MATH 4280

Lecture Notes 2: Fourier and Wavelet transforms

Fourier series

- The Fourier series of a 2π-periodic real-valued function is

$$f(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} (a_k \cos(kx) + b_k \sin(kx))$$

where

$$a_k = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(kx) dx$$

$$b_k = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(kx) dx$$

- It represents the projection of the function onto the orthogonal basis

$$\{\cos(kx), \sin(kx)\}_{k=0}^{\infty}$$

- The coefficients a_k and b_k are the coordinates in the new basis

- The Fourier series of a 2π-periodic complex-valued function is

$$f(x) = \sum_{k=-\infty}^{\infty} c_k e^{ikx}$$

- The functions $\psi_k = e^{ikx}$ provide a basis for this class of functions
- These functions are orthogonal since

$$\langle \psi_j, \psi_k \rangle = \int_{-\pi}^{\pi} e^{ijx} e^{-ikx} dx = \int_{-\pi}^{\pi} e^{i(j-k)x} dx = \left[\frac{e^{i(j-k)x}}{i(j-k)} \right]_{-\pi}^{\pi} = \begin{cases} 0 & \text{if } j \neq k \\ 2\pi & \text{if } j = k \end{cases}$$

- The Fourier series is a change of coordinates

$$f(x) = \sum_{k=-\infty}^{\infty} c_k \psi_k(x) = \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} \langle f(x), \psi_k(x) \rangle \psi_k(x).$$

where $\|\psi_k\|^2 = 2\pi$ and $\langle f(x), g(x) \rangle = \int_a^b f(x)\bar{g}(x)\, dx$

Fourier transform

- The Fourier transform pair is given by

$$f(x) = \mathcal{F}^{-1}\left(\hat{f}(\omega)\right) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}(\omega)e^{i\omega x}\,d\omega$$

Inverse Fourier transform

$$\hat{f}(\omega) = \mathcal{F}(f(x)) = \int_{-\infty}^{\infty} f(x)e^{-i\omega x}\,dx.$$

Fourier transform

where the integrals exist when $f, \hat{f} \in L^1(-\infty, \infty)$

- Derivatives of functions:

$$\begin{aligned}
\mathcal{F}\left(\frac{d}{dx}f(x)\right) &= \int_{-\infty}^{\infty} \overbrace{f'(x)}^{dv}\,\overbrace{e^{-i\omega x}}^{u}\,dx \\
&= \Big[\underbrace{f(x)e^{-i\omega x}}_{uv}\Big]_{-\infty}^{\infty} - \int_{-\infty}^{\infty} \underbrace{f(x)}_{v}\Big[\underbrace{-i\omega e^{-i\omega x}}_{du}\Big]\,dx \\
&= i\omega \int_{-\infty}^{\infty} f(x)e^{-i\omega x}\,dx \\
&= i\omega\mathcal{F}(f(x)).
\end{aligned}$$

- Linearity of Fourier transform

$$\mathcal{F}(\alpha f(x) + \beta g(x)) = \alpha\mathcal{F}(f) + \beta\mathcal{F}(g)$$

$$\mathcal{F}^{-1}(\alpha \hat{f}(\omega) + \beta\hat{g}(\omega)) = \alpha\mathcal{F}^{-1}(\hat{f}) + \beta\mathcal{F}^{-1}(\hat{g})$$

- Parseval's Theorem

$$\int_{-\infty}^{\infty} |\hat{f}(\omega)|^2 \, d\omega = 2\pi \int_{-\infty}^{\infty} |f(x)|^2 \, dx$$

- Convolution

$$(f * g)(x) = \int_{-\infty}^{\infty} f(x-\xi)g(\xi)\, d\xi$$

Definition of convolution

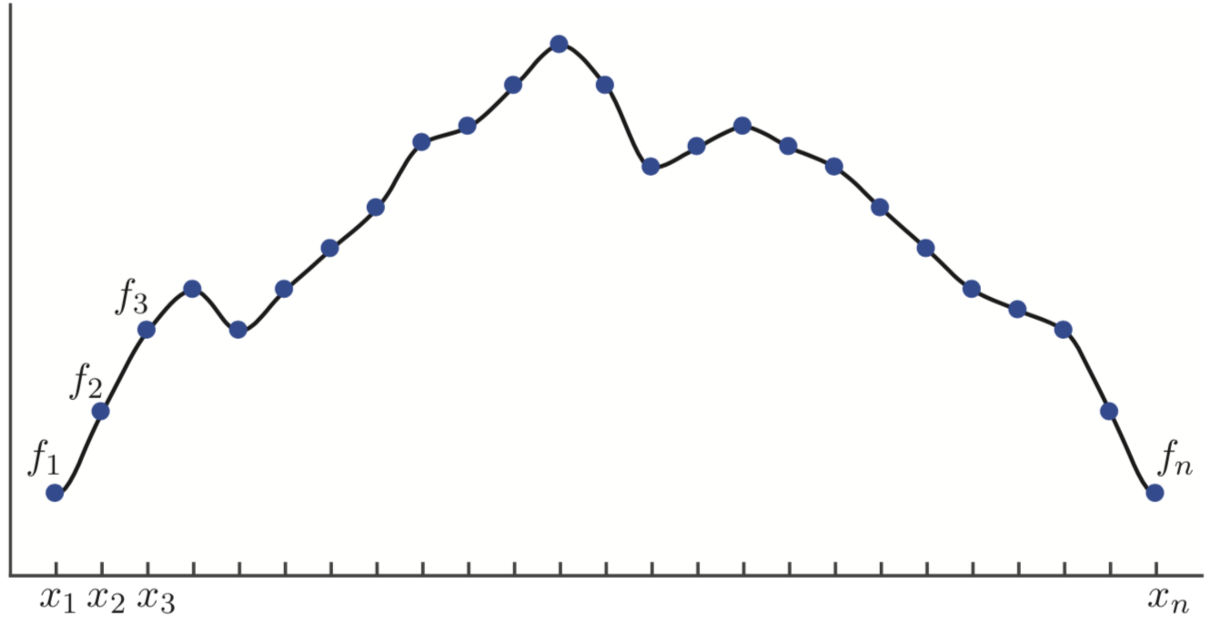
$$\mathcal{F}^{-1}\left(\hat{f}\hat{g}\right)(x) = f * g$$

Discrete Fourier transform (DFT)

