1. Consider the function $f:\mathbb{R}^3\to\mathbb{R}^3$, defined by $f(x,y,z)=\begin{pmatrix}x^3-y-z\\2x+y+z\\x+y-z\end{pmatrix}$

   (a) Compute $Jf(x,y,z)$ and show that $\mathrm{d}f_{(x,y,z)}$ is invertible for any $(x,y,z)\in\mathbb{R}^3$.

   Solution: Compute:

$$Jf(x,y,z)=\begin{bmatrix}\frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} & \frac{\partial f_1}{\partial z}\\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} & \frac{\partial f_2}{\partial z}\\ \frac{\partial f_3}{\partial x} & \frac{\partial f_3}{\partial y} & \frac{\partial f_3}{\partial z}\end{bmatrix}=\begin{bmatrix}3x^2 & -1 & -1\\ 2 & 1 & 1\\ 1 & 1 & -1\end{bmatrix}.$$

   Then we take the determinant;

$$\det Jf(x,y,z)=3x^2\begin{vmatrix}1 & 1\\ 1 & -1\end{vmatrix}-2\begin{vmatrix}-1 & -1\\ 1 & -1\end{vmatrix}+\begin{vmatrix}-1 & -1\\ 1 & 1\end{vmatrix}=-6x^2-4+0.$$

   And we can see that since this determinant has no real roots, it must be nonzero for any $(x,y,z)\in\mathbb{R}^3$, and so Jf, as well as df are both invertible in $\mathbb{R}^3$.

   (b) Find the largest open $U\subset\mathbb{R}^3$ where f has a continuously differentiable inverse function g.

   Solution: Begin by showing that f is injective in $\mathbb{R}^3$. Suppose:

$$x_1^3-y_1-z_1=x_2^3-y_2-z_2 \tag{1}$$
$$2x_1+y_1+z_1=2x_2+y_2+z_2 \tag{2}$$
$$x_1+y_1-z_1=x_2+y_2-z_2. \tag{3}$$

   However, if we add $(1)+(2)$, we get $x_1^3+2x_1=x_1(x_1^2+2)=x_2(x_2^2+2)=x_2^3+2x_2$. Let $h(x)=x^3+2x$, so that $h'(x)=3x^2+2$, positive for all x. So then h is increasing, and therefore injective, and since $h(x_1)=h(x_2)$, we must have $x_1=x_2$. Then we can transform $(2),(3)$:

$$y_1+z_1-y_2-z_2=0$$
$$y_1-z_1-y_2+z_2=0.$$

   And we transform this homogenous system into a matrix to put into RREF:

$$\begin{bmatrix}1 & 1 & -1 & -1\\ 1 & -1 & -1 & 1\end{bmatrix}\begin{bmatrix}y_1\\ z_1\\ y_2\\ z_2\end{bmatrix}=0$$

$$\begin{bmatrix}1 & 0 & -1 & 0\\ 0 & 1 & 0 & -1\end{bmatrix}\begin{bmatrix}y_1\\ z_1\\ y_2\\ z_2\end{bmatrix}=0$$

.

   And so we have $x_1=x_2, y_1=y_2, z_1=z_2$ as desired, and f is injective in $\mathbb{R}^3$, and clearly $f(\mathbb{R}^3)=\mathbb{R}^3$.
   Then since f is injective in $\mathbb{R}^3$ and df is invertible in $\mathbb{R}^3$, by the Global inversion theorem, $U=\mathbb{R}^3$ is the largest open set in which f is invertible.

2. Consider the system of equations: $(S)\begin{cases} x - y - u^2 + v^2 = 0 \\ x + y - 2uv = 0 \end{cases}$

   (a) Show that the system (S) can be solved for u, v in term of (x, y) near the point $(x, y, u, v) = (1, 1, 1, 1)$.

   Solution: We solve for the Jacobian about $(1, 1, 1, 1)$.

$$Jf(x,y,u,v) = \begin{bmatrix} 1 & -1 & -2u & 2v \\ 1 & 1 & -2v & -2u \end{bmatrix}$$

$$Jf(1,1,1,1) = \begin{bmatrix} 1 & -1 & -2 & 2 \\ 1 & 1 & -2 & -2 \end{bmatrix}.$$

   And if we break Jf into block matrices, we get the right half of Jf, $\partial_{u,v} f = \begin{bmatrix} -2 & 2 \\ -2 & -2 \end{bmatrix}$ which has nonzero determinant and must be invertible. So u, v can be implicitly defined about $(1, 1, 1, 1)$ by the Implicit Function theorem.

