

1. Let $\phi : \mathbb{R}^n \rightarrow \mathbb{R}$ be such that $\phi(x) = 0 \Leftrightarrow x = 0$ and $\phi(\lambda x) = |\lambda|\phi(x), \forall x \in \mathbb{R}^n, \forall \lambda \in \mathbb{R}$. Show that if the set $B = \{x \in \mathbb{R}^n | \phi(x) \leq 1\}$ is convex, then ϕ defines a norm on \mathbb{R}^n .

Solution: Non-degeneracy and scalar linearity are given from the definition of ϕ . So all that is left to prove is the triangle inequality and non-negativity.

Non-negativity: For the sake of contradiction, suppose that there exists some $x \in \mathbb{R}^n$ so that $\phi(x) < 0$. Then let $n \in \mathbb{N}$. Take $\phi(nx) = n\phi(x) < 0 \leq 1$. So if we take the set $\{nx : n \in \mathbb{N}\}$, which is clearly unbounded, we see that it is contained in B . However this is a contradiction since B is bounded. So $\phi(x) > 0 \forall x \neq 0$.

Triangle inequality: Let $x, y \in \mathbb{R}^n$, and take $\lambda = \frac{\phi(x)}{\phi(x) + \phi(y)} \leq 1$. Note that $\phi\left(\frac{x}{\phi(x)}\right) = \frac{\phi(x)}{\phi(x)} = 1$, and likewise for y , so we have $\frac{x}{\phi(x)}, \frac{y}{\phi(y)} \in B$ and may take:

$$\begin{aligned} \phi\left(\lambda \frac{x}{\phi(x)} + (1-\lambda) \frac{y}{\phi(y)}\right) &= \phi\left(\frac{\lambda x}{\phi(x)} + \frac{(1-\lambda)y}{\phi(y)}\right) \\ &= \phi\left(\frac{\lambda x + (1-\lambda)y}{\phi(x) + \phi(y)}\right) \\ &= \frac{\phi(\lambda x + (1-\lambda)y)}{\phi(x) + \phi(y)} \\ &\leq 1 \end{aligned}$$

$\phi(x+y) \leq \phi(x) + \phi(y)$.

And so the triangle inequality is satisfied.

2. Let E be a compact set in \mathbb{R}^n and let F be a closed set in \mathbb{R}^n such that $E \cap F = \emptyset$.

(a) Show that there exists $d > 0$ such that $\|x - y\| > d, \forall x \in E$ and $\forall y \in F$.

Solution: Take $d = \inf_{x \in E, y \in F} \|x - y\|$. Clearly this is less than or equal to any $\|x - y\|$ for $x \in E, y \in F$, and it cannot be negative since the norm is positive. So then $d \geq 0$. For contradiction suppose $d = 0$.

Since this is an inf, we can find a sequence $\{x_n - y_n\}_{n \geq 1}$, with $x_i \in E, y_i \in F$. Since $\{x_n\}_{n \geq 1}$ is a sequence in the bounded set E , we can find a convergent subsequence, say $x_{n_k} \rightarrow x$, with $x \in E$ by closure of E . But since $\|x_{n_k} - y_{n_k}\| \rightarrow 0$, we must have $y_{n_k} \rightarrow x$, meaning $x \in F$, giving us the contradiction we sought ($E \cap F = \emptyset$). Therefore d is positive, and to ensure strict inequality, we simply take $d' = \frac{d}{2} < d \leq \|x - y\|$ for any $x \in E, y \in F$.

(b) Does the result you proved in the previous question remain true if E and F are closed, but neither is compact? Justify your answer.

Solution: This does not remain true. Take the sequence $\{e_n\}_{n \geq 2}$ given by $e_n = n$, and E as its image. Take $\{f_n\}_{n \geq 2} : f_n = e_n + \frac{1}{n}$ and F as its image (These sets contain only isolated points, and are closed). Then for any $d > 0$, we can pick $N \in \mathbb{N} : \frac{1}{N} < d$; and the points e_N and f_N will have $|e_N - f_N| = \frac{1}{N} < d$, meaning we can have arbitrarily close points between the closed sets E, F , and no such d can exist.

3. Let $E = \{(x, y) | y = \sin(\frac{1}{x}), x > 0\}$. Is E open? Is it closed? What are the accumulation points of E ?

Solution: This set is not open. Take an arbitrary ball of radius r about the point $p = (\frac{1}{\pi}, 0) \in E$. Then the point $q = (\frac{1}{\pi}, \frac{r}{2}) \in B_r(p)$, but $q \notin E$ since \sin is well-defined. So any ball about p contains points not in E , and E is not open.

