

✓ Math Prep – Class Notes

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Class Notes – Day 1 – Introduction


- Basic concepts: real numbers, inequalities, intervals, functions, neighborhoods, etc.
 - Introduction to proofs and mathematical logic
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1. Basic Concepts

References:



- Chapter A.2 in Efe Ok's book
 - Chapter 1 in Craw's book
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1.1. Real Numbers

- \mathbb{N} – natural numbers
 - Intuition: count of objects
 -  **Principle of Induction:** For any $S \subseteq \mathbb{N}$, if $1 \in S$ and $n + 1 \in S$ for any $n \in S$, then $S = \mathbb{N}$.
 - Operations:
 - *Addition*
 - *Multiplication*
 - Order relations:
 - " \geq " – greater than or equal to
 - ">" – strictly greater than
 - 1 is the multiplicative identity
- $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$
 - 0 is the additive identity
- \mathbb{Z} – integers
 - *Subtraction* is the operation inverse to addition


- \mathbb{Q} – rational numbers
 - *Division* is the operation inverse to multiplication
 -  **Exercise:** Show that $\sqrt{2} \notin \mathbb{Q}$.
 - \mathbb{I} – irrational numbers
 -  **Exercise:** Give examples of irrational numbers.
 - \mathbb{R} – real numbers
 - Intuition: filling the "holes" between rational numbers
 - Real numbers are *complete*.
 - There is a real number between any two distinct rational numbers.
 - There is a rational number between any two distinct real numbers.
 -  **Exercise:** Show that $\sqrt{-1} \notin \mathbb{R}$.
 - *Supremum and Infimum*
 - *Upper bound* of a set S
 - $\sup S$ – *the least upper bound* of S
 - $\inf S = -\sup(-S)$ – *the greatest lower bound* of S
 - If $\emptyset \neq S \subset \mathbb{R}$ is bounded from above, then $\sup S \in \mathbb{R}$.
 -  **Exercise:** Find the supremum and infimum of the following sets:
 1. $S = 0.6, 0.66, 0.666, \dots$
 2. $T = 1/n : n \in \mathbb{N}$
 3. $U = \{x \in \mathbb{Q} : x^2 < 3\}$
 - Positive / negative rationals and reals:
 - $\mathbb{Q}_+ = \{x \in \mathbb{Q} : x \geq 0\}$
 - $\mathbb{Q}_{++} = \{x \in \mathbb{Q} : x > 0\}$
 - $\mathbb{R}_+ = \{x \in \mathbb{R} : x \geq 0\}$
 - $\mathbb{R}_{++} = \{x \in \mathbb{R} : x > 0\}$
 - Similarly, $\mathbb{Q}_- = \{x \in \mathbb{Q} : x \leq 0\}$, etc.
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1.2. Inequalities

-  **Proposition:** Let $a > b$ and $c \in \mathbb{R}$, then:
 - $a + c > b + c$
 - $ac > bc$ whenever $c > 0$
 - $ac < bc$ whenever $c < 0$
-  **Exercise:** Prove the *Arithmetic-Geometric Mean Inequality*.

$$\frac{a+b}{2} \geq \sqrt{ab} \quad \forall a, b \in \mathbb{R}_+.$$

1.3. Intervals

- *Open interval* – (a, b)
 - *Closed interval* – $[a, b]$
 - *Half-open intervals* – $(a, b]$ or $[a, b)$
 - Finite / infinite intervals
 -  **Exercise:** Let $I = (a, b)$ and $J = [c, d]$ be nondegenerate intervals:
 1. Under what conditions is $I \cap J$ an open interval?
 2. Under what conditions is $I \cap J$ a closed interval?
 3. Is $I \cup J$ an interval?
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1.4. Functions

