📝 *Math Prep – Class Notes*

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3. Continuity

- Open and closed sets
- Function limit and continuity
- Intermediate value theorem and extreme value theorem

Reference: Chapter 4 in Craw's book

3.1. Open and Closed Sets

- a set $S \subseteq \mathbb{R}$
- 📘 **Definition:**
    - **Open Set:** A set $U \subseteq \mathbb{R}$ is *open* if for every $a \in U$ there is a neighborhood of a contained in U.
    - **Interior Point:** a point $x \in S \subseteq \mathbb{R}$ is an interior point of S iff there is a neighborhood of x that is contained in S.
    - **Interior of a Set:** $\operatorname{int} S$ is the collection of all the interior points of set S.
    - **Limit Point:** A point $x \in \mathbb{R}$ is a limit point of set S iff there exists a sequence $\{s_n\}$ of points from S such that $s_n \to x$ as $n \to \infty$.
    - **Closed Set:** A set $C \subset \mathbb{R}$ is closed if any limit point of points from C is in C.
    - **Closure of a Set:** $\operatorname{cl} S$ (also denoted $\overline{S}$) is the set of all the limit points of set S.
    - **Boundary Point:** A point $x \in \mathbb{R}$ is a *boundary point* of set S if it is a limit point of both the set S and $\mathbb{R} \setminus S$.
    - **Boundary of a Set:** ∂S is the collection of all boundary points of S

3.2. Continuous Functions

- A real valued function: $f : D \to \mathbb{R}$, where $D \subseteq \mathbb{R}$.
- 📘 **Definition:**
    - **Limit of a Function:** A function $f : D \to \mathbb{R}$ tends to y_0 as $x \to x_0 \in \overline{D}$ if, given any $\varepsilon > 0$, there exists a $\delta > 0$ such that for all $x \in D$ with $0 < |x - x_0| < \delta$ the inequality $|f(x) - y_0| < \varepsilon$ holds. We write $\lim_{x \to x_0} f(x) = y_0$.

- **Continuous at a Point:** A function f is *continuous at* $x_0 \in D$ iff $f(x) \to f(x_0)$ as $x \to x_0$.
- **Continuous Function:** A function $f : D \to \mathbb{R}$ is continuous if it is continuous at every $x \in D$.

- **Exercise:**
  1. What is $f(9)$ if f is continuous and for any $x \neq 9$, f is given by the formula
  $$f(x) = \frac{\sqrt{x}+3}{x-9}?$$
  2. Show that the function
  $$f(x) = \begin{cases} x\sin\left(\frac{1}{x}\right) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$
  is continuous at 0.
  3. Show that the function $f(x) = \mathbf{1}_{\mathbb{Q}}(x)$ is not continuous at any $x \in \mathbb{R}$.
  4. Formulate the continuity using convergence of sequences.

3.3. One Sided Limits

- **Definition:**
  - **Left Limit:** We say that
  $$\lim_{x \to x_0-} f(x) = y_0$$
  if given $\varepsilon > 0$, there is $\delta > 0$ such that $|f(x) - f(x_0)| < \varepsilon$ for all $x \in (x_0 - \delta, x_0) \cap D$.
  - **Right Limit:** We say that
  $$\lim_{x \to x_0+} f(x) = y_0$$
  if given $\varepsilon > 0$, there is $\delta > 0$ such that $|f(x) - f(x_0)| < \varepsilon$ for all $x \in (x_0, x_0 + \delta) \cap D$.
- **Proposition:** Let $x_0 \in \partial D$. If $\lim_{x \to x_0} f(x)$ exists, then both one sided limits exist. Conversely, if both one sided limits exist and are equal, then $\lim_{x \to x_0}$ exists.
- **Continuity Test:** The function f is continuous at x_0 if both one sided limits exist and are equal to $f(x_0)$.
- **Exercise:**
  1. Show that the function $f(x) = |x|$ is contionuous.
  2. Find the left and right limit of the function $f(x) = \mathbf{1}_{\mathbb{R}_+}(x)$ at $x = 0$ and discuss its continuity.

3.4. "Algebra" of Continuity

- **Proposition:** Let f and g be continuous at x_0, and let $k \in \mathbb{R}$ be a constant. Then the functions $k \cdot f$, $f + g$, and $f \cdot g$ are continuous at x_0. Moreover, if $g(x_0) \neq 0$, then the function f/g is continuous at x_0.
- **Proposition:** Let f be continuous at x_0, and let g be continuous at $f(x_0)$. Then the composition $g \circ f$ is continuous at x_0.
- **Exercise:** Show that the function $f(x) = |x^2 - 1|$ is contionuous.

3.5. Intermediate Value Theorem

- **Theorem:** Let $f : [a, b] \to \mathbb{R}$ be a continuous function such that $f(a) < 0 < f(b)$. Then there exists $c \in (a, b)$ such that $f(c) = 0$.
- **Corollary:** Let $f : [a, b] \to \mathbb{R}$ be a continuous function. Then for any y between $f(a)$ and $f(b)$ there exists $c \in [a, b]$ such that $f(c) = y$.
- **Exercise:** Show that $f(x) = \frac{x}{x^3+1}$ attains the value $\frac{1}{3}$ at some $x \in (1, 2)$.
- **Extreme Value Theorem:** A continuous function $f : [a, b] \to \mathbb{R}$ attains both a minimum and a maximum.
- **Exercise:** A function $f : [0, 2] \to \mathbb{R}$ has values $f(0) = 1$, $f(1) = 9$, and $f(2) = 3$.

  a. Show that it has a maximum on $(0, 2)$.

  b. Give an example of a function f such that it has no minimum on $(0, 2)$.