- In most applications, we work with discrete data
- The DFT is a discrete version of Fourier transform for vectors of data

$$\mathbf{f} = [f_1 \quad f_2 \quad f_3 \quad \cdots \quad f_n]^T$$

obtained by discretizing the function f(x) at a regular spacing



- The DFT and the inverse DFT (iDFT) are given respectively by

DFT $$\hat{f}_k = \sum_{j=0}^{n-1} f_j e^{-i2\pi jk/n}$$

iDFT $$f_k = \frac{1}{n}\sum_{j=0}^{n-1} \hat{f}_j e^{i2\pi jk/n}$$

- The DFT can be represented by a matrix multiplication

$$\begin{bmatrix} \hat{f}_1 \\ \hat{f}_2 \\ \hat{f}_3 \\ \vdots \\ \hat{f}_n \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & \omega_n & \omega_n^2 & \cdots & \omega_n^{n-1} \\ 1 & \omega_n^2 & \omega_n^4 & \cdots & \omega_n^{2(n-1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \omega_n^{n-1} & \omega_n^{2(n-1)} & \cdots & \omega_n^{(n-1)^2} \end{bmatrix} \begin{bmatrix} f_1 \\ f_2 \\ f_3 \\ \vdots \\ f_n \end{bmatrix}$$

where $\omega_n = e^{-2\pi i/n}$

Fast Fourier transform (FFT)

- The DFT requires $O(n^2)$ operations
- The FFT requires $O(n \log n)$ operations, a significant improvement
- The basic idea of FFT is the fact that the DFT can be implemented efficiently when n is a power of 2
- Example: when $n = 1024 = 2^{10}$, the DFT matrix F_{1024} is

$$\hat{\mathbf{f}} = \mathbf{F}_{1024}\mathbf{f} = \begin{bmatrix} \mathbf{I}_{512} & \mathbf{D}_{512} \\ \mathbf{I}_{512} & -\mathbf{D}_{512} \end{bmatrix} \begin{bmatrix} \mathbf{F}_{512} & \mathbf{0} \\ \mathbf{0} & \mathbf{F}_{512} \end{bmatrix} \begin{bmatrix} \mathbf{f}_{\text{even}} \\ \mathbf{f}_{\text{odd}} \end{bmatrix}$$

where

$$\mathbf{D}_{512} = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & \omega & 0 & \cdots & 0 \\ 0 & 0 & \omega^2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \omega^{511} \end{bmatrix}$$

and $\omega = e^{-2\pi i/n}$

Example: denoising

- Consider a signal with two dominant frequencies $f_1 = 50$ and $f_2 = 120$

$$f(t) = \sin(2\pi f_1 t) + \sin(2\pi f_2 t)$$

- Add some Gaussian noise to the signal
- Use FFT and look at the signal in the frequency domain
- Truncate those frequencies with small amplitude
- Apply iFFT to obtain a denoised signal

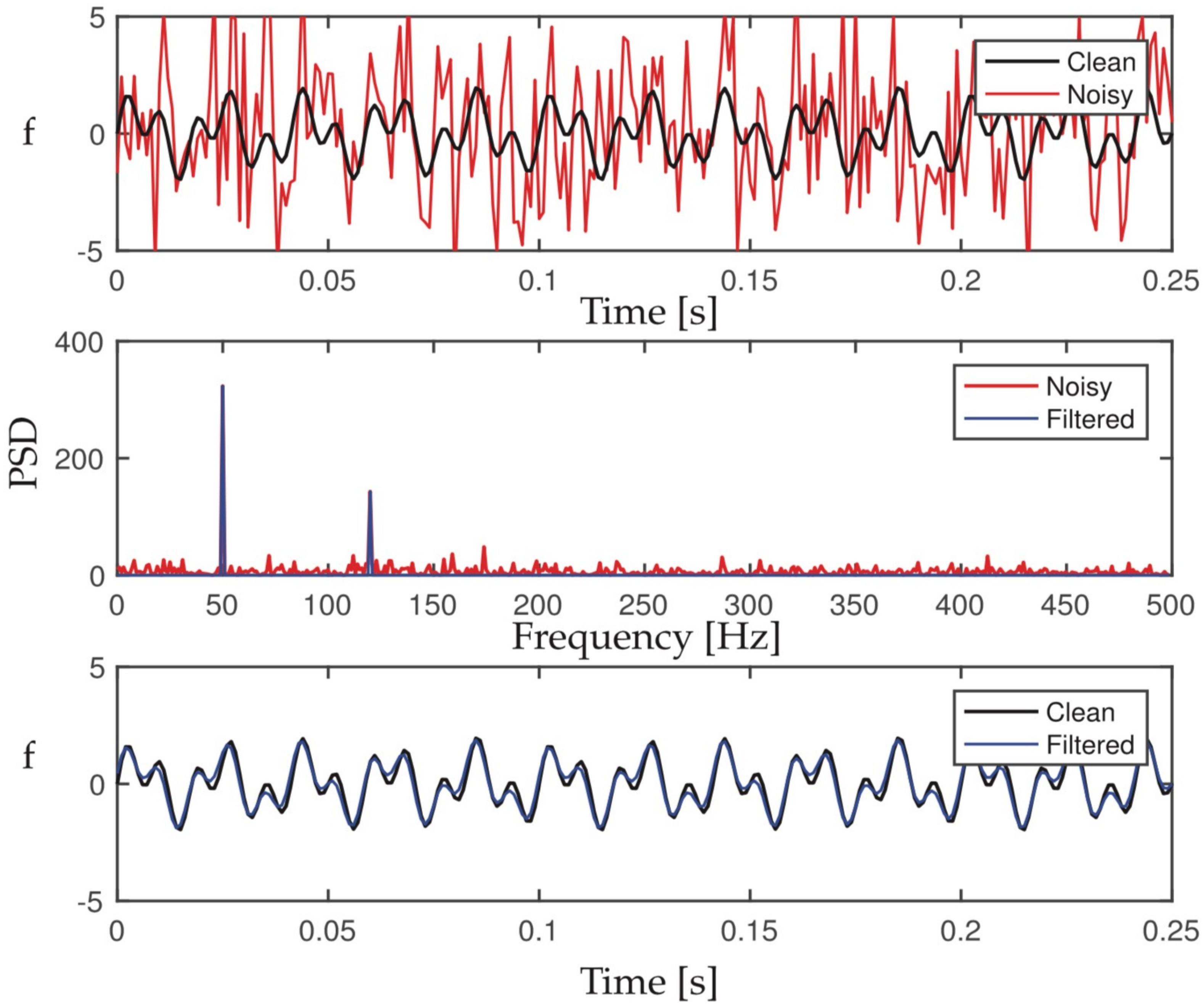
MATLAB implementation

```
dt = .001;
t = 0:dt:1;
f = sin(2*pi*50*t) + sin(2*pi*120*t); % Sum of 2 frequencies
f = f + 2.5*randn(size(t));  %  Add some noise

%% Compute the Fast Fourier Transform FFT
n = length(t);
fhat = fft(f,n);          % Compute the fast Fourier transform
PSD = fhat.*conj(fhat)/n; % Power spectrum (power per freq)
freq = 1/(dt*n)*(0:n); % Create x-axis of frequencies in Hz
L = 1:floor(n/2);     % Only plot the first half of freqs

%% Use the PSD to filter out noise
indices = PSD>100;  % Find all freqs with large power
PSDclean = PSD.*indices;  % Zero out all others
fhat = indices.*fhat;  % Zero out small Fourier coeffs. in Y
ffilt = ifft(fhat); % Inverse FFT for filtered time signal
```

