Family Name ____________________

First Name ____________________

Student Number |__|__|__|__|__|__|__|__|

Venue ____________________

Seat Number ________

UNIVERSITY OF CANTERBURY
Te Whare Wānanga o Waitaha
CHRISTCHURCH NEW ZEALAND

No electronic/communication devices are permitted.

No exam materials may be removed from the exam room.

Mathematics and Statistics

EXAMINATION

End-of-year Examinations, 2020

MATH101-20S2 (C) Methods of Mathematics

For Examiner Use Only

Question	Mark
Q1	
Q2	
Q3	
Q4	
Q5	
Q6	

Total ________

Examination Duration: 180 minutes

Exam Conditions:

Closed Book exam: Students may not bring in any written or printed materials.

No calculators are permitted.

Materials Permitted in the Exam Venue:

None.

Materials to be Supplied to Students:

1 x Write-on question paper/answer book.

This exam has 1 attachment (not an answer sheet).

Instructions to Students:

Answer all SIX questions.

There is a total of 105 marks.

Write your answers in the spaces provided.

Use blue or black pen only.

Show all your working.

Marks will be lost for poorly presented or incomplete answers.

Blank page for rough working

Questions start on page 3

Question One (18 marks)

(a) Fully factorise the expressions below:

(i) $4x^2 - 16$

(ii) $2t^2 + 2t - t - 1$

(b) A swimming pool is filled by 2 pipes of different sizes.

If the smaller pipe is operating on its own it can fill the entire pool in time T.

If the larger pipe is operating on its own it can fill the entire pool in time $\frac{1}{2}T$.

If both pipes are operating together the time taken to fill the entire pool is X where

$$\frac{1}{T} + \frac{1}{\frac{1}{2}T} = \frac{1}{X}$$

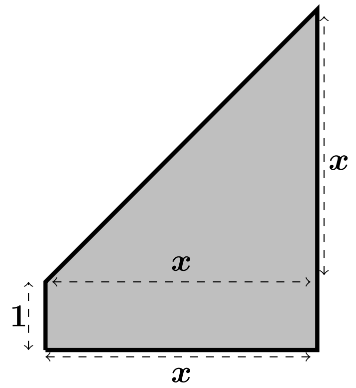
(i) Rearrange this equation to make X the subject. Simplify your answer as much as possible.

(ii) If $T = \frac{1}{2}$, measured in days, find the value of X in days.

(c) Solve the following equation for t:

$$\ln(2t+2) = \ln(t-8) + \ln(t+1)$$

(d) A piece of metal is to be cut according to the diagram below (not to scale)



The shape of the metal can be thought of as a right-angled triangle plus a rectangle, as shown on the diagram.

The height of the triangle is x m and the base of the triangle is x m. The rectangle has one side of length x m and one side of length 1 m.

If the total area of the surface of the metal shown in the diagram (the grey shaded region) is 4 m^2, find the value of x.

(e) The position of the point of a sewing machine needle over time is modeled using the function

$$p(t) = a\cos(bt) + d$$

where $p(t)$ is the height of the point of the needle in centimetres at time t and t is measured in seconds.

At $t = 0$ the point of the needle is at its maximum height of 4 cm above the platform.

The minimum height of the point of the needle is 1 cm below the maximum height.

The position of the needle completes one cycle 4 times every second.

Use this information to answer the following questions:

(i) Find the value of a in the model above.

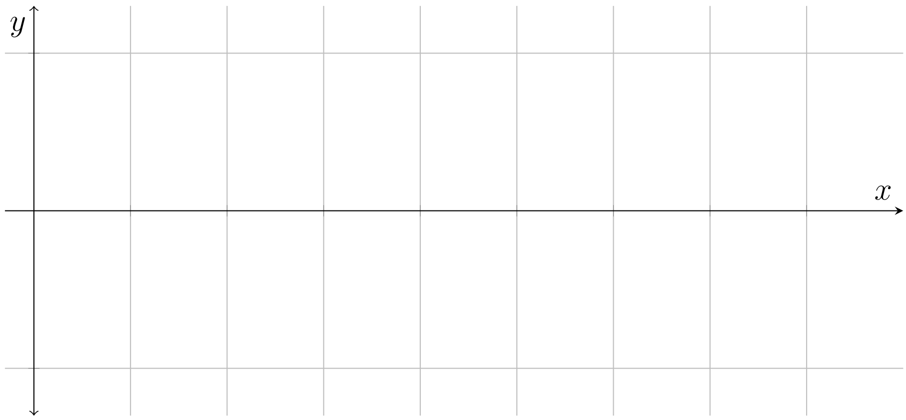
(ii) Find the period of the function $p(t)$.

