Combinatorics

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Introduction to Combinatorics

- Combinatorics is the study of collections of objects .
- Specifically, counting objects , arrangement, derangement, etc. of objects along with their mathematical properties.
- Counting objects is important in order to analyze algorithms and compute discrete probabilities .
- Originally, combinatorics was motivated by gambling
- Counting configurations is essential to elementary probability.

Introduction to Combinatorics (Cont.)

- A simple example :
  - How many arrangements are there of a deck of 52 cards?

- In addition, combinatorics can be used as a proof technique.

- A combinatorial proof is a proof method that uses counting arguments to prove as statement .

Introduction to Combinatorics (Cont.)

- Count the number of ways to put things together into various combinations.

  e.g. If a password is 6 letters and/or digits, how many passwords can there be?

Two main rules:

- Sum rule
- Product rule

Introduction to Combinatorics (Cont.)

- *Sum Rule:*

If two events are mutually exclusive, that is, they cannot be done at the same time , then we must apply the sum rule.

Theorem (Sum Rule)

If an event e_1 can be done in n_1 ways and an event e_2 can be done in n_2 ways and e_1 and e_2 are mutually exclusive, then the number of ways of both events occurring is

$$n_1 + n_2$$

Introduction to Combinatorics (Cont.)

- Let us consider two tasks:
  - m is the number of ways to do **task 1**
  - n is the number of ways to do **task 2**

  - Tasks are <u>independent</u> of each other, i.e.,
    - Performing **task 1** does not accomplish **task 2** and vice versa.

- *<u>Sum rule</u>*: the number of ways that "**either** task 1 **or** task 2 can be done, but **not both**", is $m+n$.

Introduction to Combinatorics (Cont.)

- Example: A student can choose a computer project from one of three lists. The three lists contain 23, 15, and 19 possible projects respectively. How many possible projects are there to choose from?

- Solution:

  Assuming there are no projects common to any pair of lists, a student can choose a project from the first list in 23 ways, from the second list in 15 ways and from the third list in 19 ways. Hence, there are $23 + 15 + 19 = 57$ ways in which a project can be chosen by a student or 57 possible projects to choose from.

Introduction to Combinatorics (Cont.)

- *Product Rule:* If two events are not mutually exclusive (that is, we do them separately), then we apply the product rule .

Theorem (Product Rule)

Suppose a procedure can be accomplished with two disjoint subtasks. If there are n_1 ways of doing the first task and n_2 ways of doing the second, then there are

$$n_1 \cdot n_2$$

ways of doing the overall procedure.

Introduction to Combinatorics (Cont.)

- Let us consider two tasks:
  - *m* is the number of ways to do **task 1**
  - *n* is the number of ways to do **task 2**
  - Tasks are <u>independent</u> of each other, i.e.,
    - Performing task 1does not accomplish task 2 and vice versa.
- <u>***Product rule***</u> : the number of ways that "**both** tasks 1 and 2 can be done" in *mn*.
- Generalizes to multiple tasks ...

Introduction to Combinatorics (Cont.)

- Example: How many different bit strings are there of length seven?

Sol : There are two choices for each of the seven positions in the string: thus there are $2^7 = 128$ different 7–bit (binary digit) strings.

Combinatorics Techniques

- **General Techniques**
  - Permutation
  - Combination
  - Generating Functions
  - Binomial Coefficients
  - Pigeonhole Principle

Combinatorics Techniques (Cont.)

- ***Permutation:***
  - A *permutation* of a set *S* of objects is an ordered arrangement of the elements of S where each element appears only once:

    e.g., 1 2 3,   2 1 3,   3 1 2

  - An ordered arrangement of *r* distinct elements of *S* is called an *r–permutation*.

  - The number of *r*–permutations of a set *S* with *n* elements is

$$P(n,r) = n(n-1)\ldots(n-r+1) = n!/(n-r)!$$

Combinatorics Techniques (Cont.)

- Example: A terrorist has planed an armed nuclear bomb in your city, and it is your job to disable it by cutting wires to the trigger device. There are 10 wires to the device. If you cut exactly the right three wires, in exactly the right order, you will disable the bomb, otherwise it will explode! If the wires all look the same, what are your chances of survival?

Answer

$$P(10,3) = 10 \cdot 9 \cdot 8 = 720,$$

so there is a 1 in 720 chance that you'll survive!

Combinatorics Techniques (Cont.)

Theorem

The number of r permutations of a set with n distinct elements is

$$P(n,r) = \prod_{i=0}^{r-1}(n-i) = n(n-1)(n-2)\cdots(n-r+1)$$

Combinatorics Techniques (Cont.)

It follows that

$$P(n,r) = \frac{n!}{(n-r)!}$$

In particular,

$$P(n,n) = n!$$

Again, note here that order is important. It is necessary to distinguish in what cases order is important and in which it is not.

Combinatorics Techniques (Cont.)

Example

In how many ways can the English letters be arranged so that there are exactly ten letters between a and z?

The number of ways of arranging 10 letters between *a* and *z* is P (24, 10). Since we can choose either *a* or *z* to come first, there are 2P(24, 10) arrangements of this 12- letter block.

For the remaining 14 letters, there are $P(15, 15) = 15!$ arrangements .

In all, there are $2P(24, 10) \cdot 15!$

Combinatorics Techniques (Cont.)

- *Combinations:* Where as permutations consider order, combinations are used when order does not matter .

