

1. Let $\{f_n\}$ be the sequence of functions defined by

$$f_n = \begin{cases} nx & \text{If } 0 \leq x \leq \frac{1}{n} \\ 1 & \text{If } \frac{1}{n} < x < 1 - \frac{1}{n} \\ n - nx & \text{If } 1 - \frac{1}{n} \leq x \leq 1 \end{cases}.$$

- (a) Find the pointwise limit f of the sequence.

Solution: Proceed by cases. If $x = 0$, then the first case of the function will always be taken since $0 \leq x$. So $f_n(0) = n \cdot 0 = 0$. Likewise if $x = 1$, then $f(1) = n - n \cdot 1 = n - n = 0$.

Now, if $x \in (0, 1)$, then we observe that $\frac{1}{n} \rightarrow 0$, and $1 - \frac{1}{n} \rightarrow 1$. Therefore the middle case of our piecewise function gives us $f(x) = 1$ for all x in this open interval.

- (b) Does $f_n \xrightarrow[c.u.]{{}_{[0,1]}} f$? Justify your answer.

Solution: This sequence is not uniformly convergent. Pick $\varepsilon = \frac{1}{3}$, and let $N \in \mathbb{N}$, and $n > N$. Pick $x = \frac{1}{2n}$ so that $0 \leq x \leq \frac{1}{n}$, and then $f_n(x) = nx = \frac{n}{2n} = \frac{1}{2}$. Then: $|f_n(x) - f(x)| = \left| \frac{1}{2} - 1 \right| = \frac{1}{2} > \varepsilon$.

Therefore the sequence is not uniformly convergent.

2. Let $f_n(x) = \left(\cos\left(\frac{2x}{n}\right) \right)^{n^2}$

- (a) Compute the pointwise limit f of the sequence $\{f_n\}$. **Hint:** Use the following double inequalities:

$$1 - \frac{1}{2}t^2 \leq \cos t \leq 1 - \frac{1}{2}t^2 + \frac{1}{24}t^4, \quad \forall t \in \mathbb{R}.$$

$$-t - t^2 \leq \ln(1 - t) \leq -t, \quad \forall t \in \left[0, \frac{1}{2}\right].$$

Solution: Begin with the first inequality.

$$\begin{aligned} 1 - \frac{4x^2}{2n^2} &\leq \cos\left(\frac{2x}{n}\right) \leq 1 - \frac{4x^2}{2n^2} + \frac{16x^4}{24n^4} \\ \ln\left(1 - \frac{2x^2}{n^2}\right) &\leq \ln\left(\cos\left(\frac{2x}{n}\right)\right) \leq \ln\left(1 - \left(\frac{2x^2}{n^2} - \frac{2x^4}{3n^4}\right)\right) && \ln \text{ is increasing in } \mathbb{R} \\ -\frac{2x^2}{n^2} - \frac{4x^4}{n^4} &\leq \ln\left(\cos\left(\frac{2x}{n}\right)\right) \leq -\frac{2x^2}{n^2} + \frac{2x^4}{3n^4} && \text{From the second inequality} \\ -2x^2 - \frac{4x^4}{n^2} &\leq n^2 \ln\left(\cos\left(\frac{2x}{n}\right)\right) \leq -2x^2 + \frac{2x^4}{3n^2} \\ -2x^2 - \frac{4x^4}{n^2} &\leq \ln\left(\left(\cos\left(\frac{2x}{n}\right)\right)^{n^2}\right) \leq -2x^2 + \frac{2x^4}{3n^2} \\ \exp\left(-2x^2 - \frac{4x^4}{n^2}\right) &\leq \left(\cos\left(\frac{2x}{n}\right)\right)^{n^2} \leq \exp\left(-2x^2 + \frac{2x^4}{3n^2}\right) && \exp \text{ is increasing in } \mathbb{R}. \end{aligned}$$

Intuitively, according to squeeze theorem, it appears that the limit will become $\exp(e^{-2x^2})$, however this idea needs some formalizing.

- (b) Show that $f_n \xrightarrow[c.u.]{{}_{[0,1]}} f$.

Lemma: If $f_n \xrightarrow[c.p]{[0,1]} f$, and $g : \mathbb{R} \rightarrow \mathbb{R}$ is uniformly continuous, then $g \circ f_n \xrightarrow[c.p]{[0,1]} g \circ f$.

Since f_n is uniformly convergent, it must be uniformly bounded, say that $f_n < M$ for all $n \in \mathbb{N}$ and $x \in [0, 1]$. Let $\varepsilon > 0$, and by uniform continuity of g , there exists some δ such that $|a - b| < \delta \implies |g(a) - g(b)| < \varepsilon$.

