Logic and Proof 2

Asst. Prof. Nattapol Aunsri, Ph.D.
IATE Research Unit
CCC Research Group
School of Information Technology, MFU

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Propositional Equivalences

Types of Compound Propositions

- Definition:
  - A compound proposition that is ***always true***, no matter what the truth values of the propositions that occur in it, is called a ***tautology***.
  - A compound proposition that is ***always false*** is called a **contradiction**.
  - A compound proposition that is ***neither a tautology nor a contradiction*** is called a **contingency**.

Example

Tautology		
p	$\neg p$	$p \vee \neg p$
T	F	T
F	T	T

Contradiction		
p	$\neg p$	$p \wedge \neg p$
T	F	F
F	T	F

Logical Equivalence

- Compound propositions that have the same truth values in all possible cases
- Definition : Compound propositions p and q are ***logically equivalent*** if $p \leftrightarrow q$ is a tautology (denoted by $p \equiv q$ or $p \Leftrightarrow q$)
- **De Morgan's Law**

$$\neg (p \wedge q) \equiv \neg p \vee \neg q$$

$$\neg (p \vee q) \equiv \neg p \wedge \neg q$$

Example: Logical Equivalence

$\neg(p \vee q)$ $\neg p \wedge \neg q$

TABLE 3 **Truth Tables for $\neg(p \vee q)$ and $\neg p \wedge \neg q$.**

p	q	$p \vee q$	$\neg(p \vee q)$	$\neg p$	$\neg q$	$\neg p \wedge \neg q$
T	T	T	F	F	F	F
T	F	T	F	F	T	F
F	T	T	F	T	F	F
F	F	F	T	T	T	T

TABLE 6 (1.2)

TABLE 6 Logical Equivalences.

Equivalence	*Name*
$p \wedge \mathbf{T} \equiv p$ $p \vee \mathbf{F} \equiv p$	Identity laws
$p \vee \mathbf{T} \equiv \mathbf{T}$ $p \wedge \mathbf{F} \equiv \mathbf{F}$	Domination laws
$p \vee p \equiv p$ $p \wedge p \equiv p$	Idempotent laws
$\neg(\neg p) \equiv p$	Double negation law
$p \vee q \equiv q \vee p$ $p \wedge q \equiv q \wedge p$	Commutative laws
$(p \vee q) \vee r \equiv p \vee (q \vee r)$ $(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$	Associative laws
$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$ $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$	Distributive laws
$\neg(p \wedge q) \equiv \neg p \vee \neg q$ $\neg(p \vee q) \equiv \neg p \wedge \neg q$	De Morgan's laws
$p \vee (p \wedge q) \equiv p$ $p \wedge (p \vee q) \equiv p$	Absorption laws
$p \vee \neg p \equiv \mathbf{T}$ $p \wedge \neg p \equiv \mathbf{F}$	Negation laws

TABLE 7 Logical Equivalences Involving Conditional Statements.

$$p \rightarrow q \equiv \neg p \vee q$$

$$p \rightarrow q \equiv \neg q \rightarrow \neg p$$

$$p \vee q \equiv \neg p \rightarrow q$$

$$p \wedge q \equiv \neg (p \rightarrow \neg q)$$

$$\neg (p \rightarrow q) \equiv p \wedge \neg q$$

$$(p \rightarrow q) \wedge (p \rightarrow r) \equiv p \rightarrow (q \wedge r)$$

$$(p \rightarrow r) \wedge (q \rightarrow r) \equiv (p \vee q) \rightarrow r$$

$$(p \rightarrow q) \vee (p \rightarrow r) \equiv p \rightarrow (q \vee r)$$

$$(p \rightarrow r) \vee (q \rightarrow r) \equiv (p \wedge q) \rightarrow r$$

TABLE 8 Logical Equivalences Involving Biconditionals.

