E2+TE2

SCHOOL OF ADVANCED SCIENCES
FALL SEMESTER 2021-2022 - CAT-1

S.No.	Question	Marks
1(a)	(i) Test whether the Rolle's theorem holds for the function $f(x)=\begin{cases}-x^{\frac{1}{3}}, -1\le x\le 0, \text{ then} \\ x^{\frac{1}{3}}, 0<x\le 1\end{cases}$ in the interval $[-1,1]$. Ans: Clearly, the given function is not differentiable at $x=0$, Therefore, Rolle's theorem does not apply to f in $[-1,1]$ (ii) If $f(x)=(x-1)(x-3)(x-5), a=0, b=4$. Find the value of c such that $f'(c)$ has the same value as the slope of the chord joining the points for which $x=0$ and $x=4$. Ans: We know that a polynomial function is continuous as well as differentiable for all x. Since, $f(x)=(x-1)(x-3)(x-5)$ is a polynomial in x, therefore it is continuous in [0,4] and differentiable in (0,4). Since, $f(x)$ satisfies all the conditions of Lagrange's mean value theorem, therefore there must exist some $c\in(0,4)$, such that $\frac{f(4)-f(0)}{4-0}=f'(c)$ $\Rightarrow 3c^2-18c+23=3$ $\Rightarrow 3c^2-18c+20=0$ $\therefore c=3\pm\frac{\sqrt{21}}{3}$ Or $c=4.5275252316519, 1.4724747683481$ Only, one value of c lie in the open interval. Therefore, the slope to the curve at these points is not same as the slope of chord joining the points.	5M 5M
1(b)	(i) Test whether the Rolle's theorem holds for the function $f(x)=\begin{cases}x^2, & \text{if } 0\le x\le 2 \\ 6-x, & \text{if } 2<x\le 6\end{cases}$ Ans: Clearly, the given function is not differentiable at $x=2$, Therefore, Rolle's theorem does not satisfied. (ii) If $f(x)=x(x-1)$ in $[1,2]$. Find the value of c using Lagrange's mean value theorem. Ans: We know that a polynomial function is continuous as well as differentiable for all x. Since, $f(x)=x(x-1)$ is a polynomial in x, therefore it is continuous in [1,2] and differentiable in (1,2). Since, $f(x)$ satisfies all the conditions of Lagrange's mean value theorem, therefore there must exist some $c\in(1,2)$, such that $\frac{f(2)-f(1)}{2-1}=f'(c)$ $\Rightarrow 2c-1=2$ $\Rightarrow 2c=3$ $\therefore c=\frac{3}{2}$	5M 5M
1(c)	Identify the intervals on which the function $f(x)=2x^3-9x^2+12x-5$ is (i) Increasing and decreasing, (ii) Stationary points, (iii) Absolute maximum and minimum, (iv) Points of local maxima and local minima. Ans: Given function is $f(x)=2x^3-9x^2+12x-5$	10M

	Differentiating the above polynomial and equating to zero, we get $x=1,2$ (i) Increasing on $(-\infty,1)\cup(2,\infty)$ and decreasing on $(1,2)$ (ii) Stationary points are 1, 2 (iii) Absolute maxima is 0 and minima is -1, (iv) Points of local maxima is $(1,0)$ and local minima is $(2,-1)$	
1(d)	Identify the intervals on which the function $f(x)=-2x^3-9x^2-12x+1$ is (i) Increasing and decreasing, (ii) Stationary points, (iii) Absolute maximum and minimum, (iv) Points of maxima and minima. Ans: Given function is $f(x)=-2x^3-9x^2-12x+1$ Differentiating the above polynomial and equating to zero, we get $x=-2,-1$ (i) Increasing on $(-2,-1)$ and decreasing on $(-\infty,-2)\cup(-1,\infty)$ (ii) Stationary points are $-2,-1$ (iii) Absolute maxima is 6 and minima is 5 (iv) Points of local maxima is $(-2,5)$ and local minima is $(-1,6)$	10M
2(a)	Find the volume of the solid generated by revolving the region bounded by $y=\sqrt{3},\ x=3$ and $x=3-y^2$ about the $x-$axis. Ans: Volume V = $\dfrac{9\pi}{2}$ cubic units.	.10M
2(b)	Find the volume of the solid obtained by rotating the region bounded by $y=2\sqrt{x-1}$ and $y=x-1$ about the line $x=-1$. Ans: Inner radius is $\dfrac{y^2}{4}+1+1=\dfrac{y^2}{4}+2$ and Outer radius is $y+1+1=2$ $A(y)=\pi\left[(y+2)^2-\left(\dfrac{y^2}{4}+2\right)^2\right]=\pi\left(4y-\dfrac{y^2}{16}\right)$ $V=\pi\int_0^4\left(4y-\dfrac{y^4}{16}\right)dy=\dfrac{96}{5}\pi$	10M
2(c)	Find the volume of the region obtained by rotating the region bounded by $y=6e^{-2x}$ and $y=6+4x-2x^2$ between $x=0$ and $x=1$ about the line $y=-2$. Ans: Inner radius is $2+6e^{-2x}$ and Outer radius is $8+4x-2x^2$ $A(x)=\pi\left[(\text{Outer radius})^2-(\text{Inner radius})^2\right]$ $=\pi\left[60+64x-16x^2-16x^3+4x^4-24e^{-2x}-36e^{-4x}\right]$ $V=\int_0^1\pi\left[60+64x-16x^2-16x^3+4x^4-24e^{-2x}-36e^{-4x}\right]dx$ $=\left(\dfrac{937}{15}+12e^{-2}+9e^{-4}\right)\pi$	10M
2(d)	Find the volume of the region obtained by rotating the region bounded by $y=10-6x+x^2$ and $y=-10+6x-x^2$, $x=1$ and $x=5$ about the line $y=8$.	10M
3(a)	(i) Find the values of the constant parameter λ for which the functions $u=\cos x\cos y-\lambda\sin x\sin y,\ v=\sin x\cos y+\lambda\cos x\sin y$ are functionally dependent. Ans: Given that $u=\cos x\cos y-\lambda\sin x\sin y,\ v=\sin x\cos y+\lambda\cos x\sin y$ We know that if u and v are functionally related, then $\dfrac{\partial(u,v)}{\partial(x,y)}=0$	5M 5M