(iii) Find the value of b in the model above.

(iv) Find the value of d in the model above.

(f) Sketch one complete cycle of $y = \sin(x)$ from the point where $x = 0$ on the axes below.

You will need to add a scale to the horizontal and vertical axes as well as showing the shape of the function.



Spare axes for part (f). If you use the spare axes please show clearly which sketch should be marked.

(g) Consider the function

$$f(x) = \frac{x^2 - 2x}{x}$$

(i) $f(x)$ is not defined for a particular value $x = a$. Give this value a.

(ii) Find $\lim_{x \to a} f(x)$ for your value of a from part (i).

Question Two (18 marks)

(a) The total downloads of an e-book are modelled by the formula

$$B = \frac{170t^2}{t^2 + 4}$$

where B is the total downloads of the book in millions and t is the number of years since the release of the book.

(i) Calculate the number of downloads when $t = \frac{1}{2}$.

(ii) Rearrange the formula above to make t the subject.

(There is additional space for your answer at the top of page 11 if necessary.)

(iii) What does the formula predict will happen to the total downloads (in millions) of the e-book over the long term (as $t \to \infty$)?

(iv) Give the **units** that would be used for $\dfrac{dB}{dt}$

(b) Differentiate the following with respect to x:

(i) $y = \frac{1}{2} + \frac{2}{7}x^3$

(ii) $y = e^{2x}$

(iii) $y = x\,\sin(x)$

(iv) $y = \dfrac{x+1}{x-1}$

(c) The operator of a ferry is trying to model the running costs of the ferry.

(i) Average speed is distance travelled divided by time taken.

If the ferry travels $2\,\text{km}$ in $^2/_3$ hours, find the average speed of the ferry over this time.

Remember to include the **units** with your answer.

(ii) Again given that average speed is distance travelled divided by time taken, if the ferry travels at an average speed of v km per hour show that the **time in hours** that the ferry takes to travel $5\,\text{km}$ is $\frac{5}{v}$.

The operator models the running costs (in \$) per hour when the ferry travels at an average speed v km per hour as:

$$\text{Cost per Hour} = 400 + 4v^2$$

The total cost of a journey in \$ is calculated as

$$\text{Total Cost} = \text{Cost per Hour} \times \text{Time Taken}$$

(iii) Use the formula Total Cost $=$ Cost per Hour $\times$ Time Taken with the formula for Cost per Hour given above and your answer from part (ii) for Time Taken to show that the total cost C (in \$) as a function of speed v for the 5 km trip is given by

$$C(v) = \frac{2000}{v} + 20v, \qquad v \neq 0$$

The formula for the total cost C (in \$) as a function of speed v for the 5 km trip from part (iii) is

$$C(v) = \frac{2000}{v} + 20v, \qquad v \neq 0$$

(iv) Find the value of v that will minimise $C(v)$, the cost of the 5 km trip.

Remember to include the **units** with your answer.

(v) Use calculus methods to verify that the value you have given in part (iv) gives a **minimum** cost.

Question Three (15 marks)

(a) The flight path of a model rocket is given by the function

$$y(x) = -x + 8\sqrt{x}$$

where x is the horizontal displacement in metres from the launch point and $y(x)$ is the vertical displacement (height) of the rocket above the flat ground in metres at horizontal displacement x. The rocket is launched at $x = 0$.

(i) What is the height of the rocket when it has travelled 4 m horizontally?

(ii) What is the height of the rocket when it has travelled $1/4$ m horizontally?

(iii) How far does the rocket travel horizontally before it returns to the ground?

(iv) Give the **domain** of the function $y(x)$ in the context of the situation being modelled. Use **interval notation** for your answer.