Results



Original and noisy signals

Signals in frequency domain

Denoised signal

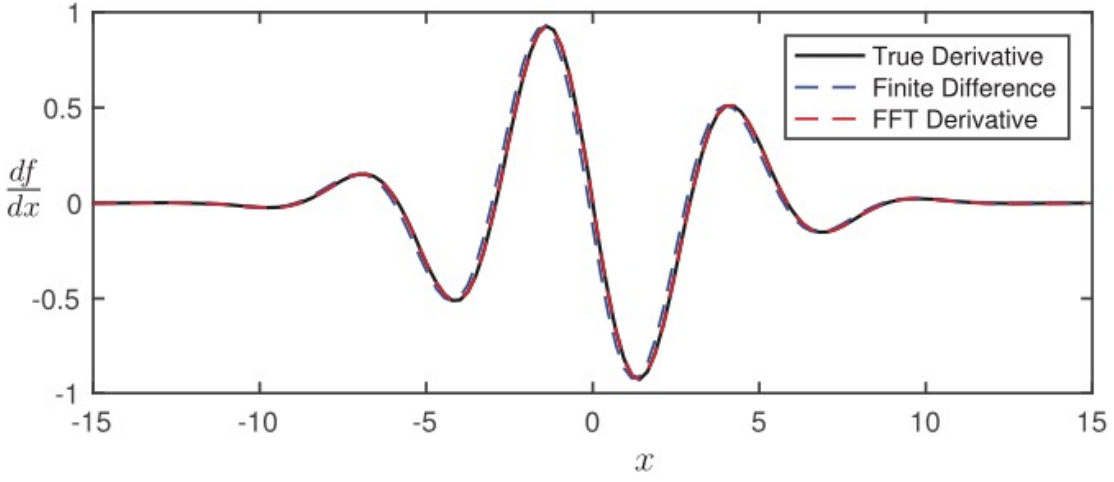
Fast computation of derivative of function

- Recall the property $\mathcal{F}(df/dx) = i\omega\mathcal{F}(f)$
- Steps of computing derivatives:

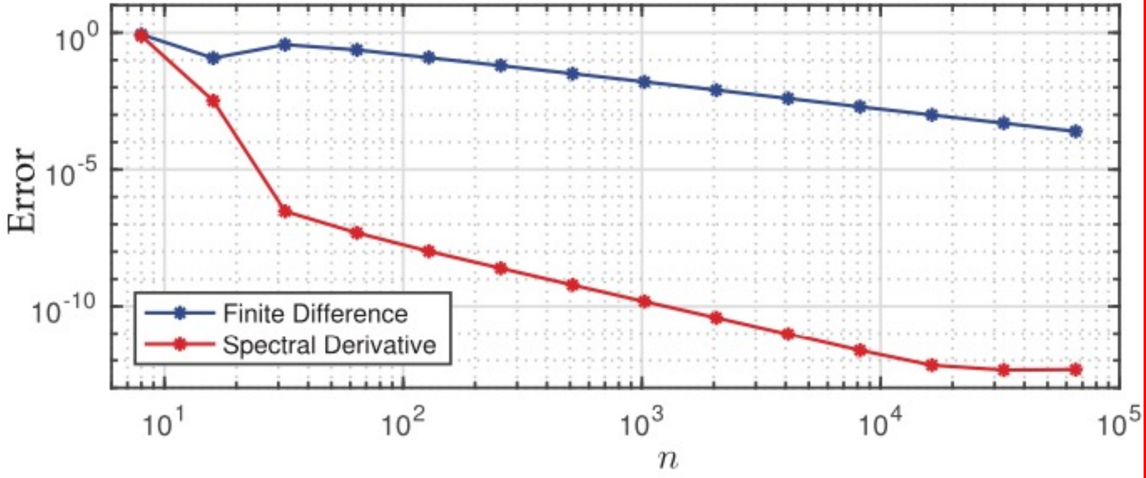
1. Compute the Fourier transform $\hat{f}$ $$\hat{f}_k = \sum_{j=0}^{n-1} f_j e^{-i2\pi jk/n}$$
2. Multiply the k-th component of $\hat{f}$ by i $2\pi k/n$
3. Compute the inverse Fourier transform $$f_k = \frac{1}{n}\sum_{j=0}^{n-1} \hat{f}_j e^{i2\pi jk/n}$$

