ON ITERATES OF RATIONAL FUNCTIONS WITH MAXIMAL NUMBER OF CRITICAL VALUES

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ABSTRACT. Let F be a rational function of one complex variable of degree $m \geq 2$. The function F is called simple if for every $z \in \mathbb{CP}^1$ the preimage $P^{-1}\{z\}$ contains at least m-1 points. We show that if F is a simple rational function of degree $m \geq 4$ and $F^{\circ l} = G_r \circ G_{r-1} \circ \cdots \circ G_1, l \geq 1$, is a decomposition of an iterate of F into a composition of indecomposable rational functions, then r=l, and there exist Möbius transformations $\mu_i, \ 1 \leq i \leq r-1$, such that $G_r = F \circ \mu_{r-1}, \ G_i = \mu_i^{-1} \circ F \circ \mu_{i-1}, \ 1 < i < r$, and $G_1 = \mu_1^{-1} \circ F$. As an application, we provide explicit solutions of a number of problems in complex and arithmetic dynamics for "general" rational functions.

1. Introduction

Let F be a rational function of one complex variable of degree $m \geq 2$. Let us recall that any representation of F in the form $F = F_r \circ F_{r-1} \circ \cdots \circ F_1$, where F_1, F_2, \ldots, F_r are rational functions, is called a decomposition of F. Two decompositions

$$F = F_r \circ F_{r-1} \circ \cdots \circ F_1$$
 and $F = G_l \circ G_{l-1} \circ \cdots \circ G_1$

are called *equivalent* if l=r and either r=1 and $F_1=G_1$, or $r\geq 2$ and there exist Möbius transformations μ_i , $1\leq i\leq r-1$, such that

$$F_r = G_r \circ \mu_{r-1}, \quad F_i = \mu_i^{-1} \circ G_i \circ \mu_{i-1}, \quad 1 < i < r, \text{ and } F_1 = \mu_1^{-1} \circ G_1.$$

A rational function F of degree $m \geq 2$ is called *indecomposable* if the equality $F = F_2 \circ F_1$, where F_1 , F_2 , are rational functions, implies that at least one of the functions F_1 , F_2 is of degree one. It is clear that any rational function F of degree $m \geq 2$ can be decomposed into a composition of indecomposable functions, although in general not in a unique way. The problem of describing all such decompositions is quite delicate, and the general theory exists only if F is a polynomial or a Laurent polynomial (see [49], [33]).

In dynamical applications, one needs to have a description of decompositions of the whole totality of iterates of a given rational function F (see e.g. [5], [19], [20], [28], [36], [44], [45]), and the main result of this paper states roughly speaking that for a general rational function of degree at least four all such decompositions are trivial. As an application, we provide explicit solutions of a number of problems in complex and arithmetic dynamics for general rational functions. Here and below, saying that some statement holds for general rational functions of degree m, we mean that if we identify the set of rational functions of degree m with an algebraic variety Rat_m obtained from \mathbb{CP}^{2m+1} by removing the resultant hypersurface, then this statement holds for all $F \in \operatorname{Rat}_m$ with exception of some proper Zariski closed subset.

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In more detail, we prove a number of results, which hold for *simple* rational functions, that is, for rational functions F of degree m such that for every $z \in \mathbb{CP}^1$ the preimage $F^{-1}\{z\}$ contains at least m-1 points. In turn, these results allow us to establish the desired results for general rational functions.

Our main result is the following statement.

Theorem 1.1. Let F be a simple rational function of degree $m \geq 4$. Then any decomposition of $F^{\circ l}$, $l \geq 1$, into a composition of indecomposable rational functions is equivalent to $F^{\circ l}$.

First applications of Theorem 1.1 concern problems related to the functional equations

$$(1) F^{\circ k} = G^{\circ l}$$

and

$$(2) F^{\circ k_1} = F^{\circ k_2} \circ G^{\circ l}.$$

According to the results of Ritt ([50]) and Levin and Przytycki ([25], [26]), for a fixed non-special rational function F of degree at least two the first of these equations describes rational functions G of degree at least two commuting with some iterate of F, while the second equation describes rational functions G with $\mu_G = \mu_F$, where μ_F denotes the measure of maximal entropy of F.

Theorem 1.1 allows us to describe solutions of (1) and (2) for a simple rational function F of degree $m \geq 4$ explicitly. In more detail, for a rational function F of degree $m \geq 2$, we denote by C(F) the semigroup of rational functions commuting with F, and by $C_{\infty}(F)$ the semigroup of rational functions commuting with some iterate of F. We also set

$$\operatorname{Aut}(F) = C(F) \cap \operatorname{Aut}(\mathbb{CP}^1), \quad \operatorname{Aut}_{\infty}(F) = C_{\infty}(F) \cap \operatorname{Aut}(\mathbb{CP}^1).$$

Notice that $C_{\infty}(F)$ obviously contains the semigroup $\langle \operatorname{Aut}_{\infty}(F), F \rangle$ generated by F and $\operatorname{Aut}_{\infty}(F)$. Further, we denote by $E_0(F)$ the subgroup of $\operatorname{Aut}(\mathbb{CP}^1)$ consisting of Möbius transformations preserving the measure of maximal entropy of F, and by E(F) the semigroup consisting of rational functions G of degree at least two with $\mu_G = \mu_F$, completed by the group $E_0(F)$. Finally, we denote by $G_0(F)$ the maximal subgroup of $\operatorname{Aut}(\mathbb{CP}^1)$ such that for every $\sigma \in G_0(F)$ the equality

$$F\circ\sigma=\nu\circ F$$

holds for some $\nu \in G_0(F)$.

