📝 *Math Prep – Class Notes*

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Class Notes – Day 1 – Introduction

- Basic concepts: real numbers, inequalities, intervals, functions, neighborhoods, etc.
- Introduction to proofs and mathematical logic

Basic Concepts

References:

- Chapter A.2 in Efe Ok's book
- Chapter 1 in Craw's book

Real Numbers

- $\mathbb{N}$ – natural numbers
  - Intuition: count of objects
  - 📜 **Principle of Induction:** For any $S \subseteq \mathbb{N}$, if $1 \in S$ and $n+1 \in S$ for any $n \in S$, then $S = \mathbb{N}$.
  - Operations:
    - *Addition*
    - *Multiplication*
  - Order relations:
    - "$\geq$" – greater than or equal to
    - "$>$" – strictly greater than
  - 1 is the multiplicative identity
- $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$
  - 0 is the additive identity
- $\mathbb{Z}$ – integers
  - *Subtraction* is the operation inverse to addition

- $\mathbb{Q}$ – rational numbers
    - *Division* is the operation inverse to multiplication
    - **Exercise:** Show that $\sqrt{2} \notin \mathbb{Q}$.
- $\mathbb{I}$ – irrational numbers
    - **Exercise:** Give examples of irrational numbers.
- $\mathbb{R}$ – real numbers
    - Intuition: filling the "holes" between rational numbers
    - Real numbers are *complete*.
    - There is a real number between any two distinct rational numbers.
    - There is a rational number between any two distinct real numbers.
    - **Exercise:** Show that $\sqrt{-1} \notin \mathbb{R}$.
- *Supremum* and *Infimum*
    - *Upper bound* of a set S
    - $\sup S$ – *the least upper bound* of S
    - $\inf S = -\sup(-S)$ – *the greatest lower bound* of S
    - If $\emptyset \neq S \subset \mathbb{R}$ is bounded from above, then $\sup S \in \mathbb{R}$.
    - **Exercise:** Find the supremum and infimum of the following sets:
        1. $S = 0.6, 0.66, 0.666, \dots$
        2. $T = 1/n : n \in \mathbb{N}$
        3. $U = x \in \mathbb{Q} : x^2 < 3$
- Positive / negative rationals and reals:
    - $\mathbb{Q}_+ = \{x \in \mathbb{Q} : x \geq 0\}$
    - $\mathbb{Q}_{++} = \{x \in \mathbb{Q} : x > 0\}$
    - $\mathbb{R}_+ = \{x \in \mathbb{R} : x \geq 0\}$
    - $\mathbb{R}_{++} = \{x \in \mathbb{R} : x > 0\}$
    - Similarly, $\mathbb{Q}_- = \{x \in \mathbb{Q} : x \leq 0\}$, etc.

Inequalities

- **Proposition:** Let $a > b$ and $c \in \mathbb{R}$, then:
    - $a + c > b + c$
    - $ac > bc$ whenever $c > 0$
    - $ac < bc$ whenever $c < 0$
- **Exercise:** Prove the *Arithmetic-Geometric Mean Inequality*.

$$\frac{a+b}{2} \geq \sqrt{ab} \quad \forall a, b \in \mathbb{R}_+.$$

Intervals

- *Open interval* – (a, b)
- *Closed interval* – $[a, b]$
- *Half-open intervals* – $(a, b]$ or $[a, b)$
- Finite / infinite intervals
- 🧩 **Exercise:** Let $I = (a, b)$ and $J = [c, d]$ be nondegenerate intervals:
  1. Under what conditions is $I \cap J$ an open interval?
  2. Under what conditions is $I \cap J$ a closed interval?
  3. Is $I \cup J$ an interval?

