📝 *Math Prep – Class Notes*

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5. Optimization

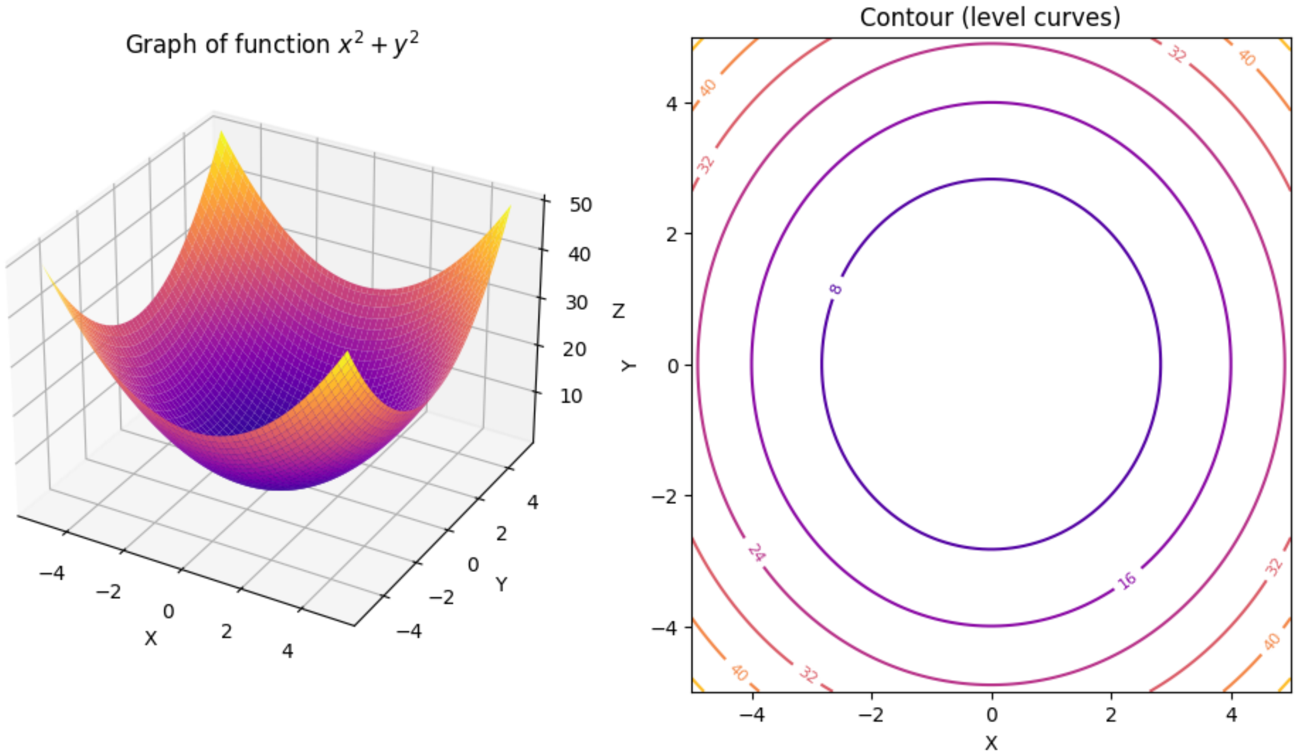
- Functions of Several Variables
- Calculus of Several Variables

Reference: Chapter 13, 14, 17 and 18 in *Simon and Blume*.

5.1. Examples of Functions of Two Variables

Plot a 3D graph of the function $f(x, y) = x^2 + y^2$

Show code



- *Linear Function:*

$$f(x, y) = ax + by = \begin{pmatrix} a & b \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

- *Affine Function:*

$$f(x, y) = ax + by + c = \begin{pmatrix} a & b & c \end{pmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix}$$

- *Quadratic Form:*

$$f(x, y) = ax^2 + bxy + cy^2 = \begin{pmatrix} x & y \end{pmatrix} \begin{pmatrix} a & \frac{b}{2} \\ \frac{b}{2} & c \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