- Let $f(x) = \cos(x)e^{-x^2/25}$
- Compare with finite difference approximation

$$\frac{df}{dx}(x_k) \approx \frac{f(x_{k+1}) - f(x_k)}{\Delta x}$$



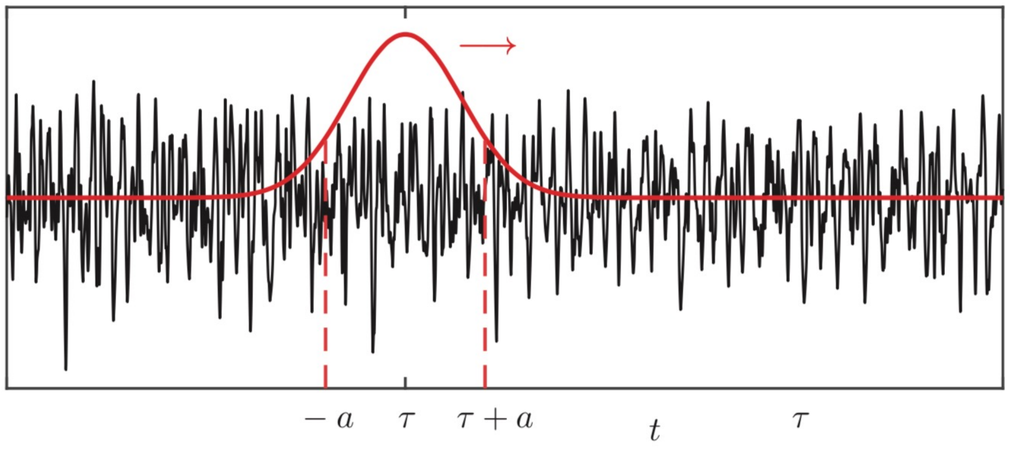
n=128



Comparison for various n

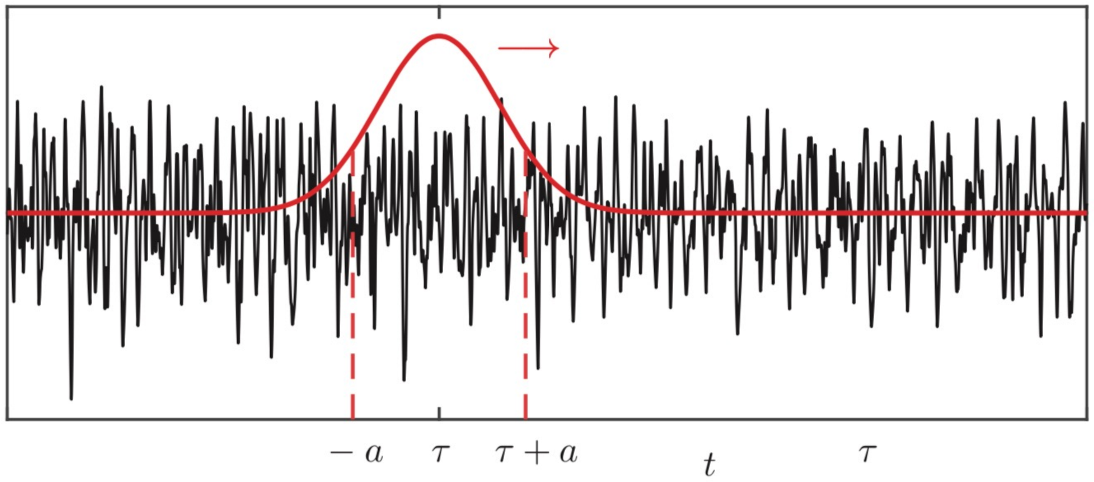
Gabor transform

- Fourier transform works for stationary signals (frequency does not change with time)
- For nonstationary signals, it is important to characterize the frequency and its evolution in time
- The Gabor transform is a windowed Fourier transform in a moving window (also called short-time Fourier transform)
- It allows localization of frequency content



The Gabor transform is defined as

$$\mathcal{G}(f)(t,\omega) = \hat{f}_g(t,\omega) = \int_{-\infty}^{\infty} f(\tau)e^{-i\omega\tau}\bar{g}(\tau - t)\,d\tau = \langle f, g_{t,\omega}\rangle$$



where

$$g_{t,\omega}(\tau) = e^{i\omega\tau} g(\tau - t)$$

$$g(t) = e^{-(t-\tau)^2/a^2}$$

The constant a determines the spread of the window, and τ determines the center of the window

The Inverse Gabor transform is given by

$$f(t) = \mathcal{G}^{-1}\left(\hat{f}_g(t,\omega)\right) = \frac{1}{2\pi\|g\|^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \hat{f}_g(\tau,\omega)g(t-\tau)e^{i\omega t}\,d\omega\,dt.$$

Discrete Gabor transform

- We discretize both in time and frequency

$$\nu = j\Delta\omega$$
$$\tau = k\Delta t.$$

- The discretized kernel function is

$$g_{j,k} = e^{i2\pi j\Delta\omega t} g(t - k\Delta t)$$

- The discrete Gabor transform is

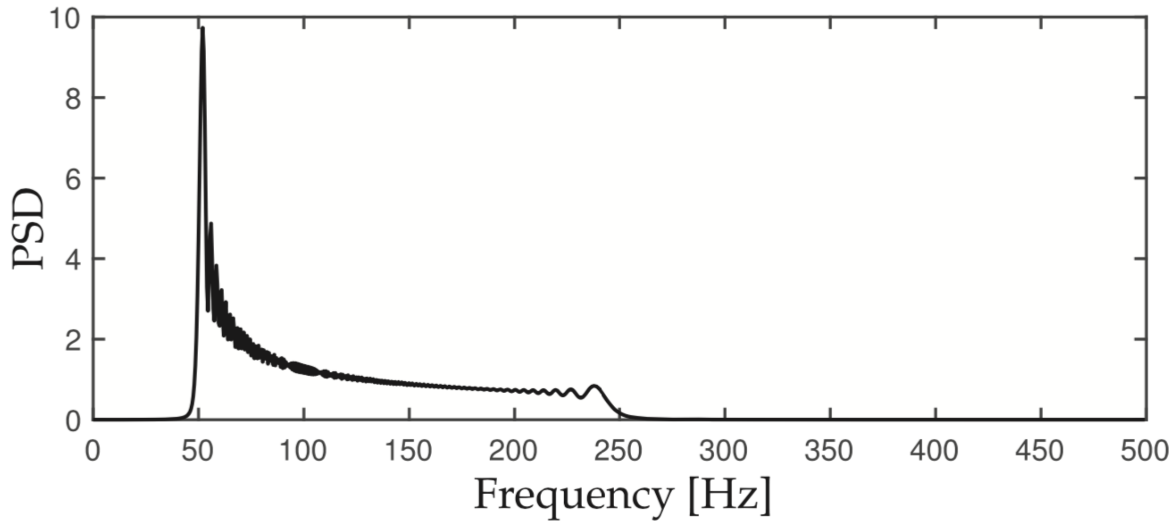
$$\hat{f}_{j,k} = \langle f, g_{j,k} \rangle = \int_{-\infty}^{\infty} f(\tau)\bar{g}_{j,k}(\tau)\, d\tau$$

