Assignment 1 - Thomas Boyko - 30191728

1. Given $x \in \mathbb{R}$ define the set of rational numbers $C_x = \{r \in \mathbb{Q} : r < x\}$. Prove that

$$x = \sup C_x.$$

More precisely, prove that

(a) C_x is non-empty and bounded above, and x is an upper bound of C_x.

Proof. Take $x - 1 < x$,so that $x \in C_x$. Then by the density of the rationals in the reals (Rudin theorem 1.20), there must exist some $q \in \mathbb{Q}$ so that $x - 1 < q < x$. So $q \in C_x$ and C_x is nonempty.
Of course x is an upper bound of C_x since for any $r \in C_x$, $r < x$ by definition of C_x.

□

(b) x is the least upper bound of C_x.

Proof. Suppose by way of contradiction that $y < x$ is the least upper bound for C_x. By theorem 1.20 from Rudin, the density of $\mathbb{Q}$ in $\mathbb{R}$, we know that since $y < x$, there must exist some $q \in \mathbb{Q}$, where $y < q < x$. But $q < x$ so $q \in C_x$, and since $q > y$, y cannot be an upper bound. And we have found our contradiction. So x is the least upper bound for C_x. □

2. A sequence of rational numbers $\{r_j\}_{j=1}^{\infty} = \{r_1, r_2, r_3, \ldots\}$ is said to be a Cauchy sequence if given any $n \in \mathbb{N}$ there exists $N \in \mathbb{N}$ such that if $j, k \geq N$ then $|x_j - x_k| < \frac{1}{n}$.
Let C denote the set of all Cauchy sequences in $\mathbb{Q}$, if r= $\{r_j\}_{j=1}^{\infty}$ and $\mathbf{q} = \{q_k\}_{k=1}^{\infty}$ are in C, we say that $\mathbf{r}$ is equivalent to $\mathbf{q}$ and write $\mathbf{r} \sim \mathbf{q}$ if the sequence $\mathbf{r} - \mathbf{q} = \{r_j - q_j\}_{j=1}^{\infty}$ converges to zero, that is, if for every $n \in \mathbb{N}$, there exists $N \in \mathbb{N}$ such that

$$j \geq N \implies |r_j - q_j| < \frac{1}{n}.$$

Prove that $\sim$ is an equivalence relation in C. That is, prove that the relation is Reflexive, Symmetric, Transitive.

Proof. (a) Reflexive:
Let $\mathbf{r}$ be a Cauchy sequence, and $n \in \mathbb{N}$. Then for any $j \geq n$, $|r_j - r_j| = 0 < 1/n$ since $n > 0$. $\mathbf{r} \sim \mathbf{r}$ and $\sim$ is reflexive.

(b) Symmetric:
Let $\mathbf{r} \sim \mathbf{q}$ be Cauchy sequences. Then for any $n \in \mathbb{N}$, there exists some $N \in \mathbb{N}$ such that $j \geq N \implies |r_j - q_j| < \frac{1}{n}$.
But $|r_j - q_j| = |-(r_j - q_j)| = |q_j - r_j| < \frac{1}{n}$
So $\mathbf{q} \sim \mathbf{r}$ and $\sim$ is symmetric.

(c) Transitive
Take $n \in \mathbb{N}$. Let $\mathbf{r} \sim \mathbf{q}, \mathbf{q} \sim \mathbf{s}$. Then for $2n \in \mathbb{N}$, there exists some $N_1, N_2 \in \mathbb{N}$, so that (Letting $N = \max\{N_1, N_2\}$ so that $N \geq N_1$ and $N \geq N_2$), we can write:

$$j \geq N \implies |r_j - q_j| < \frac{1}{2n}$$
$$j \geq N \implies |q_j - s_j| < \frac{1}{2n}.$$

And by the triangle inequality,

$$|r_j - s_j| = |(r_j - q_j) + (q_- s_j)| \leq |(r_j - q_j)| + |(q_- s_j)| < \frac{1}{2n} + \frac{1}{2n} = \frac{2}{2n} = \frac{1}{n}.$$

In particular,$|r_j - s_j| < \frac{1}{n}$ and $\mathbf{r} \sim \mathbf{s}$, hence $\sim$ is transitive.

□