1. Let $\{f_n\}$ be the sequence of functions defined by

$$f_n = \begin{cases} nx & \text{If } 0 \le x \le \frac{1}{n} \\ 1 & \text{If } \frac{1}{n} < x < 1 - \frac{1}{n} \\ n - nx & \text{If } 1 - \frac{1}{n} \le x \le 1 \end{cases}.$$

   (a) Find the pointwise limit f of the sequence.

   Solution: Proceed by cases. If $x = 0$, then the first case of the function will always be taken since $0 \le x$. So $f_n(0) = n0 = 0$. Likewise if $x = 1$, then $f(1) = n - n1 = n - n = 0$.
   Now, if $x \in (0,1)$, then we observe that $\frac{1}{n} \to 0$, and $1 - \frac{1}{n} \to 1$. Therefore the middle case of our piecewise function gives us $f(x) = 1$ for all x in this open interval.

   (b) Does $f_n \xrightarrow[{[0,1]}]{c.u} f$? Justify your answer.

   Solution: This sequence is not uniformly convergent. Pick $\varepsilon = \frac{1}{3}$, and let $N \in \mathbb{N}$, and $n > N$. Pick $x = \frac{1}{2n}$ so that $0 \le x \le \frac{1}{n}$, and then $f_n(x) = nx = \frac{n}{2n} = \frac{1}{2}$. Then: $|f_n(x) - f(x)| = \left|\frac{1}{2} - 1\right| = \frac{1}{2} > \varepsilon$.
   Therefore the sequence is not uniformly convergent.

2. Let $f_n(x) = \left(\cos\left(\frac{2x}{n}\right)\right)^{n^2}$

   (a) Compute the pointwise limit f of the sequence $\{f_n\}$. **Hint:** Use the following double inequalities:

$$1 - \frac{1}{2}t^2 \le \cos t \le 1 - \frac{1}{2}t^2 + \frac{1}{24}t^4, \quad \forall t \in \mathbb{R}.$$

$$-t - t^2 \le \ln(1-t) \le -t, \quad \forall t \in \left[0, \frac{1}{2}\right].$$

   Solution: Begin with the first inequality.

$$\begin{aligned}
1 - \frac{4x^2}{2n^2} \le \cos\left(\frac{2x}{n}\right) &\le 1 - \frac{4x^2}{2n^2} + \frac{16x^4}{24n^4} \\
\ln\left(1 - \frac{2x^2}{n^2}\right) \le \ln\left(\cos\left(\frac{2x}{n}\right)\right) &\le \ln\left(1 - \left(\frac{2x^2}{n^2} - \frac{2x^4}{3n^4}\right)\right) && \ln \text{ is increasing in } \mathbb{R} \\
-\frac{2x^2}{n^2} - \frac{4x^4}{n^4} \le \ln\left(\cos\left(\frac{2x}{n}\right)\right) &\le -\frac{2x^2}{n^2} + \frac{2x^4}{3n^4} && \text{From the second inequality} \\
-2x^2 - \frac{4x^4}{n^2} \le n^2 \ln\left(\cos\left(\frac{2x}{n}\right)\right) &\le -2x^2 + \frac{2x^4}{3n^2} \\
-2x^2 - \frac{4x^4}{n^2} \le \ln\left(\left(\cos\left(\frac{2x}{n}\right)\right)^{n^2}\right) &\le -2x^2 + \frac{2x^4}{3n^2} \\
\exp\left(-2x^2 - \frac{4x^4}{n^2}\right) \le \left(\cos\left(\frac{2x}{n}\right)\right)^{n^2} &\le \exp\left(-2x^2 + \frac{2x^4}{3n^2}\right) && \exp \text{ is increasing in } \mathbb{R}.
\end{aligned}$$

   Intuitively, according to squeeze theorem, it appears that the limit will become $\exp\left(e^{-2x^2}\right)$, however this idea needs some formalizing.

   (b) Show that $f_n \xrightarrow[{[0,1]}]{c.u} f$.

Lemma: If $f_n \xrightarrow[[0,1]]{c.u} f$, and $g:\mathbb{R}\to\mathbb{R}$ is uniformly continuous, then $g\circ f_n \xrightarrow[[0,1]]{c.u} g\circ f$.
Since f_n is uniformly convergent, it must be uniformly bounded, say that $f_n < M$ for all $n \in \mathbb{N}$ and $x \in [0,1]$. Let $\varepsilon > 0$, and by uniform continuity of g, there exists some δ such that $|a-b| < \delta \implies |g(a)-g(b)| < \varepsilon$.
Now, take $N \in \mathbb{N}$ so that $n > N \implies |f_n(x) - f(x)| < \delta$.
Therefore $|g(f_n(x)) - g(f(x))| < \varepsilon$, and $g\circ f_n \xrightarrow[[0,1]]{c.u} g\circ f$.