Example: a quadratic chirp

- We consider the signal

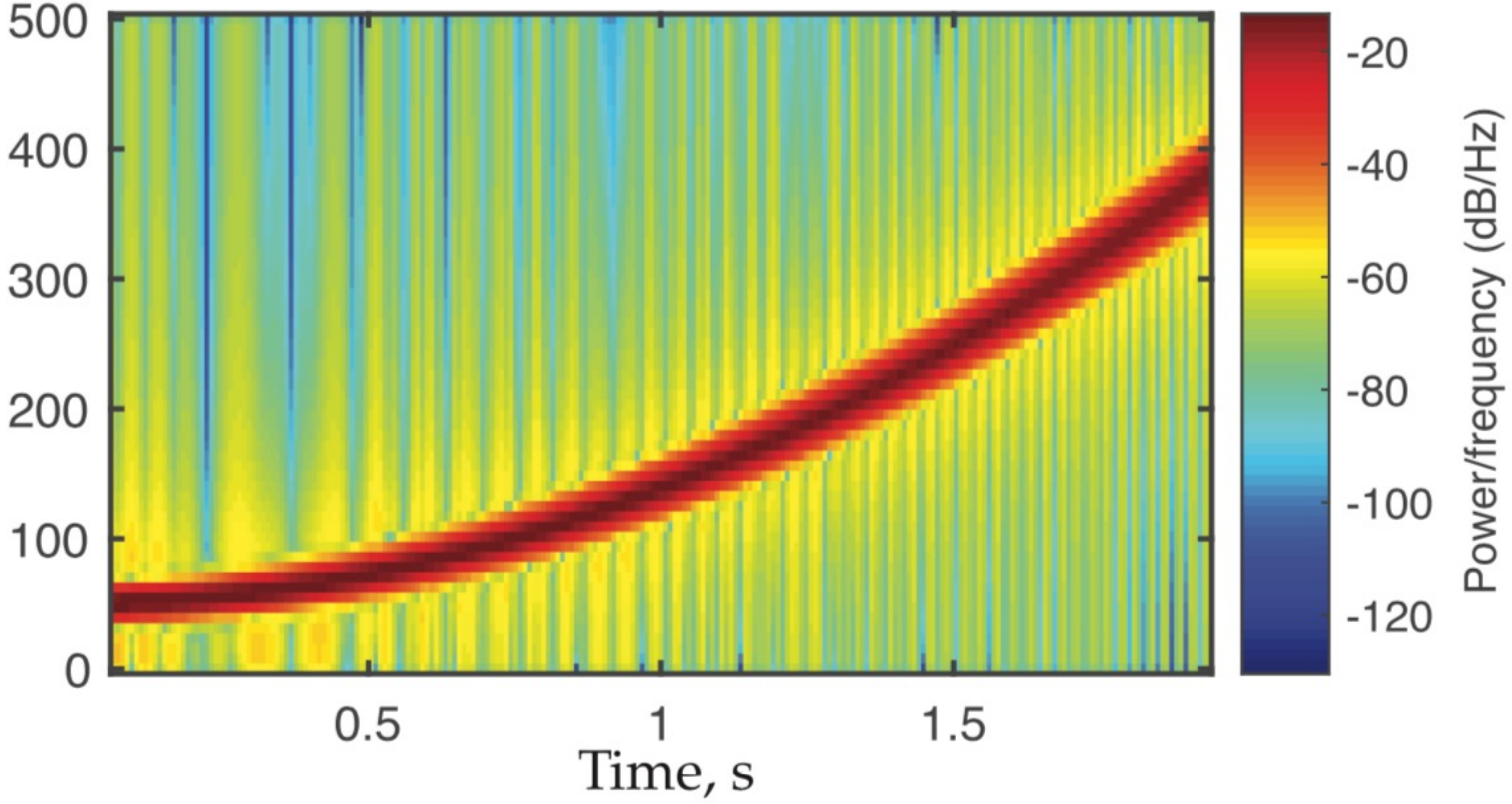
$$f(t) = \cos(2\pi t \omega(t)) \quad \text{where} \quad \omega(t) = \omega_0 + (\omega_1 - \omega_0)t^2/3t_1^2$$

- The frequency changes from ω_0 to ω_1 from $t = 0$ to $t = t_1$
- Take $\omega_0 = 50$ and $\omega_1 = 250$ and $t_1 = 2$



