Title - Thomas Boyko - 30191728

1. Prove that if p and q are distinct prime numbers, then $\sqrt{p} + \sqrt{q}$ is irrational.

   Proof. Let p, q be distinct primes and $\alpha = \sqrt{p} + \sqrt{q}$. Suppose for the sake of contradiction that $\alpha \in \mathbb{Q}$. Squaring both sides, $\alpha^2 = p + q + 2\sqrt{pq}$, and $\frac{\alpha^2 - p - q}{2} = \sqrt{pq}$. The left side of this equation is rational, so if we can show that $\sqrt{pq} \notin \mathbb{Q}$, we have a contradiction.

   Suppose for the sake of contradiction that $\sqrt{pq} = \frac{a}{b}$ for some coprime a, b with $b > 0$. Then $pq = \frac{a^2}{b^2}$ and $b^2 pq = a^2$. So $b^2 | a^2$, which means that $b | a$, and $GCD(a, b) = b$. But $GCD(a, b) = 1$, so we have our contradiction. □

2. The number e is defined as the sum of the reciprocals of the factorials, If e were rational, let n be its denominator when represented as a fraction, let x be the sum of the terms up to $\frac{1}{n!}$, and let y be the sum of the rest of the terms. Demonstrate in this case that $n!x$ is an integer and $n!e$ is an integer, and that $0 < n!y < 1$. Use this to achieve a contradiction, and fill in the steps to prove that e is irrational.

   Proof. Let $e \in \mathbb{Q}$ for the sake of contradiction. That is, $e = \frac{m}{n}$ for some $m, n \in \mathbb{Z}$, $n \neq 0$.
   Write $e = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} \ldots = x + y$, where $x = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \ldots + \frac{1}{n!}$ and y is the remaining terms of the sum.
   We see that:

   $$\begin{aligned} n!x &= n!\left(1 + \frac{1}{1!} + \frac{1}{2!} + \ldots + \frac{1}{n!}\right) \\ &= n! + \frac{n!}{1!} + \frac{n!}{2!} + \ldots + 1. \end{aligned}$$

   And since each term in this sum is an integer, we can say that $n!x \in \mathbb{Z}$.
   Since $n!e = n!x + n!y$, if we can show that $0 \leq n!y \leq 1$, the left side of the equation is an integer and the right is between two integers, which is a contradiction.
   Consider:

   $$\begin{aligned} n!y = n!\left(\frac{1}{(n+1)!} + \frac{1}{(n+2)!} + \ldots\right) \\ &= \frac{1}{(n+1)} + \frac{1}{(n+2)(n+1)} + \ldots \\ &\leq \frac{1}{n+1} + \frac{1}{(n+1)^2} \ldots \end{aligned}$$

   .

   Comparing this to the geometric series $a_n = \left(\frac{1}{3}^n\right)$, since $n \geq 2$ (if $n = 1$, e is an integer), we get $n!y \leq \frac{1}{3} + \frac{1}{3^2} + \ldots$.
   Using the formula for the sum of an infinite geometric series, $S = \frac{1}{1-r}$, we know that this sum must be less than $\frac{1}{2}$. So we have reached our contradiction.

   □

3. (a) Let z be a complex number. Prove that if z is a Gaussian integer and an Eisenstein integer, then z is an ordinary integer.

   Proof. Let $z \in \mathbb{Z}[i]$ and $z \in \mathbb{Z}[\omega]$. Then for some $a, b, c, d \in \mathbb{Z}$:

   $$\begin{aligned} z = a + bi &= c + d\omega \\ &= c + d(1 - \sqrt{3}i) \\ &= (c + d) - d\sqrt{3}i. \end{aligned}$$

   We equate the imaginary parts and the real parts. So $bi = -d\sqrt{3}i$, and $b = -d\sqrt{3}$. Since $\sqrt{3} \notin \mathbb{Q}$, the only way for this equality to hold is if $b = d = 0$. Therefore $z = a + 0i$ and so $a \in \mathbb{Z}$. □

(b) Let n be a positive integer. Let$(\zeta = e^{2i\pi/n}$. Prove that:

$$1 + \zeta + \zeta^2 + \ldots + \zeta^{n-1}.$$

Note: If $n = 1$ then the sum is equal to 1, i will take the case $n > 1$.

Proof. Let $n \in \mathbb{Z}_{>1}$ and $\zeta = e^{2i\pi/2}$, and let:

$$\alpha = 1 + \zeta + \zeta^2 + \ldots + \zeta^{n-1}.$$

So $\zeta\alpha = \zeta + \zeta^2 + \zeta^3 + \ldots + \zeta^n$.
Consider $\zeta^n = e^{n2i\pi/n} = e^{2i\pi} = 1$.
Therefore $\zeta\alpha = 1 + \zeta + \zeta^2 + \ldots + \zeta^{n-1} = \alpha$.
And so $\alpha(\zeta - 1) = 0$. So either α or $\zeta - 1$ is zero. $\zeta - 1$ can only be 0 if $\zeta = 1$, but since $n \geq 2$, this cannot be the case. Therefore $\alpha = 0$.

□

4. Factorize 85 into Gaussian primes.

   First let's take the ordinary prime factorization, $85 = 5 \times 17$. Conveniently both these primes are one more than a perfect square, so we can write $17 = (4+i)(4-i)$ and $5 = (2+i)(2-i)$.

   We can check using the results in the bonus that these are Gaussian Primes.

   So the prime factorization of $85 \in \mathbb{Z}[i]$ is $85 = (2+i)(2-i)(4-i)(4+i)$.

5. Let x be an integer. Prove that $x + i$ is a Gaussian prime number if and only if $x^2 + 1$ is an ordinary prime number

   Proof. $\Longrightarrow$: Let $x + i$ be a prime Gaussian integer. Then if $x + i = uv$ for some Gaussian integers u, v, then u or v is a unit. Take the square modulus of each side.

   $x^2 + 1 = |u|^2|v|^2$. Since one of these must be a unit, we can say that $x^2 + 1$ must be one times the square modulus of the other. Therefore $x^2 + 1$ is prime.

   $\Longleftarrow$: Let $x^2 + 1$ be a prime integer. x must be even since $x^2 + 1$ is odd. Since even squares are $\equiv 0 \pmod 4$, $x^2 + 1 \equiv 1 \pmod 4$. So we can use Fermat's Christmas Theorem, to say that $x^2 + 1 = a^2 + b^2 = (a + bi)(a - bi)$. (could not finish this part).

   □