Theorem 1.1 yields that for simple rational functions the objects introduced above are related in a very easy way.

Theorem 1.2. Let F be a simple rational function of degree $m \geq 4$. Then

$$E_0(F) = \operatorname{Aut}_{\infty}(F) = G_0(F)$$
 and $E(F) = C_{\infty}(F) = \langle \operatorname{Aut}_{\infty}(F), F \rangle$.

In turn, Theorem 1.2 implies that for a general rational function F of degree $m \ge 4$ the equality

$$(3) E(F) = \langle F \rangle$$

holds (see Section 3.2). This provides an affirmative answer to the question of Ye, who proved that (3) holds after removing from Rat_m countably many algebraic sets, and asked whether (3) remains true if to remove from Rat_m only finitely many such sets ([55]).

Further applications of Theorem 1.1 concern problems that can be reformulated in terms of semiconjugacies between rational functions (see the papers [4], [9], [11], [23], [28], [38], [44] for examples of such problems). We recall that a rational function B of degree at least two is called *semiconjugate* to a rational function A if there exists a non-constant rational function X such that

$$(4) A \circ X = X \circ B.$$

A comprehensive description of solutions of equation (4) was obtained in the series of papers [35], [37], [39], [40]. In case A is simple, Theorem 1.1 permits to reduce this description to the following uncomplicated form suitable for applications.

Theorem 1.3. Let F be a simple rational function of degree $m \geq 4$ and G, X rational functions of degree at least two such that the diagram

(5)
$$\begin{array}{ccc}
\mathbb{CP}^1 & \xrightarrow{G} & \mathbb{CP}^1 \\
X \downarrow & & \downarrow X \\
\mathbb{CP}^1 & \xrightarrow{F^{\circ r}} & \mathbb{CP}^1
\end{array}$$

commutes for some $r \geq 1$. Then there exist a Möbius transformation ν and an integer $l \geq 1$ such that the equalities

$$X = F^{\circ l} \circ \nu, \qquad G = \nu^{-1} \circ F^{\circ r} \circ \nu$$

hold.

As an example of application of Theorem 1.3, we consider the problem of describing periodic algebraic curves for endomorphisms of $(\mathbb{CP}^1)^2$ of the form

(6)
$$(z_1, z_2) \to (F_1(z_1), F_2(z_2)),$$

where F_1 , F_2 are rational functions, which reduces to describing solutions of a system of semiconjugacies. In the case where F_1 , F_2 are polynomials, a description of periodic curves was obtained by Medvedev and Scanlon ([28]), and has numerous applications in complex and arithmetic dynamics (see e. g. [3], [11], [16], [17], [18], [21], [32]). A description of periodic curves for arbitrary non-special rational functions F_1 , F_2 was obtained in the recent paper [44]. Theorem 1.3 permits to shorten considerably the results of [44] in case F_1 and F_2 are simple, leading to the following result, which can be easily used for applications.

Theorem 1.4. Let F_1 and F_2 be simple rational functions of degree $m \geq 4$, and C an irreducible algebraic curve in $(\mathbb{P}^1(\mathbb{C}))^2$ that is not a vertical or horizontal line. Then $(F_1, F_2)^{\circ d}(C) = C$ for an integer $d \geq 1$ if and only if

$$F_2^{\circ d} = \alpha \circ F_1^{\circ d} \circ \alpha^{-1}$$

for some Möbius transformation α , and C is one of the graphs

$$X_2 = (\alpha \circ \nu \circ F_1^{\circ s})(X_1), \qquad X_1 = (\nu \circ F_1^{\circ s} \circ \alpha^{-1})(X_2),$$

where $\nu \in \operatorname{Aut}(F_1^{\circ d})$ and $s \geq 0$.

Notice that Theorem 1.4 implies that for general pairs of rational functions F_1 , F_2 of degree $m \geq 4$ endomorphisms (6) have no periodic curves distinct from vertical or horizontal lines. In particular, the (F_1, F_2) -orbit of a point (x, y) in $(\mathbb{P}^1(\mathbb{C}))^2$ is Zariski dense in $(\mathbb{P}^1(\mathbb{C}))^2$, unless x is a preperiodic point of F_1 , or y is a preperiodic point of F_2 (see Section 4.3).

For proving Theorem 1.1, we use the following strategy. First, we show that if F is a simple rational function of degree $m \geq 4$ and H is an indecomposable rational function of degree at least two such that the algebraic curve

$$(7) H(y) - F(x) = 0$$

is irreducible, then the genus of this curve is greater than zero. Then we show that if (7) is reducible, then either $H = F \circ \mu$, where μ is a Möbius transformation, or deg H is equal to the binomial coefficient $\binom{m}{k}$ for some k, 1 < k < m-1. Finally, using the theorem of Sylvester [53] and Schur [52] about prime divisors of binomial coefficients, we show that there exists a prime number p such that $p \mid \binom{m}{k}$ but $p \nmid m$. This implies that if a rational function H is a compositional left factor of some iterate of a simple rational function F, that is,

$$F^{\circ l} = H \circ R$$

for some rational function R, then H necessarily has the form $H = F \circ \mu$ for some $\mu \in \operatorname{Aut}(\mathbb{CP}^1)$. Finally, from the last statement we deduce inductively the conclusion of the theorem.