(b) The quantity Q of a drug (in mg) present in the body t hours after an injection is modeled using the function

$$Q(t) = 100\,t\,e^{-0.5t}$$

Use this model to answer the following questions:

(i) What amount of the drug is present initially (at t = 0 hours)?

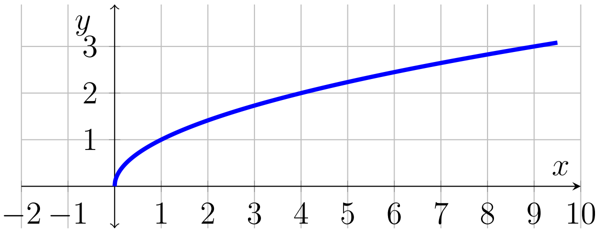
(ii) Find the amount of the drug present after 5 hours.

(Hint: $e^{-2.5} \approx 0.08$)

(iii) Find ${}^{dQ}\!/_{dt}$, the rate of change of the quantity of the drug present with respect to time, as an expression in terms of t.

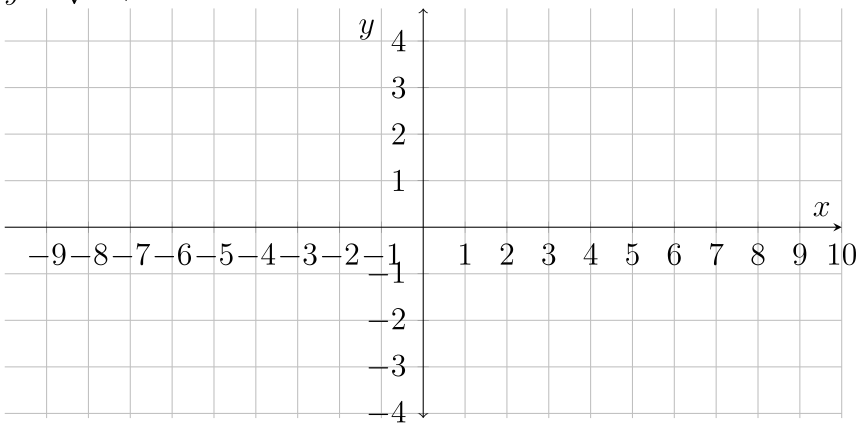
(iv) Is the quantity of the drug present increasing or decreasing when $t = 5$? Briefly justify your response using your answer to part (iii).

(c) The graph of $y = \sqrt{x}$ is shown below.

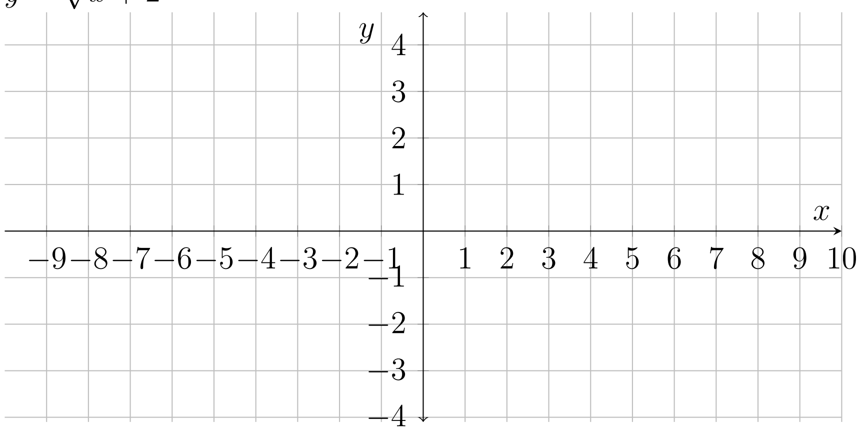


Sketch the following transformations of $y = \sqrt{x}$ on the axes provided.
There are spare axes on page 23.

