1. Exercise 6.2 #1(b) Show that $u = 1 + \sqrt{1 + \sqrt[3]{2}}$ is algebraic over $\mathbb{Q}$.

Solution: Begin with algebraic trickery.

$$\begin{aligned} u &= 1 + \sqrt{1 + \sqrt[3]{2}} \\ u - 1 &= \sqrt{1 + \sqrt[3]{2}} \\ (u + 1)^2 &= 1 + \sqrt[3]{2} \\ u^2 + 2u &= \sqrt[3]{2} \\ (u^2 + 2u)^3 &= 2 \\ u^6 + 6u^5 + 12u^4 + 8u^3 &= 2 \\ u^6 + 6u^5 + 12u^4 + 8u^3 - 2 &= 0. \end{aligned}$$

So we have found a polynomial $f(x) = x^6 + 6x^5 + 12x^4 + 8x^3 - 2$ so that $f(u) = 0$.

2. Exercise 6.2 #7 Find the minimal polynomial of $u = \sqrt{3} - i$ over $\mathbb{Q}$ and also over $\mathbb{R}$.

Solution: Use a similar strategy as above for $\mathbb{Q}$:

$$\begin{aligned} u &= \sqrt{3} - i \\ u^2 &= 3 - 2\sqrt{3}i - 1 \\ (u^2 - 2)^2 &= 12 \\ u^4 - 4u^2 + 4 &= 12 \\ u^4 - 4u^2 + 16 &= 0. \end{aligned}$$

We can use the Modular Irreducibility test (Nicholson Theorem 4.2.7), with $p = 3$, to reduce our polynomial to $f(x) = x^4 - x^2 + 1 = 0$. Then $f(0) = f(1) = f(2) = 1$ so the polynomial has no roots in $\mathbb{Z}_3$ and by the theorem, it is irreducible over $\mathbb{Q}$. Now since it is monic and has u as a root, we can say it is minimal.

In $\mathbb{R}$, we find a mininmal polynomial in $\mathbb{R}$ by attempting to eliminate the imaginary components to a polynomial with a root u.

$$\begin{aligned} u &= \sqrt{3} - i \\ u - \sqrt{3} &= i \\ u^2 - 2u\sqrt{3} + 3 &= -1 \\ u^2 - 2u\sqrt{3} + 4 &= 0. \end{aligned}$$

Then we can see the discriminant of the polynomial $g(x) = x^2 - 2\sqrt{3}x + 4$ is $2\sqrt{3} - 16$ which is negative, so the polynomial is irreducible over $\mathbb{R}$. And since it is monic, and has u as a root, it is minimal.

3. Exercise 6.2 #20 Let $\mathbb{K}$ be a field extension of $\mathbb{E}$ which is a field extension of F, and let $[\mathbb{E} : \mathbb{F}]$ be finite. Let $u \in \mathbb{K}$ be algebraic over E.

   (a) Show that $[\mathrm{E}(u) : \mathbb{E}] \leq [\mathbb{F}(u) : \mathbb{F}]$.

   Solution: Let $\mathbb{K} \supseteq \mathbb{E} \supseteq \mathbb{F}$, and $[\mathbb{E} : \mathbb{F}] = n$ be finite. Then let u be algebraic over $\mathbb{K}$ with minimal polynomial $m(x) \in \mathbb{F}[x]$. Now we proceed by cases.

   If m is irreducible in $\mathbb{E}$, then it remains the minimal polynomial for u in $\mathbb{E}$, and the degrees of the two extensions are equal.

   Now, if m is reducible in $\mathbb{E}$, then there exists $f, g \in \mathbb{E}[x]$, both with degree less than m, and $m = fg$. We take the one which has u as a root and repeat the process until we reach an irreducible polynomial $h \in \mathbb{E}(u)$ with u as a root. If this is not monic, we factor out the leading coefficient, and we will have a $h' \in \mathbb{E}(u)$ with $h'(u) = 0$ that is monic and irreducible. This has degree strictly less than m, and is minimal for u.

   After all this we have $[\mathbb{E}(u) : \mathbb{E}] = \deg h' \leq \deg m = [\mathbb{F}(u) : \mathbb{F}]$.

   (b) Show that $[\mathbb{E}(u) : \mathbb{F}(u)] \leq [\mathbb{E} : \mathbb{F}]$. (Hint: Theorem 6.1.6.) Take the same minimal polynomials we found above; and rewrite both:

$$\begin{aligned}
[\mathbb{E}(u) : \mathbb{F}(u)] &= [\mathbb{E}(u) : \mathbb{E}][\mathbb{E} : \mathbb{F}(u)] \\
&= [\mathbb{E}(u) : \mathbb{E}]\frac{[\mathbb{E} : \mathbb{F}]}{[\mathbb{F}(u) : \mathbb{F}]} \\
&= \deg h' \frac{n}{\deg m}.
\end{aligned}$$

   Then since $\deg h' \leq \deg m$, $\frac{\deg h'}{\deg m} \leq 1$. So then $[\mathbb{E}(u) : \mathbb{F}(u)] = \frac{n \deg h'}{\deg m} \leq n = [\mathbb{E} : \mathbb{F}]$.