The paper is organized as follows. In the second section, using the above approach we prove Theorem 1.1. We also provide examples of simple rational functions of degree two and three for which Theorem 1.1 is not true. In the third section, we prove Theorem 1.2 and some related results. Finally, in the fourth section, we prove Theorem 1.3 and Theorem 1.4. We also prove the above mentioned corollaries of Theorem 1.4 concerning dynamics of endomorphisms (6) for general pairs of rational functions.

2. Proof of Theorem 1.1

2.1. Calculation of genus of H(x) - F(y) = 0. Let F be a rational function of degree $m \geq 2$. We denote by $\operatorname{Mon}(F) \subseteq S_m$ the monodromy group of F, and by $\deg_z F$ the multiplicity of F at a point $z \in \mathbb{CP}^1$. The following two results are known. We include the proofs for the reader's convenience.

Lemma 2.1. Let F be a rational function of degree $m \geq 2$. Then the following conditions are equivalent.

- i) The function is simple.
- ii) The number of critical points of F is equal to the number of critical values, and the multiplicity of F at every critical point is equal to two.
- iii) The number of critical values of F is equal to 2m-2.

Proof. The equivalence $i) \Leftrightarrow ii)$ follows from the definition. Furthermore, the Riemann-Hurwitz formula

(8)
$$2m-2 = \sum_{z \in \mathbb{CP}^1} (\deg_z F - 1)$$

implies that the number of critical points of F does not exceed 2m-2, and the equality is attained if and only if the multiplicity of F at every critical point is equal to two. Since the number of critical values of F does not exceed the number of critical points, this yields the implication $iii) \Rightarrow ii$. Finally, (8) implies that if the multiplicity of F at every critical point is equal to two, then the number of critical points is equal to 2m-2. Thus, $ii) \Rightarrow iii$.

Theorem 2.2. Let F be a simple rational function of degree $m \geq 2$. Then F is indecomposable, and $Mon(F) \cong S_m$.

Proof. Assume that

$$(9) F = F_1 \circ F_2,$$

where F_1 and F_2 are rational functions of degrees m_1 and m_2 . Since F is simple, the number of critical values of F is $2m_1m_2-2$, by Lemma 2.1. On the other hand, it follows from (9) by the chain rule that the number of critical values of F does not exceed $(2m_1-2)+(2m_2-2)$. Thus,

$$2m_1m_2 - 2 \le (2m_1 - 2) - (2m_2 - 2),$$

implying that

$$2m_1m_2 - 2 - (2m_1 - 2) - (2m_2 - 2) = 2(m_1 - 1)(m_2 - 1) \le 0.$$

Therefore, at least one of the functions F_1 and F_2 has degree one.

Since F is indecomposable, the monodromy group $\operatorname{Mon}(F)$ of F is imprimitive. Moreover, for any critical value c of F, the permutation in $\operatorname{Mon}(P)$ corresponding to c is a transposition. Since a primitive permutation group containing a transposition is a full symmetric group (see [54], Theorem 13.3), we conclude that $\operatorname{Mon}(F) = S_m$.

Let F and H be rational functions of degrees n and m, and H_1 , H_2 and F_1 , F_2 pairs of polynomials without common roots such that $H = H_1/H_2$ and $F = F_1/F_2$. Let us define algebraic curves $h_{F,H}(x,y)$ and $h_F(x,y)$ by the formulas

$$h_{H,F}: H_1(x)F_2(y) - H_2(x)F_1(y) = 0,$$

and

$$h_F: \frac{F_1(x)F_2(y) - F_2(x)F_1(y)}{x - y} = 0.$$

In case these curves are irreducible, their genera can be calculated explicitly in terms of ramification of H and F as follows. Let $S = \{z_1, z_2, \ldots, z_r\}$ be a union of critical values of H and F. For $i, 1 \le i \le r$, we denote by

$$(a_{i,1}, a_{i,2}, ..., a_{i,p_i})$$

the collection of multiplicities of H at the points of $H^{-1}\{z_i\}$, and by

$$(b_{i,1}, b_{i,2}, ..., b_{i,q_i})$$

the collection of multiplicities of F at the points of $F^{-1}\{z_i\}$. In this notation, the following formulas hold (see [13] or [34]):

(10)
$$2 - 2g(h_{H,F}) = \sum_{i=1}^{r} \sum_{j_2=1}^{q_i} \sum_{j_1=1}^{p_i} GCD(a_{i,j_1}b_{i,j_2}) - mn(r-2),$$

(11)
$$4 - 2g(h_F) = \sum_{i=1}^{r} \sum_{j_2=1}^{p_i} \sum_{j_1=1}^{p_i} GCD(b_{i,j_1}b_{i,j_2}) - (r-2)m^2.$$

Theorem 2.3. Let F be a simple rational function of degree $m \geq 4$, and H a rational function of degree $n \geq 2$ such that the curve $h_{H,F}$ is irreducible. Then $g(h_{H,F}) > 0$. In particular, the functional equation $F \circ X = H \circ Y$ has no solutions in rational functions X, Y.

