1. Let V and W be finite dimensional vector spaces with given bases $\mathcal{B} = \{\vec{b}_1, \ldots, \vec{b}_n\}$ and $\mathcal{D} = \{\vec{d}_1, \ldots, \vec{d}_m\}$, respectively.

   (a) For a given $\vec{x} \in V$, there are unique scalars so that $\vec{x} = a_1\vec{b}_1 + \cdots + a_n\vec{b}_n$. Define the vector $[\vec{x}]_{\mathcal{B}} := (a_1, \ldots, a_n)^T \in \mathbb{C}^n$. Show that the map $\vec{x} \mapsto [\vec{x}]_{\mathcal{B}}$ is a linear isomorphism from V into $\mathbb{C}^n$.

   Linearity: Let $\vec{x}, \vec{y} \in V$. Then write $\vec{x} = a_1\vec{b}_1 + \cdots + a_n\vec{b}_n$ and $\vec{y} = c_1\vec{b}_1 + \cdots + c_n\vec{b}_n$. Now:

$$[\vec{x} + \vec{y}]_{\mathcal{B}} = \begin{bmatrix} a_1 + c_1 \\ \vdots \\ a_n + c_n \end{bmatrix} = \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} + \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} = [\vec{x}]_{\mathcal{B}} + [\vec{y}]_{\mathcal{B}}$$

$$[\alpha\vec{x}]_{\mathcal{B}} = \begin{bmatrix} \alpha a_1 \\ \vdots \\ \alpha a_n \end{bmatrix} = \alpha \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} = \alpha[\vec{x}]_{\mathcal{B}}.$$

   So $[\cdot]_{\mathcal{B}}$ is linear.

   Isomorphism: Since $\dim V = \dim \mathbb{C}^n = n$, it will suffice to show that this mapping is injective. We do so by showing $\ker[\cdot]_{\mathcal{B}} = \{0\}$. Clearly 0 is in the kernel since $[0]_{\mathcal{B}} = [0\vec{x}]_{\mathcal{B}} = 0[\vec{x}]_{\mathcal{B}} = 0$. For inclusion the other way, let $\vec{x} \in \ker[\cdot]_{\mathcal{B}}$. Then $[\vec{x}]_{\mathcal{B}} = 0$; meaning the basis representation of $\vec{x}$ is through zero coefficients; and

$$\vec{x} = 0\vec{b}_1 + \ldots + 0\vec{b}_n = 0.$$

   So $\ker[\cdot]_{\mathcal{B}} = \{0\}$, and this map is injective. But since the spaces are of the same dimension it must also be surjective thanks to Rank-Nullity. So the map is a linear isomorphism from V to $\mathbb{C}^n$.

   (b) Let $T : V \to W$ be a linear map. In class, we defined the matrix representation of T with respect to $\mathcal{B}$ and $\mathcal{D}$ as the $m \times n$ matrix $[T]_{\mathcal{BD}} = [[T\vec{b}_1]_{\mathcal{D}}, \ldots, [T\vec{b}_n]_{\mathcal{D}}]$. In other words, the j-the column of $[T]_{\mathcal{BD}}$ is $[T\vec{b}_j]_{\mathcal{D}}$. Show that $[T]_{\mathcal{BD}}[\vec{x}]_{\mathcal{B}} = [T\vec{x}]_{\mathcal{D}}$ for any $\vec{x} \in V$.

   Solution: Let $T : V \to W$ be linear, then write $\vec{x} = a_1\vec{b}_1 + \ldots + a_n\vec{b}_n$.

$$\begin{aligned}
[T]_{\mathcal{BD}}[\vec{x}]_{\mathcal{B}} &= \big[[T\vec{b}_1]_{\mathcal{D}} \ldots [T\vec{b}_n]_{\mathcal{D}}\big] \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} \\
&= a_1[T\vec{b}_1]_{\mathcal{D}} + \ldots + a_n[T\vec{b}_n]_{\mathcal{D}} \\
&= [a_1T\vec{b}_1 + \ldots + a_nT\vec{b}_n]_{\mathcal{D}} && \text{By linearity of } [\cdot]_{\mathcal{D}} \\
&= \big[T(a_1\vec{b}_1 + \ldots + a_n\vec{b}_n)\big]_{\mathcal{D}} && \text{By linearity of } T \\
&= [T\vec{x}]_{\mathcal{D}}.
\end{aligned}$$

(c) Show that $[T]_{\mathcal{BD}}$ is a linear isomorphism from $L(V,W)$ (the vector space of linear maps from V to W) to $M_{mn}(\mathbb{C})$ (vector space of $m \times n$ complex matrices).

