1. An $m \times n$ matrix is said to be a queen if the restriction of A to the orthogonal complement of its kernel is an isometry.

   (a) Show that A is a queen if and only if A^*A is an orthogonal projection.

   Solution: We know already that $(A^*A)^* = A^*A$, so me need only prove that $(A^*A)^2 = A^*A$.
   Suppose that A is a queen. Then for any $v \in (\ker A)^\perp = \operatorname{ran} A^*$, $Av = v$. But since $v \in \operatorname{ran} A^*$, there exists some w so that $A^*w = v$.

   (b) Show that A is a queen if and only if AA^* is an orthogonal projection.

   Solution: Conversely, suppose that A^*A is an orthogonal projection. Choose $v \in (\ker A)^\perp = \operatorname{ran} A^*$, so that we have some w, where $A^*w = v$.

$$\begin{aligned}
\|Av\|^2 &= \langle Av, Av\rangle \\
&= \langle A^*Av, v\rangle \\
&= \langle A^*AA^*w, A^*w\rangle \\
&= \langle AA^*AA^*w, w\rangle \\
&= \langle AA^*w, w\rangle \\
&= \langle A^*w, A^*w\rangle \\
&= \langle v, v\rangle \\
&= \|v\|^2 \\
\|Av\| &= \|v\|.
\end{aligned}$$

   (c) Show that a queen A is an isometry if and only if $\ker A = 0$.

   Solution: If $\ker A = \{0\}$, then $(\ker A)^\perp = V$, so the restriction of A to the orthogonal complement of its kernel is A restricted to its domain. Then A is an isometry on any vector.
   Conversely, suppose A is an isometry. Then:

$$\begin{aligned}
v \in \ker A &\iff Av = 0 \\
&\iff \|Av\| = \|v\| = 0 \\
&\iff v = 0.
\end{aligned}$$

   Therefore $\ker A = \{0\}$.

   (d) Find an example of a 4×2 queen that has non-zero kernel. Be sure to prove it's a queen!

   Solution: Take the matrix:

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

   Then clearly $A\begin{bmatrix} t \\ 0 \end{bmatrix} = 0$ for any $t \in \mathbb{C}$, so A has nonzero kernel, and all we must show is that A is a queen.
   Begin by observing that since $\ker A = \operatorname{span}\left\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right\}$, we can find:

$(\ker A)^\perp = \operatorname{span}\left\{\begin{bmatrix}0\\1\end{bmatrix}\right\}$. Then let $v = \begin{bmatrix}0\\t\end{bmatrix} \in (\ker A)^\perp$. Computing both $\|v\|, \|Av\|$, we see:

$$\|Av\| = \|\begin{bmatrix}0 & 1\\0 & 0\\0 & 0\\0 & 0\end{bmatrix}\begin{bmatrix}0\\t\end{bmatrix}\| = \|\begin{bmatrix}t\\0\\0\\0\end{bmatrix}\| = |t| = \|\begin{bmatrix}0\\t\end{bmatrix}\| = \|v\|.$$

So the restriction of A to the orthogonal complement of its kernel is an isometry, and A is a queen.

2. (a) Given a singular value decomposition $A = W\Sigma V^*$ of a square matrix A, construct a polar decomposition of A using W, V, Σ.

   (b) Using the method above, compute a polar decomposition for

$$A = \begin{bmatrix}1 & 3\\2 & 6\end{bmatrix}.$$

3. Find your favorite 4×2 matrix A of rank 2 and compute a singular value decomposition for A. All of the entries of A must be nonzero.

4. For an $m \times n$ matrix A, show that the set of nonzero eigenvalues for A^*A coincide with that of AA^*.

5. Suppose $A = W\Sigma V^*$ is a singular value decomposition for A. Show that the columns of W are eigenvectors for AA^*.