Proof. Keeping the above notation, let us observe that if z_i , $1 \le i \le r$, is not a critical value of F, then obviously

(12)
$$\sum_{j_1=1}^{p_i} GCD(a_{i,j_1}b_{i,j_2}) = p_i, \quad 1 \le j_2 \le q_i,$$

and

(13)
$$\sum_{j_2=1}^{q_i} \sum_{j_1=1}^{p_i} GCD(a_{i,j_1}b_{i,j_2}) = mp_i.$$

Assume now that z_i , $1 \le i \le r$, is a critical value of F. Then (12) still holds if $b_{i,j_2} = 1$, while if $b_{i,j_2} = 2$, $1 \le j_2 \le q_i$, we have

$$\sum_{j_1=1}^{p_i} GCD(a_{i,j_1}b_{i,j_2}) = p_i + l_i,$$

where l_i is the number of even numbers among the numbers a_{i,j_1} , $1 \leq j_1 \leq p_i$. Thus, taking into account that among the numbers b_{i,j_2} , $1 \leq j_2 \leq q_i$, exactly one is equal to two and all the other are equal to one, we conclude that

(14)
$$\sum_{j_2=1}^{q_i} \sum_{j_1=1}^{p_i} GCD(a_{i,j_1}b_{i,j_2}) = (m-2)p_i + p_i + l_i = mp_i + (l_i - p_i).$$

Since

$$2n - 2 = \sum_{z \in \mathbb{CP}^1} (\deg_z H - 1) = \sum_{i=1}^r \sum_{j_1=1}^{p_i} (a_{i,j_1} - 1) = rn - \sum_{i=1}^r p_i,$$

the equality

(15)
$$\sum_{i=1}^{r} p_i = (r-2)n + 2$$

holds, implying by (13) and (14) that

(16)
$$\sum_{i=1}^{r} \sum_{j_2=1}^{q_i} \sum_{j_1=1}^{p_i} GCD(a_{i,j_1}b_{i,j_2}) = \sum_{i=1}^{r} mp_i + \sum' (l_i - p_i) =$$

$$= m((r-2)n+2) + \sum_{i}' (l_i - p_i),$$

where the sum \sum' runs only over indices corresponding to critical values of F. It follows now from (10) that $g(h_{H,F}) = 0$ if and only if

$$2m - 2 + \sum_{i} (l_i - p_i) = 0.$$

Stating differently, $g(h_{H,F}) = 0$ if and only if the preimage $H^{-1}\{c_1, c_2, \dots, c_{2m-2}\}$, where $c_1, c_2, \dots, c_{2m-2}$ are critical values of F, contains exactly 2m-2 points where the multiplicity of H is odd.

Let us observe now that for any finite subset S of \mathbb{CP}^1 it follows from

$$2n-2=\sum_{z\in\mathbb{CP}^1}(\deg{_zH}-1)\geq\sum_{z\in H^{-1}(S)}(\deg{_zH}-1)$$

that the preimage $H^{-1}(S)$ contains at least n(|S|-2)+2 points and the equality is attained if and only if S contains the set of critical values of H. Therefore,

$$H^{-1}\{c_1, c_2, \dots, c_{2m-2}\} \ge (2m-4)n+2,$$

and the equality is attained if and only if any critical value of H is a critical value of F. On the other hand, the condition that $H^{-1}\{c_1, c_2, \ldots, c_{2m-2}\}$ contains 2m-2 points where the multiplicity of H is odd implies that

$$H^{-1}\{c_1, c_2, \dots, c_{2m-2}\} \le (2m-2) + \frac{n(2m-2) - (2m-2)}{2} = (n+1)(m-1),$$

and the equality is attained if and only if all 2m-2 points in $H^{-1}\{c_1, c_2, \ldots, c_{2m-2}\}$ with odd multiplicity have multiplicity one, while all points with even multiplicity have multiplicity two. Thus, if $g(h_{H,F}) = 0$, then

$$(2m-4)n+2 \le (n+1)(m-1),$$

implying that

$$(n-1)(m-3) \le 0.$$

Since the last inequality is satisfied only for n=1 or for m=2,3, we conclude that $g(h_{H,F})>0$.

Theorem 2.4. Let F be a simple rational function of degree $m \geq 3$. Then the curve h_F is irreducible and $g(h_F) > 0$. In particular, the equality $F \circ X = F \circ Y$, where X and Y are rational functions, implies that X = Y.

Proof. It is well-known (see e.g. [34], Corollary 2.3) that the curve $h_F(x, y)$ is irreducible if and only if the monodromy group Mon(F) is doubly transitive. Therefore, since a symmetric group is doubly transitive, the irreducibility of $h_F(x, y)$ follows from Theorem 2.2.

Further, if H = F, then n = m, r = 2m - 2, and $l_i = 1$, $1 \le i \le r$, implying by (15) and (16) that

$$\sum_{i=1}^{r} \sum_{j_2=1}^{p_i} \sum_{j_1=1}^{p_i} GCD(b_{i,j_1}b_{i,j_2}) = \sum_{i=1}^{2m-2} mp_i + \sum_{i=1}^{2m-2} (1-p_i) = \sum_{i=1}^{2m-2} (m-1)p_i + 2m - 2 = \sum_$$

$$= (m-1)((2m-4)m+2) + 2m - 2 = m^{2}(2m-4) - 2m^{2} + 8m - 4.$$

Therefore, by (11), we have:

$$q(h_F) = (m-2)^2$$

and hence $g(h_F) > 0$ whenever $m \geq 3$.

2.2. Conditions for reducibility of H(x) - F(y) = 0. The reducibility problem for algebraic curves $h_{H,F}$ has a long story and is not solved yet in its full generality (see [15] for an introduction to the topic). In this section, we consider a very particular case of this problem, which is related to the subject of this paper and can be handled without using serious group theoretic methods.