Clearly each point of E is an accumulation point.

The accumulation points of E not contained in E are of the form $(0, a)$ for $a \in [-1, 1]$. Take one such point, and some $r > 0$, and consider the r -ball about $(0, a)$. Choose $k \in \mathbb{N}$ so that $\frac{1}{2\pi k} < r$, and let $x = \frac{1}{2\pi k + \arcsin a} \leq \frac{1}{2\pi k} < r$. Then:

$$\begin{aligned}\frac{1}{x} &= 2\pi k + \arcsin a \\ \frac{1}{x} - 2\pi k &= \arcsin a \\ \sin\left(\frac{1}{x} - 2\pi k\right) &= a \\ \sin\left(\frac{1}{x}\right) &= a.\end{aligned}$$

Then the point (x, a) is in E , and $\|(x, a) - (0, a)\| = \|(x, 0)\| = \sqrt{x^2} = x < r$, so x is in the arbitrary open ball we chose around $(0, a)$, and so every open ball around p contains a distinct point in E , and as such p is an accumulation point of E .

Clearly none of these accumulation points can be in E thanks to the condition $x > 0$, so E does not contain all its limit points and is not closed.

4. Let $f : \mathbb{R}^n \rightarrow \mathbb{R}$ be a function in $C^1(\mathbb{R}^n)$, i.e., $f, \partial_{x_1}f, \dots, \partial_{x_n}f$ are continuous in \mathbb{R}^n . Suppose $f(tx) = tf(x), \forall x \in \mathbb{R}^n, \forall t \in \mathbb{R}$. Show that f is a linear function.

Solution: Take the partial derivative with respect to x_i for some $1 \leq i \leq n$.

$$\begin{aligned}f(tx) &= tf(x) \\ \partial_i f(tx) &= \partial_i tf(x) \\ t f_{x_i}(tx) &= t f_{x_i}(x) \\ f_{x_i}(tx) &= f_{x_i}(x).\end{aligned}$$

But this must mean any partial f_{x_i} is constant, and combined with $f(0) = f(0x) = 0f(x) = 0$, we can write that:

$$f(x_1, \dots, x_n) = a_1 x_1 + \dots + a_n x_n.$$

For real constants a_1, \dots, a_n , so f is linear.

5. Given $u : \mathbb{R} \rightarrow \mathbb{R}$ a function in $C^2(\mathbb{R})$, define $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ by $f(x, y) = \begin{cases} \frac{u(y)-u(x)}{y-x} & \text{if } y \neq x \\ u'(x) & \text{if } y = x \end{cases}$
Show that f is differentiable at any point (a, a) .

Solution: Take the partial derivative with respect to y ,

$$\begin{aligned} \frac{\partial f}{\partial y}(a, a) &= \lim_{h \rightarrow 0} \frac{f(a, a+h) - f(a, a)}{h} \\ &= \lim_{h \rightarrow 0} \frac{\frac{u(a+h)-u(a)}{a+h-a} - u'(a)}{h} \\ &= \lim_{h \rightarrow 0} \frac{\frac{u(a+h)-u(a)}{h} - u'(a)}{h}. \end{aligned}$$

Before we get too far, let's do the same for x :

$$\begin{aligned} \frac{\partial f}{\partial x}(a, a) &= \lim_{h \rightarrow 0} \frac{f(a+h, a) - f(a, a)}{h} \\ &= \lim_{h \rightarrow 0} \frac{\frac{u(a)-u(a+h)}{a-a-h} - u'(a)}{h} \\ &= \lim_{h \rightarrow 0} \frac{\frac{u(a+h)-u(a)}{h} - u'(a)}{h}. \end{aligned}$$

And so now we have $\frac{\partial f}{\partial x}(a, a) = \frac{\partial f}{\partial y}(a, a)$

Now we apply l'Hopital in h , noticing that the first term on the numerator tends to $u'(a)$:

$$\lim_{h \rightarrow 0} \frac{\frac{u(a+h)-u(a)}{h} - u'(a)}{h} = \lim_{h \rightarrow 0} \frac{hu'(a+h) - u(a+h) + u(a)}{h^2}.$$

And again,

$$\begin{aligned} \lim_{h \rightarrow 0} \frac{hu'(a+h) - u(a+h) + u(a)}{h^2} &= \lim_{h \rightarrow 0} \frac{hu''(a+h) + u'(a+h) - u'(a+h)}{2h} \\ &= \lim_{h \rightarrow 0} \frac{u''(a+h)}{2} \\ &= \frac{u''(a)}{2}. \end{aligned}$$

Therefore:

$$\frac{\partial f}{\partial x}(a, a) = \frac{u''(a)}{2} = \frac{\partial f}{\partial y}(a, a).$$

And thanks to $u \in C^2(\mathbb{R})$, both our partials exist and are continuous at any (a, a) and therefore f is differentiable at any (a, a) .

