📝 *Math Prep – Class Notes*

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4. Differentiability

- Derivative of a function
- Local approximation of a function

Reference: Chapter 5 in Craw's book

4.1. Definition and Basic Properties

- 📘 **Definition:** Let $U \subseteq \mathbb{R}$ be an open set and $f : U \to \mathbb{R}$. We say that function f is differentiable at $x_0 \in U$ if
$$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}$$
exists. The limit, if it exists, is written as $f'(x_0)$. We say that f is differentiable in U if it is differentiable at each $a \in U$.

- **Intuition:** If function f has derivative at x_0, then it can be locally approximated by its tangent
$$f(x) \approx f(x_0) + (x - x_0) f'(x_0)$$

- 🧩 **Exercise:**
    1. Is the definition equivalent to the following?
    $$f'(x_0) = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}$$
    2. Find the derivative of $f(x) = x^2$ using the definition of the derivative.
    3. Let f and g be functions differentiable at $a \in \mathbb{R}$. Using the definition of the derivative, find the formula for $(f + g)'(x_0)$.
    4. Let f and g be functions differentiable at $a \in \mathbb{R}$. Using the definition of the derivative, find the formula for $(fg)'(x_0)$.

- 📜 **Proposition:** If $f : U \to \mathbb{R}$ is differentiable at $x_0 \in U$, then f is continuous at x_0.

- 📜 **Proposition:** Let $f : U \to \mathbb{R}$ be differentiable at $x_0 \in U$, and let $g : V \to \mathbb{R}$ be differentiable at $y_0 = f(x_0)$. Then $g \circ f$ is differentiable at a and $(g \circ f)'(x_0) = g'(y_0) \cdot f'(x_0)$.

- 🧩 **Exercise:** Consider a function $f : U \to \mathbb{R}$ that has nonzero derivative at $x_0 \in U$. Define $y_0 = f(x_0)$.
    1. Show that there exists a neighborhood N of x_0 such that there exists inverse function f^{-1} to the function $f|_N$ (the restriction of f to N).
    2. Use the fact that $(f^{-1} \circ f)(x) = x$ for all $x \in N$ to find a formula for $(f^{-1})'(y_0)$.
- 🧩 **Exercise:** Find the derivative of the function $f(x) = \sqrt{x^2 - 1}$.
- 🧩 **Exercise:** Consider the function $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = x \sin\left(\frac{1}{x}\right)$ for $x \neq 0$ and $f(0) = 0$. Show that f is continuous and differentiable everywhere except at $x = 0$. At $x = 0$ the function is continuous, but not differentiable.

4.2. l'Hôpital's rule

- 📜 **l'Hôpital's rule:** Let f and g be functions such that $f(x_0) = g(x_0) = 0$ and both functions are differentiable at x_0, with $g'(x_0) \neq 0$. Then
$$\lim_{x \to x_0} \frac{f(x)}{g(x)} = \frac{f'(x_0)}{g'(x_0)}.$$
- 🧩 **Exercise:** Find $\lim_{x \to 0} \frac{e^{2x}-1}{e^x-1}$.

4.3. Lottle–o and Big–O Notation

- 📘 **Definition:**
    - **Little–o Notation:** We say that
    $$f(x) = o(g(x)) \quad \text{as } \ x \to a$$
    when
    $$\frac{f(x)}{g(x)} \to 0 \quad \text{as } \ x \to a.$$
    - **Big–O Notation:** We say that
    $$f(x) = O(g(x)) \quad \text{as } \ x \to a$$
    when $\dfrac{f(x)}{g(x)}$ is bounded on some neighborhood of a (with the exception of point a itself).
- 🧩 **Exercise:** Let f be continuous at x_0. Which of the following statements are true (find a proof or a contraexamle):

    a. $f(x) = f(x_0) + O(1)$ as $x \to x_0$

    b. $f(x) = f(x_0) + O(x - x_0)$ as $x \to x_0$

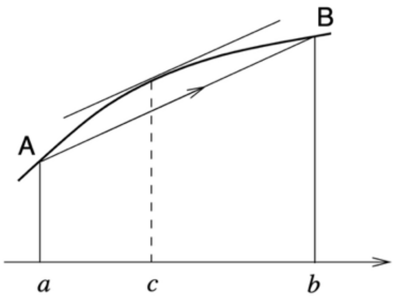
c. $f(x) = f(x_0) + o(1)$ as $x \to x_0$

4.4. Rolle and the Mean Value Theorem

- **Rolle's Theorem:** Let f be continuous on $[a, b]$ and differentiable on (a, b), and suppose that $f(a) = f(b)$. Then $f'(c) = 0$ for some $c \in (a, b)$.
- **Exercise:** A function $f : \mathbb{R} \to \mathbb{R}$ is differentiable and its derivative is never 0. Show that f is invertible.
- **The Mean Value Theorem:** Let f be continuous on $[a, b]$ and differentiable on (a, b). Then

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

  for some $c \in (a, b)$.
- Equivalently: $f(b) = f(a) + (b - a)f'(c)$.



- **Exercise:** Consider a function $f : \mathbb{R}_+ \to \mathbb{R}$ such that $f(0) = 0$ and $f'(x) \geq 1$ for all $x \geq 0$. Show that $f(x) > x$ for all $x \geq 0$.
- **Exercise:** Considef a neihborhood N of a point $a \in \mathbb{R}$ and function $f : N \to \mathbb{R}$ that is differentaible with $f'(x)$ bounded.

  a. Show that

$$f(x) = f(a) + O(x - a) \qquad \text{as } \; x \to a.$$

  b. If in addition $f'(x)$ is continuous at a, then

$$f(x) = f(a) + (x - a)f'(a) + o(x - a) \qquad \text{as } \; x \to a.$$

4.5. Taylos's Theorem

- **Second Mean Value Theorem:** Let f be twice differentiable on $[a, b]$. Then,

$$f(b) = f(a) + (b - a)f'(a) + \frac{1}{2}(b - a)^2 f''(c),$$

  for some $c \in (a, b)$.
- **Taylor's Theorem:** Let f be $(n + 1)$th differentiable on $[a, x]$. Then,

$$f(x) = P_n(x) + R_n(x),$$

where $P_n(x)$, the Taylor polynomial of degree n about a, and $R_n(x)$, the corresponding remainder, are given by

$$P_n(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!}f''(a) + \ldots \frac{f^{(n)}(a)}{n!}(x-a)^n,$$

$$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1},$$

where c is some point between a and x.

- **Exercise:** Write down the second–order approximaiton of the function $f(x) = e^x$ about $x = 0$.

4.6. Additional Exericeses

1. Consider the function $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = x^2 \sin\left(\frac{1}{x}\right)$ for $x \neq 0$ and $f(0) = 0$. Show that f is differentiable, but the derivative $f'(x)$ is not continuous at $x = 0$.
2. Let f be a function differentiable at x_0. Which of the following statements are true (find a proof or a contraexamle):

   a. $f(x) = f(x_0) + (x - x_0)f'(x_0) + O(x - x_0)$ as $x \to x_0$

   b. $f(x) = f(x_0) + (x - x_0)f'(x_0) + O\big((x - x_0)^2\big)$ as $x \to x_0$

   c. $f(x) = f(x_0) + (x - x_0)f'(x_0) + o(x - x_0)$ as $x \to x_0$
3. Find $\lim_{x\to 1} \frac{\log x^2}{\log(x+1) - \log(2)}$
4. Write down the second–order approximation of the functon $f(x) = \log(x)$ about 1.