Let F be a rational function of degree $m \geq 2$, and $U \subset \mathbb{CP}^1$ a simply connected domain containing no critical values of F. Then in U there exist $m = \deg F$ different branches of the algebraic function $F^{-1}(z)$. We will denote these branches by small letters f_1, f_2, \ldots, f_m .

Lemma 2.5. Let F be a rational function of degree $m \geq 2$ such that $Mon(F) = S_m$. Then the equality

(17)
$$c_1(z)f_1(z) + c_2(z)f_2(z) + \dots + c_m(z)f_m(z) = c_0(z),$$

where $c_i(z) \in \mathbb{C}(z)$, $0 \le i \le m$, implies that

$$c_1(z) = c_2(z) = \dots = c_m(z).$$

Proof. Assume, say, that $c_1(z) \neq c_2(z)$. Since $Mon(F) = S_m$, the transposition $\sigma = (1, 2)$ is contained in Mon(F), and considering the analytical continuation of equality (17) along a loop corresponding to σ , we obtain the equality

(18)
$$c_1(z)f_2(z) + c_2(z)f_1(z) + \dots + c_m(z)f_m(z) = c_0(z).$$

It follows now from (17) and (18) that $f_1(z) = f_2(z)$ in contradiction with the assumption that f_1, f_2, \ldots, f_m are different branches of $F^{-1}(z)$.

Let P(z) and Q(z) be non-constant rational functions, and p(z) a single-valued branch of $P^{-1}(z)$ defined in some disk $D \subset \mathbb{CP}^1$ containing no critical values of P. We denote by Q(p(z)) the algebraic function obtained by the full analytic continuation of the germ $\{D, Q(p(z))\}$, and by d(Q(p(z))) the degree of Q(p(z)), that is, the number of its branches.

Lemma 2.6. Let P(z) and Q(z) be non-constant rational functions. Then

$$d(Q(p(z))) = \deg P(z)/[\mathbb{C}(z) : \mathbb{C}(P,Q)].$$

In particular, the degree d(Q(p(z))) does not depend on the choice of the branch of $P^{-1}(z)$. Moreover, if P is indecomposable, then either d(Q(p(z))) = 1, or $d(Q(p(z))) = \deg P$.

Proof. Since any algebraic relation between Q(p(z)) and z over \mathbb{C} transforms to an algebraic relation between Q(z) and P(z) and vice verse we have:

$$(19) \quad d(Q(p(z))) = [\mathbb{C}(P,Q) : \mathbb{C}(P)] =$$

$$[\mathbb{C}(z):\mathbb{C}(P)]/[\mathbb{C}(z):\mathbb{C}(P,Q)] = \deg P(z)/[\mathbb{C}(z):\mathbb{C}(P,Q)].$$

Furthermore, by the Lüroth theorem, $\mathbb{C}(P,Q)=\mathbb{C}(R)$ for some $R\in\mathbb{C}(z),$ implying that

$$P = \widetilde{P} \circ R, \quad Q = \widetilde{Q} \circ R$$

for some $\widetilde{P},\widetilde{Q}\in\mathbb{C}(z)$. Therefore, if the function P is indecomposable, then either $\mathbb{C}(P,Q)=\mathbb{C}(P),$ or $\mathbb{C}(P,Q)=\mathbb{C}(z),$ implying by (19) that either d(Q(p(z)))=1, or $d(Q(p(z)))=\deg P.$

Theorem 2.7. Let H and F be rational functions of degrees $n \geq 2$ and $m \geq 2$ such that H is indecomposable, $Mon(F) = S_m$, and the curve $h_{H,F}$ is reducible. Then either $H = F \circ \mu$, where μ is a Möbius transformation, or $n = {m \choose k}$ for some k, 1 < k < m - 1.

Proof. Let H_1 , H_2 and F_1 , F_2 be pairs of polynomials without common roots such that $H = H_1/H_2$ and $F = F_1/F_2$. Suppose that

(20)
$$H_1(x)F_2(y) - H_2(x)F_1(y) = M(x,y)N(x,y)$$

for some non-constant polynomials M(x,y), N(x,y). Notice that since H and F are non-constants the numbers $\deg_x M$, $\deg_y M$, $\deg_x N$, $\deg_y N$ are necessarily distinct from zero.

Let $h_1, h_2, \ldots h_n$ be branches of $H^{-1}(z)$ and $f_1, f_2, \ldots f_m$ branches of $F^{-1}(z)$ defined in a simply connected domain $U \subset \mathbb{CP}^1$ containing no critical values of F or H. Since $\deg_y M > 0$, it follows from (20) that for at least one $i, 1 \leq i \leq m$, the equality

$$(21) M(h_1, f_i) = 0$$

holds. On the other hand, equality (21) cannot be satisfied for all $i, 1 \leq i \leq m$, since otherwise the equality deg $_yN=0$ holds. Let i_1,i_2,\ldots,i_k be indices for which (21) holds. Then (21) implies that

$$(22) f_{i_1} + f_{i_2} + \dots + f_{i_k} = Q(h_1)$$

for some $Q \in \mathbb{C}(z)$. Furthermore, since the set $\{i_1, i_2, \ldots, i_k\}$ is a proper subset of $\{1, 2, \ldots, m\}$, the function $Q(h_1)$ is not a rational function by Lemma 2.5. Thus, $d(Q(h_1(z))) \neq 1$, implying by Lemma 2.6 that the functions $Q(h_i(z))$, $1 \leq i \leq n$, are pairwise different.