Functions

A real function: $f : D \to T$

- $D \subseteq \mathbb{R}$ is the *domain*
- T is the *codomain* (also known as the *target space*)
- $f(D) = \{f(x) : x \in D\}$ is the *range*
- $f(X)$ is the *image* of the set $X \subseteq D$
- $f^{-1}(Y) = \{x \in D : f(x) \in Y\}$ is the *preimage* of $Y \subseteq \mathbb{R}$
- **Examples:**
  1. A linear function
  2. The distance from point $\bar{x} \in \mathbb{R}$
  3. A polynomial
  4. A trigonometric function
  5. An *indicator function* (also known as the *characteristic function* of set S)

$$\mathbf{1}_S(x) = \begin{cases} 1, & \text{if } x \in S, \\ 0, & \text{otherwise.} \end{cases}$$

Neighborhoods

- 📘 **Definition:** A neighborhood of a point $a \in \mathbb{R}$ is the interval $(a - \delta, a + \delta)$ for some $\delta > 0$.
- **Intuition:** It is a collection of points near a.
- 🧩 **Exercise:** Show that an open interval contains a neighborhood of each of its points.

Absolute Value

- 📘 **Definition:** The function $|\cdot| : \mathbb{R} \to \mathbb{R}$ defined by
$$|x| = \begin{cases} x, & \text{if } x \geq 0, \\ -x, & \text{if } x < 0 \end{cases}$$
is called *the absolute value*.
- 🧩 **Exercise:**
  1. Characterize the open interval $(a - \delta, a + \delta)$ using absolute value.
  2. Describe the set $\{x \in \mathbb{R} : |7x + 2| < 9\}$ as an interval.
  3. Prove the *triangle inequality*:
  $$|x + y| \leq |x| + |y| \quad \forall x, y \in \mathbb{R}.$$
  4. Prove that
  $$|x - y| \geq \big||x| - |y|\big| \quad \forall x, y \in \mathbb{R}.$$

Proofs

Reference:

- Appendix A1.3 in Simon and Bloom's book

Direct Proof

- Direct way of proving $A \implies B$:
$$A = A_0 \implies A_1 \implies A_1 \implies A_1 \implies \ldots \implies A_n = B$$
- **Example:**

Theorem A1.1 For any $x, y, z \in \mathbf{R}$, if $x + z = y + z$, then $x = y$.

Proof

1. $x + z = y + z$. — Hypothesis.
2. There exists $(-z)$ such that $z + (-z) = 0$. — Additive inverse property.
3. $(x + z) + (-z) = (y + z) + (-z)$. — Rule (3).
4. $x + (z + (-z)) = y + (z + (-z))$. — Additive associative property.
5. $x + 0 = y + 0$. — Step 2.
6. $x = y$. — Additive identity property. ■

Implication, converse, and contrapositive:

1. $A \implies B$ – logical statement A implies logical statement B
2. $A \iff B$ – A is *equivalent* to B (A holds *if and only if* B holds)
3. $B \implies A$ – the converse of $A \implies B$
4. $\neg A$ – negation of the logical statement A (also known as *not* A, or $\sim A$)

5. $\neg B \implies \neg A$ – the contrapositive of the implication $A \implies B$; is equivalent to $A \implies B$

Exercise: Let $a, b \in \mathbb{N}$, and consider the following logical statements:

> A: both a and b are odd;

> B: $a \cdot b$ is odd;

Formulate the stetements 1.–5. and determine their validity.

Proof by contrapositive

- Instead of proving original statement

  > $A \implies B,$

  prove its contrapositive

  > $\neg B \implies \neg A.$

- **Exercise:** Prove by contrapositive the following statement:

  > Let $a \in \mathbb{N}$. If a^2 is odd, then a is odd.

Proof by Contradiction

- Prove that $A \implies B$ by contradiction:
  1. **Assume the Opposite**: Suppose that A holds, but B does **not** hold (i.e., $\neg B$ holds).
  2. **Derive a Contradiction**: Using logical reasoning and the assumptions A and $\neg B$, derive a statement that is clearly false or contradicts a known fact.
  3. **Conclude**: Since the assumption leads to a contradiction, the original implication $A \implies B$ must be true.
- *Exercise:*
  1. Prove the statement from the last exercise by contradiction.
  2. Prove that there is infinitely many primes.