$p \leftrightarrow q \equiv (p \rightarrow q) \wedge (q \rightarrow p)$
$p \leftrightarrow q \equiv \neg p \leftrightarrow \neg q$
$p \leftrightarrow q \equiv (p \wedge q) \vee (\neg p \wedge \neg q)$
$\neg(p \leftrightarrow q) \equiv p \leftrightarrow \neg q$

Example

- Show that the following compounds are equivalent

$$\neg(p \lor (\neg p \land q))$$

$$\neg p \land \neg q$$

$$\begin{aligned}
\neg(p \lor (\neg p \land q)) &\equiv \neg p \land \neg(\neg p \land q) && \textbf{De Morgan's Law} \\
&\equiv \neg p \land [\neg(\neg p) \lor \neg q] && \textbf{De Morgan's Law} \\
&\equiv \neg p \land (p \lor \neg q) && \textbf{Double Negation Law} \\
&\equiv (\neg p \land p) \lor (\neg p \land \neg q) && \textbf{Distributive Law} \\
&\equiv F \lor (\neg p \land \neg q) && \textbf{Negation Law} \\
&\equiv (\neg p \land \neg q) \lor F && \textbf{Commutative Law for Disjunction} \\
&\equiv \neg p \land \neg q && \textbf{Identity Law}
\end{aligned}$$

In class exercise

- Check if the following is Tautology.

$$(p \wedge q) \rightarrow r] \rightarrow [p \rightarrow (q \rightarrow r)]$$

$$(p \vee \neg q) \rightarrow (p \wedge q)$$

$$(p \rightarrow q) \vee (q \wedge \neg r)$$

$$(\neg p \vee q) \leftrightarrow (p \rightarrow q)$$

Predicate Calculus: Predicates and Quantifiers

Predicates and Quantifiers

- Propositional logic can not adequately express the meaning of statements in Mathematics and in natural language.
- For example:

  "Every computer connected to the university network is functioning properly"

  "There is a computer that is under attack by an intruder"
- Predicate involves the variable , such as

  "x>3" , "x=y+3", "x+y=z", computer x is under attack.

Predicate Logic

- *Predicate*: a property that the subject of the statement can have

  Ex: $x>3$
  - x: variable
  - >3: predicate
  - $P(x)$: $x>3$
    - The value of the propositional function P at x

  $P(x_1, x_2, \ldots, x_n)$: *n*–place predicate or *n*–ary predicate

Example

- 1. Let P(x) denote the statement "$x>3$". What is the truth values of P(4) and P(2)?

  Sol. :

  P(4) is true

  P(2) is false

- 2. $Q(x,y) =$ "$x=y+3$."
  - What is the truth value of $Q(1,2)$, $Q(3,0)$?

Quantifiers

- Quantification
  - **Universal quantification**: a predicate is true for every element
  - **Existential quantification**: there is one or more element for which a predicate is true

The Universal Quantifier

- Domain: domain of discourse (universe of discourse)
- Definition : The ***universal quantification*** of $P(x)$ is the statement "$P(x)$ for all values of x in the domain", denoted by $\forall x\ P(x)$
  - "for all $x\ P(x)$" or "for every $x\ P(x)$"
  - When all elements in the domain can be listed, $P(x_1) \wedge P(x_2) \wedge \ldots \wedge P(x_n)$

The Existential Quantifier

- Definition: The ***existential quantification*** of $P(x)$ is the proposition "There exists an element x in the domain such that $P(x)$", denoted by $\exists x\, P(x)$
  - "there is an x such that $P(x)$" or "for some x $P(x)$"
  - When all elements in the domain can be listed, $P(x_1) \vee P(x_2) \vee \ldots \vee P(x_n)$

Conclusion

TABLE 1 Quantifiers.

Statement	*When True?*	*When False?*
$\forall x\, P(x)$	$P(x)$ is true for every x.	There is an x for which $P(x)$ is false.
$\exists x\, P(x)$	There is an x for which $P(x)$ is true.	$P(x)$ is false for every x.

Example

- 1. Let $Q(x)$ be the statement "x<2". What is the truth value of the quantification $\forall x Q(x)$, where the domain consists of all real numbers?
- Solution $Q(x)$ is not true for every real number x, because, for instance, $Q(3)$ is false.
- Then $\forall x Q(x)$ is false

Example

- 2. What is the truth value of $\forall x P(x)$, where P(x) is the statement $"x^2 < 10"$ and the domain consists of the positive integers not exceeding 4?
- Solution: $\forall x P(x)$ is the same of

$$P(1) \wedge P(2) \wedge P(3) \wedge P(4)$$

- Then $\forall x P(x)$ is false

Example

- 3. Let $Q(x)$ denote the statement "x=x+1". What is the truth value of the quantification $\exists x Q(x)$, where the domain consists of all real numbers?

