1. Consider the function $f:\mathbb{R}^3 \to \mathbb{R}^3$, defined by $f(x,y,z) = \begin{pmatrix} x^3-y-z \\ 2x+y+z \\ x+y-z \end{pmatrix}$

   (a) Compute $Jf(x,y,z)$ and show that $\mathrm{d}f_{(x,y,z)}$ is invertible for any $(x,y,z)\in\mathbb{R}^3$.

   Solution: Compute:

$$Jf(x,y,z) = \begin{bmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} & \frac{\partial f_1}{\partial z} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} & \frac{\partial f_2}{\partial z} \\ \frac{\partial f_3}{\partial x} & \frac{\partial f_3}{\partial y} & \frac{\partial f_3}{\partial z} \end{bmatrix} = \begin{bmatrix} 3x^2 & -1 & -1 \\ 2 & 1 & 1 \\ 1 & 1 & -1 \end{bmatrix}.$$

   Then we take the determinant;

$$\det Jf(x,y,z) = 3x^2\begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} - 2\begin{vmatrix} -1 & -1 \\ 1 & -1 \end{vmatrix} + \begin{vmatrix} -1 & -1 \\ 1 & 1 \end{vmatrix} = -6x^2-4+0.$$

   And we can see that since this determinant has no real roots, it must be nonzero for any $(x,y,z)\in\mathbb{R}^3$, and so Jf, as well as df are both invertible in $\mathbb{R}^3$.

   (b) Find the largest open $U\subset\mathbb{R}^3$ where f has a continuously differentiable inverse function g.

   Solution: Begin by showing that f is injective in $\mathbb{R}^3$. Suppose:

$$x_1^3 - y_1 - z_1 = x_2^3 - y_2 - z_2 \tag{1}$$
$$2x_1 + y_1 + z_1 = 2x_2 + y_2 + z_2 \tag{2}$$
$$x_1 + y_1 - z_1 = x_2 + y_2 - z_2. \tag{3}$$

   However, if we add $(1)+(2)$, we get $x_1^3+2x_1 = x_1(x_1^2+2) = x_2(x_2^2+2) = x_2^3+2x_2$

2. Consider the system of equations: $(S)\begin{cases} x-y-u^2+v^2=0 \\ x+y-2uv=0 \end{cases}$

   (a) Show that the system (S) can be solved for u,v in term of (x,y) near the point $(x,y,u,v)=(1,1,1,1)$.

   Solution: We solve for the Jacobian about $(1,1,1,1)$.

$$Jf(x,y,u,v) = \begin{bmatrix} 1 & -1 & -2u & 2v \\ 1 & 1 & -2v & -2u \end{bmatrix}$$

$$Jf(1,1,1,1) = \begin{bmatrix} 1 & -1 & -2 & 2 \\ 1 & 1 & -2 & -2 \end{bmatrix}.$$

   And if we break Jf into block matrices, we get the invertible right half of Jf as $\begin{bmatrix} -2 & 2 \\ -2 & -2 \end{bmatrix}$ which has nonzero determinant and must be invertible. So u,v can be implicitly defined about $(1,1,1,1)$by the Implicit Function theorem.

   (b) Compute $\partial_x u(1,1)+\partial_y v(1,1)$.

Solution: Begin with the identity from the Implicit Function Theorem:

$$\begin{aligned}
\begin{bmatrix} \partial_x u & \partial_y u \\ \partial_x v & \partial_y v \end{bmatrix} &= \begin{bmatrix} \partial_u f_1 & \partial_v f_1 \\ \partial_u f_2 & \partial_v f_2 \end{bmatrix}^{-1} \begin{bmatrix} \partial_x f_1 & \partial_y f_1 \\ \partial_x f_2 & \partial_y f_2 \end{bmatrix} \\
&= \left(\det \begin{bmatrix} -2u & 2v \\ -2v & -2u \end{bmatrix} \right)^{-1} \begin{bmatrix} -2u & -2v \\ 2v & -2u \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \\
&= \frac{1}{2(u^2+v^2)} \begin{bmatrix} -u & -v \\ v & -u \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \\
&= \frac{1}{2(u^2+v^2)} \begin{bmatrix} -u-v & u-v \\ v-u & -v-u \end{bmatrix}.
\end{aligned}$$

And so if we want the sum $\partial_x u(1,1) + \partial_y v(1,1)$ we need only take the trace of this matrix and evaluate at $(1,1)$.

$$\begin{aligned}
\partial_x u(1,1) + \partial_y v(1,1) &= \frac{1}{2(u^2+v^2)}\bigg|_{(1,1)} \\
&= -2\frac{u+v}{2(u^2+v^2)}\bigg|_{(1,1)} \\
&= -1.
\end{aligned}$$

3. Let $f : \mathbb{R}^2 \to \mathbb{R} : (x,y) \to f(x,y)$. Show that if $f \in C^1(\mathbb{R}^2, \mathbb{R})$, then f can't be injective on $\mathbb{R}^2$. Hint: Use the implicit functions theorem.

