Math Prep

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Final Test (120 minutes)

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1. (20 points) Consider the function $f(x) = x^3 - 3x$:

   a. (5 points) Find the range of f.

   b. (5 points) Find the image of the interval $I = [0, 2]$ under f.

   c. (5 points) Find the image of the interval $I = (-1, 1)$ under f.

   d. (5 points) Find the local minima and maxima of f.

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2. (20 points) For each of the following sequences determine if it is bounded, does it diverge to ∞ or $-\infty$, or is it convergent, and if so, find its limit:

   a. (5 points) $a_n = \frac{\log(n^2)}{n}$

   b. (5 points) $b_n = (\log n)^2 - n$

   c. (5 points) $c_n = \exp\big(1 - \log(n)\big)$

   d. (5 points) $d_n = \log\big(1 + 2^n\big)$

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3. (20 points) Consider the objective function

   $$f(x, y) = x^2 + y^2 - 20x - 20y + \log\sqrt{2}$$

   and the function

   $$g(x, y) = x^2 + y^2 - 50.$$

   a. (5 points) Find the point of minimum of $f(x, y)$ subject to $g(x, y) \leq 0$.

   b. (5 points) Find the point of minimum of $f(x, y)$ subject to $g(x, y) \geq 0$.

   c. (5 points) Find the point of maximum of $f(x, y)$ subject to $g(x, y) \leq 0$.

   d. (5 points) Find the point of maximum of $f(x, y)$ subject to $g(x, y) \geq 0$.

   Note: You don't have to verify the second-order conditions, but make sure not to confuse minimum with maximum.

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4. (10 points) Consider the function $f(x) = x^2 - 2x$ and an interval $I = (a, b]$. For what

values of a and b ($a < b$):

a. (5 points) The image $f(I)$ is closed.

b. (5 points) The preimage $f^{-1}(I)$ is closed.

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5. (15 points) Prove that:

   a. (10 points) The *harmonic mean* is smaller than the *arithmetic mean*:
$$\frac{2}{\frac{1}{a}+\frac{1}{b}} \leq \frac{a+b}{2}, \qquad \forall a, b \in \mathbb{R}_{+}.$$

   b. (5 points) The *harmonic mean* is smaller than the *geometric mean*:
$$\frac{2}{\frac{1}{a}+\frac{1}{b}} \leq \sqrt{ab}, \qquad \forall a, b \in \mathbb{R}_{+}.$$

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6. (15 points) You know that function f is twice differentiable and $f(10) = 3,\ f'(10) = 1,\ f''(10) = 7$. What is the strongest statement about f involving little$-o$ or big$-O$ that you can guarantee to hold:

   a. (5 points) Under no additional assumptions.

   b. (5 points) Given that $f''(x)$ is continuous at $x = 10$.

   c. (5 points) Given that $f''(x)$ is bounded on a neighborhood of $x = 10$.

   Remember the second-order version of the Taylor's theorem:

   > **Second Mean Value Theorem:** Let f be twice differentiable between points x_0 and x. Then,
   > $$f(x) = f(x_0) + (x - x_0)f'(x_0) + \frac{1}{2}(x - x_0)^2 f''(c),$$
   > for some c between x_0 and x.