Continuing equality (22) analytically along an arbitrary closed curve γ in \mathbb{CP}^1 , we obtain an equality where on the left side is a sum of branches of $F^{-1}(z)$ over a subset of $\{1, 2, \ldots, m\}$ containing k elements, while on the right side is a branch $Q(h_i), 1 \leq i \leq n$, of the function $Q(h_1(z))$. Furthermore, to different subsets of $\{1, 2, \ldots, m\}$ correspond different branches of $Q(h_1(z))$, for otherwise we obtain a contradiction with Lemma 2.5. Since the equality $\operatorname{Mon}(F) = S_m$ implies that for an appropriately chosen γ we can obtain on the left side a sum of branches of $F^{-1}(z)$ over any k-element subset of $\{1, 2, \ldots, m\}$, while the transitivity of $\operatorname{Mon}(H)$ implies that for an appropriately chosen γ we can obtain on the right side any branch of $Q(h_1(z))$, we conclude that the degree of $Q(h_1(z))$ is equal to the number of k-element subsets of $\{1, 2, \ldots, m\}$, for some $k, 1 \leq k \leq n$, that is, $n = {m \choose k}$.

To finish the proof, let us observe that if k = 1, then $n = {m \choose k}$ implies that n = m. Furthermore, equality (22) implies the equality

$$z = F \circ f_{i_1} = (F \circ Q)(h_1).$$

Thus, the function $F \circ Q$ is inverse to h_1 , that is, $H = F \circ Q$. Finally, Q is a Möbius transformation since n = m. The same conclusion is true if k = m - 1, since we can switch between M(x, y) and N(x, y).

Remark 2.8. Danny Neftin kindly informed us that the results of the recent papers [30] and [24] permit to obtain analogues of Theorem 2.7 in the cases where H or F is decomposable.

2.3. **Prime divisors of** $\binom{m}{k}$. The classical theorem of Sylvester [53] and Schur [52] states that in the set of integers $a, a+1, \ldots, a+b-1$, where a>b, there is a number divisible by a prime greater than b. For a natural number x, let us denote by $\mathcal{P}(x)$ the greatest prime factor of x. Then the theorem of Sylvester and Schur may be reformulated as follows ([7]): for any $m \geq 2k$ the inequality $\mathcal{P}(\binom{m}{k}) > k$ holds. Furthermore, the last inequality may be sharpened to the inequality

(23)
$$\mathcal{P}\left(\binom{m}{k}\right) \ge \frac{7}{5}k$$

(see [10], [22]). We will prove that this implies the following corollary.

Theorem 2.9. Let $m \ge 4$ be a natural number, and k a natural number such that 1 < k < m-1. Then there exists a prime number p such that $p \mid {m \choose k}$ but $p \nmid m$.

Proof. Since $\binom{m}{k} = \binom{m}{m-k}$, it is enough to prove the theorem under the assumption that $m \geq 2k$. Applying the Sylvester-Schur theorem, we conclude that there is a number $s, m-k+1 \leq s \leq m$, such that $\mathcal{P}\binom{m}{k} = \mathcal{P}(s) = p > k$. Moreover, if s is strictly less than m, then p cannot be a divisor of m for otherwise

$$(24) p|(m-s)$$

in contradiction with

$$(25) p > k.$$

Since however s can be equal to m, we modify slightly this argument. Namely, we apply the Sylvester-Schur theorem in its strong form (23) to the binomial coefficient $\binom{m-1}{k-1}$ related with $\binom{m}{k}$ by the equality

(26)
$$\binom{m}{k} = \frac{m(m-1)\dots(m-k+1)}{k(k-1)\dots 1} = \frac{m}{k} \binom{m-1}{k-1}.$$

Notice that (26) implies that every prime factor p of $\binom{m-1}{k-1}$ satisfying (25) remains a prime factor of $\binom{m}{k}$.

Since $m \ge 2k$ implies $m-1 \ge 2(k-1)$, applying (23) to $\binom{m-1}{k-1}$ we conclude that there is a number $s, m-k+1 \le s \le m-1$, such that

$$\mathcal{P}\left(\binom{m-1}{k-1}\right) = \mathcal{P}(s) = p \ge \frac{7}{5}(k-1).$$

Furthermore, if k > 3 then

$$p \ge \frac{7}{5}(k-1) > k,$$

implying that $p \mid \binom{m}{k}$. On the other hand, $p \nmid m$ since otherwise (24) holds in contradiction with (25).

For $k \leq 3$ the theorem can be proved by an elementary argument. If k=2 then

$$\binom{m}{k} = \frac{m(m-1)}{2}.$$

Therefore, since GCD(m, m-1) = 1, the statement of the theorem is true, whenever $(m-1) \nmid 2$, and the last condition is always satisfied if m > 3. Similarly, if k = 3, then

$$\binom{m}{k} = \frac{m(m-1)(m-2)}{2 \cdot 3},$$

and the statement of the theorem is true whenever $(m-1) \nmid 6$. The last condition fails to be true for m > 3 only if m is equal to 4 or 7. However, the pair m = 4, k = 3 does not satisfy the condition 1 < k < m - 1. On the other hand, for the pair m = 7, k = 3 we have $\binom{m}{k} = \binom{7}{3} = 5 \cdot 7$, and the statement of the theorem is satisfied for p = 5.

