📝 *Math Prep – Class Notes*

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6. Linear Algebra

- Linear Vector Space
- Linear Mapping

Reference: Chapters 1–3 in Axler's book

6.1. Vector Spaces

6.1.1. List of Length n

- 📘 **Definition:**
  $\mathbb{R}^n$ is the set of all lists of length n of elements of $\mathbb{R}$:
  $$\mathbb{R}^n = \{(x_1, \ldots, x_n) : x_1, \ldots, x_n \in \mathbb{R}\}.$$
  We say that x_k is the k *th coordinate* of $(x_1, \ldots, x_n)$.
- **Examples:**
  - $\mathbb{R}^2 = \{(x, y) : x, y \in \mathbb{R}\}$
    ($\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$, the Cartesian product of two copies of $\mathbb{R}$)
  - $\mathbb{R}^3 = \{(x, y, z) : x, y, z \in \mathbb{R}\}$
- **Operations:**
  - *Addition:* For any $x, y \in \mathbb{R}^n$,
    $$(x_1, \ldots, x_n) + (y_1, \ldots, y_n) = (x_1 + y_1, \ldots, x_n + y_n)$$
  - *Scalar Multiplication:* For any $x \in \mathbb{R}^n$ and $k \in \mathbb{R}$,
    $$k(x_1, \ldots, x_n) = (kx_1, \ldots, kx_n)$$
- 🧩 **Exercise:**
  Show that
  1. $x + y = y + x$ for any $x, y \in \mathbb{R}^n$.
  2. $(-1)x + x = 0$, where $0 = (0, \ldots, 0) \in \mathbb{R}^n$.
  3. $x + (y + z) = (x + y) + z$ for any $x, y, z \in \mathbb{R}^n$.

6.1.2. Definition of Vector Space

- **Definition:**
  A *real vector space* is a set V along with an addition on V and a scalar multiplication on V such that the following properties hold:
  - *Commutativity:* $u + v = v + u$ for any $u, v \in V$;
  - *Associativity:* $(u + v) + w = u + (v + w)$ and $(ab)v = a(bv)$ for all $u, v, w \in V$ and for all $a, b \in \mathbb{R}$;
  - *Additive identity:* There exists an element $0 \in V$ such that $v + 0 = v$ for all $v \in V$;
  - *Additive inverse:* For every $v \in V$, there exists $w \in V$ such that $v + w = 0$;
  - *Multiplicative identity:* $1v = v$ for all $v \in V$;
  - *Distributive properties:* $a(u + v) = au + av$ and $(a + b)v = av + bv$ for all $a, b \in \mathbb{R}$ and all $u, v \in V$.
- **Example:**
  1. *Space of real sequences:*
  $$\mathbb{R}^\infty = \left\{(x_1, x_2, \dots) : x_k \in \mathbb{R} \text{ for all } k \in \mathbb{N}\right\}.$$
  Addition and scalar multiplication on $\mathbb{R}^\infty$ are defined by $\{x_n\} + \{y_n\} = \{x_n + y_n\}$ and $k\{x_n\} = \{kx_n\}$.
  2. *Space of functions from D to $\mathbb{R}$:*
  $$\mathbb{R}^D = \left\{\text{all the functions } f : D \to \mathbb{R}\right\}.$$
  Addition and scalar multiplication on $\mathbb{R}^D$ are defined by $(f + g)(x) = f(x) + g(x)$ and $(kf)(x) = kf(x)$.
- **Proposition:**
  Let V be a vector space, then:
  - $0v = 0$ for every $v \in V$ (here $0 \in \mathbb{R}$ on the LHS, and $0 \in V$ on the RHS).
  - $a0 = 0$ for every $a \in \mathbb{R}$ (here $0 \in V$).
  - $(-1)v = -v$ for every $v \in V$.
- **Exercise:**
  1. Prove that $-(-v) = v$ for every $v \in V$.
  2. If $av = 0$ for $a \in \mathbb{R}$ and $v \in V$, then $a = 0$ or $v = 0$.
  3. Let $v, w \in V$. Why does there exist a unique $x \in V$ such that $v + 3x = w$?
  4. Is $\emptyset$ a vector space?
  5. Show that the set V^S of all functions from a set S to a vector space V is a vector space itself (together with the addition and scalar multiplication).

