📝 *Math Prep – Class Notes*

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6. Linear Algebra

- Linear Vector Space
- Linear Mapping

Reference: Chapters 1–3 in Axler's book

6.1. Vector Spaces

6.1.1. List of Length n

- 📘 **Definition:**
  $\mathbb{R}^n$ is the set of all lists of length n of elements of $\mathbb{R}$:
  $$\mathbb{R}^n = \{(x_1, \ldots, x_n) : x_1, \ldots, x_n \in \mathbb{R}\}.$$
  We say that x_k is the k *th coordinate* of $(x_1, \ldots, x_n)$.
- **Examples:**
  - $\mathbb{R}^2 = \{(x, y) : x, y \in \mathbb{R}\}$
    ($\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$, the Cartesian product of two copies of $\mathbb{R}$)
  - $\mathbb{R}^3 = \{(x, y, z) : x, y, z \in \mathbb{R}\}$
- **Operations:**
  - *Addition:* For any $x, y \in \mathbb{R}^n$,
    $$(x_1, \ldots, x_n) + (y_1, \ldots, y_n) = (x_1 + y_1, \ldots, x_n + y_n)$$
  - *Scalar Multiplication:* For any $x \in \mathbb{R}^n$ and $k \in \mathbb{R}$,
    $$k(x_1, \ldots, x_n) = (kx_1, \ldots, kx_n)$$
- 🧩 **Exercise:**
  Show that
  1. $x + y = y + x$ for any $x, y \in \mathbb{R}^n$.
  2. $(-1)x + x = 0$, where $0 = (0, \ldots, 0) \in \mathbb{R}^n$.
  3. $x + (y + z) = (x + y) + z$ for any $x, y, z \in \mathbb{R}^n$.

6.1.2. Definition of Vector Space

- 📘 **Definition:**
  A *real vector space* is a set V along with an addition on V and a scalar multiplication on V such that the following properties hold:
  - *Commutativity:* $u + v = v + u$ for any $u, v \in V$;
  - *Associativity:* $(u + v) + w = u + (v + w)$ and $(ab)v = a(bv)$ for all $u, v, w \in V$ and for all $a, b \in \mathbb{R}$;
  - *Additive identity:* There exists an element $0 \in V$ such that $v + 0 = v$ for all $v \in V$;
  - *Additive inverse:* For every $v \in V$, there exists $w \in V$ such that $v + w = 0$;
  - *Multiplicative identity:* $1v = v$ for all $v \in V$;
  - *Distributive properties:* $a(u + v) = au + av$ and $(a + b)v = av + bv$ for all $a, b \in \mathbb{R}$ and all $u, v \in V$.
- **Example:**
  1. *Space of real sequences:*
  $$\mathbb{R}^{\infty} = \left\{(x_1, x_2, \ldots) : x_k \in \mathbb{R} \text{ for all } k \in \mathbb{N}\right\}.$$
  Addition and scalar multiplication on $\mathbb{R}^{\infty}$ are defined by $\{x_n\} + \{y_n\} = \{x_n + y_n\}$ and $k\{x_n\} = \{kx_n\}$.
  2. *Space of functions from D to $\mathbb{R}$:*
  $$\mathbb{R}^D = \left\{\text{all the functions } f : D \to \mathbb{R}\right\}.$$
  Addition and scalar multiplication on $\mathbb{R}^D$ are defined by $(f + g)(x) = f(x) + g(x)$ and $(kf)(x) = kf(x)$.
- 📜 **Proposition:**
  Let V be a vector space, then:
  - $0v = 0$ for every $v \in V$ (here $0 \in \mathbb{R}$ on the LHS, and $0 \in V$ on the RHS).
  - $a0 = 0$ for every $a \in \mathbb{R}$ (here $0 \in V$).
  - $(-1)v = -v$ for every $v \in V$.
- 🧩 **Exercise:**
  1. Prove that $-(-v) = v$ for every $v \in V$.
  2. If $av = 0$ for $a \in \mathbb{R}$ and $v \in V$, then $a = 0$ or $v = 0$.
  3. Let $v, w \in V$. Why does there exist a unique $x \in V$ such that $v + 3x = w$?
  4. Is $\emptyset$ a vector space?
  5. Show that the set V^S of all functions from a set S to a vector space V is a vector space itself (together with the addition and scalar multiplication).

