Homework 2 - Thomas Boyko - 30191728

1. Find all values $\alpha \in \mathbb{C}$ for which the equation $\cos z = \alpha$ has real solutions, where z is a complex unknown. For such values of α, are there any non-real solutions?

   Let $z \in \mathbb{R}$. Solving this for z gives $z = \arccos \alpha$, which has an infinitude of solutions with $\alpha \in \mathbb{R}$.

   We can write $\alpha = \arccos z + 2\pi k$, for $k \in \mathbb{Z}$.

2. Consider the complex function of a real variable:

$$f(t) = \frac{t^3 + t^\alpha}{t^3 + t}.$$

   Where $\alpha \in \mathbb{C}$. Study the limit $\lim_{t \to -\infty} f(t)$.

   We can rewrite the exponential:

$$\begin{aligned}
\lim_{t\to-\infty} f(t) &= \lim_{t\to-\infty} \frac{t^3 + t^\alpha}{t^3 + t} \\
&= \lim_{t\to-\infty} \frac{t^3 + e^{\alpha \ln t}}{t^3 + t} \\
&= \lim_{t\to-\infty} \frac{t^3 + e^{x \ln t + iy \ln t}}{t^3 + t} && \text{Writing } \alpha = x + iy \\
&= \lim_{t\to-\infty} \frac{t^3 + e^{x \ln t} e^{iy \ln t}}{t^3 + t} && (*) \\
&= \lim_{t\to-\infty} \frac{t^3 + t^x(\cos(y \ln t) + i \sin(y \ln t))}{t^3 + t}.
\end{aligned}$$

   We can split into cases; $\operatorname{Re} z < 3$, $\operatorname{Re} z = 3$. $\operatorname{Re} z > 3$.

   If $\operatorname{Re} z < 3$, we have a dominant term of t^3 on both the numerator and the denominator:

$$\lim_{t\to-\infty} \frac{t^3 + t^x(\cos(y \ln t) + i \sin(y \ln t))}{t^3 + t} = 1$$

   If $\operatorname{Re} z = 3$, and $\operatorname{Im} z = 0$, our limit becomes:

$$\lim_{t\to-\infty} \frac{t^3 + t^3 \cos(0)}{t^3 + t} = 2.$$

   However if $\operatorname{Im} z \neq 0$, we have:

$$\lim_{t\to-\infty} \frac{t^3 + t^x(\cos(y \ln t) + i \sin(y \ln t))}{t^3 + t}$$

   which has oscillating real and imaginary part with constant modulus, so the limit does not exist.

   And we see from the modulus of $f(t)$

$$\begin{aligned}
\lim_{t\to-\infty} \left| \frac{t^3 + e^{x \ln t} e^{iy \ln t}}{t^3 + t} \right| &= \lim_{t\to-\infty} \left| \frac{t^3 + t^x e^{iy \ln t}}{t^3 + t} \right| && \text{From } (*) \\
&= \lim_{t\to-\infty} \left| \frac{t^3 + t^x}{t^3 + t} \right| && \text{Since } e^{iy \ln t} = 1 \text{, and by linearity of limits}
\end{aligned}$$

   .

   And since the numerator has a dominant term of $t^{3+x} > t^3$, the limit of the modulus is ∞ and therefore the limit of our function $f(t)$ will be $\infty_{\mathbb{C}}$.

3.

a) Calculate the limit:

$$\lim_{z\to 0}\frac{|z|^2\operatorname{Im} z}{\operatorname{Re} z}.$$

This limit does not exist. To show this we will find two parametric curves $\gamma(t), \delta(t)$ so that the limits do not agree.

First write $f(t) = \frac{|z|^2\operatorname{Im} z}{\operatorname{Re} z}$.

Now choose the functions:

$$\gamma(t) = t, \quad \delta(t) = it.$$

And now we study the limits:

$$\lim_{t\to 0} f(\gamma(t)), \quad \lim_{t\to 0} f(\delta(t)).$$

First we look to γ.

$$\lim_{t\to 0} f(\gamma(t)) = \lim_{t\to 0} t^2\frac{0}{t} = 0.$$

And then we look to δ.

$$\lim_{t\to 0} f(\delta(t)) = \lim_{t\to 0}\frac{t^2}{\operatorname{Re} it} = \frac{0}{0}.$$

Which is of indeterminate form. So this limit does not exist, since our function disagrees on two paths to 0.

b) Calculate the limit:

$$\lim_{z\to\infty_{\mathbb{C}}}\frac{\operatorname{Re} z}{|z|^2}.$$

Let's find a candidate limit; take $\gamma_m(t) = t + imt$. Then:

$$f(\gamma(t)) = \frac{t}{t^2 + m^2t^2} = \frac{1}{tm^2 + t}$$

And $\lim_{t\to\infty} f(\gamma(t)) = 0$ regardless of choice of m.

So let's write $f(re^{i\theta} = \frac{r\cos\theta}{r^2} = \frac{\cos\theta}{r}$. Take the function $g(r) = \frac{1}{r}$, and note that $\lim_{r\to\infty} g(r) = 0$ and $|f(re^{i\theta}| = \left|\frac{\cos\theta}{r}\right| \leq \left|\frac{1}{r}\right|$ since $\cos\theta \in [-1, 1]$. So by squeeze theorem we can say that:

$$\lim_{z\to\infty_{\mathbb{C}}}\frac{\operatorname{Re} z}{|z|^2} = 0.$$