Fourier transform does not see the change in the frequency

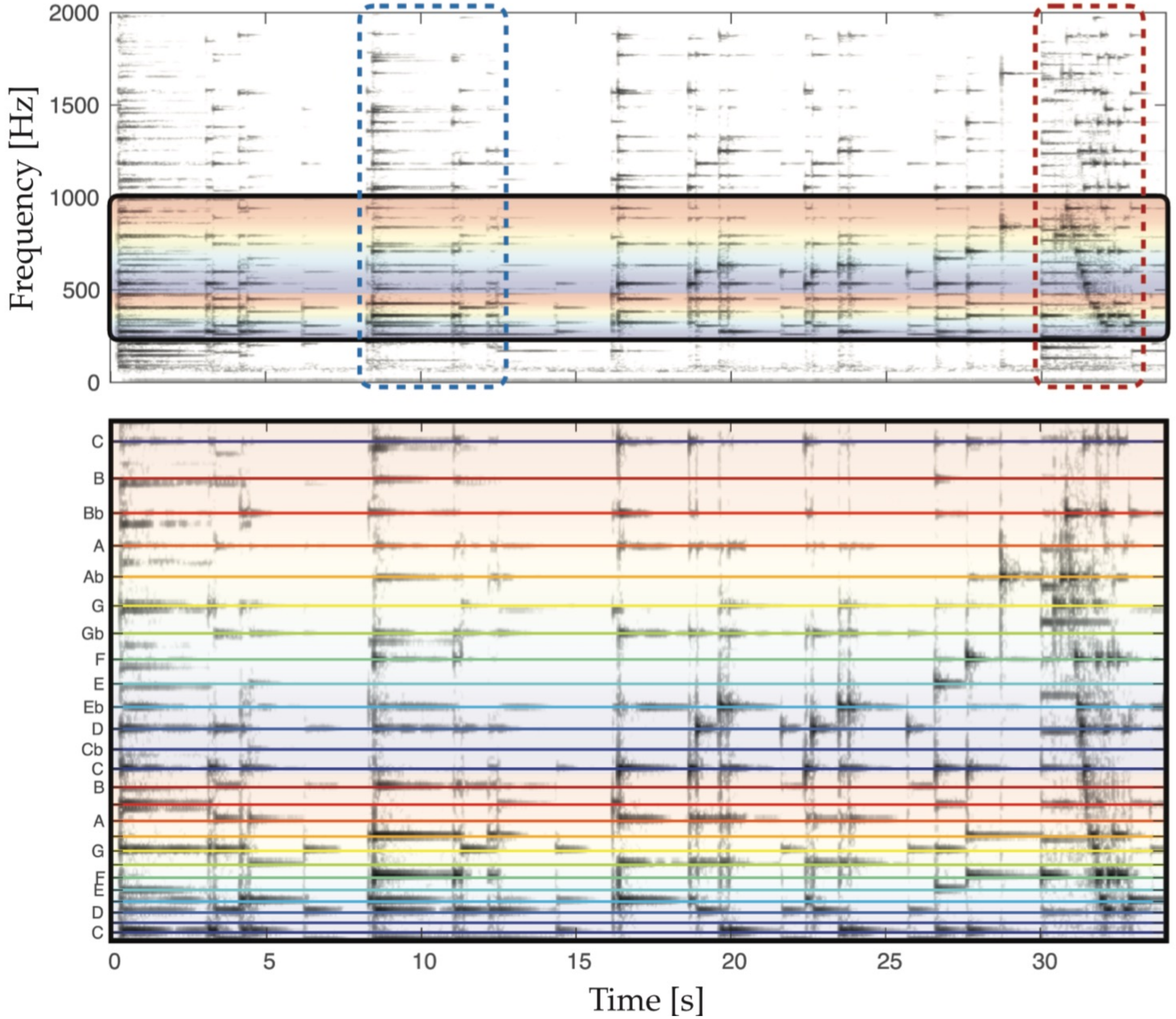
- Using the Gabor transform, we can plot the following spectrogram



- We see clearly frequency shifts in time

Example: music

- The spectrogram can be used to analyze music
- Identify key markers for classification



Uncertainty principles

- It limits the ability to simultaneously attain high resolution in both the time and frequency domains
- Mathematically,

$$\left(\int_{-\infty}^{\infty} x^2 |f(x)|^2 \, dx\right)\left(\int_{-\infty}^{\infty} \omega^2 |\hat{f}(\omega)|^2 \, d\omega\right) \geq \frac{1}{16\pi^2}$$

- For real-valued functions, this is the second moment, which measures the variance of Gaussian functions
- Thus, a function and its Fourier transform cannot both be arbitrarily localized

Wavelet transform

- Wavelets extend the concept of Fourier analysis to more general orthogonal bases
- They can partially overcome the limitation resulting from the uncertainty principle by using a multi-resolution decomposition
- The idea starts with a function $\psi(t)$, called the mother wavelet
- A family of scaled and translated functions can be generated

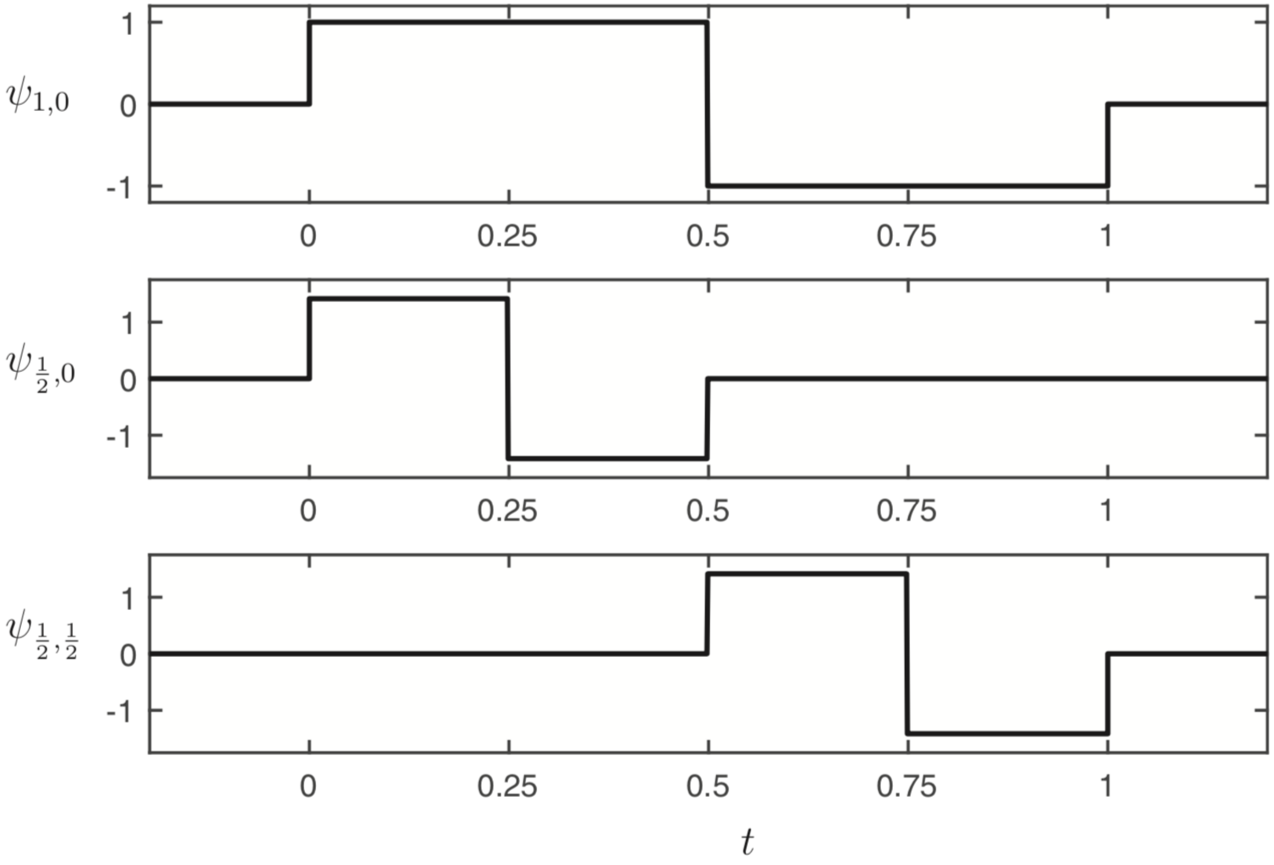
$$\psi_{a,b}(t) = \frac{1}{\sqrt{a}}\psi\left(\frac{t-b}{a}\right)$$