Proof of Theorem 1.1. Let

$$(27) F^{\circ l} = F_r \circ F_{r-1} \circ \dots \circ F_1$$

be a decomposition of $F^{\circ l}$, $l \geq 1$, into a composition of indecomposable rational functions. Since F is indecomposable by Theorem 2.2, for l=1 the theorem is true. On the other hand, since by Theorem 2.4 the equality $F \circ X = F \circ Y$ implies

that X = Y, to prove the inductive step it is enough to show that equality (27) implies that

$$(28) F_r = F \circ \mu$$

for some Möbius transformation μ .

Clearly, equality (27) implies that the algebraic curve

(29)
$$F(x) - F_r(y) = 0$$

has a factor of genus zero. Therefore, (29) is reducible by Theorem 2.3. Since $\operatorname{Mon}(F) = S_m$ by Theorem 2.2, it follows now from Theorem 2.7 that either (28) holds, or $\deg F_r = \binom{m}{k}$ for some k, 1 < k < m - 1. However, the last case is impossible since (27) implies that any prime divisor of $\deg F_r$ is a prime divisor of $\deg F$ in contradiction with Theorem 2.9.

Corollary 2.10. Let F be a simple rational function of degree $m \geq 4$, and G_i , $1 \leq i \leq r$, rational functions of degree at least two such that

$$F^{\circ l} = G_r \circ G_{r-1} \circ \dots G_1$$

for some $l \ge 1$. Then there exist Möbius transformations ν_i , $1 \le i < r$, and integers $s_i \ge 1$, $1 \le i \le r$, such that

$$G_r = F^{\circ s_r} \circ \nu_{r-1}, \quad G_i = \nu_i^{-1} \circ F^{\circ s_i} \circ \nu_{i-1}, \quad 1 < i < r, \quad and \quad G_1 = \nu_1^{-1} \circ F^{\circ s_1}.$$

Proof. To prove the corollary, it is enough to decompose each G_i , $1 \le i \le r$, into a composition of indecomposable rational functions and to apply Theorem 1.1.

Remark 2.11. Theorem 1.1 is not true if the degree m of F is equal to 2 or 3. Indeed, for example, for the rational functions

$$P = \frac{z^2 - 1}{z^2 + 1}, \quad Q = -\frac{1}{2z^2 - 1}, \quad R = \frac{1}{2}\left(z + \frac{1}{z}\right)$$

the equality

$$P \circ P = Q \circ R = -\frac{2z^2}{z^4 + 1}$$

holds. On the other hand, the equality

$$(30) R = \mu \circ P$$

for a Möbius transformation μ is impossible, since critical points of P and hence of $\mu \circ P$ are 0 and ∞ , while critical points of R are -1 and 1.

Further, setting

$$P = \frac{6x}{x^3 - 2}$$

and

$$Q = -\frac{23328x}{x^3 + 216\sqrt[3]{2}x^2 + 3888\sqrt{2}/3}x - 93312}, \quad R = \frac{36x\left(2^{2/3}x^2 - 4x + 2\sqrt[3]{2}\right)}{2^{2/3}x^2 + 2x + 2\sqrt[3]{2}},$$

one can check that

$$P \circ P = Q \circ R = -\frac{18(x^3 - 2)^2 x}{x^9 - 6x^6 - 96x^3 - 8}.$$

On the other hand, equality (30) cannot be satisfied for a Möbius transformation μ , since ∞ is a critical point of P, but not a critical point of R.

3. Proof of Theorem 1.2

3.1. Groups and semigroups related to simple rational functions. We start this section by recalling some basic facts concerning the groups and semigroups defined in the introduction. We will say that a rational function F is special if F is either a Lattès map, or it is conjugate to $z^{\pm n}$ or $\pm T_n$.

It is obvious that C(F) is a semigroup, and it follows from the inclusions

$$C(F^{\circ k}), \ C(F^{\circ l}) \subseteq C(F^{\circ LCM(k,l)})$$

that $C_{\infty}(F)$ is also a semigroup. Further, by the Ritt theorem (see [50], and also [8], [42]), commuting rational functions of degree at least two are either special or have a common iterate. Thus, any element G of the semigroup $C(F) \setminus \operatorname{Aut}(F)$ for a non-special rational function F satisfies condition (1) for some $k, l \geq 1$. Furthermore, since equality (1) implies that G commutes with $F^{\circ k}$, the Ritt theorem yields that the semigroup $C_{\infty}(F) \setminus \operatorname{Aut}_{\infty}(F)$ coincides with the set of rational functions sharing an iterate with F. A method for describing C(F) for an arbitrary non-special rational function F was given in the recent paper [42]. A satisfactory description of $C_{\infty}(F)$ is still not known (see [47] for some particular results).

