Math 211 Quick reference

1 Vectors

Definition 1.1

The formula for projection is as follows:

$$proj_{\overrightarrow{v}}\overrightarrow{u} = \frac{\overrightarrow{u} \cdot \overrightarrow{v}}{\|\overrightarrow{u}\|}\overrightarrow{u}$$

Definition 1.2

To find the cross product of two vectors $\overrightarrow{u}$ and $\overrightarrow{b}$, we take the determinant of the matrix:

$$\overrightarrow{u} \times \overrightarrow{v} = \begin{vmatrix} i & j & k \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$$

Where i, j, k are columns of the identity matrix I_3.
This gives us a vector of the form

$$\overrightarrow{u} \times \overrightarrow{v} = \begin{bmatrix} \begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix}, & -\begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix}, & \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix} \end{bmatrix}^T$$

2 Linear Transformations

As far as this cheat sheet goes, the most important parts of this unit will be the matrices of tranformation for simple transfomations such as reflections in a line with slope m, or rotations by θ.

Definition 2.1

First, the matrix of rotation (counterclockwise by θ)

$$\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

Definition 2.2

Next, the matrix of reflection in a line with slope m:

$$\frac{1}{m^2+1}\begin{bmatrix} 1-m^2 & 2m \\ 2m & m^2-1 \end{bmatrix}$$

Definition 2.3

Finally, the matrix of projection onto the line $y = mx$ is:

$$\frac{1}{m^2+1}\begin{bmatrix} 1 & m \\ m & m^2 \end{bmatrix}$$

3 Complex Numbers

Complex numbers have fewer formulas than other units, so rather than provide any formulas (there are few), I'm including methods to convert between forms they can take, as well as procedures for different operations that can be taken on them

First, we note that the standard form for complex numbers is $a + ib$ We denote this number z, $z \in \mathbb{C}$, and $a, b \in \mathbb{R}$. Complex numbers in this form can be visualised on a cartesian plane, with the real component a plotted on the x-axis, and the complex component b on the y-axis.

Complex numbers can be imagined in a similar way on a cartesian plane, however if we consider polar form, our variables change from a, b to θ, r.

We can define these two variables using a, b though.

$$r = \sqrt{a^2 + b^2}$$

$$\cos\theta = \frac{a}{r}$$

$$\sin\theta = \frac{b}{r}$$

Think of r as a scalar multiple, the length of the vector from the origin to a,b. Then consider θ to be the angle between the x-axis and said vector.

Given all this, we have two ways to write z in terms of r and θ.

$$z = re^{i\theta}$$

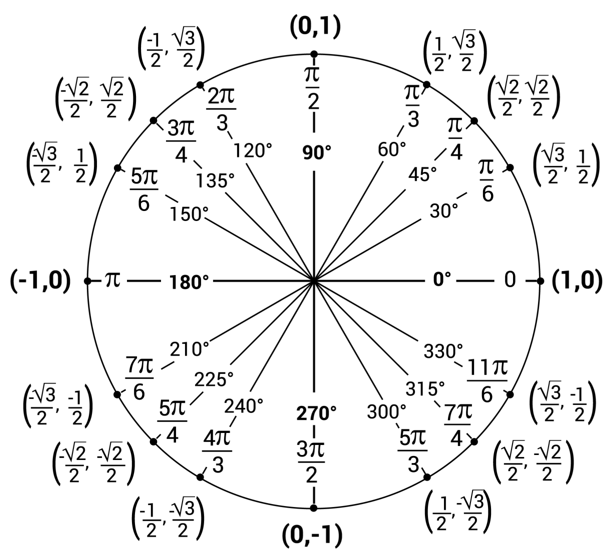
$$z = r(\cos\theta + i\sin\theta)$$

Finally, it's important to note that

$$r\cos\theta = a$$

$$ri\sin\theta = bi$$

Because of the numerous references I've already had to make to my unit circle, I'm including one here.



De Moivre's theorem is critical to this unit for finding roots of complex numbers. It is stated as follows:

$$\left(e^{i\theta}\right)^n = e^{in\theta}$$

When we take the k^{th} root of z, there are always k roots. These roots can be found through the following process:

Say we have a known complex number w and an unknown root of w, z.

$$z^k = w$$

$$w = se^{i\phi}$$

$$z = re^{i\theta}$$

$$r^k e^{ik\theta} = se^{i\phi}$$

Using this, we can write in general that:

$$r^k = s$$

However, we cannot write that $ik\theta = i\phi$. Since there must be k roots, we write that

$$n\theta = \phi + 2\pi l$$

$$\text{For } l = 0, 1, 2, \cdots, n-1.$$

To provide full solutions for roots like these, we combine our values θ, r with our polar form for z.

The quadratic equation can be used to find complex roots of some polyonomials. Simply use the the equation how you normally would. However, where you would have claimed there were no solutions in Math 20, you can now take the negative root and present the solution in the form of $a + bi$. In some cases, the value b will be imaginary or complex. Simply proceed as normal, the solution should simplify to a single complex number.

Here's the formula, for reference:

$$\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

4 Spectral Theory

An **Eigenvalue** is a value λ such that a $n \times n$ matrix multiplied by a vector X is equal to λX. An **Eigenvector** in this case, would be X.

To find eigenvalues of a matrix A, we solve the equation for the **characteristic polynomial**:

$$det(xI - A) = 0$$

To find the eigenvectors after we've found a λ, we can use the same equation, with the eigenvector X unknown.

$$(\lambda I - A)X = 0$$

Of course, a zero vector would multiply with any matrix to equal itself, but we consider this to be a trivial solution, not a true eigenvector. However, we can have an eigenvalue of 0, with the eigenvector being nonzero.

Similar matrices A and B are matrices such that a third invertible matrix P exists and:

$$A = PDP^{-1}$$

Assuming D is diagonal, it will be a matrix of size $n \times n$, where the only entries are the eigenvalues of A on the main diagonal.

For each eigenvalue λ_n in D, a column v_1 in P will be the eigenvector corresponding to λ_n.

Applications of Spectral Theory

One common application of spectral theory is in statistics. A **Markov Matrix** is one where the sum of the values on every column is equal to 1. Markov matrices can be used to represent the chance of differentchanges between a list of states. They can represent weather, population changes, and more.

Example 4.1

Consider a fox who lives in a forest with three regions, A, B, and C. The fox follows the following rules day-by-day.

- *The fox will never hunt in the same region twice in a row.*
- *The fox will always hunt in C after A.*
- *After hunting in B or C, the fox is twice as likely to pick A opposed to the other choice.*

What is the Markov Matrix for these criteria?

After we have found a Markov Matrix, we can use it to determine the long-term state of an environment. We can find this using the formula $A - I = 0$. This gives us a vector with a parameter, and we choose the parameter such that the entries of the vector sum to 1.