Haar wavelet

- The simplest and the earliest example is the Haar wavelet

$$\psi(t) = \begin{cases} 1 & 0 \le t < 1/2 \\ -1 & 1/2 \le t < 1 \\ 0 & \text{otherwise.} \end{cases}$$

- Three Haar wavelets are shown
- Choosing the next higher frequency layer using a bisection, the resulting Haar wavelets are orthogonal



Continuous wavelet transform (CWT)

- The continuous wavelet transform (CWT) is given by

$$\mathcal{W}_\psi(f)(a,b) = \langle f, \psi_{a,b} \rangle = \int_{-\infty}^{\infty} f(t)\bar{\psi}_{a,b}(t)\,dt$$

- The inverse continuous wavelet transform (iCWT) is given by

$$f(t) = \frac{1}{C_\psi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \mathcal{W}_\psi(f)(a,b)\psi_{a,b}(t)\frac{1}{a^2}\,da\,db$$

where $C_\psi = \int_{-\infty}^{\infty} \frac{|\hat{\psi}(\omega)|^2}{|\omega|}\,d\omega$

Discrete wavelet transform (DWT)

- The discrete wavelet transform (DWT) is given by

$$\mathcal{W}_\psi(f)(j,k) = \langle f, \psi_{j,k} \rangle = \int_{-\infty}^{\infty} f(t)\bar{\psi}_{j,k}(t)\,dt$$

- Here $\psi_{j,k}(t)$ is a discrete family of wavelets

$$\psi_{j,k}(t) = \frac{1}{a^j}\psi\left(\frac{t-kb}{a^j}\right)$$

- If the family is orthonormal, then we can expand a function as follows

$$f(t) = \sum_{j,k=-\infty}^{\infty} \langle f(t), \psi_{j,k}(t) \rangle \psi_{j,k}(t)$$

Simple illustration of wavelet transform

1D Haar wavelet transform

The 1D Haar wavelet transform is the following matrix

$$H = \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{pmatrix}$$

Given a vector $x = \begin{pmatrix} a \\ b \end{pmatrix}$. The result of the transform is

$$Hx = \begin{pmatrix} \frac{a+b}{\sqrt{2}} \\ \\ \frac{a-b}{\sqrt{2}} \end{pmatrix}$$

Observe that

- the first component provides the "mean"
- the second component provides the "difference" or "edge"
- the inverse transform is given by H^T, that is, if y is a vector containing the mean and the difference, then the original vector is $x = H^T y$

Two level image decomposition

Consider a vector x of length N, where N is even. We consider it as a 1D image with N pixels.

Two level decomposition:

- divide the vector into $N/2$ blocks, each block has size 2
- apply the 1D Haar wavelet transform to each block of size 2
- define a vector x_M of length $N/2$ consisting the mean values
- define a vector x_D of length $N/2$ consisting the differences

Example: consider

$$x = (100, 200, 44, 50, 20, 20, 4, 2)^T$$

We have

$$x_M = \sqrt{2}\,(150, 47, 20, 3)^T, \qquad x_D = \sqrt{2}\,(50, 3, 0, -1)^T$$

Notice that larger values in x_D correspond to larger jumps, and hence edges. Also, x_M corresponds to the main image.

2D Haar wavelet transform

The 2D Haar wavelet transform is based on the following matrix

$$H = \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{pmatrix}$$

Given a 2×2 matrix A, the transform is defined as

$$HAH^T$$

Consider the matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

The result of the transform is

$$HAH^T = \begin{pmatrix} \frac{1}{2}(a+b+c+d) & \frac{1}{2}(a+c-b-d) \\ \\ \frac{1}{2}(a+b-c-d) & \frac{1}{2}(a+d-b-c) \end{pmatrix}$$

Recall that for the matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

The result of the transform is

$$HAH^T = \begin{pmatrix} \frac{1}{2}(a+b+c+d) & \frac{1}{2}(a+c-b-d) \\ \\ \frac{1}{2}(a+b-c-d) & \frac{1}{2}(a+d-b-c) \end{pmatrix}$$

Few remarks:

- the upper-left entry corresponds to the "mean"
- the upper-right entry is formed by taking the mean along each column, and then the difference of the columns. Hence large value corresponds to large change in the horizontal direction
- the lower-left entry is formed by taking the mean along each row, and then the difference of the rows. Hence large value corresponds to large change in the vertical direction
- the lower-right entry measures the changes along the diagonals

Two level image decomposition

Consider a matrix X with size $N \times N$, where N is even. We consider it as a 2D image with N^2 pixels.

Two level decomposition:

- divide the matrix into $N/2 \times N/2$ blocks, each block has size 2×2
- apply the 2D Haar wavelet transform to each block of size 2×2
- define a matrix x_M of size $N/2 \times N/2$ consisting the mean values
- define a matrix x_H of size $N/2 \times N/2$ consisting the horizontal differences
- define a matrix x_V of size $N/2 \times N/2$ consisting the vertical differences
- define a matrix x_D of size $N/2 \times N/2$ consisting the diagonal differences

Example: Let

$$X = \begin{pmatrix} 100 & 200 & 20 & 10 \\ 100 & 400 & 40 & 20 \\ 20 & 10 & 1 & 2 \\ 20 & 40 & 4 & 1 \end{pmatrix}$$