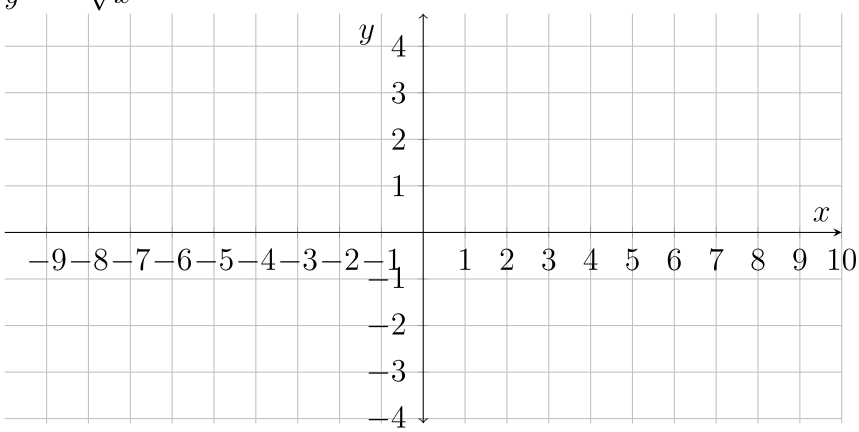
(i) $y = \sqrt{x+2}$



(ii) $y = \sqrt{x} + 2$



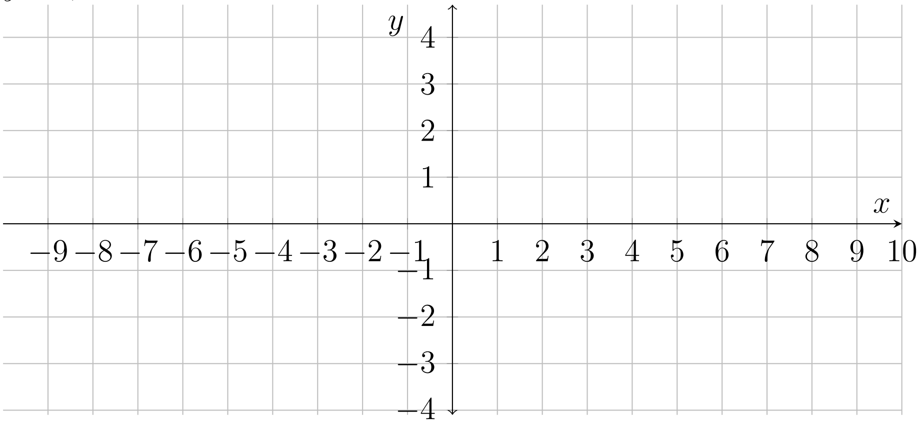
(iii) $y = -\sqrt{x}$



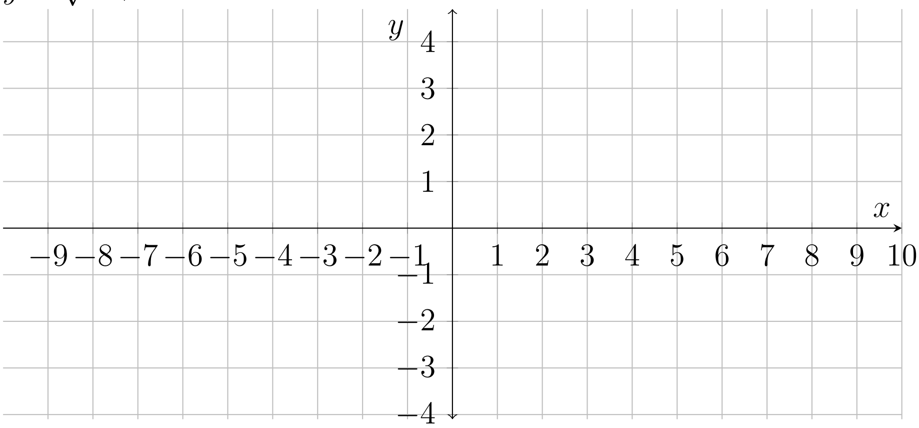
Spare axes for Question 3 part (c).

If you use the spare axes please show clearly which sketch should be marked.

