1. A sequence $\{a_n\}_{n=1}^{\infty}$ of real numbers is said to be Cesàro convergent or Cesàro summable if the sequence of means

$$\alpha_n = \frac{1}{n}\sum_{k=1}^{n} a_k$$

converges in $\mathbb{R}$.

(a) Prove that if a_n is convergent to a limit $a \in \mathbb{R}$, then α_n also converges to a.

Proof. Let $\varepsilon > 0$, since a_n is convergent to a, there exists some $N \in \mathbb{N}$ so that for any $n > \mathbb{N}$, $|a_n - a| < \varepsilon'$ (We choose ε below).
Consider:

$$\begin{aligned}
|\alpha_n - a| &= \left|\sum_{k=1}^{n}\frac{a_k}{n} - a\right| \\
&= \frac{1}{n}\left|\sum_{k=1}^{n} a_k - an\right| \quad \text{Since } n \text{ is positive, } \frac{1}{n} \text{ must also be} \\
&= \frac{1}{n}\left|\sum_{k=1}^{n}(a_k - a)\right| \\
&\leq \frac{1}{n}\sum_{k=1}^{n}|a_k - a| \qquad \text{By the triangle inequality} = \frac{1}{n}\left(\sum_{k=1}^{N-1}|a_k - a| + \sum_{k=N}^{n}|a_k - a|\right)
\end{aligned}$$

.

Since a_n is convergent, $|a_n| < M$, it is bounded, and so $|a| < M$. Then $|a_n - a| < 2M$ for any n. We use this, choose $\varepsilon' = \max\left\{\frac{n}{n-N+1}\left(\varepsilon - \frac{2(N-1)M}{n}, 1 - \frac{2(N-1)M}{n}\right)\right\}$, so that $\varepsilon' > 0$ and $\varepsilon' \leq \varepsilon - \frac{2(N-1)M}{n}$. Now we have

$$\begin{aligned}
\frac{1}{n}\left(\sum_{k=1}^{N-1}|a_k - a| + \sum_{k=N}^{n}|a_k - a|\right) &\leq \frac{2(N-1)M}{n} + \varepsilon'\frac{n-N+1}{n} \\
&\leq \frac{2(N-1)M}{n} + \frac{n}{n-N+1}\left(\varepsilon - \frac{2(N-1)M}{n}\right) \\
&= \frac{2(N-1)M}{n} + \varepsilon - \frac{2(N-1)M}{n} \\
&= \varepsilon.
\end{aligned}$$

And $|\alpha_n - a| < \varepsilon$ as desired.

(b) Show an example (no need for proof) that the converse is not true. That is, provide a sequence $\{a_n\}$ such that $\{\alpha_n\}$ converges, but $\{a_n\}$ diverges.

Solution: Take the sequence $\{(-1)^n\}_{n=1}^{\infty}$, which does not converge ($\liminf a_n = -1 \neq 1 = \limsup a_n$. Then our sum becomes $\sum_{k=1}^{\infty}\frac{(-1)^n}{k}$. We know this to be the alternating harmonic series, a convergent series.

2. Let $\{u_n\}_{n=1}^{\infty}$ be a sequence of positive real numbers. We define the sequence of products

$$p_1 = u_1, \quad p_2 = u_1 u_2, \quad p_n = u_1 u_2 \cdots u_n = \prod_{k=1}^{n} u_k.$$

We say that the infinite product $\prod_{k=1}^{\infty} u_k$ is convergent if and only if there exists a positive real number p such that $\lim_{n\to\infty} p_n = p$.

(a) Prove that if $\{u_n\}_{n=1}^{\infty}$ is a sequence of positive real numbers such that

$$\text{for all } \varepsilon > 0 \quad \exists N \in \mathbb{N} : m \geq n \geq N \quad \Longrightarrow \quad \left|\prod_{k=n}^{m} u_k - 1\right| < \varepsilon,$$

then $\prod u_k$ is convergent. This is called the *Cauchy condition for products*.