6.1.3. Subspaces

- 📘 **Definition:**
  A subset U of V is a *subspace* of V if U is also a vector space with the same addition and scalar multiplication as on V.
- 🧩 **Exercise:** Show that the additive identity in a subspace is the same as in the original space.
- 📜 **Proposition:**
  A subset U of a vector space V is its subspace if:
  1. *Additive identity:* $0 \in U$.
  2. *Closed under addition:* $u + v \in U$ for any $u, v \in U$.
  3. *Closed under scalar multiplication:* $ku \in U$ for any $k \in \mathbb{R}$ and $u \in U$.
- 📜 **Proposition:**
  A nonempty subset U of a vector space V is its subspace if $u + kv \in U$ for any $u, v \in U$ and $k \in \mathbb{R}$.
- 🧩 **Exercise:**
  1. Consider the vector space $V = \mathbb{R}^2$, which of the following sets are subspaces of V:

     a. $U = \{(x, 1) \in \mathbb{R}^2 : x \in \mathbb{R}\}$

     b. $U = \{x \in \mathbb{R}^2 : x_1 + x_2 = 0\}$

     c. $U = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 0\}$
  2. Consider the vector space $V = \mathbb{R}^\infty$, which of the following sets are subspaces of V:

     a. $U = \{x \in \mathbb{R}^\infty : x_n \text{ is eventually } 0\}$

     b. $U = \{x \in \mathbb{R}^\infty : x_n \text{ is eventually } 1\}$

     c. $U = \{x \in \mathbb{R}^\infty : x_n \text{ is frequently } 0\}$

     d. $U = \{x \in \mathbb{R}^\infty : x_n \text{ is unbounded }\}$

     e. $U = \{x \in \mathbb{R}^\infty : x_n \text{ converges to } 0\}$

     f. $U = \{x \in \mathbb{R}^\infty : x_n \text{ converges to a rational number }\}$
  3. Consider the vector space $V = \mathbb{R}^{[0,1]}$, which of the following sets are subspaces of V:

     a. $U = \{f \in \mathbb{R}^{[0,1]} : f(1) = 0\}$

     b. $U = \{f \in \mathbb{R}^{[0,1]} : f \text{ is continuous at } 1\}$

     c. $U = \{f \in \mathbb{R}^{[0,1]} : f \text{ is continuous }\}$

d. $U = \{ f \in \mathbb{R}^{[0,1]} : f \text{ is increasing } \}$

e. $U = \{ f \in \mathbb{R}^{[0,1]} : f(x) = 0 \text{ at infinitely many points } \}$

f. $U = \{ f \in \mathbb{R}^{[0,1]} : f \text{ is differentiable on } (0, 1)\}$

- 📘 **Definition:**
  Let A, B be two nonempty subsets of a vector space V. Then the sum of A and B is defined as
  $$A + B = \{a + b : a \in A, b \in B\}.$$
- 🧩 **Exercise:** What is the sum of the following subspaces?
  a. $U = \{(x, 0, 0) \in \mathbb{R}^3 : x \in \mathbb{R}\}, V = \{(0, y, 0) \in \mathbb{R}^3 : y \in \mathbb{R}\}$
  b. $U = \{(x, x, y, y) \in \mathbb{R}^4 : x, y \in \mathbb{R}\}, V = \{(x, x, x, y) \in \mathbb{R}^4 : x, y \in \mathbb{R}\}$
- 📜 **Proposition:**
  Suppose $V_1, \ldots, V_m$ are subspaces of V. Then $V_1 + \ldots + V_m$ is the smallest subspace of V containing $V_1, \ldots, V_m$.