6. Let $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ be a function that is defined in an open set Ω in \mathbb{R}^2 . Show that if $\partial_x f(x, y)$, $\partial_y f(x, y)$ and $\partial_{xy} f(x, y)$ are continuous in Ω , then $\partial_{yx} f(x, y)$ exists in Ω and we have $\partial_{yx} f(x, y) = \partial_{xy} f(x, y)$, $\forall (x, y) \in \Omega$ Hint: Consider the expression $\Delta(s, t) = f(a + s, b + t) - f(a + s, b) - f(a, b + t) + f(a, b)$.

Solution: Let $(x, y) \in \Omega$, with t, s real and small enough that $(x + s, y + t) \in \Omega$ and write $g(y) = f(x + s, y) - f(x, y)$. Apply the Mean Value Theorem in the interval $(y, y + t)$ to obtain some $\mu \in (0, 1)$ so that:

$$\begin{aligned} \Delta(s, t) &= (f(x + s, y + t) - f(x, y + t)) - (f(x, y) - f(x + s, y)) \\ &= g(y + t) - g(y) \\ &= \frac{d}{dy} g(y + \mu t)(y + t - y) \\ &= t \frac{\partial}{\partial y} (f(x + s, y + \mu t) - f(x, y + \mu t)). \end{aligned}$$

Now we take the function $h(x) = f_y(x, y + \mu t)$, and we use MVT again in conjunction with our above expression to find some $\tau \in (0, 1)$:

$$\begin{aligned} \Delta(s, t) &= t(f_y(x + s, y + \mu t) - f_y(x, y + \mu t)) \\ &= t(h(x + s) - h(x)) \\ &= t(x + s - x) \frac{d}{dx} (h(x + \tau s)) \\ &= ts \frac{d}{dx} (f_y(x + \tau s, y + \mu t)) \\ &= ts f_{yx}(x + \tau s, y + \mu t) \\ \frac{\Delta(s, t)}{st} &= f_{yx}(x + \tau s, y + \mu t). \end{aligned}$$

If we perform the same process on x before y , we will obtain some λ, θ so that:

$$\frac{\Delta(s, t)}{st} = f_{xy}(x + \theta s, y + \lambda t).$$

Then we simply take the limit:

$$f_{yx}(x, y) = \lim_{t, s \rightarrow 0} f_{yx}(x + \tau s, y + \mu t) = \lim_{t, s \rightarrow 0} \frac{\Delta(s, t)}{st} = \lim_{t, s \rightarrow 0} f_{xy}(x + \theta s, y + \lambda t) = f_{xy}(x, y).$$

7. Compute the degree 3 Taylor polynomial $T_3(x, x_2)$ of the function $f : \mathbb{R}^2 \rightarrow \mathbb{R}$, defined by $f(x_1, x_2) = \frac{4x_1 + 6x_2 - 1}{2x_1 + 3x_2}$ at the point $(-1, 1)$.

Solution: Begin by computing all necessary partials, and evaluating at $(-1, 1)$:

$f(x_1, x_2) = \frac{4x_1 + 6x_2 - 1}{2x_1 + 3x_2},$	$f(-1, 1) = 1$
$\frac{\partial f}{\partial x_1} = \frac{2}{(2x_1 + 3x_2)^2},$	$\frac{\partial f}{\partial x_1}(-1, 1) = 2$
$\frac{\partial f}{\partial x_2} = \frac{3}{(2x_1 + 3x_2)^2},$	$\frac{\partial f}{\partial x_2}(-1, 1) = 3$
$\frac{\partial^2 f}{\partial x_1 x_2} = \frac{-12}{(2x_1 + 3x_2)^3},$	$\frac{\partial^2 f}{\partial x_1 x_2}(-1, 1) = -12$
$\frac{\partial^2 f}{\partial x_1 x_1} = \frac{-8}{(2x_1 + 3x_2)^3},$	$\frac{\partial^2 f}{\partial x_1 x_1}(-1, 1) = -8$
$\frac{\partial^2 f}{\partial x_2 x_2} = \frac{-18}{(2x_1 + 3x_2)^3},$	$\frac{\partial^2 f}{\partial x_2 x_2}(-1, 1) = -18$
$\frac{\partial^3 f}{\partial x_1 x_1 x_1} = \frac{48}{(2x_1 + 3x_2)^4},$	$\frac{\partial^3 f}{\partial x_1 x_1 x_1}(-1, 1) = 48$
$\frac{\partial^3 f}{\partial x_2 x_2 x_2} = \frac{162}{(2x_1 + 3x_2)^4},$	$\frac{\partial^3 f}{\partial x_2 x_2 x_2}(-1, 1) = 162$
$\frac{\partial^3 f}{\partial x_1 x_2 x_2} = \frac{72}{(2x_1 + 3x_2)^4},$	$\frac{\partial^3 f}{\partial x_1 x_2 x_2}(-1, 1) = 72$
$\frac{\partial^3 f}{\partial x_1 x_1 x_2} = \frac{108}{(2x_1 + 3x_2)^4},$	$\frac{\partial^3 f}{\partial x_1 x_1 x_2}(-1, 1) = 108.$

Then begin expanding the first three terms of the Taylor expansion

$$\begin{aligned}
 f((-1, 1) + x) &= f(-1, 1) + \left(\sum_{k=1}^3 \frac{1}{k!} \left((x_1 + 1) \frac{\partial}{\partial x_1} + (x_2 - 1) \frac{\partial}{\partial x_2} \right)^k f(-1, 1) \right) \\
 &= 1 + \frac{1}{1!} \left((x_1 + 1) \frac{\partial}{\partial x_1} + (x_2 - 1) \frac{\partial}{\partial x_2} \right)^1 f(-1, 1) \\
 &\quad + \frac{1}{2!} \left((x_1 + 1) \frac{\partial}{\partial x_1} + (x_2 - 1) \frac{\partial}{\partial x_2} \right)^2 f(-1, 1) \\
 &\quad + \frac{1}{3!} \left((x_1 + 1) \frac{\partial}{\partial x_1} + (x_2 - 1) \frac{\partial}{\partial x_2} \right)^3 f(-1, 1) \\
 &= 1 + (x_1 + 1) \frac{\partial f}{\partial x_1}(-1, 1) + (x_2 - 1) \frac{\partial f}{\partial x_2}(-1, 1) \\
 &\quad + \frac{1}{2} \left((x_1 + 1)^2 \frac{\partial^2}{\partial x_1 x_1} + 2(x_1 + 1)(x_2 - 1) \frac{\partial^2}{\partial x_1 x_2} \right. \\
 &\quad \left. + (x_2 - 1)^2 \frac{\partial^2}{\partial x_2 x_2} \right) f(-1, 1) \\
 &\quad + \frac{1}{6} \left((x_1 + 1)^3 \frac{\partial^3}{\partial x_1 x_1 x_1} + 3(x_1 + 1)^2(x_2 - 1) \frac{\partial^3}{\partial x_1 x_1 x_2} \right. \\
 &\quad \left. + 3(x_1 + 1)(x_2 - 1)^2 \frac{\partial^3}{\partial x_1 x_2 x_2} + (x_2 - 1)^3 \frac{\partial^3}{\partial x_2 x_2 x_2} \right) f(-1, 1) \\
 &= 1 + 2(x_1 + 1) + 3(x_2 - 1) \\
 &\quad + \frac{1}{2} (-8(x_1 + 1)^2 - 24(x_1 + 1)(x_2 - 1) - 18(x_2 - 1)^2) \\
 &\quad + \frac{1}{6} (48(x_1 + 1)^3 + 3 \cdot 108(x_1 + 1)^2(x_2 - 1) \\
 &\quad + 3 \cdot 72(x_1 + 1)(x_2 - 1)^2 + 162(x_2 - 1)^3) \\
 &= 2x_1 - 3x_2 - 4(x_1 + 1)^2 - 12(x_1 + 1)(x_2 - 1) - 9(x_2 - 1)^2 \\
 &\quad + 8(x_1 + 1)^3 + 54(x_1 + 1)^2(x_2 - 1) \\
 &\quad + 36(x_1 + 1)(x_2 - 1)^2 + 27(x_2 - 1)^3.
 \end{aligned}$$