Homework 3 - Thomas Boyko - 30191728

1. Compute $\oint_C \vec{F} d\vec{r}$ where C is the ellipse $4x^2 + y^2 = 4$ (with counterclockwise motion) and $\vec{F}(x,y) = \left(\frac{3x^2}{2\sqrt{x^3+y^2}} + y^2, \qquad \frac{y}{\sqrt{x^3+y^2}} + x^2\right)^T$.

   Trying to find an "almost potential" for $\vec{F}$, we begin by integrating F_1, F_2:

$$\int \frac{3x^2}{2\sqrt{x^3+y^2}} + y^2\, dx = \sqrt{x^3+y^2} + xy^2 + f_1(y).$$

$$\int \frac{y}{\sqrt{x^3+y^2}} + x^2\, dy = \sqrt{x^3+y^2} + x^2y + f_2(x).$$

   And we can see that there is no f_1, f_2 which satisfy both equations.

   So we will take $\Phi = \sqrt{x^3+y^2}$ and $\vec{G} = (xy^2, x^2y)^T$, so that $\nabla\Phi + \vec{G} = \vec{F}$. Now $\oint_C \vec{F}\, d\vec{r} = \oint_C \vec{G}\, d\vec{r}$.

   Can we find a potential Ψ for $\vec{G}$?

$$\Psi = \int xy^2\, dx = \frac{x^2y^2}{2} + f_3(y)$$

$$\Psi = \int x^2y\, dy = \frac{x^2y^2}{2} + f_4(x).$$

   And we can quickly see that if we take $\Phi = \frac{x^2y^2}{2}$ then $\nabla\Psi = \vec{G}$.

   So we have a potential for $\vec{G}$, meaning that $\vec{G}$ is conservative, and since our integral is over a closed curve, $\oint \vec{F} d\vec{r} = \oint \vec{G} d\vec{r} = 0$.

2. Compute the surface integral $\iint_S (z+3) dS$ where S is the part of the paraboloid $z = 2x^2 + 2y^2 - 3$ that lies below the plane $z = 1$.

   We begin by parameterizing S. Since S is a function-type, we can write $\vec{r}(x,y) = (x, y, 2x^2 + 2y^2 - 3)^T$.

   As for our bounds on x, y, we can just say that $(x,y) \in D$ where D is the circle of radius 2 centered at the origin.

   Now we find $\|\vec{n}\|$.

$$\vec{n} = \begin{pmatrix} 4x \\ 4y \\ -1 \end{pmatrix}$$

$$\|\vec{n}\| = \sqrt{16x^2 + 16y^2 + 1}$$

   We can write our integral now as:

$$2\iint_D (x^2+y^2)\sqrt{16x^2+16y^2+1}\, dA.$$

   Converting to cylindrical coordinates:

$$2\int_0^2 \int_0^{2\pi} r^3\sqrt{16r^2+1}\, d\theta\, dr = 4\pi \int_0^2 r^3\sqrt{16r^2+1} dr.$$

   Let $u = 16r^2 + 1$, so $dr = \frac{du}{32r}$, and $r^2 = \frac{u-1}{16}$. Our endpoints now go from $u(0) = 1$ to $u(2) = 65$.

$$\frac{\pi}{128}\int_1^{65} (u-1)\sqrt{u}\, du = \frac{\pi}{128}\left(\frac{2}{5}u^{\frac{5}{2}} - \frac{2}{3}u^{\frac{3}{2}}\right)_{u=1}^{65}$$

$$= \frac{\pi}{128}\left(\frac{2}{5}65^{\frac{5}{2}} - \frac{2}{3}65^{\frac{3}{2}} + \frac{4}{15}\right).$$

3. Compute the surface integral $\iint_S y dS$ where S is the part of the cylinder $4x^2 + y^2 = 4$ bounded between the planes $z = -3$ and $z = 4y + x + 3$.

   First we parameterize S as:

   $$\vec{r}(z, \theta) = \begin{pmatrix} \cos\theta & 2\sin\theta & z \end{pmatrix}^T.$$

   Within a region D given by $\theta \in [0, 2\pi]$ and $z \in [-3, 8\sin\theta + \cos\theta + 7]$.

   Taking the cross product of the derivatives of $\vec{r}$ we get:

   $$\vec{n} = \begin{pmatrix} 2\cos\theta & \sin\theta & 0 \end{pmatrix}^T.$$

   And so $\|\vec{n}\| = \sqrt{3\cos^2\theta + 1}$, which gives us the integral

   $$\begin{aligned} \int_0^{2\pi} \int_{-3}^{8\sin\theta+\cos\theta+7} 2\sin\theta\sqrt{3\cos^2\theta + 1} dz d\theta &= \int_0^{2\pi} (8\sin\theta + \cos\theta + 10)\sqrt{\cos^2\theta + 1} d\theta \\ &\approx 96.8845. \end{aligned}$$

   I was not able to compute this final step but Wolfram was.

4. Compute the surface integral $\iint_S xz dS$ where S is the part of the plane $z = 4y + x + 3$ that lies inside the cylinder $4x^2 + y^2 = 4$.

   First we parameterize S with $\vec{r}(x, y) = (x, y, 4y + x + 3)^T$ with $0 \leq 4x^2 + y^2 \leq 4$. This gives our normal $\vec{n} = (1, 4, -1)^T$, and $\|\vec{n}\| = \sqrt{18} = 3\sqrt{2}$.

   Our bounds on x, y become $x \in [-1, 1]$ and $y \in [-2\sqrt{1 - x^2}, 2\sqrt{1 - x^2}]$.

   We transform our integral into:

   $$\begin{aligned} 3\sqrt{2} \int_{-1}^{1} \int_{-2\sqrt{1-x^2}}^{2\sqrt{1-x^2}} 4xy + x^2 + 3x \, dy \, dx &= 3\sqrt{2} \int_{-1}^{1} 4x^2\sqrt{1 - x^2} + 12x\sqrt{1 - x^2} \, dx \\ &= 3\sqrt{2} \int_{-1}^{1} (4x^2 + 12)\sqrt{x^2 - 1}. \end{aligned}$$

   Wolfram helpfully evaluates this lengthy inverse trig substitution to give a final answer of $3\sqrt{2}\frac{\pi}{2}$