Linearity: Let T, S be linear from V to W. Then:

$$\begin{aligned}
[T+S]_{\mathcal{BD}} &= \left[[(T+S)\vec{b}_1]_{\mathcal{D}} \dots [(T+S)\vec{b}_n]_{\mathcal{D}}\right] \\
&= \left[[(T\vec{b}_1 + S\vec{b}_1)]_{\mathcal{D}} \dots [(T\vec{b}_n + S\vec{b}_n)]_{\mathcal{D}}\right] \\
&= \left[[T\vec{b}_1]_{\mathcal{D}} \dots [T\vec{b}_n]_{\mathcal{D}}\right] + \left[[S\vec{b}_1]_{\mathcal{D}} \dots [S\vec{b}_n]_{\mathcal{D}}\right] \quad \text{By Linearity of } [\cdot]_{\mathcal{D}} \\
&= [S]_{\mathcal{BD}} + [T]_{\mathcal{BD}}.
\end{aligned}$$

And then letting $\alpha \in \mathbb{C}$,

$$\begin{aligned}
\alpha[T]_{\mathcal{BD}} &= \alpha\left[[T\vec{b}_1]_{\mathcal{D}} \dots [T\vec{b}_n]_{\mathcal{D}}\right] \\
&= \left[\alpha[T\vec{b}_1]_{\mathcal{D}} \dots \alpha[T\vec{b}_n]_{\mathcal{D}}\right] \qquad \text{Linearity of } [\cdot]_{\mathcal{D}} \\
&= \left[[\alpha T\vec{b}_1]_{\mathcal{D}} \dots [\alpha T\vec{b}_n]_{\mathcal{D}}\right] \\
&= [\alpha T]_{\mathcal{BD}}.
\end{aligned}$$

Injective: Clearly $0 \in \ker[\cdot]_{\mathcal{BD}}$; take any transformation T and $[0]_{\mathcal{BD}} = [T - T]_{\mathcal{BD}} = [T]_{\mathcal{BD}} - [T]_{\mathcal{BD}} = 0$.
Then let $\mathcal{B} = \{b_1, \dots, b_n\}$ and $T \in \ker[\cdot]_{\mathcal{BD}}$. Then:

$$\begin{gathered}
[T]_{\mathcal{BD}} = 0 \\
[[Tb_1]_{\mathcal{D}} \dots [Tb_n]_{\mathcal{D}}] = [0 \dots 0].
\end{gathered}$$

Then $[Tb_i]_{\mathcal{D}} = 0$ for any basis vector b_i. In particular this means that $TB_i = 0$, since $[\cdot]_{\mathcal{D}}$ is an isomorphism. Now for any arbitrary $v \in V$, write $v = a_1 b_1 + \dots a_n b_n$. Then $Tv = T(a_1 b_1 + \dots a_n b_n) = a_1 T b_1 + \dots + a_n T b_n = 0 + \dots + 0 = 0$ and $T = 0$.
Therefore $\ker[\cdot]_{\mathcal{BD}} = \{0\}$.

Surjective: The argument that $\dim L(V,W) = \dim M_{mn}(\mathbb{C})$ proves difficult, so instead we show directly that $[L(V,W)]_{BD} = M_{mn}(\mathbb{C})$.
Let $A \in M_{mn}(\mathbb{C})$, and write $A = \left[\vec{a}_1 \dots \vec{a}_n\right]$, where $\vec{a}_j$ are column vectors in $\mathbb{C}^m$. Then take the inverse map for $[\cdot]_D$ (which was shown to exist in 1(a)), denote it $[\cdot]_D^{-1}$ and define T on the basis vectors in $\mathcal{B}$ such that $Tb_j = [\vec{a}_j]_{\mathcal{D}}^{-1}$. Then:

$$\begin{aligned}
[T]_{\mathcal{BD}} &= [[Tb_1]_{\mathcal{D}} \dots [Tb_n]_{\mathcal{D}}] \\
&= \left[\left[[\vec{a}_1]_{\mathcal{D}}^{-1}\right]_{\mathcal{D}} \dots \left[[\vec{a}_n]_{\mathcal{D}}^{-1}\right]_{\mathcal{D}}\right] \\
&= \left[\vec{a}_1 \dots \vec{a}_n\right] \\
&= A.
\end{aligned}$$

