📝 Math Prep – Class Notes

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2. Sequences

- Sequences and criteria for convergence.

Reference: Chapters 2 and 3 in Craw's book

2.1. Sequences

- 📘 **Definition:** A *sequence* is a map $a : \mathbb{N} \to \mathbb{R}$.
- Common Notation: $\{a_n\}$, $(a_n)_n$
- *Limit* of the sequence: the value of the sequence "in the long run"
- **Example:**
  1. $\{a_n\} = (5, 3, 9, 7, \dots, 6, 0, 0, 0, \dots)$
  2. $a_n = n + \frac{1}{n}$
  3. $a_n = \sin(n)$
  4. The sequence of approximations of the slope of the function $f(x) = x^2$ at $x_0 = 1$: $a_n = n \cdot [f(1 + 1/n) - f(1)]$.
- A sequence $(a_n)_n$ can be:
  - *convergent* – converge to a value $l \in \mathbb{R}$, denoted $a_n \to l$ as $n \to \infty$
  - *divergent* – oscillate or tend to infinity
- 📘 **Definition:** A sequence $\{a_n\}$ converges to $l \in \mathbb{R}$ if for any $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that $|a_n - l| < \varepsilon$ for any $n \geq N$.
- 📘 **Definition:** We say that property $P(n)$ holds *eventually* iff there is N, such that $P(n)$ holds for all $n \geq N$. It holds *frequently* if given N there is some $n \geq N$ such that $P(n)$ holds.
- 🧩 **Exercise**:
  1. Use the terms *eventually* and *frequently* to define when a sequence is convergent / divergent.
  2. Consider the sequence $a_n = 1 + c^n$. For what values of $c \in \mathbb{R}$ are the following

statements true:

- a) a_n is frequently negative
- b) a_n is eventually positive
- c) for any $K \in \mathbb{R}$, eventually $a_n > K$
- d) for any $K \in \mathbb{R}$, frequently $a_n > K$

3. Show that if a_n is frequently positive and frequently negative, then the only possible limit it can have is 0.
4. Show that if $a_n \to l$ and $a_n \to m$ as $n \to \infty$, then $l = m$.
5. Translate the following logical expression into words and define its negation:

> $$\forall \varepsilon > 0,\ \exists N,\ \forall n \geq N:\ |a_n - l| < \varepsilon$$

2.2. Sums, Products and Quotients

- 🧩 **Exercise:** Given that $a_n \to l$ and $b_n \to m$ as $n \to \infty$, prove the following limits:

  1. $a_n + c \to l + c$ as $n \to \infty$, for any $c \in \mathbb{R}$
  2. $a_n + b_n \to l + m$ as $n \to \infty$
  3. $2a_n \to 2l$ as $n \to \infty$
  4. $a_n b_n \to lm$ as $n \to \infty$
  5. $a_n^2 \to l^2$ as $n \to \infty$

- 📜 **Theorem:** If $a_n \to l$ and $b_n \to m \neq 0$ as $n \to \infty$, then $\frac{a_n}{b_n} \to \frac{l}{m}$ as $n \to \infty$.
- 🧩 **Exercise:** Find limit of the sequence $\{a_n\}$ such that:

  1. $a_n = \frac{4-n}{1+n^2}$
  2. $2a_n + \frac{1}{n} \to -2$ as $n \to \infty$
  3. $a_n^2 \to 3$ as $n \to \infty$ and a_n is convergent.

2.3. Squeezing

- 📜 **Squizing Lemma:** Let $a_n \leq b_n \leq c_n$, and suppose that $a_n \to l$ and $c_n \to l$ as $n \to \infty$. Then $\{b_n\}$ is convergent and $b_n \to l$ as $n \to \infty$.
- 🧩 **Exercise:** Find the limit of sequences

  1. $b_n = \frac{1+n}{1+n^2}\cos(n^3\pi)$
  2. $b_n = \sqrt{n+1} - \sqrt{n}$

- 📜 **Proposition:** Let f be a function continuous at l, and suppose that $a_n \to l$ as $n \to \infty$. Then $f(a_n) \to f(l)$ as $n \to \infty$.

- **Exercise:**

    1. Find the limit of $a_n = \log\left(1 + \frac{1}{n}\right)$.
    2. If $\{a_n\}$ diverges, does it imply that $\{a_n^2\}$ diverges?
    3. Suppose that $\{a_n\}$ diverges and $a_n \geq 0$. Show that $\{a_n^2\}$ diverges then.

2.4. Bounded Sequences

- **Definition:** We say that $\{a_n\}$ is a *bounded* sequence if there is some $K \in \mathbb{R}$ such that $|a_n| \leq K$ for all n.
- **Proposition:** An eventually bounded sequence is bounded.
- **Proposition:** A convergent sequence is bounded.
- **Exercise:** Is the sequence $a_n = n \sin\left(\frac{10^9}{n}\right)$ bounded?

2.5. Infinite Limits

- **Definition:** We say that $a_n \to \infty$ as $n \to \infty$ (a_n diverges to ∞) iff given K, eventually $a_n \geq K$.
- **Exercise:** Let $a_n \to \infty$ and $b_n \to l$. Does $a_n b_n \to \infty$?

2.6. Monotone Convergence

- **The Monotone Convergence Theorem:** Let $\{a_n\}$ be a bounded, monotone sequence. Then $\{a_n\}$ converges.

- **Exercise:**

    1. Consider the sequence $\{a_n\}$ defined recursively by $a_1 = 1$ and $a_{n+1} = (4a_n + 2)/(a_n + 3)$ for $n \geq 1$. Show that it is convergent and find its limit.
    2. Let $\{a_n\}$ be an increasing sequence. Then $a_n \to \sup_{n\in\mathbb{N}} a_n$.