6.1.3. Subspaces

- 📘 **Definition:**
  A subset U of V is a *subspace* of V if U is also a vector space with the same addition and scalar multiplication as on V.
- 🧩 **Exercise:** Show that the additive identity in a subspace is the same as in the original space.
- 📜 **Proposition:**
  A subset U of a vector space V is its subspace if:
  1. *Additive identity:* $0 \in U$.
  2. *Closed under addition:* $u + v \in U$ for any $u, v \in U$.
  3. *Closed under scalar multiplication:* $ku \in U$ for any $k \in \mathbb{R}$ and $u \in U$.
- 📜 **Proposition:**
  A nonempty subset U of a vector space V is its subspace if $u + kv \in U$ for any $u, v \in U$ and $k \in \mathbb{R}$.
- 🧩 **Exercise:**
  1. Consider the vector space $V = \mathbb{R}^2$, which of the following sets are subspaces of V:

     a. $U = \{(x, 1) \in \mathbb{R}^2 : x \in \mathbb{R}\}$

     b. $U = \{x \in \mathbb{R}^2 : x_1 + x_2 = 0\}$

     c. $U = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 0\}$
  2. Consider the vector space $V = \mathbb{R}^\infty$, which of the following sets are subspaces of V:

     a. $U = \{x \in \mathbb{R}^\infty : x_n \text{ is eventually } 0\}$

     b. $U = \{x \in \mathbb{R}^\infty : x_n \text{ is eventually } 1\}$

     c. $U = \{x \in \mathbb{R}^\infty : x_n \text{ is frequently } 0\}$

     d. $U = \{x \in \mathbb{R}^\infty : x_n \text{ is unbounded }\}$

     e. $U = \{x \in \mathbb{R}^\infty : x_n \text{ converges to } 0\}$

     f. $U = \{x \in \mathbb{R}^\infty : x_n \text{ converges to a rational number }\}$
  3. Consider the vector space $V = \mathbb{R}^{[0,1]}$, which of the following sets are subspaces of V:

     a. $U = \{f \in \mathbb{R}^{[0,1]} : f(1) = 0\}$

     b. $U = \{f \in \mathbb{R}^{[0,1]} : f \text{ is continuous at } 1\}$

     c. $U = \{f \in \mathbb{R}^{[0,1]} : f \text{ is continuous }\}$

d. $U = \{f \in \mathbb{R}^{[0,1]} : f \text{ is increasing }\}$

e. $U = \{f \in \mathbb{R}^{[0,1]} : f(x) = 0 \text{ at infinitely many points }\}$

f. $U = \{f \in \mathbb{R}^{[0,1]} : f \text{ is differentiable on } (0, 1)\}$

- 📘 **Definition:**
  Let A, B be two nonempty subsets of a vector space V. Then the sum of A and B is defined as
  $$A + B = \{a + b : a \in A, b \in B\}.$$
- 🧩 **Exercise:** What is the sum of the following subspaces?

  a. $U = \{(x, 0, 0) \in \mathbb{R}^3 : x \in \mathbb{R}\}, V = \{(0, y, 0) \in \mathbb{R}^3 : y \in \mathbb{R}\}$

  b. $U = \{(x, x, y, y) \in \mathbb{R}^4 : x, y \in \mathbb{R}\}, V = \{(x, x, x, y) \in \mathbb{R}^4 : x, y \in \mathbb{R}\}$
- 📜 **Proposition:**
  Suppose $V_1, \ldots, V_m$ are subspaces of V. Then $V_1 + \ldots + V_m$ is the smallest subspace of V containing $V_1, \ldots, V_m$.

6.2. Finite Dimensional Vector Spaces

6.2.1. Span and Linear Independence

- 📘 **Definition:**
  A *linear combination* of $v_1, \ldots, v_m \in V$ is a vector of the form
  $$a_1 v_1 + \ldots + a_m v_m,$$
  where $a_1, \ldots, a_m \in \mathbb{R}$. The collection of all such linear combinations is called the *span* of the vectors:
  $$\operatorname{span}(v_1, \ldots, v_m) = \left\{a_1 v_1 + \ldots + a_m v_m : a_1, \ldots, a_m \in \mathbb{R}\right\}.$$
- 📜 **Proposition:**
  The span of $v_1, \ldots, v_m$ is the smallest subspace of V containing $v_1, \ldots, v_m$.
- 📘 **Definition:**
  If $\operatorname{span}(v_1, \ldots, v_m) = V$, we say that the list $v_1, \ldots, v_m$ *spans* V.
- 📘 **Definition:**
  A vector space is called *finite dimensional* if there is a finite list of vectors that spans the space.
- 📘 **Definition:**
  A list of vectors $v_1, \ldots, v_m \in V$ is said to be *linearly independent* if the only scalars

$a_1, \ldots, a_m \in \mathbb{R}$ that satisfy

$$a_1 v_1 + \ldots + a_m v_m = 0$$

are $a_1 = a_2 = \ldots = a_m = 0$. Otherwise, the list is called *linearly dependent*.