So every arbitrary matrix has a preimage in the space of linear transformations, and therefore the mapping is onto. Since $[\cdot]_{\mathcal{BD}}$ is bijective and linear, it must then be an isomorphism.

2. Let V, W and U be finite dimensional vector spaces with given bases: $\mathcal{B} = \{\vec{b}_1, \ldots, \vec{b}_n\}, \mathcal{D} = \{\vec{d}_1, \ldots, \vec{d}_m\}$, and $\mathcal{F} = \{f_1, \ldots, f_k\}$, respectively. Suppose $T : V \to W$ and $S : W \to U$ are linear. Prove or disprove the following statement for the composition linear map $ST : V \to U$:

$$[ST]_{\mathcal{BF}} = [S]_{\mathcal{DF}}[T]_{\mathcal{BD}}.$$

Solution: We make great use of the property shown in 1(b). Where it is used will be marked with **(∗)**. Let $v \in V$ be arbitrary and recall that $[v]_{\mathcal{B}}$ is unique since $[\cdot]_{\mathcal{B}}$ is an isomorphism.

$$\begin{aligned} [ST]_{\mathcal{BF}}[v]_{\mathcal{B}} &= [STv]_{\mathcal{F}} && (*) \\ &= [S]_{\mathcal{DF}}[Tv]_{\mathcal{D}} && (*) \\ &= [S]_{\mathcal{DF}}[T]_{\mathcal{BD}}[v]_{\mathcal{B}} && (*) \end{aligned}$$

So we have shown that these matrices $[ST]_{\mathcal{BF}}$ and $[S]_{\mathcal{DF}}[T]_{\mathcal{BD}}$ agree upon all vectors in the image of $[\cdot]_{\mathcal{B}}$. However since this particular mapping is onto, we know this to be all of $\mathbb{C}^n$. This means the matrices agree upon all of $\mathbb{C}^n$ and therefore they must be equal.

3. Let V be a finite dimensional vector space and $T : V \to V$ be linear. Show that $\sigma(T) = \sigma([T]_{\mathcal{B}})$ where $\mathcal{B}$ is any basis for V.

Linearity of inverse: To show this we use the fact that $[\cdot]_{\mathcal{B}}^{-1}$ is linear from $L(V, V)$ to $M_{mn}(\mathbb{C})$. Included is a brief demonstration of this fact. Let $T, S \in L(V, V)$ and $\alpha \in \mathbb{C}$.

$$\begin{aligned} [T]_{\mathcal{B}}^{-1} + [S]_{\mathcal{B}}^{-1} &= [[[T]_{\mathcal{B}}^{-1} + [S]_{\mathcal{B}}^{-1}]_{\mathcal{B}}]_{\mathcal{B}}^{-1} \\ &= [[[T]_{\mathcal{B}}^{-1}]_{\mathcal{B}} + [[S]_{\mathcal{B}}^{-1}]_{\mathcal{B}}]_{\mathcal{B}}^{-1} \\ &= [T + S]_{\mathcal{B}}^{-1}. \end{aligned}$$

$$\begin{aligned} \alpha[T]_{\mathcal{B}}^{-1} &= [[\alpha[T]_{\mathcal{B}}^{-1}]_{\mathcal{B}}]_{\mathcal{B}}^{-1} \\ &= [\alpha[[T]_{\mathcal{B}}^{-1}]_{\mathcal{B}}]_{\mathcal{B}}^{-1} \\ &= [\alpha T]_{\mathcal{B}}^{-1}. \end{aligned}$$