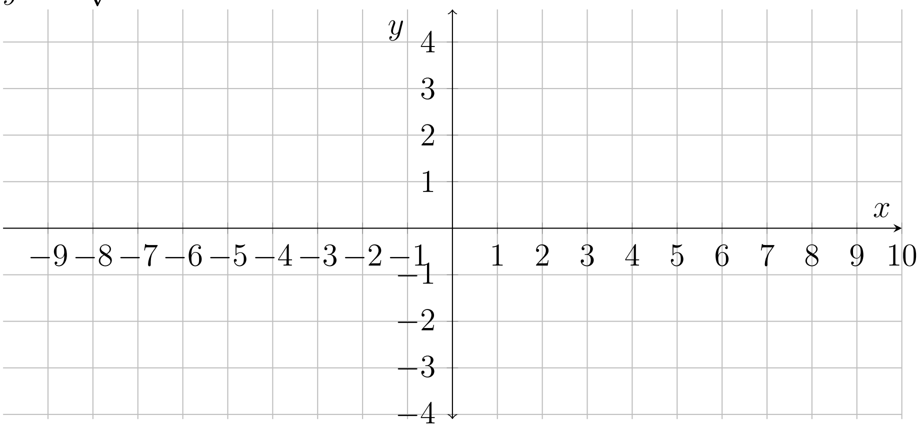
(i) $y = \sqrt{x+2}$



(ii) $y = \sqrt{x} + 2$



(iii) $y = -\sqrt{x}$



Question Four (17 marks)

(a) Consider the function

$$f(x) = 2 + 4x^3 - x^4$$

(i) Find $f'(x)$

(ii) Find the x-values where the critical points of f occur.

You do NOT need to calculate the y-values of the critical points.

(iii) Use the **first derivative test** to find the open intervals on which f is decreasing and on which f is increasing.

You MUST show your working for the first derivative test clearly to get any marks.

(iv) Hence identify the nature of the critical points from part (ii).

(v) Still considering the function $f(x) = 2 + 4x^3 - x^4$ use the **second derivative** of f to find the open intervals on which f is concave up and on which f is concave down.

You MUST show your working including the second derivative function clearly to get any marks.

(vi) Give the x values of any points of inflection of f.

(vii) Show **how** you would calculate the y-values of the inflection points. You do **not** need to complete the calculations.

(b) Consider the function $f(x) = x^5 + 2x$

(i) Find the gradient function $f'(x)$

(ii) Find the value of x at which the gradient is a minimum value.

(iii) What is the slope of the tangent line to the function at the value of x from your answer to part (ii)?

(c) Consider the function $f(x) = e^{-x}$

(i) The list below gives four possible statements about tangent lines to this function.

(A) For any value of a the tangent line to this function at $x = a$ will be flat.

(B) For any value of a the tangent line to this function at $x = a$ will slope up.

(C) For any value of a the tangent line to this function at $x = a$ will slope down.

(D) Depending on the value of a the tangent line to this function at $x = a$ could slope up, or slope down, or be flat.

Write the label (A, B, C, D) of the only true statement about tangent lines to this function $f(x) = e^{-x}$ from the list above and briefly **justify your answer** You may use a sketch as part of your justification but you must explain why the sketch is relevant.

(ii) Is this function concave up or concave down? Briefly **justify your answer**.

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The questions continue on the following pages

Question Five (19 marks)

(a) Find the following integrals.