Now $\frac{\partial(u,v)}{\partial(x,y)}=\begin{vmatrix}\frac{\partial u}{\partial x} & \frac{\partial u}{\partial y}\\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y}\end{vmatrix}=0$

$$\frac{\partial(u,v)}{\partial(x,y)}=\begin{vmatrix}-\sin x\cos y-\lambda\cos x\sin y & -\cos x\sin y-\lambda\sin x\cos y\\ \cos x\cos y-\lambda\sin x\sin y & -\sin x\sin y+\lambda\cos x\cos y\end{vmatrix}=0$$

$\Rightarrow(-\sin x\cos y-\lambda\cos x\sin y)(-\sin x\sin y+\lambda\cos x\cos y)$

$-(-\cos x\sin y-\lambda\sin x\cos y)(\cos x\cos y-\lambda\sin x\sin y)=0$

$\Rightarrow(-\sin x\cos y)(-\sin x\sin y)=\sin^2 x\sin y\cos y,$

$\Rightarrow(-\sin x\cos y)(\lambda\cos x\cos y)=-\lambda\sin x\cos x\cos^2 y$

$\Rightarrow(-\lambda\cos x\sin y)(-\sin x\sin y)=\lambda\sin x\cos x\sin^2 y,$

$\Rightarrow(-\lambda\cos x\sin y)(\lambda\cos x\cos y)=-\lambda^2\cos^2 x\sin y\cos y$

$\sin^2 x\sin y\cos y-\lambda\sin x\cos x\cos^2 y+\lambda\sin x\cos x\sin^2 y-\lambda^2\cos^2 x\sin y\cos y$

$(-\cos x\sin y-\lambda\sin x\cos y)(\cos x\cos y-\lambda\sin x\sin y)$

$\Rightarrow(-\cos x\sin y)(\cos x\cos y)=-\cos^2 x\sin y\cos y,$

$\Rightarrow(-\cos x\sin y)(-\lambda\sin x\sin y)=\lambda\sin x\cos x\sin^2 y$

$\Rightarrow(-\lambda\sin x\cos y)(\cos x\cos y)=-\lambda\sin x\cos x\cos^2 y,$

$\Rightarrow(-\lambda\sin x\cos y)(-\lambda\sin x\sin y)=\lambda^2\sin^2 x\sin y\cos y$

$\Rightarrow-\cos^2 x\sin y\cos y+\lambda\sin x\cos x\sin^2 y-\lambda\sin x\cos x\cos^2 y+\lambda^2\sin^2 x\sin y\cos y$

Now $\sin^2 x\sin y\cos y-\lambda\sin x\cos x\cos^2 y+\lambda\sin x\cos x\sin^2 y-\lambda^2\cos^2 x\sin y\cos y$

$+\cos^2 x\sin y\cos y-\lambda\sin x\cos x\sin^2 y+\lambda\sin x\cos x\cos^2 y-\lambda^2\sin^2 x\sin y\cos y=0$

$\sin^2 x\sin y\cos y-\lambda^2\cos^2 x\sin y\cos y+\cos^2 x\sin y\cos y-\lambda^2\sin^2 x\sin y\cos y=0$

$\Rightarrow\sin y\cos y(\sin^2 x+\cos^2 x)-\lambda^2\sin y\cos y(\cos^2 x+\sin^2 x)=0$

$\Rightarrow\sin y\cos y-\lambda^2\sin y\cos y=0$

$\Rightarrow\sin y\cos y(1-\lambda^2)=0$

$\Rightarrow 1-\lambda^2=0$

$\boxed{\therefore\lambda=\pm1}$

(ii) If $x=e^u\cos v,\ y=e^u\sin v$, Find J and J', Verify $JJ'=1$ or not?

