Q1

Let S be a subset (not necessarily a subspace) of a finite dimensional inner product space V. Show that $(S^{\perp})^{\perp} = \text{span } S$, where

$$\text{span } S := \left\{ \sum_{j=1}^{m} \alpha_j s_j : \alpha_j \in \mathbb{C}, s_j \in S, m \in \mathbb{N} \right\}$$

is the smallest subspace of V containing S (think of this as the set of all possible linear combinations of vectors from S).

Solution: Write $E = \operatorname{span} S$, and let $\{b_1, \ldots, b_n\}$ be a basis for $E^{\perp}$.

$$v \in E \iff .$$

Q2

Let V and W be finite dimensional inner product spaces and suppose $\ker A = \{0\}$. Find a left inverse for A in terms of A and A^*.

Solution: Begin with the identity,

$$\{0\} = \ker A = \ker A^* A.$$

So the composition of transformations $A^*A : V \to V$ has zero kernel and is injective, and by rank-nullity it must too surjective. Then this map is invertible, and if we take $(A^*A)^{-1}A^*A = I$, we see that $(A^*A)^{-1}A^*$ is a left inverse for A.

Q3

Let V be a finite dimensional inner product space.

(a) We can think of any $x \in V$ as a linear map from $\mathbb{C} \to V$ by setting $x(\lambda) := \lambda x$. You do not have to prove that this is linear. Show that $x^* : V \to \mathbb{C}$ satisfies

$$x^* y = \langle y, x \rangle.$$

Use this to deduce that the map xy^* is given by $xy^* v = \langle v, y \rangle x$. HINT: The inner product on $\mathbb{C}$ is assumed to be $\langle z, w \rangle = z\overline{w}$.

(b) Show that if $T : V \to \mathbb{C}$ is any linear map, then there is a vector y so that $T = y^*$.

Q4

Let V and W be finite dimensional vector spaces. You may find problem 3 useful here.

(a) Suppose $T : V \to W$ satisfies rank $T = 1$. Show that there are vectors $x \in W$ and $y \in V$ so that $T = xy^*$.

(b) Suppose $T : V \to W$ satisfies rank $T = k$. Show that T is the sum of k rank one operators. Hint: $PT = T$ where P is the orthogonal projection onto ran T.

Q5

Suppose that A and B are unitarily equivalent $n \times n$ matrices. That is, there is a unitary matrix U so that $U^*AU = B$. Show that E is an invariant subspace for B if and only if UE is invariant for A. Recall that a subspace E of V is invariant for T if $Tv \in E$ for all $v \in E$.