(i) $\displaystyle\int \left(\frac{1}{3}\,x^2 + 5\right)\,dx$

(ii) $\displaystyle\int \left(\frac{2}{x} + 3e^x\right)\,dx$

(iii) $\displaystyle\int \left(\frac{1}{2x^2}\right) dx$

(iv) $\displaystyle\int \left(\frac{1}{2}\sqrt{x}\right) dx$

(v) $\displaystyle\int \left(x \sin\left(3x^2\right)\right)\, dx$

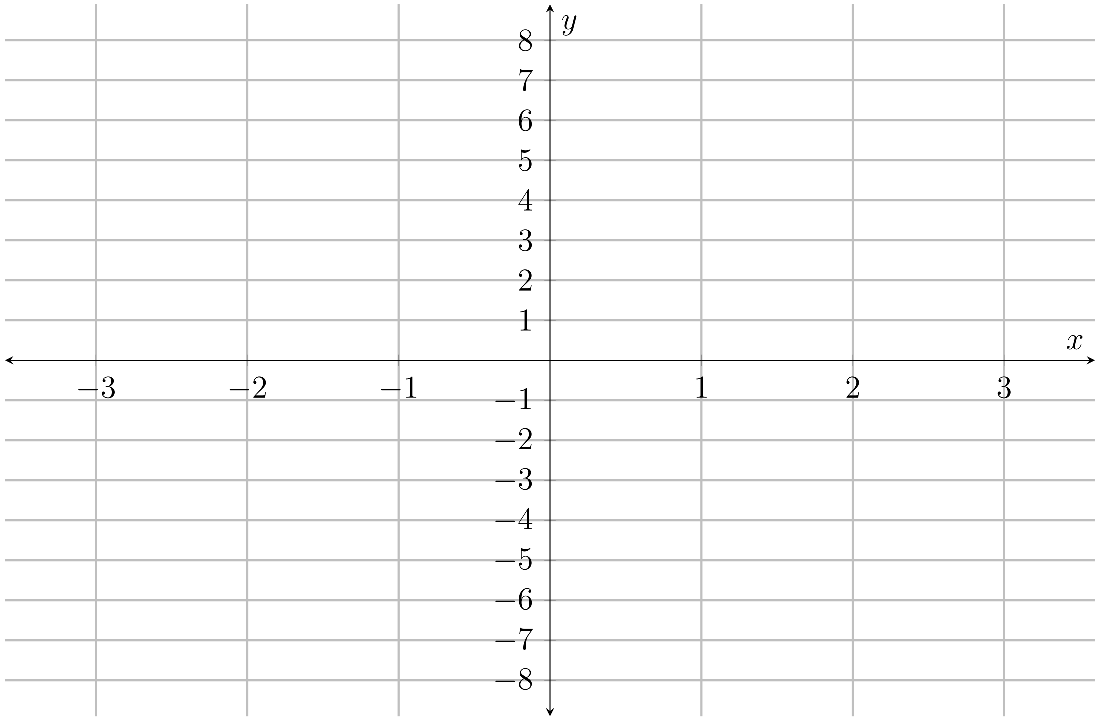
(b) Calculate the following definite integrals:

(i) $\displaystyle\int_{\pi}^{2\pi} (2\sin(x))\ dx$

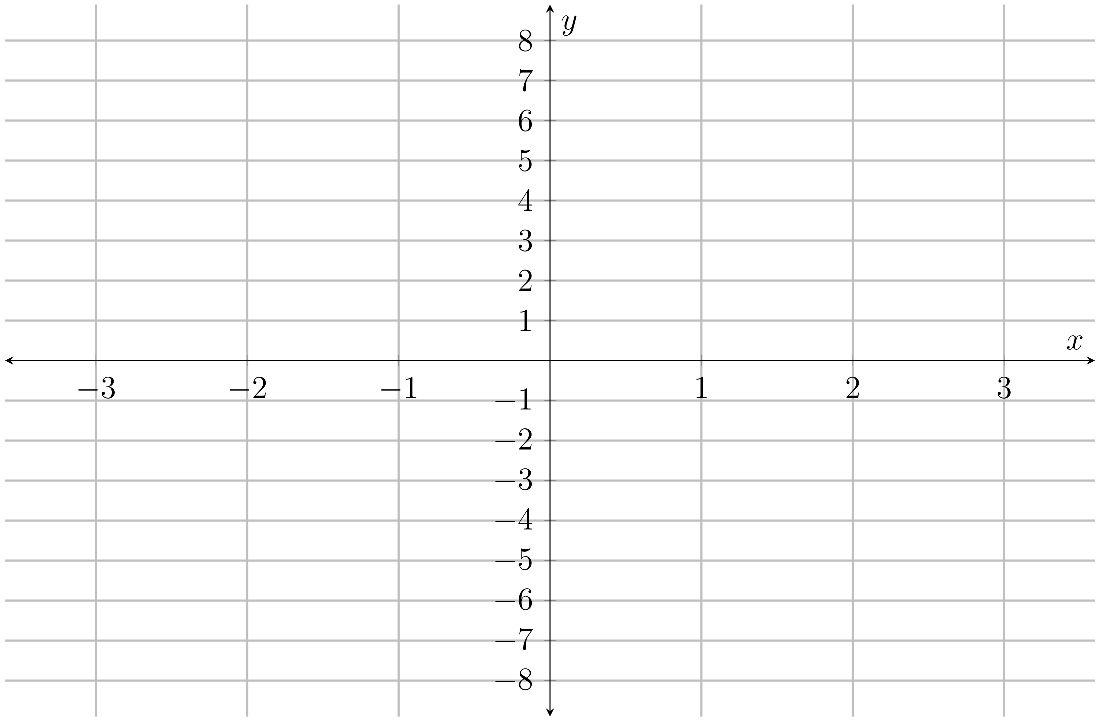
(ii) $\displaystyle\int_{0}^{\frac{\pi}{3}} \left(\sec^2(x)\right)\ dx$

(c) Consider the graph of the function $f(x) = -x^3$.

