Assignment 3 - Thomas Boyko - 30191728

1. An urn contains two red balls, four blue balls, and three green balls. You randomly select a ball from the urn, record its color, and do not return it to the urn. You do this process twice (for a total of two recorded colors). Let X represent the number of red balls that you observed.

   Suppose $X \sim Hypergeometric(2, 9, 2)$

   (a) Create a probability distribution table for X.

   ```
   dhyper(0:2,2,7,2)
   ```

x	0	1	2
P(X=x)	0.58333333	0.38888889	0.02777778

   (b) How many red balls should you expect to select in this process?

   $$E[X] = \frac{nr}{N} = \frac{2(2)}{9} = \frac{4}{9}$$

   So the expected value is 0.44444.

2. A certain region of the Florida coast experiences 4.04 hurricanes per year.

   Suppose $X \sim Poisson(4.04)$.

   (a) What is the probability that this region experiences exactly 10 hurricanes in a two-year period? Modify the distribution to consider a two year period rather than one (See part b). This gives $\lambda = 8.08$.

   ```
   dpois(10,8.08)
   #0.1012165
   ```

   (b) How many hurricanes would we expect this region to experience in a two-year period?

   $$E[2X] = 2E[X] = 2\lambda = 8.08$$

3. Let $f(x) = \begin{cases} 2x - 1 & 1 \leq x \leq 2 \\ 0 & elsewhere \end{cases}$

   Show that $f(x)$ is not a valid pdf.

   Consider:

   $$\begin{aligned} \int_1^2 2x - 1dx &= x^2 - x|_1^2 \\ &= (2^2 - 2) - (1^2 - 1) \\ &= 2 - 0 \\ &= 2 \neq 1 \end{aligned}$$

   So f is not a valid pdf since the integral over all nonzero values is not 1.

4. The moment generating function for a Poisson random variable X with parameter λ is $M_X(t) = e^{\lambda(e^t - 1)}$ Use this moment generating function to show that $E[X] = \lambda$.

   $$\begin{aligned} E[X] &= \frac{d}{dx} M_X(t) \\ &= \frac{d}{dx}(e^{\lambda(e^t-1)}) \\ &= (\lambda e^t e^{\lambda(e^t-1)}) \\ t = 0: \qquad &= \lambda e^0 e^{\lambda(e^0-1)} \\ &= \lambda 1 e^{\lambda(0)} \\ &= \lambda \end{aligned}$$

5. Let $f(x) = \begin{cases} 2x & 0 \leq x \leq 1 \\ 0 & elsewhere \end{cases}$

(a) Find $E[X]$.

$$\begin{aligned} E[X] &= \int_0^1 xf(x)dx \\ &= \int_0^1 2x^2 dx \\ &= \frac{2x^3}{3}|_0^1 \\ &= \frac{2}{3} \end{aligned}$$

So the expected value is $\frac{2}{3}$.

(b) Find $P(X \leq \frac{1}{2})$

$$\begin{aligned} P(X \leq \frac{1}{2}) &= F(\frac{1}{2}) \\ &= \int_0^{\frac{1}{2}} 2xdx \\ &= x^2|_0^{\frac{1}{2}} \\ &= \frac{1}{4} \end{aligned}$$

So $P(X \leq \frac{1}{2}) = \frac{1}{4}$.

6. The duration of a pulse of light from a device follows a uniform distribution with a = 10 seconds and b = 20 seconds. Let X represent the duration of the pulse (in seconds). If the duration of a specific pulse is at least 13 seconds long, what is the probability that the duration of the pulse is at most 18 seconds long?

We can say: $X \sim Uniform(10, 20)$ and $f(x) = \frac{1}{10}$.

$$P(X \leq 18|X \geq 13) = \frac{P(13 \leq X \leq 18)}{P(X \geq 13)} = \frac{F(18) - F(13)}{1 - F(13)}$$

```
(punif(18,10,20)-punif(13,10,20))/(1-punif(13,10,20))
#0.7142857
```

So there is a 0.7142857 chance that the pulse is at most 18 seconds long, given that the pulse is at least 13 seconds long

7. The finishing time for a particular half-marathon is normally distributed with $\mu = 131.9$ minutes and $\sigma = 20.3$ minutes.

   So we say $X \sim Normal(131.9, 20.3)$.

   (a) What is the probability that a given runner's finishing time is less than two hours?

   ```
   pnorm(120,131.9,20.3)
   #0.2788682
   ```

   So the probability that a runner's finishing time is less than two hours is 0.27887.

   (b) What is the fastest a runner can finish the half-marathon and still be in the slowest 20% of finishers?

   ```
   qnorm(0.2,131.9,20.3)
   #114.8151
   ```

   So the fastest a runner can finish while being in the slowest 20% of runners is 113.8151 minutes.

   (c) Suppose 10 random finishers are selected. What is the probability that exactly four of the finishers selected took less than two hours to finish the half-marathon?

   Each runner's time being under two hours can be considered a Bernouli trial with $p = 0.27887$. So we can model our 10 trials with a Binomial Distribution.

   Suppose $Y \sim Binomial(0.27887, 10)$.

   ```
   dbinom(4,0.2788682,10)
   #0.1786089
   ```

   So the probability that exactly four of the ten runners selected took less than two hours to complete the half-marathon is 0.17861.

8. The number of times per hour a printer is used at a busy office can be modeled with a Poisson distribution with $\lambda = 4.3$. Let W represent the amount of time (in hours) that passes between consecutive uses.

   (a) What type of distribution can be used to model the behavior of W? State any relevant parameter values as well.

   $W \sim Exponential\left(\frac{1}{4.3}\right)$.

   (b) What is the probability that fewer than 30 minutes pass between consecutive uses of this printer?

   ```
   pexp(0.5,4.3)
   #0.8835158
   ```

   So the probability that fewer than 30 minutes pass between consecutive uses of the printer is 0.10977.