Now, take $N \in \mathbb{N}$ so that $n > N \implies |f_n(x) - f(x)| < \delta$.

Therefore $|g(f_n(x)) - g(f(x))| < \varepsilon$, and $g \circ f_n \xrightarrow[c.p]{[0,1]} g \circ f$.

Solution: The exponential is continuous, so it must be uniformly continuous on a compact set. Since $-2x^2 + \frac{2x^4}{3n^2} \xrightarrow[c.u]{[0,1]} -2x^2$, and $-2x^2 - \frac{4x^4}{n^2} \xrightarrow[c.u]{[0,1]} -2x^2$, then by the above lemma both the bounds found for f_n in part (a) must converge uniformly to $\exp(-2x^2)$.

Then by squeeze theorem:

$$f_n \xrightarrow[c.u]{[0,1]} e^{-2x^2}.$$

3. Let $a \in \mathbb{R}_+$. Compute the limit

$$\lim_{n \rightarrow \infty} \int_a^\pi \frac{\sin(nx)}{nx} dx.$$

What happens if $a = 0$?

Solution: We begin by considering our sequence of functions within the integral, each of which is a quotient and composition of continuous functions, and is itself continuous (for all but $x = 0$). Call this $g_n(x) = \frac{\sin(nx)}{nx}$. Note that since $-1 \leq \sin(nx) \leq 1$, we can find (for nonzero x) that $-\frac{1}{nx} \leq g_n(x) \leq \frac{1}{nx}$. Both the sequences bounding g have a zero limit at infinity, so by the squeeze theorem on sequences of functions, we can say that for nonzero x , $g_n \rightarrow 0$. Now since we have already shown that our sequence g_n is bounded, and since each g_n is integrable, we can say:

$$\begin{aligned} \lim_{n \rightarrow \infty} \int_a^\pi \frac{\sin(nx)}{nx} dx &= \int_a^\pi \lim_{n \rightarrow \infty} \frac{\sin(nx)}{nx} dx \\ &= \int_a^\pi 0 dx \\ &= 0 - 0 \\ &= 0. \end{aligned}$$

4. Construct a sequence of functions defined in $[0, 1]$, each of which is discontinuous at every point of $[0, 1]$ and which converges uniformly to a function that is continuous at every point

Solution: Take the series $\{f_n\}$ defined by:

$$f_n(x) = \begin{cases} \frac{1}{n} & \text{if } x \in \mathbb{Q} \\ 0 & \text{Otherwise} \end{cases}$$

Claim: $\{f_n\}$ converges uniformly to the constant function 0, which we know to be continuous on the real line, as well as $[0, 1]$.

Let $\varepsilon > 0$, and choose N such that $0 < \frac{1}{N} < \varepsilon$. Then any $n \geq N$ will have $0 < \frac{1}{n} < \frac{1}{N} < \varepsilon$. Now by cases, if $x \in \mathbb{Q}$, then we have

$$|f_n(x) - f(x)| = \left| \frac{1}{n} - 0 \right| = \frac{1}{n} < \frac{1}{N} < \varepsilon.$$

And for $x \notin \mathbb{Q}$,

$$|f_n(x) - f(x)| = |0 - 0| = 0 < \varepsilon.$$

Now we must show that each of these functions is continuous nowhere in $[0, 1]$. Suppose by way of contradiction that f_n is continuous at some $c \in [0, 1]$. Then for $\varepsilon = \frac{1}{n+1}$, there must be some δ such that if $|x - c| < \delta$, $|f_n(x) - f_n(c)| < \frac{1}{n+1}$. Take $B_\delta(c)$ the δ -ball about c , and proceed by cases on c .

$c \in \mathbb{Q}$: If c is rational, find some $d \notin \mathbb{Q}$ inside $B_\delta(c)$. Then we will have $f_n(c) = \frac{1}{n}$ and $f_n(d) = 0$

$c \notin \mathbb{Q}$: If c is irrational, find some $d \in \mathbb{Q}$ inside $B_\delta(c)$. Then we will have $f_n(d) = \frac{1}{n}$ and $f_n(c) = 0$

Regardless of case, we will get $|f_n(c) - f_n(d)| = \frac{1}{n} > \frac{1}{n+1}$, and we have found our contradiction.

Therefore $\{f_n\}$ is a sequence of functions which are continuous nowhere, convergent to the zero function which is continuous everywhere.

5. Consider the series of functions $\sum_{n \geq 1} \frac{x}{n(n+x)}$.

(a) Show that the series converges uniformly in the interval $[0, b]$ for any $b > 0$.

