1. (a) Prove that the series

$$\sum_{n=2}^{\infty} \frac{1}{(\log_2 n)^{p(\log_2 n)}}$$

is convergent for all $p > 1$. Here $\log_2 x$ denotes the logarithm base 2 of x. You may assume that $\log_2 n$ is increasing in n.

Proof. We attempt to satisfy the criterion in Rudin Theorem 3.27. Rewrite the series; and let $c_n : \mathbb{N} \to \mathbb{R}$:

$$c_n = \begin{cases} 1 & n = 1 \\ \frac{1}{(\log_2 n)^{p(\log_2 n)}} & n \geq 2 \end{cases}.$$

Then our summation becomes

$$1 + \sum_{n=2}^{\infty} \frac{1}{(\log_2 n)^{p(\log_2 n)}} = \sum_{n=1}^{\infty} c_n.$$

Next we want to show that the general term is decreasing so we can use Theorem 3.27. Let $1 < x < y$ be natural numbers, and thanks to $\log_2$ being increasing in $\mathbb{N}$,

$$\begin{aligned} x &\leq y \\ \log_2 x &\leq \log_2 y \\ (\log_2 x)^p &\leq (\log_2 y)^p \\ (\log_2 x)^{p \log_2 x} &\leq (\log_2 y)^{p \log_2 y} \\ \frac{1}{(\log_2 x)^{p \log_2 x}} &\geq \frac{1}{(\log_2 y)^{p \log_2 y}}. \end{aligned}$$

Note above we can only take the inverse when $\log_2$ is positive, so the series is decreasing for $x \geq 2$, but we have defined $c_1 = c_2$ so that our series decreases regardless. Now that our sum is indexed from 1 and we have shown that the general term is decreasing, we can apply Rudin Theorem 3.27. Our series of c_n converges if and only if the following series converges.

$$\sum_{k=0}^{\infty} 2^k c_{2^k} = 1 \cdot 2^0 + \sum_{k=1}^{\infty} \frac{2^k}{(\log_2 2^k)^{p \log_2 2^k}}.$$

By Rudin Theorem 3.3 (b), this is convergent $\iff \sum_{k=1}^{\infty} \frac{2^k}{(\log_2 2^k)^{p \log_2 2^k}}$ is convergent.

Now consider:

$$\begin{aligned} \limsup_{k\to\infty} \sqrt[k]{\left|\frac{2}{k^p}\right|^k} &= \limsup_{k\to\infty} \left|\frac{2}{k^p}\right| \\ &= \limsup_{k\to\infty} \frac{2}{k^p} \\ &= \lim_{k\to\infty} \frac{2}{k^p} && \text{By Rudin Theorem 3.18} \\ &= 2 \lim_{k\to\infty} \frac{1}{k^p} \\ &= 2(0) && \text{By Rudin Theorem 3.20} \\ &= 0 \\ &< 1. \end{aligned}$$

So by Rudin Theorem 3.33 this series is convergent, and so our original series must be.

(b) For $a > 0$ find the sum of the series

$$\sum_{k=2}^{\infty}\left(\frac{a}{a+1}\right)^k$$

(show your work)

Solution: Since $a > 0$,we can say $0 < a < a+1$ and $0 < \frac{a}{a+1} < 1$, satisfying one condition of Rudin Theorem 3.26. Then we must reindex the summation in order to use the theorem:

$$\begin{aligned}
\sum_{k=2}^{\infty}\left(\frac{a}{a+1}\right)^k &= \sum_{k=1}^{\infty}\left(\frac{a}{a+1}\right)^k - \left(\frac{a}{a+1}\right) - 1 \\
&= \frac{1}{1-\frac{a}{a+1}} - \frac{a}{a+1} - 1 \\
&= \frac{1}{\frac{a+1}{a+1} - \frac{a}{a+1}} - \frac{a}{a+1} - 1 \\
&= \frac{1}{\frac{1}{a+1}} - \frac{a}{a+1} - 1 \\
&= a+1 - \frac{a}{a+1} - 1 \\
&= \frac{(a+1)^2}{a+1} - \frac{a}{a+1} - 1 \\
&= \frac{a^2+2a+1}{a+1} - \frac{a}{a+1} - \frac{a+1}{a+1} \\
&= \frac{a^2}{a+1}.
\end{aligned}$$

2. (a) Prove that $f(x) = \sin\left(x^2\right)$ is not uniformly continuous in $[0, \infty)$.