4. Exercise 6.3 #4(a) and 4(b). Find the splitting field $\mathbb{E}$ of $f(x) = x^3 + 1$ over $\mathbb{F} = \mathbb{Z}_2$ and factor $f(x)$ completely in $\mathbb{F}[x]$. Then, do the same thing but replace $\mathbb{F} = \mathbb{Z}_2$ with $\mathbb{F} = \mathbb{Z}_3$ (see the statement in the textbook).

   Solution: For $\mathbb{Z}_2$, we check the elements. $f(0) = 0^3 + 1 = 1$, and $f(1) = 1^3 + 1 \equiv 0$, so 1 is a root of the polynomial. Rewrite $f(x) = (x + 1)(x^2 + x + 1)$. Then let $f'(x) = x^2 + x + 1$, and check that this is also irreducible in $\mathbb{Z}_2$:

$$f'(0) = 0^2 + 0 + 1 = 1 \neq 0, \quad f'(1) = 1^2 + 1 + 1 = 1 \neq 0.$$

   So this polynomial has no roots in $\mathbb{Z}_2$ and therefore is irreducible. Let α be such that $f'(\alpha) = 0$. Then there extists (By Kronecker's Theorem) a field extension of F in which α is a root, and $\alpha^2 + \alpha \equiv 1 \pmod 2$. So we can factor $f'(x) = (x + \alpha)(x + \alpha + 1)$. And so $f(x) = (x + 1)(x + \alpha)(x + \alpha + 1)$, so x splits over $\mathbb{Z}_2(\alpha)$. And since f' is monic and irreducible, it is the minimal polynomial for α. Having degree 2, we can say that $[\mathbb{Z}_2(\alpha) : \mathbb{Z}_2] = 2$, and since $|\mathbb{Z}_2| = 2$, by the multiplication theorem $|\mathbb{Z}_2(a)| = 4$. Then we can finally say by the characterization of finite fields that $\mathbb{Z}_2(a) \cong \mathbb{F}_4$.

   Now, in $\mathbb{Z}_3$ we can see that $f(2) = 9 \equiv 0 \pmod 3$, so $2 \equiv -1$ is a root of f. Rewrite, $f(x) = (x + 1)(x^2 + 2x + 1) = (x + 1)^3$. So the splitting field for f over $\mathbb{Z}_3$ is $\mathbb{Z}_3$.

5. Exercise 6.3 #9 Let $f(x)$ and $g(x)$ be polynomials in $F[x]$. Show that $f(x)$ and $g(x)$ are relatively prime (have no common nonconstant factors) in $F[x]$ if and only if they have no common root in any extension E of F.

   Solution:

   $\Longrightarrow$ **:** Let $f, g \in F[x]$ be coprime, and suppose for the sake of contradiction that they have a common root a in some extension E of F. Then $f(x) = f'(x)(x - a)$ and $g(x) = g'(x)(x - a)$. Then $x - a$ is a common nonconstant factor, a contradiction! Therefore f, g must have no common factor in $F[x]$.

$\Longleftarrow$: Suppose that f, g share no root in any $E \supseteq F$. Then suppose for the sake of contradiction that there exists some $h \in F[x]$, and $g = g'h$, $f = f'h$. This polynomial must have a root in some extension of F, say $u \in K$. Then $g(u) = g'(u)h(u) = 0 = f'(u)h(u) = f(u)$. So u is a root of both f, g, a contradiction. Then by contradiction f, g must be coprime.

6. Exercise 6.3 #17 If E over F is an algebraic extension and every polynomial in $F[x]$ splits over E, show that E is algebraically closed. (Hint: Theorem 6.2.6)

Solution: Let $E \supseteq F$ be an algebraic extension, where every polynomial in $F[x]$ splits over E. Then suppose for the sake of contradiction that E is not algebraically closed. So there exists some $f(x) \in E[x]$, where f has degree greater than one, and is irreducible. Write $f(x) = a_0 + a_1 x + \ldots + a_n x^n$, with each $a_i \in E$. Take the field extension $F(a_0, \ldots, a_n) = F(a_0)(\ldots)(a_n)$. Since E is an algebraic extension, each a_i has a finite degree monic polynomial, each adjoined element on F produces another finite degree extension (Theorem 6.2.6). Since this extension is finite, it must be algebraic.