Proof. Let N_1 be such that if $m \geq n \geq N_1$, then $\left|\prod_{k=n}^{m} u_k - 1\right| < 1$.
Then let N_2 be such that if $m \geq n \geq N_2$, then $\left|\prod_{k=n}^{m} u_k - 1\right| < \varepsilon$.
Take $N = \max\{N_1, N_2\}$

$$\begin{aligned}
\left|\prod_{k=N}^{m} u_k\right| &= \left|\prod_{k=N}^{m} u_k + 1 - 1\right| \\
&\leq \left|\prod_{k=N}^{m} u_k - 1\right| + 1 && \text{By the triangle inequality} \\
&< \varepsilon + 1 = 2 && \text{By the Cauchy condition.}
\end{aligned}$$

Now we use this to find an upper bound for partial products.

$$\begin{aligned}
\left|\prod_{k=1}^{m} u_k\right| &= \left|\frac{\prod_{k=N}^{m} u_k}{\prod_{k=1}^{N-1} u_k}\right| \\
&= \frac{\prod_{k=N}^{m} u_k}{\prod_{k=1}^{N-1} u_k} && \text{Since each term is positive, partial products must be positive.} \\
&< \frac{2}{\prod_{k=1}^{N-1} u_k} && \text{See above :).}
\end{aligned}$$

Then let $M = \prod_{k=1}^{N-1} u_k$, which is dependent on neither m, n, so our above inequality becomes $\frac{\prod_{k=N}^{m} u_k}{M} < 2$, and we get that $\prod_{k=N}^{m} u_k < 2M$.
By the Cauchy condition,

$$\left|\prod_{k=n}^{m} u_k - 1\right| < \frac{\varepsilon}{M}.$$

Combining this inequality with our above boundedness statement,

$$|p_n - p_m| = \left|\prod_{k=1}^{n} u_k + \prod_{k=1}^{m} u_k\right| = \left|\prod_{k=n}^{m} u_k - 1\right| \prod_{k=1}^{n-1} u_k < \frac{\varepsilon M}{M} = \varepsilon.$$

So we have shown that p_n is a Cauchy sequence, and since it is a real sequence, it must converge thanks to the completeness of the reals.

(b) Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of nonnegative real numbers, and let $u_n = 1 + a_n$. **Prove that if** $\sum a_n$ is convergent then $\prod u_n$ is convergent . $Hint$: Apply the inequality $\ln(1+x) \leq x$, valid for all $x > -1$. Prove that the partial products are increasing and bounded. You can use that $e^x \to 1$ as $x \to 0$ (only if needed).

Proof. Suppose that $\sum_{k=1}^{\infty} a_k$ of nonnegative terms is convergent to $a \in \mathbb{R}$.
Since each a_i is bounded below by zero, $u_i = a_i + 1$ is bounded below by 1.
Then $u_i > 1 \implies u_{i+1} \prod_{k=1}^{i} u_k = \prod_{k=1}^{i+1} u_k > \prod_{k=1}^{i} u_k$, so the partial products are increasing.
Likewise, $a_i > 0 \implies a_{i+1} + \sum_{k=1}^{i} a_k = \sum_{k=1}^{i+1} a_k > \sum_{k=1}^{i} a_i$ so our partial sums are increasing.
As a result, since $\sum a_k$ is convergent, we know that it must be bounded. So there exists some M so that $0 < \sum_{k=1}^{i} a_k < M$ for any i.

$$\begin{aligned}
\prod_{k=1}^{n} u_k &= \exp\left(\ln\left(\prod_{k=1}^{n} u_n\right)\right) \\
&= \exp\left(\sum_{k=1}^{n} \ln(a_n + 1)\right) \\
&\leq \exp\left(\sum_{k=1}^{n} a_n\right) && e^x \text{ is increasing, and } \ln(1+x) \leq x \\
&< \exp(M) && \text{Again since } e^x \text{ is increasing.}
\end{aligned}$$

So any partial product is less than e^M, which is finite since M is finite. and since the sequence of partial products is increasing and bounded, it must be convergent.