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1. 1

We must find the bias of each estimator. Begin with $\hat{\lambda}_1$.

$$\begin{aligned} B(\hat{\lambda}_1) &= E\left[\frac{Y_1 + Y_2}{2}\right] - \lambda \\ &= \frac{E[Y_1] + E[Y_2]}{2} - \lambda \\ &= \lambda - \lambda \\ &= 0. \end{aligned}$$

So $\hat{\lambda}_1$ is unbiased, so its MSE is simply its variance. Now for $\hat{\lambda}_2$.

$$\begin{aligned} B(\hat{\lambda}_2) &= E[\bar{Y}] - \lambda \\ &= \frac{1}{25} E\left[\sum_{i=1}^{25} Y_i\right] - \lambda \\ &= \frac{1}{25} \sum_{i=1}^{25} E[Y_i] - \lambda \\ &= E[Y_i] - \lambda \\ &= \lambda - \lambda \\ &= 0. \end{aligned}$$

And the efficiency of these two random variables is simply the ratio of their variances.

$$\begin{aligned} eff(\hat{\lambda}_1, \hat{\lambda}_2) &= \frac{V\left[\frac{Y_1 + Y_2}{2}\right]}{V[\bar{Y}]} \\ &= \frac{\frac{1}{4} V[Y_1 + Y_2]}{\frac{1}{25^2} V\left[\sum_{i=1}^{25} Y_i\right]} \\ &= \frac{\frac{\lambda}{2}}{\frac{1}{25} V\left[\sum_{i=1}^{25} Y_i\right]} && \text{By independence} \\ &= \frac{\frac{\lambda}{2}}{\frac{1}{25} \lambda} \\ &= \frac{25}{2}. \end{aligned}$$

So $\hat{\lambda}_2$ is more efficient than $\hat{\lambda}_1$.

2. 2

3. 3

$$\begin{aligned} \frac{\sum_{i=1}^n (X_i - \bar{X})^2 + \sum_{i=1}^n (Y_i - \bar{Y})^2}{2(n-1)} &= \frac{(n-1)S_X^2 + (n-1)S_Y^2}{2(n-1)} \\ &= \frac{S_X^2 + S_Y^2}{2}. \end{aligned}$$

And since we know $S_X^2 \xrightarrow{p} \sigma^2$ and $S_Y^2 \xrightarrow{p} \sigma^2$, using summation and constant properties of consistency we know that $\frac{S_X^2 + S_Y^2}{2} \xrightarrow{p} \sigma^2$, and our above expression converges in probability to σ^2 .

4. 4

5. 5

We start with a probability statement. Let $\epsilon > 0$.

$$\begin{aligned} P(|Y_{(1)} - \beta| \leq \epsilon) &= P(\beta - \epsilon \leq Y_{(1)} \leq \beta + \epsilon) \\ &= F_{Y_{(1)}}(\beta + \epsilon) - F_{Y_{(1)}}(\beta - \epsilon). \end{aligned}$$

We can see that the second term of this must be zero since $\beta - \epsilon < \beta$, ϵ is positive. So our expression becomes:

$$P(|Y_{(1)} - \beta| \leq \epsilon) = 1 - \left(\frac{\beta}{\beta + \epsilon} \right)^{\alpha n}.$$

And we apply the limit to find

$$\lim_{n \rightarrow \infty} P(|Y_{(1)} - \beta| \leq \epsilon) = \lim_{A \rightarrow \infty} 1 - \left(\frac{\beta}{\beta + \epsilon} \right)^{\alpha n} = 1 - 0 = 1.$$

(The second last equality comes from $\beta + \epsilon > \beta$, so the fraction is less than 1, and its limit tends to 0. So $Y_{(1)}$ is consistent for β

6. 6

7. 7

(a) We find an MoM estimator for β , by equating our first sample and theoretical moments.

$$\bar{X} = E[X] = \int_{\beta}^{\infty} x e^{\beta-x} dx = \beta + 1.$$

So we have $\bar{X} = \beta + 1$, and this gives $\hat{\beta}_{MoM} = \beta + 1$.

(b) We begin with our likelihood function:

$$\begin{aligned} L(\beta) &= \prod_{i=1}^n e^{\beta-x_i} \\ &= e^{\beta n} e^{\sum_{i=1}^n x_i} \\ &= e^{\beta n} e^{-n\bar{x}}. \end{aligned}$$

And from this our log-likelihood function:

$$\begin{aligned} l(\beta) &= \ln(e^{\beta n - n\bar{x}}) \\ &= n(\beta - \bar{x}). \end{aligned}$$

Setting this equal to zero,

$$\begin{aligned} 0 &= n(\beta - \bar{x}) \\ \hat{\beta}_{MLE} &= \bar{x}. \end{aligned}$$

8. 8

9. 9

We begin with our likelihood function for θ .

$$\begin{aligned} L(\theta) &= \prod_{i=1}^n (\theta + 1) y^{\theta} \\ &= (\theta + 1)^n \prod_{i=1}^{\infty} y^{\theta} \\ &= (\theta + 1)^n \left(\prod_{i=1}^{\infty} y \right)^{\theta}. \end{aligned}$$

Taking \ln for our log-likelihood,

$$\begin{aligned} l(\theta) &= \ln \left((\theta + 1)^n \prod_{i=1}^n y^{\theta} \right) \\ &= n \ln(\theta + 1) + \sum_{i=1}^n n \ln \theta \\ &= n \ln(\theta + 1) + \theta \sum_{i=1}^n \ln y. \end{aligned}$$

And now we differentiate w.r.t θ ,

$$\begin{aligned}\frac{\partial l}{\partial \theta} &= \frac{n}{\theta + 1} + \sum_{i=1}^n \ln y \\ 0 &= \frac{n}{\theta + 1} + \sum_{i=1}^n \ln y \\ \frac{n}{\theta + 1} &= - \sum_{i=1}^n \ln y \\ \frac{\theta + 1}{n} &= - \frac{1}{\sum_{i=1}^n \ln y} \\ \theta + 1 &= - \frac{n}{\sum_{i=1}^n \ln y} \\ \hat{\theta}_{MLE} &= - \frac{n}{\sum_{i=1}^n \ln y} - 1.\end{aligned}$$

10. 10

11. 11

Our null hypothesis H_0 is that the true population proportion of overweight children is $p \geq 0.15$, and our alternative H_a is that the true population proportion $p < 0.15$.

We can do this with `prop.test` in R

```
prop.test(13,100,.15,alternative="less",conf.level=.95,correct=F)
```

Our test gives us a p-value of .02877, so we reject the null hypothesis and conclude the true value of p is less than 15%.

12. 12

13. 13

- (a) We can say by looking at the histogram, that the average internet usage of the Canadians surveyed appears normally distributed.
- (b) Our null hypothesis H_0 is that the mean personal time spent online by the average Canadian in a week is less than or equal to 12.7 hours, the amount observed in 2005. This means our alternative H_a is that the average Canadian spends strictly more than 12.7 hours a week online in their personal time.

For this we use the R code:

```
> t.test(data,alternative="less",mean=12.7)
```

Which gives us the p-value of 0.9973, a test statistic of 2.8548, and since the p-value is much higher than our $\alpha = .05$, we fail to reject the null hypothesis and cannot say that the true mean is less than 12.7.

- (c) R gives us our confidence interval from `t.test`, -Inf 20.69673. Of course we cannot have any negative hours of time spent on the Internet, so our 95% CI for the true mean is:

(0, 20.69673).

14. 14