Problem Set 2 - Thomas Boyko - 30191728

1. Consider the subset of $\mathbb{R}$ defined by

$$\mathbb{Q}(\sqrt{2}) = \{a + \sqrt{2}b : a, b \in \mathbb{Q}\},$$

with the usual addition and multiplication. Show that this is a field.

Solution Begin with the axioms for addition. For all the following, let $a + b\sqrt{2}, c + d\sqrt{2} \in \mathbb{Q}(\sqrt{2})$. (

(i) Closure: $a + b\sqrt{2} + c + d\sqrt{2} = (a + c) + (b + d)\sqrt{2}$, and since $a + c, b + d \in \mathbb{Q}$, $\mathbb{Q}(\sqrt{2})$ is closed under $+$.

(ii) Identity: Clearly $0 = 0 + 0\sqrt{2}$ is identity.

(iii) Commutativity: $a + b\sqrt{2} + c + d\sqrt{2} = (a + b) + (c + d)\sqrt{2}$ by commutativity of addition in $\mathbb{R}$.

(iv) Inverses: $a + b\sqrt{2} + (-a - b\sqrt{2}) = 0$

Then the multiplication axioms:

(i) Closure: $(a + b\sqrt{2})(c + d\sqrt{2}) = ab + ad\sqrt{2} + bc\sqrt{2} + 2bd = (ab + 2bd) + (ad + bc)\sqrt{2}$

(ii) Identity: we inherit $1 = 1 + 0\sqrt{2}$, the identity from $\mathbb{R}$.

(iii) Commutativity: Follows from commutativity of addition and multiplication in $\mathbb{R}$.

(iv) Inverses: $\frac{1}{a+b\sqrt{2}} = \frac{a-b\sqrt{2}}{a^2-2b^2} = \frac{a}{a^2-2b^2} + \frac{-b}{a^2-2b^2}\sqrt{2}$. Clearly this is a multiplicative inverse for $a+b\sqrt{2}$.

2. If z is a complex number prove that there exists a unique $r \geq 0$ and a complex $|w| = 1$ so that $z = rw$.

Proof. Let $r = |z|$. We know already that this is real and ≥ 0. Then let $w = \frac{z}{|z|}$. So clearly this is complex and $z = rw$. Now we must show uniqueness of these two variables.

TODO is this the right process? Suppose $z = z' = x + iy$, then clearly $|z| = \sqrt{x^2 + y^2} = |z'|$, so r is unique, and since r is unique w is.????

3. Let E° be the set of all interior points for a set E; E° is the *interior* of E. Prove:

(a) E° is open

Proof. E° is clearly open, every point must be an interior point.

(b) E is open $\iff$ $E^\circ = E$.

Proof. $\implies$: Suppose E is open. Then every point of E is an interior point. By definition, $E^\circ \subseteq E$ Take $p \in E$. Since E is open, p is an interior point and must be in E°. So $E = E^\circ$.
$\impliedby$: Easy; E° is open, so if $E^\circ = E$, then E must be open.

(c) If $G \subseteq E$ and G is open, $G \subseteq E^\circ$.

Proof. Let $G \subseteq E$ be open, and take $g \in G$. g must be an interior point of G, which is a subset of E. Therefore g is an interior point in E; and $g \in E^\circ$. So $G \subseteq E^\circ$.

(d) The complement of E° is the closure of the complement of E.

Proof. The complement of E° is the set of all points in X which are not interior points of E. So a open ball about any point x in $(E^\circ)^c$ contains some point not in E. So either x is a limit point of E, or x is not in E. And we have described all points in the closure of E^c.

(e) Do E and E° have the same interiors?
Yes, any interior point of E^{circ} is an interior point of E and vice versa.

(f) Do E and E° have the same closures?
Yes, any closure point of E^{circ} is an closure point of E and vice versa.

4. Prove that every open set in $\mathbb{R}$ is the union of at most countable collection of disjoint segments. (Use ex.22)