Solution: $\subseteq$: Let $\lambda \in \sigma(T)$, and let $\vec{v}$ be an associated eigenvector. We show that $[\vec{v}]_{\mathcal{B}}$ is an eigenvector for λ under $[T]_{\mathcal{B}}$.

$$\begin{aligned} [T]_{\mathcal{B}}[\vec{v}]_{\mathcal{B}} &= [T\vec{v}]_{\mathcal{B}} && \text{By 1(b)} \\ &= [\lambda\vec{v}]_{\mathcal{B}} \\ &= \lambda[\vec{v}]_{\mathcal{B}} && [\cdot]_{\mathcal{B}} \text{ is linear .} \end{aligned}$$

$\supseteq$: Let τ be an eigenvalue of $[T]_B$ with associated eigenvector $\vec{y}$. Since $[\cdot]_{\mathcal{B}}$ is an isomorphism, $\vec{y}$ has a unique preimage under the mapping, say $\vec{x}$ so that $[\vec{x}]_{\mathcal{B}} = \vec{y}$. Recall that $[\cdot]_{\mathcal{B}}$ also must have an inverse. Denote this $[\cdot]_{\mathcal{B}}^{-1} : M_{mn}(\mathbb{C}) \to L(V, W)$.

$$\begin{aligned} T\vec{x} &= [[T\vec{x}]_{\mathcal{B}}]_{\mathcal{B}}^{-1} \\ &= [[T]_{\mathcal{B}}[\vec{x}]_{\mathcal{B}}]_{\mathcal{B}}^{-1} && \text{Again by 1(b)} \\ &= [[T]_{\mathcal{B}}\vec{y}]_{\mathcal{B}}^{-1} \\ &= [\tau\vec{y}]_{\mathcal{B}}^{-1} \\ &= \tau[\vec{y}]_{\mathcal{B}}^{-1} && [\cdot]_{\mathcal{B}} \text{ is linear} \\ &= \tau\vec{x}. \end{aligned}$$

Therefore $\sigma(T) = \sigma([T]_{\mathcal{B}})$.

4. Let A be an $n \times n$ complex matrix with $\sigma(A) = \{1\}$. Show that A is diagonalizable if and only if A is the identity matrix.

$\implies$: Let A be a diagonalizable matrix and $\sigma(A) = \{1\}$. Then there exists some invertible S so that $S^{-1}AS = D = \text{diag}\{1, \ldots, 1\} = I$. Multiply both sides:

$$\begin{aligned} S^{-1}AS &= I \\ SS^{-1}ASS^{-1} &= SIS^{-1} \\ A &= SS^{-1} \\ A &= I. \end{aligned}$$

$\impliedby$: Conversely, if $A = I$, then take the invertible matrix I, so that $IAI^{-1} = A = I$, and since I is diagonal, A is diagonalizable.

5. Determine whether or not the derivative map $D : P_n \to P_n$ given by $Dp(z) = p'(z)$ is diagonalizable.

Claim: The $k+1$-th derivative of a polynomial of degree $k \in \mathbb{C}[x]$ is identically zero.

Proof of claim: Proceed by induction on the degree of p.

Base case: If p has degree 0, then p is constant and has zero derivative, and as such, any subsequent derivative will be zero.

Inductive hypothesis: Suppose that the k-th derivative of any $p \in \mathbb{C}[x]$ with $\deg p = k - 1$ is identically zero.

Inductive step: Let $f(x) \in \mathbb{C}[x]$ be of degree k. Write $f(x) = a_0 + a_1 x + \ldots + a_k x^k$ for complex a_i . Then $Df(x) = a_1 + 2a_2 + \ldots + ka_n x^{k-1}$. And since this polynomial Df is of degree $k-1$, by the inductive hypothesis the k-th derivative of this must be zero, and $D^{k+1}f = D(D^k f) = 0$. Therefore by induction on $\deg f$, the derivative operator is nilpotent on $\mathbb{C}[x]$ $\square$

Solution: The derivative map is nilpotent on P_n for $n \geq 1$ (For $n = 0$, $D^1 = D = 0$, a diagonal operator), and since nilpotent operators are not diagonalizable, the derivative operator is not diagonalizable for $n \geq 1$.