- *Monomial:* (of degree $m + n$)

$$f(x, y) = x^m y^n, \quad m, n \in \mathbb{N}_0$$

- *Polynomial:* is a finite sum of monomials
- **Example:**
    - Polynomial of degree 1 is an affine function
    - Polynomial of degree 2 is a quadratic function

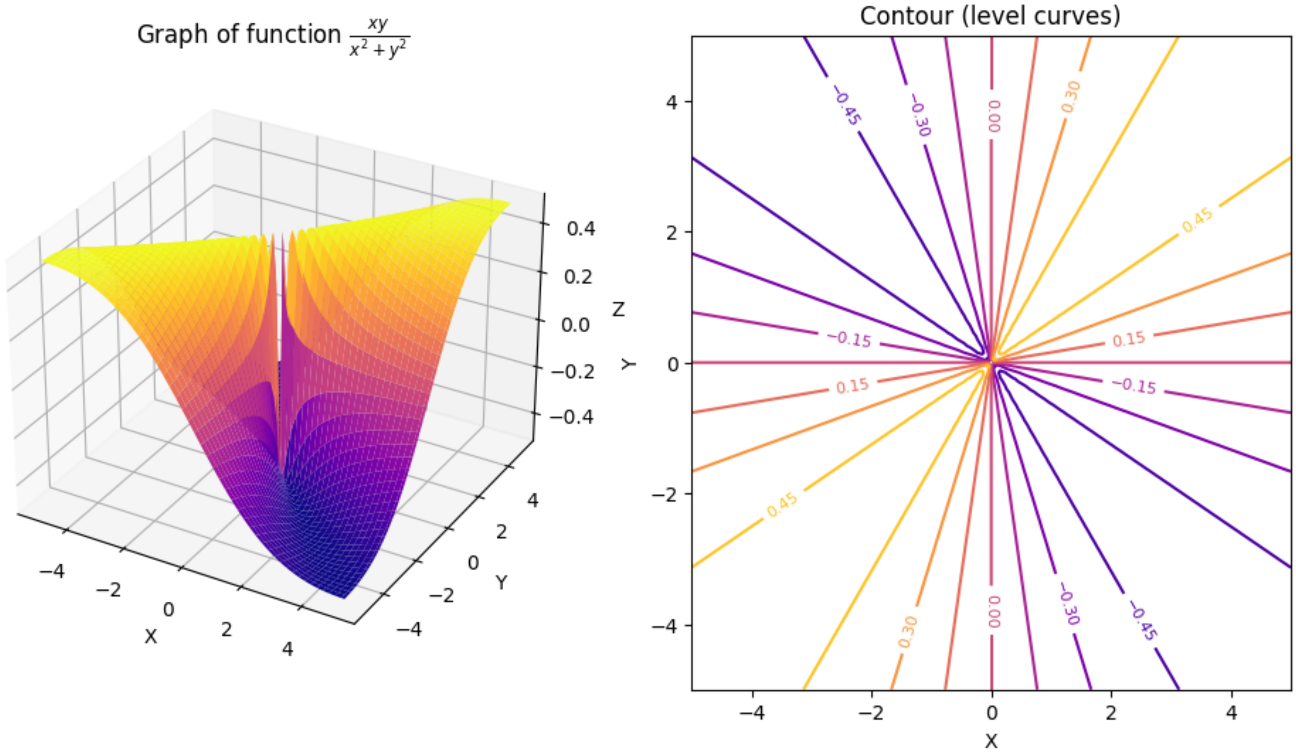
$$f(x, y) = ax^2 + bxy + cy^2 + dx + ey + f = \begin{pmatrix} x & y & 1 \end{pmatrix} \begin{pmatrix} a & \frac{b}{2} & \frac{d}{2} \\ \frac{b}{2} & c & \frac{e}{2} \\ \frac{d}{2} & \frac{e}{2} & f \end{pmatrix}$$

5.2. Continuity

- A function $f(x, y)$ might be continuous in each the variables individually and yet not be continuous.