Solution: For both our functions of the form $h_c(x) = -2x^2 + \frac{cx^4}{n^2}$, we know that they are continuous on the compact set $[0,1]$, and so they must be bounded on that interval, say by M_c. The exponential is continuous, so it must be uniformly continuous on the compact set $[-M_c, M_c]$, which contains $\exp(h_c([0,1]))$. Since $-2x^2 + \frac{2x^4}{3n^2} \xrightarrow[[0,1]]{c.u} -2x^2$, and $-2x^2 - \frac{4x^4}{n^2} \xrightarrow[[0,1]]{c.u} -2x^2$, then by the above lemma both the bounds found for f_n in part (a) must converge uniformly to $\exp\left(-2x^2\right)$. Then by squeeze theorem:

$$f_n \xrightarrow[[0,1]]{c.u} e^{-2x^2}.$$

3. Let $a \in \mathbb{R}_+$. Compute the limit

$$\lim_{n\to\infty} \int_a^\pi \frac{\sin(nx)}{nx}\,dx.$$

What happens if $a = 0$?

Solution: We begin by considering our sequence of functions within the integral, each of which is a quotient and composition of continuous functions, and is itself continuous (for all but $x = 0$). Call this $g_n(x) = \frac{\sin(nx)}{nx}$. Note that since $-1 \le \sin(nx) \le 1$, we can find (for nonzero x) that $-\frac{1}{nx} \le g_n(x) \le \frac{1}{nx}$. Both the sequences bounding g have a zero limit at infinity, so by the squeeze theorem on sequences of functions, we can say that for nonzero x, $g_n \to 0$. Now since we have already shown that our sequence g_n is bounded, and since each g_n is integrable, we can say:

$$\begin{aligned}
\lim_{n\to\infty} \int_a^\pi \frac{\sin(nx)}{nx}\,dx &= \int_a^\pi \lim_{n\to\infty} \frac{\sin(nx)}{nx}\,dx \\
&= \int_a^\pi 0\,dx \\
&= 0 - 0 \\
&= 0.
\end{aligned}$$

4. Construct a sequence of functions defined in $[0,1]$, each of which is discontinuous at every point of $[0,1]$ and which converges uniformly to a function that is continuous at every point

Solution: Take the series $\{f_n\}$ defined by:

$$f_n(x) = \begin{cases} \frac{1}{n} & \text{If } x \in \mathbb{Q} \\ 0 & \text{Otherwise} \end{cases}$$

Claim: $\{f_n\}$ converges uniformly to the constant function 0, which we know to be continuous on the real line, as well as $[0,1]$.

Let $\varepsilon > 0$, and choose N such that $0 < \frac{1}{N} < \varepsilon$. Then any $n \geq N$ will have $0 < \frac{1}{n} < \frac{1}{N} < \varepsilon$. Now by cases, if $x \in \mathbb{Q}$, then we have

$$|f_n(x) - f(x)| = \left|\frac{1}{n} - 0\right| = \frac{1}{n} < \frac{1}{N} < \varepsilon.$$

And for $x \notin \mathbb{Q}$,

$$|f_n(x) - f(x)| = |0 - 0| = 0 < \varepsilon.$$

Now we must show that each of these functions is continuous nowhere in $[0,1]$. Suppose by way of contradiction that f_n is continuous at some $c \in [0,1]$. Then for $\varepsilon = \frac{1}{n+1}$, there must be some δ such that if $|x-c| < \delta$, $|f_n(x) - f_n(c)| < \frac{1}{n+1}$. Take $B_\delta(c)$ the δ-ball about c, and proceed by cases on c.

$c \in \mathbb{Q}$: If c is rational, find some $d \notin \mathbb{Q}$ inside $B_\delta(c)$. Then we will have $f_n(c) = \frac{1}{n}$ and $f_n(d) = 0$

$c \notin \mathbb{Q}$: If c is irrational, find some $d \in \mathbb{Q}$ inside $B_\delta(c)$. Then we will have $f_n(d) = \frac{1}{n}$ and $f_n(c) = 0$

Regardless of case, we will get $|f_n(c) - f_n(d)| = \frac{1}{n} > \frac{1}{n+1}$, and we have found our contradiction.

Therefore $\{f_n\}$ is a sequence of functions which are continuous nowhere, convergent to the zero function which is continuous everywhere.

5. Consider the series of functions $\sum_{n\geq 1} \frac{x}{n(n+x)}$.

   (a) Show that the series converges uniformly in the interval $[0,b]$ for any $b > 0$.

