# Take Home Midterm - Thomas Boyko - 30191728

1. Let $m, n$ be positive, coprime integers i.e. $GCD(m, n) = 1$. Show that if $k|mn$ then there exists integers $a, b$ such that $k = ab$ with $a|m$ and $b|n$.

   *Proof.* Let $m, n \in \mathbb{Z}$ and $GCD(m, n) = 1$. Let $k \in \mathbb{Z}$ so that $k|mn$. Write $m$ and $n$ according to the Fundamental Theorem of Arithmetic.

   $$m = p_1^{\alpha_1} p_2^{\alpha_2} \dots p_i^{\alpha_i}$$
   $$n = q_1^{\beta_1} q_2^{\beta_2} \dots q_j^{\beta_j}.$$

   With $\alpha, \beta \in 0, 1, 2 \dots$ and all $p \neq q$ (since $GCD(m, n) = 1$. Now we can write:

   $$mn = p_1^{\alpha_1} p_2^{\alpha_2} \dots p_i^{\alpha_i} q_1^{\beta_1} q_2^{\beta_2} \dots q_j^{\beta_j}.$$

   And since $k|mn$:

   $$k = p_1^{\gamma_1} p_2^{\gamma_2} \dots p_i^{\gamma_i} q_1^{\delta_1} q_2^{\delta_2} \dots q_j^{\delta_j}.$$

   With $0 \leq \gamma \leq \alpha$ and $0 \leq \delta \leq \beta$.

   If we choose $a = p_1^{\gamma_1} p_2^{\gamma_2} \dots p_i^{\gamma_i}$ and $b = q_1^{\delta_1} q_2^{\delta_2} \dots q_j^{\delta_j}$, then $ab = p_1^{\gamma_1} p_2^{\gamma_2} \dots p_i^{\gamma_i} q_1^{\delta_1} q_2^{\delta_2} \dots q_j^{\delta_j} = k$, with both $a|m$ and $b|n$.

   So there exist $a, b \in \mathbb{Z}$ so that $ab = k$ and $a|m$, $b|n$. $\square$

2. Sam-I-am wants exactly 600 of his daily calories to come from green eggs and ham. Each slice of ham has 102 calories, and each green egg has 18 calories. Find all combinations of green eggs and ham, through which this can be done?

   To begin we find $GCD(102, 18)$ and checking that it divides 600.

   $$102 = 5 \times 18 + 12$$
   $$18 = 12 \times 1 + 6$$
   $$12 = 6 \times 2 + 0.$$

   So $GCD(102, 18) = 6$, and $6|600$. Now we can setup the Diophantine Equation:

   $$18e + 102h = 600.$$

   We obtain $(e_0, h_0) = (600, -100)$ by reversing the steps we used to find $GCD$, and from this we can find general solutions:

   $$h = -100 + 3n, \quad e = 600 - 17n.$$

   Now we should check which values of $n$ give us positive numbers of both eggs and ham. If $n \leq 33$, we have negative $h$. And if $n \geq 36$ we have negative $e$. So the only valid solutions are when $n = 35$ and $n = 34$. Substituting this back into our formula for $h$ and $e$, we find that either $e = 22$ and $h = 2$ or $e = 5$ and $h = 5$.

   So Sam-I-am can have either 22 eggs and two slices of ham, or he can have 5 eggs and 5 slices of ham.

3. Find all integers $n$ such that $106n$ and $n$ has the same last two digits.

   *Proof.* We can create an equivalence which considers only the last two digits of $106n$ and $n$: $106n \equiv n \pmod{100}$. Note that $106 \equiv 6 \pmod{100}$. So we are given the new equivalence: $6n \equiv n \pmod{100} \implies 5n \equiv 0 \pmod{100}$. Converting this to a divisibility statement, we get: $100|5n$. So $100k = 5n$ for some $k \in \mathbb{Z}$.

   Finally, we obtain $n = 20k$. So for any $n$ of the form $n = 20k$, $n \equiv 106n \pmod{100}$. $\square$

4. Let $a, b$ and $n$ be positive integers. Prove that if $a^n | b^n$ then $a|b$.

*Proof.* Let $a, b, n \in \mathbb{Z}^+$, and $a^n | b^n$.

Write the prime decomposition of $a, b$:

$$\begin{aligned} a &= p_1^{\alpha_1} p_2^{\alpha_2} \dots p_i^{\alpha_i} \\ b &= p_1^{\beta_1} p_2^{\beta_2} \dots p_i^{\beta_i} \\ a^n &= p_1^{n\alpha_1} p_2^{n\alpha_2} \dots p_i^{n\alpha_i} \\ b^n &= p_1^{n\beta_1} p_2^{n\beta_2} \dots p_i^{n\beta_i}. \end{aligned}$$

Where all $p$ are primes, and all $\alpha, \beta \in \{0, 1, 2, \ldots\}$. Since $a^n | b^n$, $n\alpha_j \leq n\beta_j$ for all $1 \leq j \leq i$. Dividing by $n$ we can see that $\alpha_j \leq \beta_j$, which means $a|b$.

So for positive integers $a, b, n$, if $a^n | b^n$ then $a|n$.

□

5. Show that $x^{10} + y^{10} - 11z^{10} = 5$ has no integer solution.

*Proof.* Let the equation above be $E$. To show that our equation has no integer solutions, we must find a modulus $m$ so that $E \pmod{m}$ has no solutions.

Choose $m = 11$. Then our equation becomes $x^{10} + y^{10} \equiv 5 \pmod{11}$. Using Fermat's little Theorem, we can say that $x^{10} \equiv 1 \pmod{11}$ and we obtain $1 + 1 \equiv 2 \equiv 5 \pmod{11}$. So no matter our choices for $(x, y, z)$, our left hand side of our equation $E$ will be equivalent to 2 while the right is 5. So the equation has no solution $\pmod{11}$ and therefore no solutions in $\mathbb{Z}$. □