- 🧩 **Exercise:**
    1. Show that if $v_1, \ldots, v_m$ are linearly independent, then none of the vectors can be written as a linear combination of the others.
    2. Prove that any subset of a linearly independent set is also linearly independent.
    3. Prove that any superset of a linearly dependent set is also linearly dependent.

6.2.2. Bases

- 📘 **Definition:**
  A *basis* of a finite dimensional vector space V is a list of vectors in V that is both linearly independent and spans V. In other words, a basis is a minimal spanning set for V.
- **Note:**
  The general definition of bases (allowing for V to be infinite dimensional) is that $B \subset V$ is a bases of V if the following two conditions are satisfied:
    1. Any finite list $b_1, \ldots, b_k \in B$ is linearly independent.
    2. Any vector $v \in V$ can be expressed as a linear combination of some list $b_1, \ldots, b_k \in B$.
- 📜 **Proposition:**
  Every finite dimensional vector space has a basis.
- **Note:**
  Every infinite dimensional vector space also has a basis, but the proof is beyond the scope of this course.
- 📜 **Proposition:**
  If $b_1, \ldots, b_n$ is a basis of V, then every vector $v \in V$ can be uniquely written as a linear combination of $b_1, \ldots, b_n$.
- 🧩 **Exercise:**
    1. Prove that any two bases of a finite-dimensional vector space have the same number of vectors.
    2. Let $V = \mathbb{R}^3$ and consider the vectors $v_1 = (1, 0, 0), v_2 = (1, 1, 0), v_3 = (1, 1, 1)$. Show that $\{v_1, v_2, v_3\}$ is a basis of $\mathbb{R}^3$.

6.2.3. Dimension

- **Proposition:**
  If lists $b_1, \dots, b_m$ and $b'_1, \dots, b'_n$ are two basis of a vector space V, then $m = n$.
- **Definition:**
  The *dimension* of a finite-dimensional vector space V is the number lenth of its basis.
- **Proposition:**
  If V is a vector space with dimension n, then any list of more than n vectors in V is linearly dependent.
- **Proposition:**
  If V is a vector space with dimension n, then any linearly independent list of n vectors spans V.
- **Exercise:**
  1. Show that the space $\mathbb{R}^n$ has dimension n.
  2. Let $V = \mathbb{R}^3$. Find the dimension of the subspace $W = \{(x, y, z) \in \mathbb{R}^3 : x + y + z = 0\}$.
  3. Prove that the dimension of the span of any linearly independent list of vectors equals the number of vectors in the list.

6.3. Matrices

6.3.1. Basic Operations with Matrices

- **Definition:** A *matrix* is a rectangular array of real numbers arranged in rows and columns. An $m \times n$ matrix A has m rows and n columns, we write $A \in \mathbb{R}^{m \times n}$.
- **Matrix Addition:** Two matrices of the same size can be added together by adding their corresponding elements:
  $$A + B = \left(a_{ij}\right) + \left(b_{ij}\right) = \left(a_{ij} + b_{ij}\right).$$
- **Matrix Scalar Multiplication:** A matrix A can be multiplied by a scalar $\lambda \in \mathbb{R}$ by multiplying each element of the matrix by α:
  $$\lambda A = \left(\lambda a_{ij}\right).$$
- **Matrix Multiplication:** The product of two matrices $A \in \mathbb{R}^{m \times n}$ and $B \in \mathbb{R}^{n \times p}$ is the matrix $C \in \mathbb{R}^{m \times p}$, where the element c_{ij} is given by:

$$c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}.$$

- 📘 **Definition:** The *identity matrix* I_n is a square matrix of size $n \times n$ where all the elements on the main diagonal (from the top left to the bottom right) are 1, and all other elements are 0:

$$I_n = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix}.$$

- Properties of Matrix Multiplication:
  1. **Associativity:** Matrix multiplication is associative:
  $$A(BC) = (AB)C.$$
  2. **Distributivity:** Matrix multiplication is distributive over addition:
  $$A(B + C) = AB + AC.$$
  3. **Non-Commutativity:** In general, matrix multiplication is not commutative, meaning that $AB \neq BA$.
  4. **Identity Matrix:** Let $A \in \mathbb{R}^{m \times n}$, then:
  $$AI_n = A = I_m A.$$