   Solution:

$$\begin{aligned}\frac{x}{n(n+x)} &= \frac{x}{n^2+nx} \\ &\leq \frac{x}{n^2} \\ &\leq \frac{b}{n^2}.\end{aligned}$$

   Define $u_n = \frac{b}{n^2}$, then by the Weierstrass Comparison test, since $\sum_{n\geq 1} u_n$ is convergent as a p-series with $p = 2$, $\frac{x}{n(n+x)} \leq u_n$, this series must converge.

   (b) Let $F(x) = \sum_{n\geq 1} \frac{x}{n(n+x)}$. Show that $F'(x) = \sum_{n\geq 1} \frac{1}{n(n+x)^2}$, $x \geq 0$.

Solution:

$$\begin{aligned}
F'(x) &= \frac{d}{dx}\left(\sum_{k=1}^{\infty}\frac{x}{k(k+x)}\right) \\
&= \frac{d}{dx}\left(\lim_{n\to\infty}\sum_{k=1}^{n}\frac{x}{k(k+x)}\right) \\
&= \lim_{n\to\infty}\left(\frac{d}{dx}\sum_{k=1}^{n}\frac{x}{k(k+x)}\right) \\
&= \lim_{n\to\infty}\sum_{k=1}^{n}\frac{d}{dx}\frac{x}{k(k+x)} \\
&= \lim_{n\to\infty}\sum_{k=1}^{n}\frac{k(x+k)-kx}{k^2(k+x)^2} \\
&= \lim_{n\to\infty}\sum_{k=1}^{n}\frac{k}{k^2(k+x)^2} \\
&= \lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{k(k+x)^2} \\
&= \sum_{k=1}^{\infty}\frac{1}{k(k+x)^2}.
\end{aligned}$$

6. Consider the series of functions $\sum_{n\geq 1} \frac{x}{1+n^2x^2}$. Show that the series doesn't converge uniformly in $\mathbb{R}_+$.

   Hint: You could start by showing that $\frac{x}{1+n^2x^2} \geq \int_n^{n+1} \frac{x}{1+t^2x^2}\,dt, \quad \forall x \in \mathbb{R}$.

   Solution: Begin with the hint. If we take $P_0 = \{n, n+1\}$, the trivial partition on $[n, n+1]$, then we will have the upper sum:

$$U\left(P_0, \frac{x}{1+t^2x^2}\right) = \sum_{k=1}^{1} \sup_{t\in[n,n+1]} \left(\frac{x}{1+t^2x^2}\right)((n+1)-n) = \frac{x}{1+n^2x^2}.$$

   But from the definition of the Riemann integral, we have (For σ the set of all partitions of $[n, n+1]$):

$$\begin{aligned} \int_n^{n+1} \frac{x}{1+t^2x^2}\,dt &= \inf_{P\in\sigma} U\left(P, \frac{x}{1+t^2x^2}\right) \\ &\leq U\left(P_0, \frac{x}{1+t^2x^2}\right) \\ &= \frac{x}{1+n^2x^2}. \end{aligned}$$

   Suppose by way of contradiction that the series does converge uniformly. Then there exists some m such that, for any $x \in \mathbb{R}$

$$\left|\sum_{n=1}^{\infty} \frac{x}{1+n^2x^2} - \sum_{n=1}^{m} \frac{x}{1+n^2x^2}\right| < \frac{1}{2}.$$

   However,

$$\begin{aligned} \left|\sum_{n=1}^{\infty} \frac{x}{1+n^2x^2} - \sum_{n=1}^{m} \frac{1}{1+n^2x^2}\right| &= \left|\sum_{n=m+1}^{\infty} \frac{x}{1+x^2n^2}\right| \\ &= \sum_{n=m+1}^{\infty} \frac{x}{1+x^2n^2} \\ &\geq \sum_{n=m+1}^{\infty} \int_n^{n+1} \frac{x}{1+t^2x^2}\,dt \\ &= \int_{m+1}^{\infty} \frac{x}{1+t^2x^2}\,dt && \text{Let } u = tx \\ &= \int_{(m+1)x}^{\infty} \frac{1}{1+u^2}\,du && du = xdt \\ &= (\arctan u)_{u=(m+1)x}^{\infty} \\ &= \frac{\pi}{2} - \arctan((m+1)x) \\ &= \frac{\pi}{2} - \arctan(1) && \text{Pick } x = \frac{1}{m+1} \\ &= \frac{\pi}{2} - \frac{\pi}{4} \\ &= \frac{\pi}{4} \\ &> \frac{1}{2}. \end{aligned}$$

   A contradiction, so our series cannot converge uniformly.