> Plot a 3D graph of the function $f(x, y) = \frac{xy}{x^2+y^2}$

Show code



- 📘 **Definition:**
  - A real-valued function $f(x, y)$ is *continuous* at (x_0, y_0) if for any sequence $\{(x_n, y_n)\}$ that converges to (x_0, y_0), the sequence $f(x_n, y_n)$ converges to $f(x_0, y_0)$.
  - A function is said to be *continuous* if it is continuous at every point of its domain.
- **Note:** A sequence of points $(x_n, y_n) \in \mathbb{R}^2$ converges to (x_0, y_0) if $x_n \to x_0$ and $y_n \to y_0$ as $n \to \infty$.
- 🧩 **Exercise:**
  1. Rewrite the definition of continuity using the $\varepsilon > 0$ and $\delta > 0$ formulation instead of sequences.

2. Show that any linear function $f(x, y)$ is continuous.
3. Show that the function

$$f(x, y) = \begin{cases} \frac{xy}{x^2+y^2} & \text{if } (x, y) \neq (0, 0), \\ 0 & \text{if } (x, y) = (0, 0) \end{cases}$$

is not continuous at $(0, 0)$.

- **Proposition:** Let functions $f(x, y)$ and $g(x, y)$ be continuous at the point (x_0, y_0). Then the functions $f + g$ and $f \cdot g$ are continuous at (x_0, y_0), and if $g(x_0, y_0) \neq 0$, then the function f/g is also continuous at (x_0, y_0).
- **Exercise:**
  1. Show that every polynomial is continuous.
  2. What can you say about the continuity of a composite function $f \circ g$?

5.3. Partial Derivatives

- **Definition:** The partial derivatives of a function $f : \mathbb{R}^2 \to \mathbb{R}$ are defined as

$$\frac{\partial}{\partial x} f(x, y) = \lim_{h \to 0} \frac{f(x + h, y) - f(x, y)}{h},$$
$$\frac{\partial}{\partial y} f(x, y) = \lim_{h \to 0} \frac{f(x, y + h) - f(x, y)}{h}.$$

- **Alternative Notation:**
$\frac{\partial}{\partial x} f(x, y)$ is sometimes written as $\partial_x f(x, y)$, $f_x(x, y)$, or $f_1(x, y)$.
- **Exercise:** Find the partial derivatives of the following functions:
  1. $f(x, y) = x^2 + y$
  2. $f(x, y) = xy$
  3. $f(x, y) = Ax^\alpha y^\beta$, for given parameters $A, \alpha, \beta \in \mathbb{R}_{++}$
  4. $f(x, y) = \frac{x+y}{x-y}$

5.4. Total Derivative

- **Definition:** Consider a function $f : \mathbb{R}^2 \to \mathbb{R}$ and a point $(x, y) \in \mathbb{R}^2$. Small changes in x, y, and f are denoted dx, dy, and df, and are called *differentials*. The change in f as a function of the change in x and y is

$$df(x, y) = \frac{\partial}{\partial x} f(x, y)\ dx + \frac{\partial}{\partial y} f(x, y)\ dy,$$

and is called *the total differential*.

- **Exercise:**

1. Calculate the total differential of the function $f(x, y) = xy$ at the point $(x, y) = (3, 5)$.
2. Approximate the function $f(x, y) = x^{\frac{1}{3}} y^{\frac{2}{3}}$ using differentials at the point $(x, y) = (1, 1)$.
3. Write down the total differential of the revenue function $R(p, q) = pq$.