Ans: Given that $x=e^u\cos v,\ y=e^u\sin v$

then $J=\frac{\partial(x,y)}{\partial(u,v)}=\begin{vmatrix}\frac{\partial x}{\partial u} & \frac{\partial x}{\partial v}\\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v}\end{vmatrix}$

$$J=\begin{vmatrix}e^u\cos v & -e^u\sin v\\ e^u\sin v & e^u\cos v\end{vmatrix}=(e^u\cos v)(e^u\cos v)-(e^u\sin v)(-e^u\sin v)$$

$\Rightarrow e^{2u}\cos^2 v+e^{2u}\sin^2 v=e^{2u}$

$\therefore J=e^{2u}$

$J'=\frac{\partial(u,v)}{\partial(x,y)}=\begin{vmatrix}\frac{\partial u}{\partial x} & \frac{\partial u}{\partial y}\\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y}\end{vmatrix}$

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<td>3(b)</td>
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(i) Find the values of the constant parameter k for which $u = kx^2 + 4y^2 + z^2, v = 3x + 2y + z, w = 2yz + 3zx + 6xy$ are functionally related, and obtain the corresponding relation.

Ans: Given that $u = kx^2 + 4y^2 + z^2$, $v = 3x + 2y + z, w = 2yz + 3zx + 6xy$

We know that if u and v are functionally related, then $\frac{\partial(u,v,w)}{\partial(x,y,z)} = 0$

Now $\frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix} = 0$

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} 2kx & 8y & 2z \\ 3 & 2 & 1 \\ 3z+6y & 2z+6x & 2y+3x \end{vmatrix} = 0$$

$$\Rightarrow 2kx\left[2(2y+3x)-(2z+6x)\right] - 8y\left[3(2y+3x)-(3z+6y)\right]$$
$$+2z\left[3(2z+6x)-2(3z+6x)\right] = 0$$
$$\Rightarrow 2kx[4y+6x-2z-6x] - 8y[6y+9x-3z-6y] + 2z[6z+18x-6z-12x] = 0$$
$$\Rightarrow 2kx[4y-2z] - 8y[9x-3z] + 2z[18x-12x] = 0$$
$$\Rightarrow 2kx[4y-2z] - 8y[9x-3z] + 12xz = 0$$
$$\Rightarrow 8kxy - 4kxz - 72xy + 24yz + 12xz = 0$$
$$\Rightarrow 8kxy - 72xy - 4kxz + 12xz + 24yz = 0$$
$$\Rightarrow 8xy(k-9) - 4xz(k-3) + 24yz = 0$$

Therefore $k = 3$ or 9

(ii) If $f(x,y) = \begin{cases} \frac{x^2-xy}{x+y}; (x,y) \neq (0,0) \\ 0; (x,y) = (0,0) \end{cases}$, Then $f_x(0,0), f_y(0,0)$.

Ans: $f_x(0,0) = 1;\ f_y(0,0) = 0$

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<td>3(c)</td>
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(i) Verify the variables x and y are given by $x = \frac{s+t}{s}, y = \frac{s+t}{t}$ are functionally dependent, and obtain the relationship $f(x,y)$.

(ii) If $f(x,y) = \begin{cases} xy\tan\left(\frac{y}{x}\right); (x,y) \neq (0,0) \\ 0; (x,y) = (0,0) \end{cases}$, Then find $xf_x + yf_y$ at (0,0)

Ans: $xf_x + yf_y = 2f$

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<td>3(d)</td>
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(i) If the thermodynamics variables P,V,T are connected by a relation $f(P,V,T) = 0$, Then find the value of $\frac{\partial P}{\partial T} \times \frac{\partial T}{\partial V} \times \frac{\partial V}{\partial P}$.

Ans: $\left(\frac{\partial P}{\partial T}\right) \times \left(\frac{\partial T}{\partial V}\right) \times \left(\frac{\partial V}{\partial P}\right) = -1$

(ii) The power consumed in an electric resistor is given by $P = \frac{E^2}{R}$ (in Watts). If $E = 80$ Volts and $R = 5$ ohms, by how much the power consumption will charge if E is increased

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	by 3 Volts and R is increased by 0.1. Ans: 121.6 Watts.	