(i) Sketch the graph of the function on the axes below.



Spare axes for part (i). If you use the spare axes please show clearly which sketch should be marked.



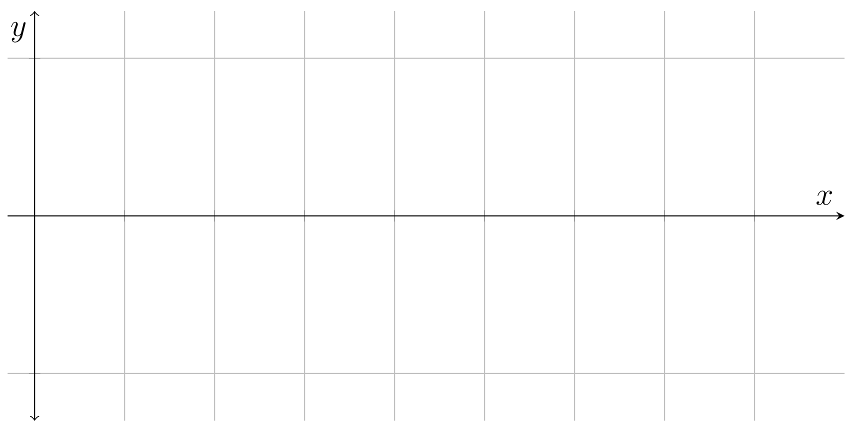
(ii) On your graph, shade the area between the curve $y = f(x)$, the x-axis, and the lines $\boldsymbol{x = -2}$ and $\boldsymbol{x = 1}$.

(iii) Find the total **geometric area** between the curve $y = f(x)$, the x-axis, and the lines $\boldsymbol{x = -2}$ and $\boldsymbol{x = 1}$ (the area shaded in part (ii)).

(d) (i) Calculate the value of the definite integral $\int_0^{\frac{\pi}{4}} (\cos(x))\ dx$

(ii) Sketch one complete cycle of $\cos(x)$ and shade the area representing $\int_0^{\frac{\pi}{4}} (\cos(x))\ dx$ on the axes below.

You will need to add a scale to the horizontal and vertical axes as well as showing the shape of the function and shading the indicated area.



Spare axes for part (ii). If you use the spare axes please show clearly which sketch should be marked.

Question Six (18 marks)

(a) Consider the following functions:

A $f(x) = x^{1/2}$

B $f(x) = \ln\left(\frac{x}{2}\right)$

C $f(x) = \frac{1}{2}x$

D $f(x) = x + \frac{1}{2}$

E $f(x) = \frac{1}{2x}$

F $f(x) = e^{2x}$

For each question below give the letter (A, B, C, D, E, or F) of the only function that matches the description given.

(i) The domain of $f(x)$ is $(0, \infty)$

(ii) The graph $y = f(x)$ has an y-intercept at $(0, 1)$

(iii) The range of $f(x)$ is $[0, \infty)$

(iv) The graph $y = f(x)$ has an x-intercept at $(-0.5, 0)$

(b) World consumption of a certain metal is changing at the rate of $135e^{0.02t}$ thousand metric tons per year, where t is the number of years since 2015.

At the start of 2015 the total consumption of this metal is 0.

(i) Find a formula for the total amount of metal (in thousands of metric tons) consumed in t years since 2015 if consumption continues to change as described above.

(ii) Use your answer from part (i) to calculate the number of years after 2015 it will take for consumption since 2015 to total 6750 thousand metric tons.