Solution: $\exists x Q(x)$ is false because $Q(x)$ is

false for every real number x.

Example

- 4. What is the truth value of $\exists x P(x)$, where P(x) is the statement $"x^2 > 10"$ and the domain consists of the positive integers not exceeding 4?
- Solution: since $\exists x P(x)$ is the same as

$$P(1) \vee P(2) \vee P(3) \vee P(4)$$

- Because P(4) is true then $\exists x P(x)$ is true.

Logical Equivalence involving Quantifiers

- Statements involving predicates and quantifiers are ***logically equivalent*** if and only if they have the same truth value no matter which predicates are substituted and which domain is used
  - E.g. $\forall x\,(P(x) \wedge Q(x))$ and $\forall x\, P(x) \wedge \forall x\, Q(x)$

Negating Quantified Expressions

$$\neg \forall x P(x) \equiv \exists x \neg P(x)$$

Negation of the statement *"Every student in your class has taken a course in Calculus"*
"There is a student in your class who has not taken a course in Calculus"

$$\neg \exists x P(x) \equiv \forall x \neg P(x)$$

De Morgan's Laws for Quantifiers

$$
\begin{aligned}
\neg\forall x P(x) \quad &\equiv \neg\left(P(x_1) \wedge P(x_2) \wedge \ldots \wedge P(x_n)\right) \\
&\equiv \neg P(x_1) \vee \neg P(x_2) \vee \ldots \vee \neg P(x_n) \\
&\equiv \exists x \neg P(x)
\end{aligned}
$$

$$
\begin{aligned}
\neg\exists x P(x) \quad &\equiv \neg\left(P(x_1) \vee P(x_2) \vee \ldots \vee P(x_n)\right) \\
&\equiv \neg P(x_1) \wedge \neg P(x_2) \wedge \ldots \wedge \neg P(x_n) \\
&\equiv \forall x \neg P(x)
\end{aligned}
$$

De Morgan's Laws for Quantifiers

TABLE 2 De Morgan's Laws for Quantifiers.

Negation	*Equivalent Statement*	*When Is Negation True?*	*When False?*
$\neg\exists x P(x)$	$\forall x \neg P(x)$	For every x, $P(x)$ is false.	There is an x for which $P(x)$ is true.
$\neg\forall x P(x)$	$\exists x \neg P(x)$	There is an x for which $P(x)$ is false.	$P(x)$ is true for every x.

1.4 Nested Quantifiers

- Two quantifiers are nested if one is within the scope of the other
  - $\forall x \exists y\, (x+y=0)$
  - $\forall x\ \forall y\, ((x>0) \wedge (y<0) \rightarrow (xy<0))$
- Thinking of quantification as loops
  - $\forall x \forall y\, P(x, y)$
  - $\forall x \exists y\, P(x, y)$
  - $\exists x \forall y\, P(x, y)$
  - $\exists x \exists y\, P(x, y)$

TABLE 1 Quantifications of Two Variables.

Statement	*When True?*	*When False?*
$\forall x \forall y P(x, y)$ $\forall y \forall x P(x, y)$	$P(x, y)$ is true for every pair x, y.	There is a pair x, y for which $P(x, y)$ is false.
$\forall x \exists y P(x, y)$	For every x there is a y for which $P(x, y)$ is true.	There is an x such that $P(x, y)$ is false for every y.
$\exists x \forall y P(x, y)$	There is an x for which $P(x, y)$ is true for every y.	For every x there is a y for which $P(x, y)$ is false.
$\exists x \exists y P(x, y)$ $\exists y \exists x P(x, y)$	There is a pair x, y for which $P(x, y)$ is true.	$P(x, y)$ is false for every pair x, y.

Order of Quantifiers

- The order of the quantifiers is important, unless all the quantifiers are universal quantifiers or all are existential quantifiers

$$\forall x \forall y P(x, y) \equiv \forall y \forall x P(x, y)$$

$$\exists x \exists y P(x, y) \equiv \exists y \exists x P(x, y)$$

Order of Quantifiers

$$\forall x \exists y P(x, y)$$

Not equivalent to

$$\exists y \forall x P(x, y)$$

END

Q & A