Bonus Assignment - Thomas Boyko - 30191728

1. Let X and Y be continuous random variables and let

$$f(x,y) = \begin{cases} \frac{x}{5} + cy & 0 \leq x \leq 1, 0 \leq y \leq 5 \\ 0 & elsewhere \end{cases}$$

   Find the value of the constant c that makes $f(x,y)$ a valid pdf.

   For f to be a valid pdf, the double integral over all nonzero values of x, y must equal 1.

$$\begin{aligned} 1 &= \int_0^1 \int_0^5 \frac{x}{5} + cydydx \\ &= \int_0^1 \frac{25c}{2} dx \\ &= \frac{25cx}{2} |_0^1 \\ &= \frac{25c}{2} \\ \frac{2}{25} &= c \end{aligned}$$

2. Let X and Y be continuous random variables with a joint probability density function defined as follows:

$$f(x,y) = \begin{cases} 2e^{-(x+y)} & 0 \leq x, 0 \leq y, x \leq y \\ 0 & elsewhere \end{cases}$$

   Are X and Y independent random variables? Show why/why not.

   X, Y are independent $\iff f(x,y) = f(x)f(y)$.

$$\begin{aligned} f(x)f(y) &= \int_0^\infty f(x,y)dy \cdot \int_0^y f(x,y)dx \\ &= \int_0^\infty 2e^{-(x+y)}dy \cdot \int_0^y 2e^{-(x+y)}dx \\ &= 2e^{-x} \cdot (2e^{-y} - 2e^{-2y}) \neq 2e^{-(x+y)} = f(x,y) \end{aligned}$$

   Since $f(x,y) \neq f(x)f(y)$, the variables are not independent.

3. Let X and Y be continuous random variables with a joint probability density function defined as follows:

$$f(x,y) = \begin{cases} xy & 0 \leq x \leq 1, 0 \leq y \leq 2 \\ 0 & elsewhere \end{cases}$$

   Find $P(X \leq Y)$.

$$\begin{aligned} P(X \leq Y) &= \int_0^2 \int_0^y xydxdy \\ &= \int_0^2 \frac{yx^2}{2} |_0^y dy \\ &= \int_0^2 \frac{y^3}{2} dy \\ &= \frac{y^4}{8} |_0^1 \\ &= \frac{1}{8} \end{aligned}$$

   So $P(X \leq Y) = \frac{1}{8}$.

4. Suppose a game consists of rolling two fair four-sided die (a red one and a blue one) and observing the number on the uppermost face of each die. In this game, you win \$2 if the blue die shows the same number as the red die, you win nothing (\$0) if the blue die shows a higher number than the red die, and you lose \$1 on any other outcome.

   Define two random variables as follows:

   - Let X be the number shown on the uppermost face of the blue die.
   - Let Y be the amount of money you win (or lose) playing this game once.

   (a) Find the joint probability distribution of X and Y and display it as a joint probability distribution table. Please be sure to clearly label which variable corresponds to the row values and which variable corresponds to the column values.

$\downarrow Y \rightarrow X$	1	2	3	4	Total
-1	0	1/16	2/16	3/16	6/16
0	3/16	2/16	1/16	0	6/16
2	1/16	1/16	1/16	1/16	4/16
Total	4/16	4/16	4/16	4/16	1

   (b) Find $COV(X, Y)$.

$$\begin{aligned} COV[X,Y] &= E[XY] - E[X]E[Y] \\ &= \end{aligned}$$