{G^{op}}) = 1_G$. Take $g^{op} \in G^{op}$, which we know to have a preimage g^{-1} under F.

$$\begin{aligned} (1_G)^{op}g^{op} &= F(1_G)g^{op} \\ &= F(1_G)F(g^{-1}) \\ &= F(1_G g^{-1}) \\ &= F(g^{-1}) \\ &= g^{op}. \end{aligned}$$

   And so $1_{G^{op}} = (1_G)^{op} = F(1_G)$ (Since identity of right composition follows from the same argument). Now for $g \in G$,

$$\begin{aligned} F'(1_{G^{op}}) &= F'((1_G)^{op}) \\ &= 1_G^{-1} \\ &= 1_G. \end{aligned}$$

   So F and F' are functors which serve as inverses for one another, and $G \cong G'$. ■

   (b) Find a monoid which is not isomorphic to its opposite.

Solution: Take $\mathbb{N}$,

8. Find a universal property of the indiscrete topology, defined on any set by letting the open sets be exactly $\emptyset, S$.

   Solution: We observe that if $f : X \to S$ is a function on a topological space X, that the preimage of $\emptyset$ is $\emptyset$ and the preimage of S is X. So the preimage of our two open sets are both open, and f is continuous. So given any function from $f : X \to S$ we define a function $\bar{f} : X \to I(S)$ which is continuous:

$$\begin{tikzcd} S & I(S) \arrow[l, hook', "\iota"] \\ & X \arrow[ul, "f"] \arrow[u, dashed, "\bar{f}"'] \end{tikzcd}$$