5.5. Unconstrained Optimization

- Consider a set $U \subseteq \mathbb{R}^2$.
- **Definition:** A point $(x^*, y^*) \in U$ is a point of:
  - *max* of f on U if $f(x^*, y^*) \geq f(x, y)$ for all $(x, y) \in U$,
  - *strict max* of f if $f(x^*, y^*) \geq f(x, y)$ holds strictly for every $(x, y) \neq (x^*, y^*)$,
  - *local max* of f if there is a ball $B_r(x^*, y^*)$ such that $f(x^*, y^*) \geq f(x, y)$ for all $(x, y) \in B_r(x^*, y^*) \cap U$,
  - *strict local max* of f if there is a ball $B_r(x^*, y^*)$ such that $f(x^*, y^*) \geq f(x, y)$ for all $(x, y) \in B_r(x^*, y^*) \cap U$ and the inequality holds strictly whenever $(x, y) \neq (x^*, y^*)$.
- **Note:** An *open ball about* (x^*, y^*) with radius r is the set
$$B_r(x^*, y^*) = \left\{(x, y) \in \mathbb{R}^2 : (x - x^*)^2 + (y - y^*)^2 < r^2\right\}.$$
- **Proposition:** *(Theorem 17.1)* Let $f : U \to \mathbb{R}$ be a C^1 function. If (x^*, y^*) is a local max or local min of f in U and if (x^*, y^*) is an interior point of U, then (x^*, y^*) is a critical point of f.
- **Note:**
  1. A point $(x^*, y^*) \in U$ is a critical point of f if
  $$\frac{\partial f}{\partial x}(x^*, y^*) = 0 \quad \text{and} \quad \frac{\partial f}{\partial y}(x^*, y^*) = 0.$$
  2. A function f is C^1 if it is continuously differentiable, i.e., both of its partial derivatives $\frac{\partial}{\partial x} f(x, y)$ and $\frac{\partial}{\partial y} f(x, y)$ exist and are continuous.

5.6. Constrained Optimization

- **Definition:**(general formulation) A *constrained optimization problem* is
$$\max f(x_1, \ldots, x_n)$$
subject to
$$g_1(x_1, \ldots, x_n) \leq b_1, \ldots, g_k(x_1, \ldots, x_n) \leq b_k,$$
and
$$h_1(x_1, \ldots, x_n) \leq c_1, \ldots, h_m(x_1, \ldots, x_n) \leq c_m.$$

- **Proposition:** *(Theorem 18.1)* Let f and h be C^1 functions of two variables. Suppose that (x^*, y^*) is a solution of the problem

$$\begin{aligned} &\max\ f(x, y) \\ &\text{s.t.}\ \ h(x, y) = c. \end{aligned}$$

Suppose further that (x^*, y^*) is not a critical point of h. Then, there is $\mu^* \in \mathbb{R}$ such that (x^*, y^*, μ^*) is a critical point of the Lagrangian function

$$L(x, y, \mu) = f(x, y) - \mu\big(h(x, y) - c\big).$$

- **Proposition:** *(Theorem 18.3)* Suppose that f and g are C^1 functions on $\mathbb{R}^2$. If (x^*, y^*) maximizes f subject to the constraint $g(x, y) \leq b$, then at least one of the following conditions must be satisfied:

  (i) (x^*, y^*, λ^*) is a critical point of the Lagrangian

$$L(x, y, \lambda) = f(x, y) - \lambda \cdot [g(x, y) - b],$$

  (ii) (x^*, y^*) is a critical point of $f(x, y)$ and $g(x^*, y^*) < b$,

  (iii) (x^*, y^*) is a critical point of $g(x, y)$.

- Equivalent the original statement:

> **Theorem 18.3:** Suppose that f and g are C^1 functions on $\mathbb{R}^2$ and that (x^*, y^*) maximizes f on the constraint set $g(x, y) \leq b$. If $g(x^*, y^*) = b$, suppose that
>
> $$\frac{\partial g}{\partial x}(x^*, y^*) \neq 0 \quad \text{or} \quad \frac{\partial g}{\partial y}(x^*, y^*) \neq 0.$$
>
> In any case, form the Lagrangian function
>
> $$L(x, y, \lambda) = f(x, y) - \lambda \cdot [g(x, y) - b].$$
>
> Then, there is a multiplier λ^* such that:
> (a) $\frac{\partial L}{\partial x}(x^*, y^*, \lambda^*) = 0$,
> (b) $\frac{\partial L}{\partial y}(x^*, y^*, \lambda^*) = 0$,
> (c) $\lambda^* \cdot [g(x^*, y^*) - b] = 0$,
> (d) $\lambda^* \geq 0$,
> (e) $g(x^*, y^*) \leq b$.