   (b) Compute $\partial_x u(1, 1) + \partial_y v(1, 1)$.

   Solution: Begin with the identity from the Implicit Function Theorem:

$$\begin{aligned}
\begin{bmatrix} \partial_x u & \partial_y u \\ \partial_x v & \partial_y v \end{bmatrix} &= \begin{bmatrix} \partial_u f_1 & \partial_v f_1 \\ \partial_u f_2 & \partial_v f_2 \end{bmatrix}^{-1} \begin{bmatrix} \partial_x f_1 & \partial_y f_1 \\ \partial_x f_2 & \partial_y f_2 \end{bmatrix} \\
&= \left(\det \begin{bmatrix} -2u & 2v \\ -2v & -2u \end{bmatrix}\right)^{-1} \begin{bmatrix} -2u & -2v \\ 2v & -2u \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \\
&= \frac{1}{2(u^2+v^2)} \begin{bmatrix} -u & -v \\ v & -u \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \\
&= \frac{1}{2(u^2+v^2)} \begin{bmatrix} -u-v & u-v \\ v-u & -v-u \end{bmatrix}.
\end{aligned}$$

   And so if we want the sum $\partial_x u(1, 1) + \partial_y v(1, 1)$ we need only take the trace of this matrix and evaluate at $(1, 1)$.

$$\begin{aligned}
\partial_x u(1,1) + \partial_y v(1,1) &= \frac{1}{2(u^2+v^2)}\Big|_{(1,1)} \\
&= -2\frac{u+v}{2(u^2+v^2)}\Big|_{(1,1)} \\
&= -1.
\end{aligned}$$

3. Let $f : \mathbb{R}^2 \to \mathbb{R} : (x, y) \to f(x, y)$. Show that if $f \in C^1(\mathbb{R}^2, \mathbb{R})$, then f can't be injective on $\mathbb{R}^2$. Hint: Use the implicit functions theorem.

   Solution: Assume, for the purpose of deriving a contradiction, that f is injective on $\mathbb{R}^2$. Let $(a, b) \in \mathbb{R}^2$, and note that $c = f(a, b)$ is only attained at (a, b). Take the derivative;

$$df = [\partial_x f \quad \partial_y f].$$

   If $\partial_y f(a, b) \neq 0$, there exists some function $g : \mathbb{R} \to \mathbb{R}$ (By Implicit Function Theorem), so that $c = f(x, g(x))$, for all (x, y) close to (a, b) within a neighborhood U of (a, b). But then we have multiple distinct points mapping to p, and then $\partial_Y f(a, b) = 0$.

   Otherwise, if $\partial_y f = 0$, we check $\partial_x f$. If this is nonzero, we repeat the above argument with ImFT for x, to get $\partial_x f(a, b) = 0$.

   Since (a, b) was chosen arbitrarily, we must have $df = 0$ for any $(x, y) \in \mathbb{R}^2$. The only functions for which this holds true are constant, and since constant functions are not injective, f cannot be injective.

4. Let $E = C([a,b],\mathbb{R})$ equipped with the norm of uniform convergence, let $u \in C(\mathbb{R},\mathbb{R})$, and consider the mapping $\phi : E \to E$, defined by $\phi(v) = u \circ v$. Is ϕ continuous? Make sure to justify your answer.

Solution: Let $\varepsilon > 0$, and $v, w \in E$. Recall that the image of compact sets under continuous functions is compact, and the union of compact sets is compact. Then since continuous functions are uniformly continuous on compact sets, u must be uniformly continuous on $v([a,b]) \cup w([a,b])$. Let $x \in [a,b]$, and let δ be chosen so that $|w(x) - v(x)| < \delta \implies |u(w(x)) - u(v(x))| < \varepsilon$. Suppose

$$\|w - v\| = \sup_{x \in [a,b]} |w(x) - v(x)| < \delta. \tag{$*$}$$

Then we must have $|w(x) - v(x)| < \delta$ for any $x \in [a,b]$. But by continuity of u, we have

$$|\phi(w) - \phi(v)| = |u(w(x)) - u(v(x))| < \varepsilon.$$

for any $x \in [a,b]$. Then recall that since $u, v, w \in E$ are continuous, the composition, difference and absolute value $|u \circ w - u \circ v|$ is continuous. Therefore the supremum of this function is attained in the compact set $[a,b]$, and when we take the supremum $\sup_{x \in [a,b]} |u(w(x)) - u(v(x))|$, we can say that it is attained for some $x_0 \in [a,b]$. And from ($*$), we have:

$$\begin{aligned}
\|\phi(w) - \phi(v)\| &= \|u \circ w - u \circ v\| \\
&= \sup_{x \in [a,b]} |u(w(x)) - u(v(x))| \\
&= |u(w(x_0)) - u(v(x_0))| \\
&< \varepsilon.
\end{aligned}$$

And ϕ is continuous as desired.

