1 The Spectrum and its Topology

Definition 1.1 (*Spectrum of a Ring*)

Let R be a ring. Define the spectrum of R,

$$Spec(R) = \{\mathfrak{p} \trianglelefteq R : \mathfrak{p} \text{ is prime}\}.$$

Definition 1.2

Upon $Spec(R)$ we define a topology by:

$$\mathcal{T} = V(I) = \{\mathfrak{p} \in Spec(R) : I \subseteq \mathfrak{p}\}.$$

Proposition 1.1

$V(I)$ induces a topology on $Spec(R)$, define:

$$\mathcal{T} = \{V(I) : I \trianglelefteq R\}.$$

1. $\varnothing, Spec(R) \in \mathcal{T}$
2. $\bigcup_{i=1}^{n} V(I_i) \in \mathcal{T}$
3. $\bigcap_{\alpha} V(I_i) \in \mathcal{T}$

Proof:

1. Take $V(\{0\}) = \{\mathfrak{p} \trianglelefteq R \text{ prime} : \{0\} \subseteq \mathfrak{p}\}$. But naturally, the additive identity is contained within every prime ideal (which must be an abelian group w.r.t +), so we have $V(\{0\}) = Spec(R)$, and $Spec(R) \in \mathcal{T}$

   Now take $V(R) = \{\mathfrak{p} \trianglelefteq R \text{ prime} : R \subseteq \mathfrak{p}\}$. But by definition, no prime ideal can contain R, so $V(R) = \varnothing$ and $\varnothing \in \mathcal{T}$
2. It is sufficient to show that $V(I) \cup V(J) \in \mathcal{T}$ for any $V(I), V(J) \in \mathcal{T}$. Any finite union can be proven inductively using this result.
3. Take ■