6.2. Finite Dimensional Vector Spaces

6.2.1. Span and Linear Independence

- 📘 **Definition:**
  A *linear combination* of $v_1, \ldots, v_m \in V$ is a vector of the form
  $$a_1 v_1 + \ldots + a_m v_m,$$
  where $a_1, \ldots, a_m \in \mathbb{R}$. The collection of all such linear combinations is called the *span* of the vectors:
  $$\operatorname{span}(v_1, \ldots, v_m) = \left\{a_1 v_1 + \ldots + a_m v_m : a_1, \ldots, a_m \in \mathbb{R}\right\}.$$
- 📜 **Proposition:**
  The span of $v_1, \ldots, v_m$ is the smallest subspace of V containing $v_1, \ldots, v_m$.
- 📘 **Definition:**
  If $\operatorname{span}(v_1, \ldots, v_m) = V$, we say that the list $v_1, \ldots, v_m$ *spans* V.
- 📘 **Definition:**
  A vector space is called *finite dimensional* if there is a finite list of vectors that spans the space.
- 📘 **Definition:**
  A list of vectors $v_1, \ldots, v_m \in V$ is said to be *linearly independent* if the only scalars $a_1, \ldots, a_m \in \mathbb{R}$ that satisfy

$$a_1 v_1 + \ldots + a_m v_m = 0$$

are $a_1 = a_2 = \ldots = a_m = 0$. Otherwise, the list is called *linearly dependent*.

- 🧩 **Exercise:**
  1. Show that if $v_1, \ldots, v_m$ are linearly independent, then none of the vectors can be written as a linear combination of the others.
  2. Prove that any subset of a linearly independent set is also linearly independent.
  3. Prove that any superset of a linearly dependent set is also linearly dependent.

6.2.2. Bases

- 📘 **Definition:**
  A *basis* of a vector space V is a list of vectors in V that is both linearly independent and spans V. In other words, a basis is a minimal spanning set for V.
- 📜 **Proposition:**
  Every vector space has a basis.
- 📜 **Proposition:**
  If $v_1, \ldots, v_m$ is a basis of V, then every vector $v \in V$ can be uniquely written as a linear combination of $v_1, \ldots, v_m$.
- 🧩 **Exercise:**
  1. Prove that any two bases of a finite-dimensional vector space have the same number of vectors.
  2. Let $V = \mathbb{R}^3$ and consider the vectors $v_1 = (1, 0, 0)$, $v_2 = (1, 1, 0)$, $v_3 = (1, 1, 1)$. Show that $\{v_1, v_2, v_3\}$ is a basis of $\mathbb{R}^3$.

6.2.3. Dimension

- 📘 **Definition:**
  The *dimension* of a vector space V, denoted by $\dim V$, is the number of vectors in any basis of V. If V is spanned by a finite list of vectors, then V is called *finite-dimensional*, and $\dim V$ is a finite number. Otherwise, V is *infinite-dimensional*.
- 📜 **Proposition:**
  If V is a vector space with dimension n, then any list of more than n vectors in V is linearly dependent.
- 📜 **Proposition:**
  If V is a vector space with dimension n, then any linearly independent list of n vectors spans V.

- 🧩 **Exercise:**
  1. Show that the space $\mathbb{R}^n$ has dimension n.
  2. Let $V = \mathbb{R}^3$. Find the dimension of the subspace $W = \{(x, y, z) \in \mathbb{R}^3 : x + y + z = 0\}$.
  3. Prove that the dimension of the span of any linearly independent list of vectors equals the number of vectors in the list.