6.3.2. Transposition

- 📘 **Definition:** The *transpose* of a matrix $A \in \mathbb{R}^{m \times n}$ is the matrix $A^T \in \mathbb{R}^{n \times m}$ obtained by swapping rows and columns:

$$(a_{ij})^T = (a_{ji}).$$

- Properties of Transpose
  1. **Transpose of a Sum:**
  $$(A + B)^T = A^T + B^T.$$
  2. **Transpose of a Product:**
  $$(AB)^T = B^T A^T.$$
  3. **Transpose of a Transpose:**
  $$(A^T)^T = A.$$

6.3.3. Inverse of a Matrix

- 📘 **Definition:** A matrix $A \in \mathbb{R}^{n\times n}$ is *invertible* if there exists a matrix $A^{-1} \in \mathbb{R}^{n\times n}$ such that:
$$AA^{-1} = A^{-1}A = I_n.$$
- Properties of Inverses:
  1. **Inverse of a Product:**
  $$(AB)^{-1} = B^{-1}A^{-1}.$$
  2. **Inverse of a Transpose:**
  $$(A^T)^{-1} = (A^{-1})^T.$$
- 🧩 **Exercise:**
  1. Verify that matrix multiplication is not commutative by providing an example of two matrices A and B such that $AB \neq BA$.
  2. Compute the product of the following matrices:
  $$A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$
  3. Show that for any matrix A, $(A^T A)$ is symmetric.

6.4. Solving Systems of Linear Equations

6.4.1. General Method

- 📘 **Definition:**
A *system of linear equations* is a collection of linear equations in several variables. A general system of m equations in n variables can be written as:
$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1, \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2, \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m, \end{aligned}$$
where the a_{ij} are constants, the x_j are the unknowns, and the b_i are constants.
- A system of linear equations can be written in matrix form as:
$$Ax = b,$$
where A is the coefficient matrix, x is the column vector of variables, and b is the column vector of constants:

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix}, \quad x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}, \quad b = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}.$$

- 📘 Standard methods to solve the system $Ax = b$ are:
  1. *Gaussian elimination:* A method that transforms the system into an equivalent one that is easier to solve using row operations.
  2. *LU decomposition:* Factor the matrix A as a product of a lower triangular matrix L and an upper triangular matrix U, and solve the system in two steps.
  3. *Matrix inversion:* If A is square and invertible, the solution can be found as $x = A^{-1}b$.
- 📜 **Proposition:**
  If the matrix A is invertible (i.e., $\text{rank}(A) = n$ for an $n \times n$ system), the system $Ax = b$ has a unique solution given by:
  $$x = A^{-1}b.$$
- 📘 **Definition:** *(Consistent and Inconsistent Systems)*
  - A system is *consistent* if it has at least one solution.
  - A system is *inconsistent* if it has no solution.
- 📜 **Proposition:**
  The system $Ax = b$ is consistent if and only if b lies in the column space (range) of A.

6.4.2. Solving Systems When A Is Not Invertible

- When the matrix A is not invertible, it means either the system has infinitely many solutions or no solution at all.
- 📘 **Definition:** *(Rank-Deficient Systems)*
  If $\text{rank}(A) < n$ (i.e., A is not invertible), there are two possible outcomes:
  - If the system is consistent (i.e., b lies in the column space of A), then there are infinitely many solutions. In this case, we find a particular solution and describe the general solution using the null space of A.
  - If the system is inconsistent, no exact solution exists, but we can find an approximate solution using the method of least squares.
- 📜 **Proposition:** *(General Solution for Rank-Deficient Systems)*
  If A is not invertible but the system $Ax = b$ is consistent, the general solution can be written as:

$$x = x_p + x_h,$$

where x_p is a particular solution and x_h is a solution to the homogeneous system $Ax_h = 0$.

6.4.3. Using Projection Matrices and Least Squares Approximation

- If the system $Ax = b$ is inconsistent, we can find an approximate solution by projecting b onto the column space of A. The vector x that minimizes the residual $\|b - Ax\|_2$ is given by the *least squares solution*.
- The least squares solution can be found using the **normal equations**:
$$A^T Ax = A^T b.$$
If $A^T A$ is invertible, the approximate solution is:
$$x = (A^T A)^{-1} A^T b.$$
- 📘 **Definition:**
A *projection matrix* P_A projects vectors onto the column space of A. It is defined as:
$$P_A = A(A^T A)^{-1} A^T.$$
- The vector Ax is the projection of b onto the column space of A, and the least squares solution is the x such that Ax is as close as possible to b.

6.4.4. Homogeneous Systems

- 📘 **Definition:** *(Homogeneous Systems)*
A system is called *homogeneous* if $b = 0$, i.e., all constants on the right-hand side are zero. A homogeneous system always has the trivial solution $x = 0$. If A has rank less than n, the system has infinitely many solutions.
- 📜 **Proposition:**
The general solution to a homogeneous system is of the form:
$$x = x_h,$$
where x_h is a linear combination of the basis vectors of the null space of A.