Solution:

$$\begin{aligned} \frac{x}{n(n+x)} &= \frac{x}{n^2 + nx} \\ &\leq \frac{x}{n^2} \\ &\leq \frac{b}{n^2}. \end{aligned}$$

Define $u_n = \frac{b}{n^2}$, then by the Weierstrass Comparison test, since $\sum_{n \geq 1} u_n$ is convergent as a p -series with $p = 2$, $\frac{x}{n(n+x)} \leq u_n$, this series must converge.

(b) Let $F(x) = \sum_{n \geq 1} \frac{x}{n(n+x)}$. Show that $F'(x) = \sum_{n \geq 1} \frac{1}{n(n+x)^2}$, $x \geq 0$.

Solution:

$$\begin{aligned} F'(x) &= \frac{d}{dx} \left(\sum_{k=1}^{\infty} \frac{x}{k(k+x)} \right) \\ &= \frac{d}{dx} \left(\lim_{n \rightarrow \infty} \sum_{k=1}^n \frac{x}{k(k+x)} \right) \\ &= \lim_{n \rightarrow \infty} \left(\frac{d}{dx} \sum_{k=1}^n \frac{x}{k(k+x)} \right) \\ &= \lim_{n \rightarrow \infty} \sum_{k=1}^n \frac{d}{dx} \frac{x}{k(k+x)} \\ &= \lim_{n \rightarrow \infty} \sum_{k=1}^n \frac{k(x+k) - kx}{k^2(k+x)^2} \\ &= \lim_{n \rightarrow \infty} \sum_{k=1}^n \frac{k}{k^2(k+x)^2} \\ &= \lim_{n \rightarrow \infty} \sum_{k=1}^n \frac{1}{k(k+x)^2} \\ &= \sum_{k=1}^{\infty} \frac{1}{k(k+x)^2}. \end{aligned}$$

6. Consider the series of functions $\sum_{n \geq 1} \frac{x}{1+n^2x^2}$. Show that the series doesn't converge uniformly in \mathbb{R}_+ .

Hint: You could start by showing that $\frac{x}{1+n^2x^2} \geq \int_n^{n+1} \frac{x}{1+t^2x^2} dt, \quad \forall x \in \mathbb{R}$.

Solution: Begin with the hint. If we take $P_0 = \{n, n+1\}$, the trivial partition on $[n, n+1]$, then we will have the upper sum:

$$U\left(P_0, \frac{x}{1+t^2x^2}\right) = \sum_{k=1}^1 \sup_{t \in [n, n+1]} \left(\frac{x}{1+t^2x^2} \right) ((n+1) - n) = \frac{x}{1+n^2x^2}.$$

But from the definition of the Riemann integral, we have (For σ the set of all partitions of $[n, n+1]$):

$$\begin{aligned} \int_n^{n+1} \frac{x}{1+t^2x^2} dt &= \inf_{P \in \sigma} U\left(P, \frac{x}{1+t^2x^2}\right) \\ &\leq U\left(P_0, \frac{x}{1+t^2x^2}\right) \\ &= \frac{x}{1+n^2x^2}. \end{aligned}$$

Suppose by way of contradiction that the series does converge uniformly. Then there exists some m such that, for any $x \in \mathbb{R}$

$$\left| \sum_{n=1}^{\infty} \frac{x}{1+n^2x^2} - \sum_{n=1}^m \frac{x}{1+n^2x^2} \right| < \frac{1}{2}.$$

However,

$$\begin{aligned} \left| \sum_{n=1}^{\infty} \frac{x}{1+n^2x^2} - \sum_{n=1}^m \frac{1}{1+n^2x^2} \right| &= \left| \sum_{n=m+1}^{\infty} \frac{x}{1+x^2n^2} \right| \\ &= \sum_{n=m+1}^{\infty} \frac{x}{1+x^2n^2} \\ &\geq \sum_{n=m+1}^{\infty} \int_n^{n+1} \frac{x}{1+t^2x^2} dt \\ &= \int_{m+1}^{\infty} \frac{x}{1+t^2x^2} dt && \text{Let } u = tx \\ &= \int_{(m+1)x}^{\infty} \frac{1}{1+u^2} du && du = xdt \\ &= (\arctan u)_{u=(m+1)x}^{\infty} \\ &= \frac{\pi}{2} - \arctan((m+1)x) \\ &= \frac{\pi}{2} - \arctan(1) && \text{Pick } x = \frac{1}{m+1} \\ &= \frac{\pi}{2} - \frac{\pi}{4} \\ &= \frac{\pi}{4} \\ &> \frac{1}{2}. \end{aligned}$$

A contradiction, so our series cannot converge uniformly.