Proof. Choose $\varepsilon = 1$, and let $\delta > 0$. Then let $k \in \mathbb{Z}$, and $k > \frac{1}{\delta^2}$ which we can do by the Archimedian Property.

We attempt to choose x, y so that the function's value on one is 0, and on the other is ± 1. Then let $x^2 = k\pi$ for some $k \in \mathbb{N}$, and $y^2 = k\pi + \frac{\pi}{2}$, and our final choice is

$$y = \sqrt{k\pi + \frac{\pi}{2}}, \quad x = \sqrt{k\pi}.$$

Then regardless of our choice of k,

$$|f(x) - f(y)| = \left|\sin\left(\left(\sqrt{k\pi}\right)^2\right) - \sin\left(\left(\sqrt{k\pi + \frac{\pi}{2}}\right)^2\right)\right| = \left|\sin(k\pi) - \sin\left(k\pi + \frac{\pi}{2}\right)\right|.$$

If n is odd, then $|\sin(k\pi) - \sin\left(k\pi + \frac{\pi}{2}\right)| = |\pm 1 - 0| = 1$, and if k is even,
$|\sin(k\pi) - \sin\left(k\pi + \frac{\pi}{2}\right)| = |0 - \pm 1| = 1$.
We now have guaranteed that $|f(x) - f(y)| = 1$ for any k. It aids us to note that thanks to our choice of k, we can say that $\frac{1}{k} < \delta^2$ and $\frac{1}{\sqrt{k}} < \delta$. So then we proceed on $|y - x|$.

$$\begin{aligned}
|y - x| &= y - x && \text{Since } y > x \\
&= \sqrt{\pi k + \frac{\pi}{2}} - \sqrt{\pi k} \\
&= \frac{\left(\sqrt{\pi k + \frac{\pi}{2}} - \sqrt{\pi k}\right)\left(\sqrt{\pi k + \frac{\pi}{2}} + \sqrt{\pi k}\right)}{\left(\sqrt{\pi k + \frac{\pi}{2}} + \sqrt{\pi k}\right)} \\
&= \frac{\pi k + \frac{\pi}{2} - \pi k}{\left(\sqrt{\pi k + \frac{\pi}{2}} + \sqrt{\pi k}\right)} \\
&= \frac{\frac{\pi}{2}}{\sqrt{\pi k + \frac{\pi}{2}} + \sqrt{\pi k}} \\
&= \frac{\sqrt{\pi}}{2\left(\sqrt{k + \frac{1}{2}} + \sqrt{k}\right)} \\
&< \frac{\sqrt{\pi}}{2\left(\sqrt{k} + \sqrt{k}\right)} && \sqrt{x} \text{ is monotinically increasing} \\
&< \frac{\sqrt{\pi}}{4\sqrt{k}} \\
&< \frac{1}{\sqrt{k}} && \text{Since } \frac{\sqrt{\pi}}{4} < 1 \\
&< \delta.
\end{aligned}$$

Therefore $\sin x^2$ is not uniformly continuous on $[0, \infty)$

(b) Show an example of a continuous function in $(0, 1)$ which is not uniformly continuous (no proof necessary).

Solution: $f(x) = \sin\left(\frac{1}{x^2}\right)$ is continuous in $(0, 1)$ since it is the composition of continuous functions. However it is not uniformly continuous (as shown in class).