A real function: $f : D \rightarrow T$

- $D \subseteq \mathbb{R}$ is the *domain*
- T is the *codomain* (also known as the *target space*)
- $f(D) = \{f(x) : x \in D\}$ is the *range*
- $f(X)$ is the *image* of the set $X \subseteq D$
- $f^{-1}(Y) = \{x \in D : f(x) \in Y\}$ is the *preimage* of $Y \subseteq \mathbb{R}$
- **Examples:**
 1. A linear function
 2. The distance from point $\bar{x} \in \mathbb{R}$
 3. A polynomial
 4. A trigonometric function
 5. An *indicator function* (also known as the *characteristic function* of set S)

$$\mathbf{1}_S(x) = \begin{cases} 1, & \text{if } x \in S, \\ 0, & \text{otherwise.} \end{cases}$$

1.5. Neighborhoods

-  **Definition:** A neighborhood of a point $a \in \mathbb{R}$ is the interval $(a - \delta, a + \delta)$ for some $\delta > 0$.
 - **Intuition:** It is a collection of points near a .
 -  **Exercise:** Show that an open interval contains a neighborhood of each of its points.
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1.6. Absolute Value

-  **Definition:** The function $|\cdot| : \mathbb{R} \rightarrow \mathbb{R}$ defined by

$$|x| = \begin{cases} x, & \text{if } x \geq 0, \\ -x, & \text{if } x < 0 \end{cases}$$

is called *the absolute value*.

-  **Exercise:**

1. Characterize the open interval $(a - \delta, a + \delta)$ using absolute value.
2. Describe the set $\{x \in \mathbb{R} : |7x + 2| < 9\}$ as an interval.
3. Prove the *triangle inequality*.

$$|x + y| \leq |x| + |y| \quad \forall x, y \in \mathbb{R}.$$

4. Prove that

$$|x - y| \geq \left| |x| - |y| \right| \quad \forall x, y \in \mathbb{R}.$$

2. Proofs

Reference:

- Appendix A1.3 in Simon and Bloom's book

2.1. Direct Proof

- Direct way of proving $A \implies B$:

$$A = A_0 \implies A_1 \implies A_1 \implies A_1 \implies \dots \implies A_n = B$$

- **Example:**

Theorem A1.1 For any $x, y, z \in \mathbb{R}$, if $x + z = y + z$, then $x = y$.

Proof

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|--|---------------------------------------|
| 1. $x + z = y + z$. | Hypothesis. |
| 2. There exists $(-z)$ such that
$z + (-z) = 0$. | Additive inverse property. |
| 3. $(x + z) + (-z) = (y + z) + (-z)$. | Rule (3). |
| 4. $x + (z + (-z)) = y + (z + (-z))$. | Additive associative property. |
| 5. $x + 0 = y + 0$. | Step 2. |
| 6. $x = y$. | Additive identity property. ■ |

- Implication, converse, and contrapositive:

1. $A \implies B$ – logical statement A implies logical statement B
2. $A \iff B$ – A is *equivalent* to B (A holds *if and only if* B holds)
3. $B \implies A$ – the converse of $A \implies B$
4. $\neg A$ – negation of the logical statement A (also known as *not* A , or $\sim A$)

5. $\neg B \implies \neg A$ – the contrapositive of the implication $A \implies B$; is equivalent to $A \implies B$

- **Exercise:** Let $a, b \in \mathbb{N}$, and consider the following logical statements:

┃ A: both a and b are odd;

┃ B: $a \cdot b$ is odd;

Formulate the statements 1.–5. and determine their validity.

2.2. Proof by contrapositive

- Instead of proving original statement

┃ $A \implies B$,


prove its contrapositive

┃ $\neg B \implies \neg A$.

-  **Exercise:** Prove by contrapositive the following statement:

┃ Let $a \in \mathbb{N}$. If a^2 is odd, then a is odd.

2.3. Proof by Contradiction

- Prove that $A \implies B$ by contradiction:
 1. **Assume the Opposite:** Suppose that A holds, but B does **not** hold (i.e., $\neg B$ holds).
 2. **Derive a Contradiction:** Using logical reasoning and the assumptions A and $\neg B$, derive a statement that is clearly false or contradicts a known fact.
 3. **Conclude:** Since the assumption leads to a contradiction, the original implication $A \implies B$ must be true.
-  **Exercise:**
 1. Prove the statement from the last exercise by contradiction.
 2. Prove that there is infinitely many primes.