4. Let $E = C([a,b], \mathbb{R})$ equipped with the norm of uniform convergence, let $u \in C(\mathbb{R},\mathbb{R})$, and consider the mapping $\phi : E \to E$, defined by $\phi(v) = u \circ v$. Is ϕ continuous? Make sure to justify your answer.

Solution: Let $\varepsilon > 0$, and $v, w \in E$. Recall that the image of compact sets under continuous functions is compact, and the union of compact sets is compact. Then since continuous functions are uniformly continuous on compact sets, u must be uniformly continuous on $v([a,b]) \cup w([a,b])$. Let $x \in [a,b]$, and let δ be chosen so that $|w(x) - v(x)| < \delta \implies |u(w(x)) - u(v(x))| < \varepsilon$. Suppose

$$\|w - v\| = \sup_{x \in [a,b]} |w(x) - v(x)| < \delta. \tag{$*$}$$

Then we must have $|w(x) - v(x)| < \delta$ for any $x \in [a,b]$. But by continuity of u, we have

$$|\phi(w) - \phi(v)| = |u(w(x)) - u(v(x))| < \varepsilon.$$

for any $x \in [a,b]$. Then recall that since $u, v, w \in E$ are continuous, the composition, difference and absolute value $|u \circ w - u \circ v|$ is continuous. Therefore the supremum of this function is attained in the compact set $[a,b]$, and when we take the supremum $\sup_{x \in [a,b]} |u(w(x)) - u(v(x))|$, we can say that it is attained for some $x_0 \in [a,b]$. And from $(*)$, we have:

$$\begin{aligned}
\|\phi(w) - \phi(v)\| &= \|u \circ w - u \circ v\| \\
&= \sup_{x \in [a,b]} |u(w(x)) - u(v(x))| \\
&= |u(w(x_0)) - u(v(x_0))| \\
&< \varepsilon.
\end{aligned}$$

And ϕ is continuous as desired.

5. Find in $C([0,1], \mathbb{R})$ the distance from the function $u(t) = t$ to the subspace $\mathbb{P}_0$ of polynomials of degree 0. Make sure to justify your answer.

Solution: Let $u(t) = t$, and take the distance:

$$\begin{aligned}
d(u, \mathbb{P}_0) &= \inf_{p \in \mathbb{P}_0} d(u, p) \\
&= \inf_{c \in \mathbb{R}} \|u - c\| && p \text{ is simply a real constant} \\
&= \inf_{c \in \mathbb{R}} \sup_{t \in [0,1]} |u(t) - c| \\
&= \inf_{c \in \mathbb{R}} \sup_{t \in [0,1]} |t - c|
\end{aligned}$$

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6. Let $f \in C([a, b], \mathbb{R})$ be such that $\int_a^b f(x) x^n \, \mathrm{dx} = 0, \quad \forall n \in \mathbb{N}$ Show that f is identically zero. Hint: Use Weierstrass Theorem.

Solution: First, we claim that if p is any real polynomial, then $\int_a^b f(x)p(x)\,dx = 0$. Write $p = \sum_{i=0}^n a_i x^i$. Then:

$$\begin{aligned}
\int_a^b f(x)p(x)\,dx &= \int_a^b f(x) \sum_{i=0}^n a_i x^i \,dx \\
&= \sum_{i=0}^n \int_a^b a_i x^i f(x)\,dx \\
&= \sum_{i=0}^n a_i \int_a^b x^i f(x)\,dx \\
&= \sum_{i=1}^n a_i \cdot 0 \\
&= 0.
\end{aligned}$$

By Weierstrass, there exists a sequence of real polynomials convergent to f. Let $\{p_n\}$ be such a sequence, and take:

$$\begin{aligned}
\int_a^b f^2(x)\,dx &= \int_a^b \lim_{n \to \infty} p_n(x) f(x)\,dx \\
&= \lim_{n \to \infty} \int_a^b f(x) p_n(x)\,dx && f p_n \in C([a, b]) \\
&= \lim_{n \to \infty} 0 \\
&= 0.
\end{aligned}$$

And if we recall that $\langle f, g \rangle = \int_a^b f(x)g(x)\,dx$ is an inner product on $C([a, b])$, we know that $\langle f, f \rangle = 0 \iff f \equiv 0$, so f must be identically zero.