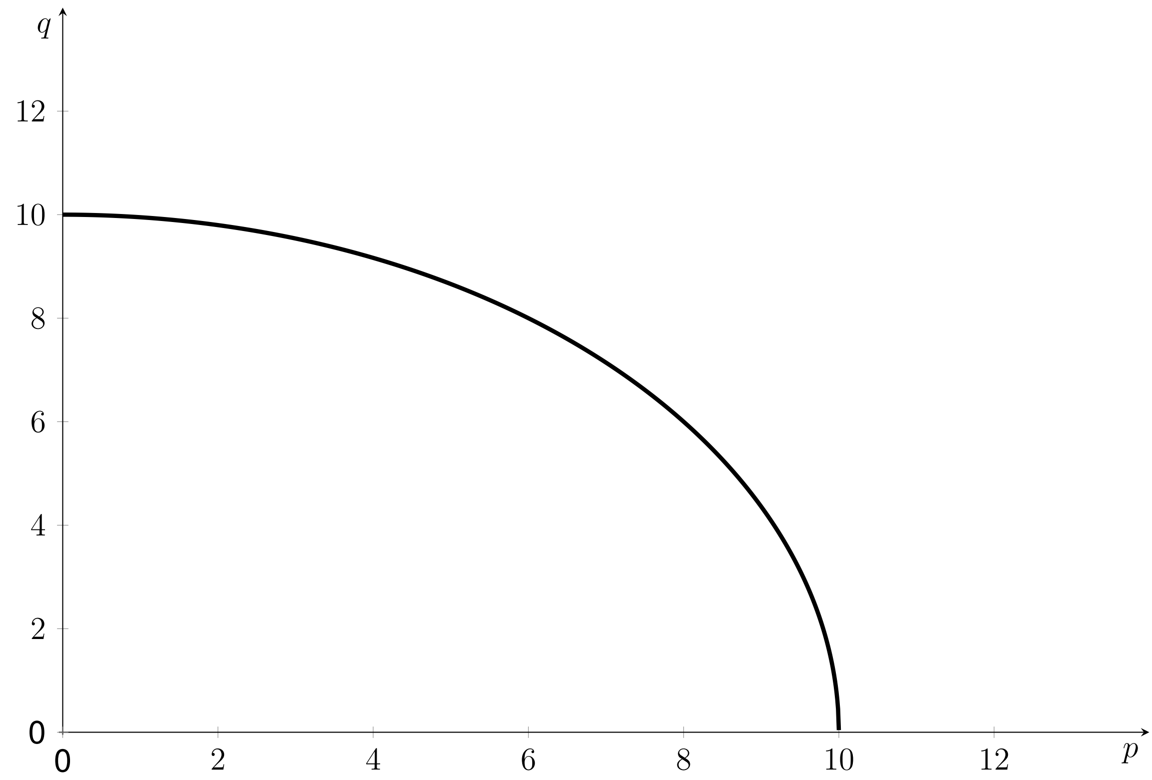
Use the approximation $\ln(2) \approx 0.7$

(c) A company models the demand curve for their product as

$$q = \sqrt{A - p^2}$$

where p is the price in \$, q is the demand in thousands of units, and A is a parameter of the model.

A graph of q against p for a particular value of A is shown below.



(i) Use the graph to show that the value of the parameter A used in this model is $A = 100$. You must show your working/reasoning clearly.

The demand model from page 40 with $A = 100$ is

$$q = \sqrt{100 - p^2}$$

where p is the price in \$, q is the demand in thousands of units.

(ii) Find an expression for $\dfrac{dq}{dp}$, the rate of change of q with respect to p.

The price p is cyclical over the year and is modeled as

$$p = 2\cos\left(\frac{\pi t}{6}\right) + 4$$

where t is the time in months since the start of the year.

(iii) Find an expression for $\frac{dp}{dt}$, the rate of change of p with respect to time t.

(iv) Use your answer from part (iii) to show that the rate of change of p with respect to time t is $-\frac{\pi}{6}$ when $t = 1$.

(v) Use your answers from parts (ii) and (iii) to find an expression in terms of p and t for the rate of change of demand (q) with respect to time (t). You **do not** need to simplify your answer.

(vi) Assuming that $p^2 < 100$ and demand q is always positive, is demand q increasing or decreasing at time $t = 1$? Briefly **justify your response**.

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END OF EXAMINATION