Definition

An k-combination of elements of a set is an unordered selection of k elements from the set. A combination is simply a subset of cardinality k.

Combinatorics Techniques (Cont.)

Theorem

The number of k-combinations of a set with cardinality n with $0 \leq k \leq n$ is

$$C(n,k) = \binom{n}{k} = \frac{n!}{(n-k)!k!}$$

Note : the notation, $\binom{n}{k}$ is read, " n choose k ".

Combinatorics Techniques (Cont.)

- A useful fact about combinations is that they are symmetric.

$$\binom{n}{1} = \binom{n}{n-1}$$

$$\binom{n}{2} = \binom{n}{n-2}$$

Etc.

Combinatorics Techniques (Cont.)

- This is formalized in the following corollary.

Corollary

Let n, k be nonnegative integers with $k \leq n$, then

$$\binom{n}{k} = \binom{n}{n-k}$$

Combinatorics Techniques (Cont.)

- Example:

Example

In the Powerball lottery, you pick five numbers between 1 and 55 and a single "powerball" number between 1 and 42. How many possible plays are there?

Order here doesn't matter, so the number of ways of choosing five regular numbers is

$$\binom{55}{5}$$

Combinatorics Techniques (Cont.)

We can choose among 42 power ball numbers . These events are not mutually exclusive, thus we use the product rule .

$$42\binom{55}{5} = 42\frac{55!}{(55-5)!5!} = 146,107,962$$

So the odds of winning are

$$\frac{1}{146,107,962} < .00000006845$$

Combinatorics Techniques (Cont.)

- *Combinations vs Permutations:*

Essentially <u>unordered permutations</u> …

$$P(n,r) = C(n,r)P(r,r)$$

$$C(n,r) = \binom{n}{r} = \frac{P(n,r)}{P(r,r)} = \frac{n!/(n-r)!}{r!} = \frac{n!}{r!(n-r)!}$$

Note that $C(n,r) = C(n,\ n-r)$

Combinatorics Techniques (Cont.)

- Combination Example: How many distinct 7-card hands can be drawn from a standard 52-card deck?
  - The order of cards in a hand doesn't matter.

Answer $C(52,7) = P(52,7)/P(7,7)$

$$= 52 \cdot \underset{17}{\cancel{51}} \cdot \underset{10}{\cancel{50}} \cdot \underset{7}{\cancel{49}} \cdot \underset{\cancel{2}}{\underset{\cancel{8}}{\cancel{48}}} \cdot 47 \cdot 46 \,/\, \cancel{7} \cdot \cancel{6} \cdot \cancel{5} \cdot \cancel{4} \cdot \cancel{3} \cdot 2 \cdot 1$$

$$52 \cdot 17 \cdot 10 \cdot 7 \cdot 47 \cdot 46 = 133{,}784{,}560$$

Combinatorics Techniques (Cont.)

Theorem

The number of different permutations of n objects where there are n_1 indistinguishable objects of type 1, n_2 of type 2, ..., and n_k of type k is

$$\frac{n!}{n_1!n_2!\cdots n_k!}$$

An equivalent way of interpreting this theorem is the number of ways to distribute n distinguishable objects into k distinguishable boxes so that n_i objects are placed into box i for i = 1, 2 , . . . , k .

Combinatorics Techniques (Cont.)

- Example: How many permutations of word "Mississippi" are there?

<u>Sol,</u> "Mississippi" contains 4 distinct letters, M , i, s and p ; with1 , 4 , 4 , 2 Occurrences respectively.

Therefore there are $$\frac{11!}{1!4!4!2!}$$

permutations.

Combinatorics Techniques (Cont.)

- ***Binomial Coefficients:***
  - The number of r –combinations $\binom{n}{r}$ s also called a binomial coefficient .
  - They are the coefficients in the expansion of the expression (multivariate polynomial), $(x + y)^n$. A binomial is a sum of two terms.

Combinatorics Techniques (Cont.)

Theorem (Binomial Theorem)

Let x, y be variables and let n be a nonnegative integer. Then

$$(x+y)^n = \sum_{j=0}^{n} \binom{n}{j} x^{n-j} y^j$$

Combinatorics Techniques (Cont.)

Expanding the s um mation, we have

$$
\begin{aligned}
(x+y)^n &= \binom{n}{0}x^n + \binom{n}{1}x^{n-1}y + \binom{n}{2}x^{n-2}y^2 + \cdots \\
&\quad + \binom{n}{n-1}xy^{n-1} + \binom{n}{n}y^n
\end{aligned}
$$

For example,

$$
\begin{aligned}
(x+y)^3 &= (x+y)(x+y)(x+y) \\
&= (x+y)(x^2+2xy+y^2) \\
&= x^3+3x^2y+3xy^2+y^3
\end{aligned}
$$

Combinatorics Techniques (Cont.)

- Example:

Example

What is the coefficient of the term x^8y^{12} in the expansion of $(3x+4y)^{20}$?

By the Binomial Theorem , we have

$$(3x+4y)^n = \sum_{j=0}^{20} \binom{20}{j} (3x)^{20-j}(4y)^j$$

Combinatorics Techniques (Cont.)

So when j = 12, we have n = 20

$$\binom{20}{12}(3x)^8(4y)^{12}$$

so the coefficient is

$$\frac{20!}{12!8!}3^8 4^{12} = 13866187326750720.$$

END

Q & A