Math Prep – Exercise

Author: Pavel Kocourek

Problem Set 4

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1. Let $f(x) = o(x-a)$ as $x \to a$ and $g(x) = o(x-a)$ as $x \to a$. Show that:

   a. $f(x) + g(x) = o(x-a)$ as $x \to a$

   b. $f(x)g(x) = o\left((x-a)^2\right)$ as $x \to a$.

2. Let $f(x) = 3 + 4x + o(x-a)$ as $x \to a$ and
   $g(x) = 1 + 7x + 2x^2 + o\left((x-a)^2\right)$ as $x \to a$. Show that:

   a.
   $$\begin{aligned} f(x) + g(x) &= 4 + 11x + 2x^2 + o(x-a) \\ &= 4 + 11x + 2a^2 + o(x-a) \end{aligned}$$
   as $x \to a$.

   b.
   $$\begin{aligned} f(x)g(x) &= (3+4x)(1+7x+2x^2) + o(x-a) \\ &= 3 + 25x + 34x^2 + 8x^3 + o(x-a) \\ &= 3 + 25x + 34a^2 + 8a^3 + o(x-a) \end{aligned}$$
   as $x \to a$.

3. Let U and V be two vector spaces. Define addition and scalar multiplication on the set $U \times V$, and show that $U \times V$, together with these operations, forms a vector space.

4. Let V be a vector space and S be a set. Define addition and scalar multiplication on the set V^S of all functions $f : S \to V$, and show that V^S, together with these operations, forms a vector space.

5. Consider the vector space of all sequences $V = \mathbb{R}^\infty$. Which of the following subsets form a subspace of V:

   a. $U = \left\{x \in \mathbb{R}^\infty : \text{the sequence } y_n := \sum_{k=1}^n |x_k| \text{ is bounded}\right\}$

   b. $U = \left\{x \in \mathbb{R}^\infty : x_k x_{k+1} = 0 \ \forall k \in \mathbb{N}\right\}$

6. Characterize the sum $U + W$ of the subspaces U and W (both of which are subspaces of some vector space V):

a. $U = \{u \in \mathbb{R}^3 : u_1 + u_2 + u_3 = 0\}$ and $W = \{w \in \mathbb{R}^3 : w_1 = w_2 = w_3\}$

b. $U = \{u \in \mathbb{R}^\infty : u_k \text{ is convergent}\}$ and $W = \{w \in \mathbb{R}^\infty : w_k \text{ is eventually } 0\}$

c. $U = \{u \in \mathbb{R}^{[0,1]} : u(0) = 0\}$ and $W = \{w \in \mathbb{R}^{[0,1]} : w(1) = 0\}$

d. $U = \{u \in \mathbb{R}^{[0,1]} : u(x) \text{ is continuous at } x = 0\}$ and
$W = \{w \in \mathbb{R}^{[0,1]} : w(x) \text{ is continuous at } x = 1\}$

e. $U = \{u \in \mathbb{R}^{[0,1]} : u(0) = 0\}$ and
$W = \{w \in \mathbb{R}^{[0,1]} : w(x) \text{ is continuous at } x = 0\}$