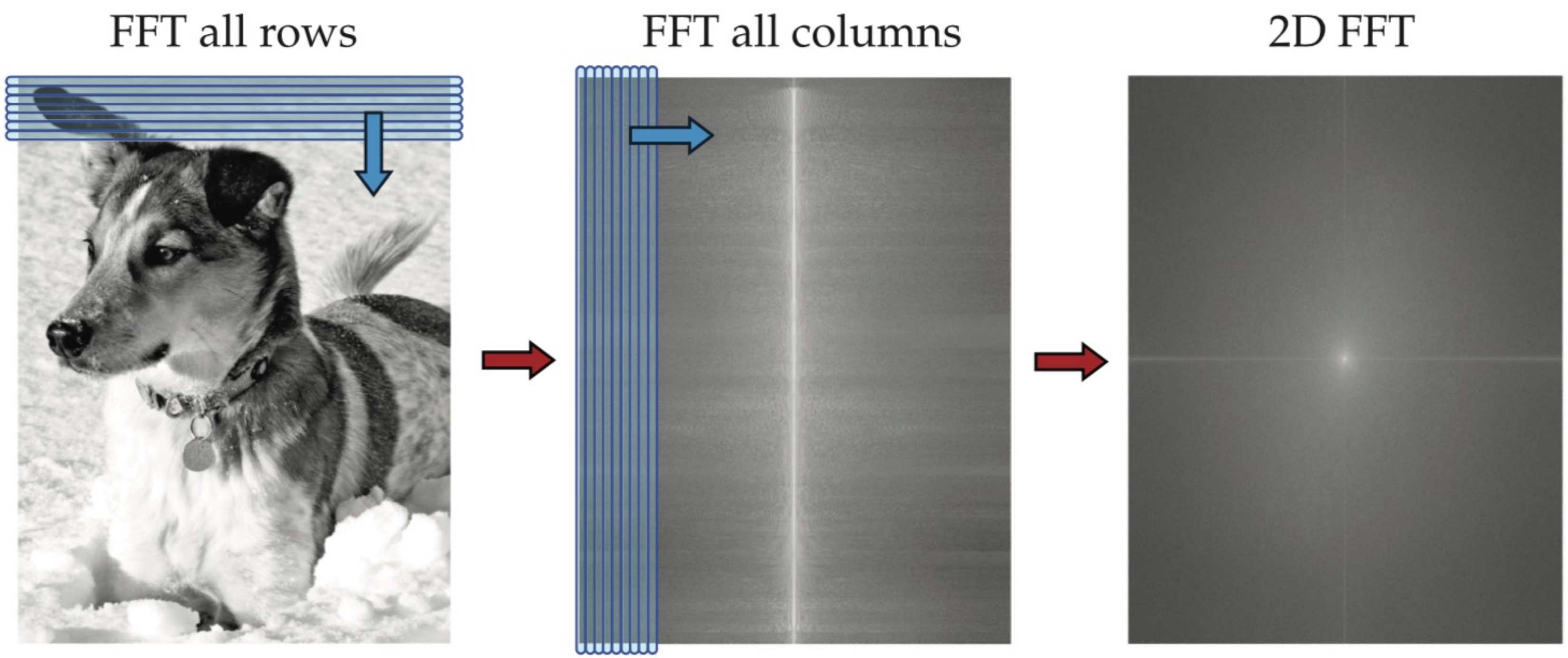
Then we have

$$x_M = \begin{pmatrix} 400 & 45 \\ 45 & 4 \end{pmatrix}, \quad x_H = \begin{pmatrix} -200 & 15 \\ -5 & 1 \end{pmatrix}$$

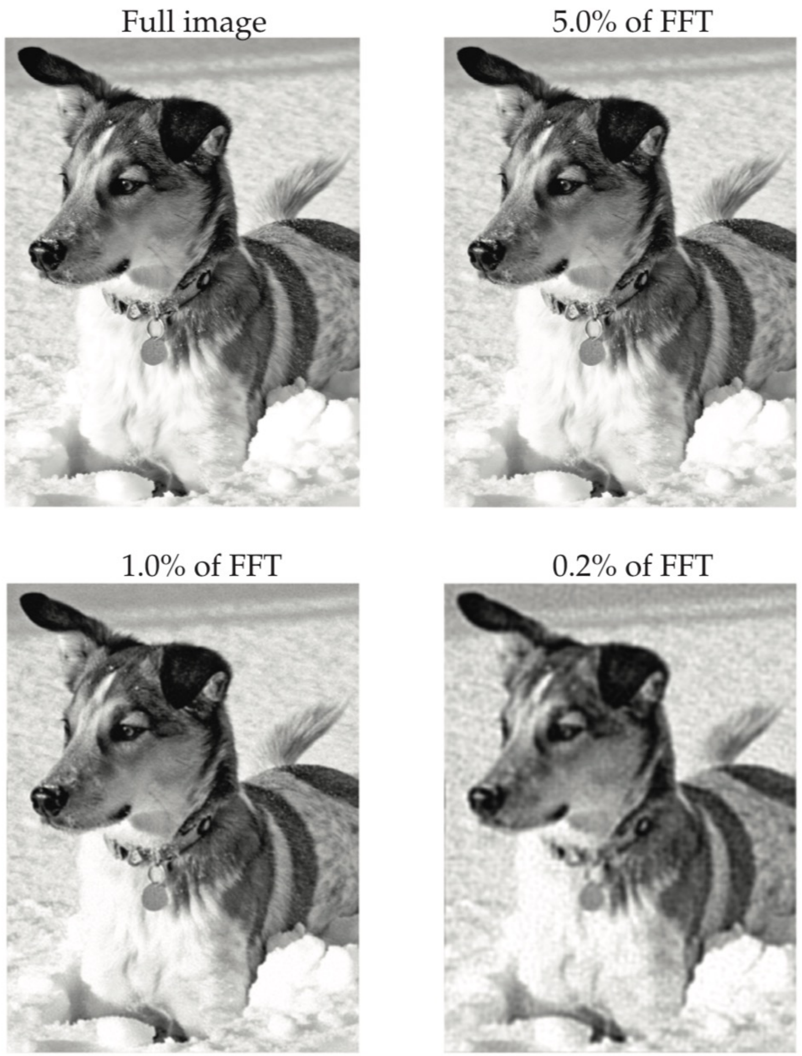
$$x_V = \begin{pmatrix} -100 & -15 \\ -15 & -1 \end{pmatrix}, \quad x_D = \begin{pmatrix} 100 & -5 \\ 15 & -2 \end{pmatrix}$$

2D Fourier transform

- Given a matrix X, its Fourier transform is obtained by applying the 1D Fourier transform to each row, then applying to each column
- The order is irrelevant



Example: image compression by FFT



Keeping 5% of the largest Fourier coefficients

Keeping 1% of the largest Fourier coefficients

Keeping 0.2% of the largest Fourier coefficients