1. Let $\phi : \mathbb{R}^n \to \mathbb{R}$ be such that $\phi(x) = 0 \Leftrightarrow x = 0$ and $\phi(\lambda x) = |\lambda|\phi(x), \forall x \in \mathbb{R}^n, \forall \lambda \in \mathbb{R}$ Show that if the set $B = \{x \in \mathbb{R}^n | \phi(x) \leq 1\}$ is convex, then ϕ defines a norm on $\mathbb{R}^n$.

   Solution: Non-degeneracy and scalar linearity are given from the description of ϕ. So all that is left to prove is the triangle inequality and non-negativity.

2. Let E be a compact set in $\mathbb{R}^n$ and let F be a closed set in $\mathbb{R}^n$ such that $E \cap F = \emptyset$. a. Show that there exists d > 0 such that $\|x - y\| > d$, $\forall x \in E$ and $\forall y \in F$. b. Does the result you proved in the previous question remain true if E and F are closed, but neither is compact? Justify your answer.

3. Let $E = \left\{(x, y) | y = \sin\left(\frac{1}{x}\right), x > 0\right\}$. Is E open? Is it closed? What are the accumulation points of E?

4. Let $f : \mathbb{R}^n \to \mathbb{R}$ be a function in $C^1(\mathbb{R}^n)$, i.e., $f, \partial_{x_1} f, \ldots, \partial_{x_n} f$ are continuous in $\mathbb{R}^n$. Suppose $f(tx) = tf(x), \forall x \in \mathbb{R}^n, \forall t \in \mathbb{R}$ Show that f is a linear function.

5. Given $u : \mathbb{R} \to \mathbb{R}$ a function in $C^2(\mathbb{R})$, define $f : \mathbb{R}^2 \to \mathbb{R}$ by $f(x, y) = \begin{cases} u(y) - u(x) & \text{if } y \neq x \\ u'(x) & \text{if } y = x \end{cases}$

   Show that f is differentiable at any point (a, a).

6. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a function that is defined in an open set Ω in $\mathbb{R}^2$. Show that if $\partial_x f(x, y), \partial_y f(x, y)$ and $\partial_{xy} f(x, y)$ are continuous in Ω, then $\partial_{yx} f(x, y)$ exists in Ω and we have $\partial_{yx} f(x, y) = \partial_{xy} f(x, y), \forall (x, y) \in \Omega$ Hint: Consider the expression $\Delta(s, t) = f(a + s, b + t) - f(a + s, b) - f(a, b + t) + f(a, b)$.

7. Compute the degree 3 Taylor polynomial $T_3(x, x_2)$ of the function $f : \mathbb{R}^2 \to \mathbb{R}$, defined by $f(x_1, x_2) = \frac{4x_1 + 6x_2 - 1}{2x_1 + 3x_2}$ at the point $(-1, 1)$.