- 🧩 **Exercise:**
  1. Solve the system
$$\begin{aligned} x + 2y - z &= 1, \\ 2x - y + 3z &= 5, \\ -x + 4y + z &= 2. \end{aligned}$$
  2. Find the least squares solution to the overdetermined system:

$$
\begin{aligned}
2x + 3y &= 5, \\
4x + 6y &= 9, \\
5x + 8y &= 12.
\end{aligned}
$$

3. Find all solutions to the homogeneous system:

$$
\begin{aligned}
x_1 - 2x_2 + 3x_3 &= 0, \\
2x_1 - x_2 + x_3 &= 0.
\end{aligned}
$$

6.5. Normed Spaces and Inner Product Spaces

6.5.1. Normed Spaces

- 📘 **Definition:** A *norm* on a vector space V is a function $| \cdot | : V \to \mathbb{R}$ that satisfies the following properties for all $x, y \in V$ and all scalars $\alpha \in \mathbb{R}$:

  1. **Positivity:** $|x| \geq 0$ and $|x| = 0$ if and only if $x = 0$.
  2. **Homogeneity:** $|\alpha x| = |\alpha| \cdot |x|$.
  3. **Triangle Inequality:** $|x + y| \leq |x| + |y|$.

  The pair $(V, | \cdot |)$ is called a *normed vector space*.

- 📘 **Definition:** A *Banach space* is a normed vector space that is *complete*, meaning that every Cauchy sequence in the space converges to a limit within the space.

- Examples of Normed Spaces

  1. **Euclidean Space $\mathbb{R}^n$:** The norm of a vector $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ is given by the Euclidean norm:

  $$|x|_2 = \left(\sum_{i=1}^{n} x_i^2 \right)^{1/2}.$$

  2. **p-Norms in $\mathbb{R}^n$:** For $1 \leq p < \infty$, the p-norm of a vector $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ is defined as:

  $$|x|_p = \left(\sum_{i=1}^{n} |x_i|^p \right)^{1/p}.$$

  Special cases include the **Manhattan norm** ($p = 1$) and the **Euclidean norm** ($p = 2$).

  3. **Infinity Norm in $\mathbb{R}^n$:** The infinity norm (or maximum norm) is defined as:

  $$|x|_\infty = \max_{1 \leq i \leq n} |x_i|.$$

6.5.2. Inner Product Spaces

- 📘 **Definition:** An *inner product* on a vector space V is a function $\langle \cdot, \cdot \rangle : V \times V \to \mathbb{R}$ that satisfies the following properties for all $x, y, z \in V$ and all scalars $\alpha \in \mathbb{R}$:
    1. **Symmetry:** $\langle x, y \rangle = \langle y, x \rangle$.
    2. **Linearity in the First Argument:** $\langle \alpha x + y, z \rangle = \alpha \langle x, z \rangle + \langle y, z \rangle$.
    3. **Positivity:** $\langle x, x \rangle \geq 0$ and $\langle x, x \rangle = 0$ if and only if $x = 0$.
- A vector space with an inner product is called an *inner product space*.
- 📘 **Definition:** An inner product space that is complete is called a *Hilbert Space*.
- Examples of Inner Product Spaces
    1. **Euclidean Space $\mathbb{R}^n$:** The standard inner product of two vectors $x = (x_1, x_2, \ldots, x_n)$ and $y = (y_1, y_2, \ldots, y_n)$ in $\mathbb{R}^n$ is given by:
    $$\langle x, y \rangle = \sum_{i=1}^{n} x_i y_i.$$
    2. **Function Space $L^2[a, b]$:** For functions $f, g \in L^2[a, b]$ (the space of square-integrable functions on $[a, b]$), the inner product is defined as:
    $$\langle f, g \rangle = \int_a^b f(x)g(x)\, dx.$$
- 🧩 **Exercise:**
    1. Prove that if $\langle x, y \rangle = 0$ for all $y \in V$, then $x = 0$.
    2. Show that the standard inner product on $\mathbb{R}^n$ satisfies the parallelogram identity:
    $$\langle x + y, x + y \rangle = \langle x, x \rangle + 2\langle x, y \rangle + \langle y, y \rangle$$
    3. If $f(x) = 1$ for all $x \in [a, b]$, compute the inner product $\langle f, f \rangle$ in $L^2[a, b]$.
    4. Show that the function $f(x) = \sqrt{\langle x, x \rangle}$ defined on V is a norm.