Exercise 1

Consider the measure space $(\mathbb{R}, \mathcal{B}(\mathbb{R}), \lambda)$, where λ denotes Lebesgue measure on $\mathbb{R}$.

1. *Show that $\lambda(\{a\}) = 0$ for any $a \in \mathbb{R}$.*
2. *Show that for any $a < b$ in $\mathbb{R}$, the follwing holds:*
$$\lambda((a,b)) = \lambda((a,b]) = \lambda([a,b)) = \lambda([a,b]) = b - a.$$
3. *Show that for any a in $\mathbb{R}$, the follwing holds:*
$$\lambda((-\infty,a)) = \lambda((-\infty,a]) = \lambda([a,\infty)) = \lambda([a,\infty]) = \infty.$$

Student's note For this assignment we take the convention $0 \notin \mathbb{N}$. I am not sure whether this is usual for the course (:

1. Let $a \in \mathbb{R}$. Take the sequence of decreasing sets:
$$\left(a - \frac{1}{n}, a + \frac{1}{n}\right).$$
So that:
$$\{a\} = \bigcap_{n\in\mathbb{N}} \left(a - \frac{1}{n}, a + \frac{1}{n}\right).$$
Then we have:
$$\begin{aligned}
\lambda(\{a\}) &= \lambda\left(\bigcap_{n\in\mathbb{N}} \left(a - \frac{1}{n}, a + \frac{1}{n}\right)\right) \\
&= \lim_{n\to\infty} \lambda\left(\left(a - \frac{1}{n}, a + \frac{1}{n}\right)\right) && \text{From prop. 1.3.4 (vi)} \\
&= \lim_{n\to\infty} a + \frac{1}{n} - \left(a - \frac{1}{n}\right) \\
&= \lim_{n\to\infty} \frac{1}{2n} \\
&= 0.
\end{aligned}$$

2. We split into 4 parts:
   (a) $\lambda((a,b)) = b - a$
   (b) $\lambda((a,b]) = b - a$
   (c) $\lambda([a,b)) = b - a$
   (d) $\lambda([a,b]) = b - a$

   Solution:

   (a) This is true by definition
   (b) Let $a < b$ be reals. We use the same strategy again. Take the sequence of decreasing sets:
$$\left(a, b + \frac{1}{n}\right).$$
   So that:
$$(a,b] = \bigcap_{n\in\mathbb{N}} \left(a, b + \frac{1}{n}\right).$$

Then we have:

$$\begin{aligned}
\lambda((a,b]) &= \lambda\left(\bigcap_{n\in\mathbb{N}}\left(a, b+\frac{1}{n}\right)\right) \\
&= \lim_{n\to\infty}\lambda\left(\left(a, b+\frac{1}{n}\right)\right) && \text{From prop. 1.3.4 (vi)} \\
&= \lim_{n\to\infty} b+\frac{1}{n}-a \\
&= b-a.
\end{aligned}$$

(c) Let $a < b$ be reals. We use the same strategy again. Take the sequence of decreasing sets:

$$\left(a-\frac{1}{n}, b\right).$$

So that:

$$[a,b) = \bigcap_{n\in\mathbb{N}}\left(a-\frac{1}{n}, b\right).$$

Then we have:

$$\begin{aligned}
\lambda([a,b)) &= \lambda\left(\bigcap_{n\in\mathbb{N}}\left(a-\frac{1}{n}, b\right)\right) \\
&= \lim_{n\to\infty}\lambda\left(\left(a-\frac{1}{n}, b\right)\right) && \text{From prop. 1.3.4 (vi)} \\
&= \lim_{n\to\infty} b-\frac{1}{n}-a \\
&= b-a.
\end{aligned}$$

(d) Let $a < b$ be reals. We use the same strategy again. Take the sequence of decreasing sets:

$$\left(a-\frac{1}{n}, b\right].$$

So that:

$$[a,b] = \bigcap_{n\in\mathbb{N}}\left(a-\frac{1}{n}, b\right].$$

Then we have:

$$\begin{aligned}
\lambda([a,b)) &= \lambda\left(\bigcap_{n\in\mathbb{N}}\left(a-\frac{1}{n}, b\right]\right) \\
&= \lim_{n\to\infty}\lambda\left(\left(a-\frac{1}{n}, b\right]\right) && \text{From prop. 1.3.4 (vi)} \\
&= \lim_{n\to\infty} b-\frac{1}{n}-a \\
&= b-a.
\end{aligned}$$

3. We split into 4 parts:
   (a) $\lambda((-\infty, a)) = \infty$
   (b) $\lambda((-\infty, a]) = \infty$
   (c) $\lambda((a, \infty)) = \infty$
   (d) $\lambda([a, \infty)) = \infty$

   Solution:

   (a) Rewrite:
   $$(-\infty, a) = \bigcup_{n\in\mathbb{N}} [a-n, a-n+1).$$
   Then since these are all disjoint subsets of $\mathbb{R}$, we have:
   $$\begin{aligned}
   \lambda((-\infty, a)) &= \lambda(\bigcup_{n\in\mathbb{N}} [a-n, a-n+1) \\
   &= \sum_{n=1}^{\infty} \lambda([a-n, a-n+1)) \\
   &= \sum_{n=1}^{\infty} (a-n+1)-(a-n) && \text{From 2.} \\
   &= \sum_{n=1}^{\infty} 1 \\
   &= \infty.
   \end{aligned}$$

   (b) Rewrite:
   $$(-\infty, a) = \bigcup_{n\in\mathbb{N}} [a-n, a-n+1).$$
   Then since these are all disjoint subsets of $\mathbb{R}$, we have:
   $$\begin{aligned}
   \lambda((-\infty, a)) &= \lambda(\bigcup_{n\in\mathbb{N}} [a-n, a-n+1) \\
   &= \sum_{n=1}^{\infty} \lambda([a-n, a-n+1)) \\
   &= \sum_{n=1}^{\infty} (a-n+1)-(a-n) && \text{From 2.} \\
   &= \sum_{n=1}^{\infty} 1 \\
   &= \infty.
   \end{aligned}$$

(c) Rewrite:

$$(a,\infty) = \bigcup_{n\in\mathbb{N}} (a+n-1, a+n].$$

Then since these are all disjoint subsets of $\mathbb{R}$, we have:

$$\begin{aligned}
\lambda\,(a,\infty) &= \lambda\left(\bigcup_{i\in I}(a+n-1, a+n]\right) \\
&= \sum_{n=1}^{\infty} \lambda\,((a+n-1, a+n]) \\
&= \sum_{n=1}^{\infty} (a-n+1)-(a-n) && \text{From 2.} \\
&= \sum_{n=1}^{\infty} 1 \\
&= \infty.
\end{aligned}$$

(d) Rewrite:

$$(a,\infty) = \bigcup_{n\in\mathbb{N}} (a+n-1, a+n].$$

Then since these are all disjoint subsets of $\mathbb{R}$, we have:

$$\begin{aligned}
\lambda\,[a,\infty) &= \lambda\left(\bigcup_{i\in I}[a+n-1, a+n)\right) \\
&= \sum_{n=1}^{\infty} \lambda\,([a+n-1, a+n)) \\
&= \sum_{n=1}^{\infty} (a-n+1)-(a-n) && \text{From 2.} \\
&= \sum_{n=1}^{\infty} 1 \\
&= \infty.
\end{aligned}$$