6.4. Solving Systems of Linear Equations

6.4.1. General Method

- 📘 **Definition:**
  A *system of linear equations* is a collection of linear equations in several variables. A general system of m equations in n variables can be written as:
  $$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1, \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2, \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m, \end{aligned}$$
  where the a_{ij} are constants, the x_j are the unknowns, and the b_i are constants.
- A system of linear equations can be written in matrix form as:
  $$Ax = b,$$
  where A is the coefficient matrix, x is the column vector of variables, and b is the column vector of constants:
  $$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix}, \quad x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}, \quad b = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}.$$
- 📘 Standard methods to solve the system $Ax = b$ are:
  1. *Gaussian elimination:* A method that transforms the system into an equivalent one that is easier to solve using row operations.
  2. *LU decomposition:* Factor the matrix A as a product of a lower triangular matrix L and an upper triangular matrix U, and solve the system in two steps.
  3. *Matrix inversion:* If A is square and invertible, the solution can be found as $x = A^{-1}b$.
- 📜 **Proposition:**

If the matrix A is invertible (i.e., $\mathrm{rank}(A) = n$ for an $n \times n$ system), the system $Ax = b$ has a unique solution given by:

$$x = A^{-1}b.$$

- 📘 **Definition:** *(Consistent and Inconsistent Systems)*
    - A system is *consistent* if it has at least one solution.
    - A system is *inconsistent* if it has no solution.
- 📜 **Proposition:**
  The system $Ax = b$ is consistent if and only if b lies in the column space (range) of A.

6.4.2. Solving Systems When A Is Not Invertible

- When the matrix A is not invertible, it means either the system has infinitely many solutions or no solution at all.
- 📘 **Definition:** *(Rank-Deficient Systems)*
  If $\mathrm{rank}(A) < n$ (i.e., A is not invertible), there are two possible outcomes:
    - If the system is consistent (i.e., b lies in the column space of A), then there are infinitely many solutions. In this case, we find a particular solution and describe the general solution using the null space of A.
    - If the system is inconsistent, no exact solution exists, but we can find an approximate solution using the method of least squares.
- 📜 **Proposition:** *(General Solution for Rank-Deficient Systems)*
  If A is not invertible but the system $Ax = b$ is consistent, the general solution can be written as:

  $$x = x_p + x_h,$$

  where x_p is a particular solution and x_h is a solution to the homogeneous system $Ax_h = 0$.

6.4.3. Using Projection Matrices and Least Squares Approximation

- If the system $Ax = b$ is inconsistent, we can find an approximate solution by projecting b onto the column space of A. The vector x that minimizes the residual $\|b - Ax\|_2$ is given by the *least squares solution*.
- The least squares solution can be found using the **normal equations**:

  $$A^T Ax = A^T b.$$

  If $A^T A$ is invertible, the approximate solution is:

  $$x = (A^T A)^{-1} A^T b.$$

- 📘 **Definition:**

A *projection matrix* P_A projects vectors onto the column space of A. It is defined as:

$$P_A = A(A^T A)^{-1} A^T.$$

- The vector Ax is the projection of b onto the column space of A, and the least squares solution is the x such that Ax is as close as possible to b.

6.4.4. Homogeneous Systems

- **Definition:** *(Homogeneous Systems)*
  A system is called *homogeneous* if $b = 0$, i.e., all constants on the right-hand side are zero. A homogeneous system always has the trivial solution $x = 0$. If A has rank less than n, the system has infinitely many solutions.
- **Proposition:**
  The general solution to a homogeneous system is of the form:
  $$x = x_h,$$
  where x_h is a linear combination of the basis vectors of the null space of A.

- **Exercise:**
  1. Solve the system
  $$\begin{aligned} x + 2y - z &= 1, \\ 2x - y + 3z &= 5, \\ -x + 4y + z &= 2. \end{aligned}$$
  2. Find the least squares solution to the overdetermined system:
  $$\begin{aligned} 2x + 3y &= 5, \\ 4x + 6y &= 9, \\ 5x + 8y &= 12. \end{aligned}$$
  3. Find all solutions to the homogeneous system:
  $$\begin{aligned} x_1 - 2x_2 + 3x_3 &= 0, \\ 2x_1 - x_2 + x_3 &= 0. \end{aligned}$$