5. Find in $C([0,1],\mathbb{R})$ the distance from the function $u(t) = t$ to the subspace $\mathbb{P}_0$ of polynomials of degree 0. Make sure to justify your answer.

Solution: Let $u(t) = t$, and take the distance:

$$\begin{aligned}
d(u, \mathbb{P}_0) &= \inf_{p \in \mathbb{P}_0} d(u,p) \\
&= \inf_{c \in \mathbb{R}} \|u - c\| && p \text{ is simply a real constant} \\
&= \inf_{c \in \mathbb{R}} \sup_{t \in [0,1]} |u(t) - c| \\
&= \inf_{c \in \mathbb{R}} \sup_{t \in [0,1]} |t - c|.
\end{aligned}$$

Write $f_c(t) = |t - c|$. This continuous function will attain its supremum in the compact $[0,1]$. Since $|t - c|$ is decreasing on $(-\infty, c]$ and increasing on $[c, \infty)$, the supremum must be either at 0 or 1. So we can rewrite $d(u, \mathbb{P}) = \inf_{c \in \mathbb{R}} \max\{|c|, |1-c|\}$.

We claim that $\frac{1}{2}$ is this infimum. First we show that this is a lower bound; let $c \in \mathbb{R}$. If $c < \frac{1}{2}$, then $|c| < \frac{1}{2}, |1-c| > \frac{1}{2}$, and so $\frac{1}{2} < \max\{|c|, |1-c|\} = |1-c|$.

Now if $c > \frac{1}{2}$, we have $|1-c| < \frac{1}{2}, |c| > \frac{1}{2}$ and so $\frac{1}{2} < \max\{|c|, |1-c|\} = |c|$.

Finally, if $c = \frac{1}{2}$, we have $\max\left\{|\frac{1}{2}|, |1 - \frac{1}{2}|\right\} = \frac{1}{2}$. All this is to say that $\frac{1}{2}$ is the greatest lower bound for this set of maximums, and therefore $d(u, \mathbb{P}_0) = \frac{1}{2}$

6. Let $f \in C([a,b],\mathbb{R})$ be such that $\int_a^b f(x)x^n \,\mathrm{dx} = 0, \quad \forall n \in \mathbb{N}$ Show that f is identically zero. Hint: Use Weierstrass Theorem.

Solution: First, we claim that if p is any real polynomial, then $\int_a^b f(x)p(x)\,dx = 0$. Write $p = \sum_{i=0}^n c_i x^i$. Then:

$$\begin{aligned}
\int_a^b f(x)p(x)\,dx &= \int_a^b f(x)\sum_{i=0}^n c_i x^i\,dx \\
&= \sum_{i=0}^n \int_a^b c_i x^i f(x)\,dx \\
&= \sum_{i=0}^n c_i \int_a^b x^i f(x)\,dx \\
&= \sum_{i=1}^n c_i \cdot 0 \\
&= 0.
\end{aligned}$$

By Weierstrass, there exists a sequence of real polynomials convergent to f. Let $\{p_n\}$ be such a sequence, and take:

$$\begin{aligned}
\int_a^b f^2(x)\,dx &= \int_a^b \lim_{n\to\infty} p_n(x)f(x)\,dx \\
&= \lim_{n\to\infty}\int_a^b f(x)p_n(x)\,dx && fp_n \in C([a,b]) \\
&= \lim_{n\to\infty} 0 \\
&= 0.
\end{aligned}$$

Define $F(x)$ so that $\frac{d}{dx}F(x) = f^2(x)$ by the Fundamental Theorem of Calculus. Then $\int_a^b f^2(x)\,dx = F(a) - F(b) = 0$. Then $F(a) = F(b)$. But since $f^2(x) \geq 0$, F is increasing, and we must have $F(x) = c$ for some constant real $c \in \mathbb{R}$. Then since $f^2(x) = \frac{d}{dx}F(x) = \frac{d}{dx}c = 0$, we have f^2 is identically 0, and then f must also be identically 0.