Let us recall that by the results of Freire, Lopes, Mañé ([12]) and Lyubich ([27]), for every rational function F of degree $n \geq 2$ there exists a unique probability measure μ_F on \mathbb{CP}^1 , which is invariant under F, has support equal to the Julia set J_F , and achieves maximal entropy $\log n$ among all F-invariant probability measures. The measure μ_F can be described as follows. For $a \in \mathbb{CP}^1$ let $z_i^k(a)$, $i = 1, \ldots, n^k$, be the roots of the equation $F^{\circ k}(z) = a$ counted with multiplicity, and $\mu_{F,k}(a)$ be the measure defined by

$$\mu_{F,k}(a) = \frac{1}{n^k} \sum_{i=1}^{n^k} \delta_{z_i^k(a)}.$$

Then for every $a \in \mathbb{CP}^1$ with two possible exceptions, the sequence $\mu_{F,k}(a)$, $k \geq 1$, converges in the weak topology to μ_F . In particular, this yields that

(31)
$$\operatorname{Aut}_{\infty}(F) \subseteq E_0(F),$$

since the equality

$$F^{\circ n} = \alpha^{-1} \circ F^{\circ n} \circ \alpha$$

for some $\alpha \in \operatorname{Aut}(\mathbb{CP}^1)$ and $n \geq 1$ implies that for every set $S \subset \mathbb{CP}^1$ and $k \geq 1$ the equality

$$|S \cap F^{-nk}(a)| = |\alpha(S) \cap F^{-nk}(\alpha(a))|$$

holds. Moreover, the above description of μ_F implies that any G sharing an iterate with F belongs to E(F). Thus,

(32)
$$C_{\infty}(F) \subseteq E(F).$$

The fact that E(F) is a semigroup can be established using the Lyubich operator or the balancedness property of μ_F (see [6], [47]). Further, it follows from the results of Levin and Przytycki (see [25], [26], and also [55]) that if F is non-special, then $G \in E(F) \setminus E_0(F)$ if and only if G satisfies equation (2), which generalizes equation (1). A complete description of E(F) is known only if F is a polynomial, in which case $E(F) \setminus E_0(F)$ coincides with the set of polynomials sharing a Julia set with F (see [1], [2], [51] and also [43], [47]). Some partial results in the rational case can be found in [41], [55].

The group $G_0(F)$ obviously is a subgroup of the group G(F) consisting of Möbius transformations σ such that

$$(33) F \circ \sigma = \nu \circ F$$

for some Möbius transformations ν . It is easy to see that G(F) is indeed a group and that the map

$$\gamma:\sigma\to\nu_\sigma$$

is a homomorphism from G(F) to the group $\operatorname{Aut}(\mathbb{CP}^1)$. The group G(F) is finite and its order is bounded in terms of $m = \deg F$, unless

$$(34) \alpha \circ F \circ \beta = z^m$$

for some $\alpha, \beta \in \text{Aut}(\mathbb{CP}^1)$ (see [39], Section 4 or [46], Section 2). Thus, the group $G_0(F)$ is also finite, unless (34) holds.

Lemma 3.1. Let F be a simple rational function of degree $m \geq 3$. Then the group $G_0(F)$ is finite, and the restriction of γ to $G_0(F)$ is an automorphism of $G_0(F)$.

Proof. Since equality (34) is impossible for simple F of degree $m \geq 3$, the group $G_0(F)$ is finite. Furthermore, since the equality $F = F \circ \sigma$ contradicts to Theorem 2.4, the kernel of γ is trivial. Since $\gamma(G_0(F)) \subseteq G_0(F)$ by the definition, this implies that the restriction of γ to $G_0(F)$ is an automorphism $G_0(F)$.

Corollary 3.2. Let F be a simple rational function of degree $m \geq 3$. Then $G_0(F) \subseteq \operatorname{Aut}(F^{\circ s})$, where $s = |\operatorname{Aut}(G_0(F))|$.

Proof. For $s = |\operatorname{Aut}(G_0(F))|$ the iterate $\gamma^{\circ s}$ is the identical automorphism of $G_0(F)$. Therefore, since

$$F^{\circ s} \circ \sigma = \gamma^{\circ s}(\sigma) \circ F^{\circ s}, \quad \sigma \in G_0(F),$$

every element of $G_0(F)$ commutes with $F^{\circ s}$.

Lemma 3.3. Let F be a rational function, and σ a Möbius transformation such that

$$(35) \qquad (\sigma \circ F)^{\circ l} = F^{\circ l}$$

for some l > 1. Then $\sigma \in \operatorname{Aut}(F^{\circ l})$.

Proof. Clearly, equality (35) implies the equality

(36)
$$(\sigma \circ F)^{\circ (l-1)} \circ \sigma = F^{\circ (l-1)}.$$

Composing now F with the both parts of equality (36), we obtain the equality

$$(37) (F \circ \sigma)^{\circ l} = F^{\circ l}.$$

It follows now from (35) and (37) that

$$F^{\circ l} \circ \sigma = (\sigma \circ F)^{\circ l} \circ \sigma = \sigma \circ (F \circ \sigma)^{\circ l} = \sigma \circ F^{\circ l}.$$

Lemma 3.4. Let F be a simple rational function of degree $m \ge 4$. Then F is not a special function.

Proof. The proof follows easily from the analysis of ramifications of special functions. Since below we prove a more general result (Lemma 4.3), we omit the details. \Box

Theorem 3.5. Let F be a simple rational function of degree $m \geq 4$. Then

$$E_0(F) = G_0(F) = \operatorname{Aut}_{\infty}(F) = \operatorname{Aut}(A^{\circ s}),$$

where $s = |\operatorname{Aut}(G_0(F))|$.

Proof. By Corollary 3.2 and (31), we have:

$$G_0(F) \subseteq \operatorname{Aut}(A^{\circ s}) \subseteq \operatorname{Aut}_{\infty}(F) \subseteq E_0(F).$$

Thus, to prove the theorem we only must show that $E_0(F) \subseteq G_0(F)$. In turn, for this purpose it is enough to establish that for every $\sigma \in E_0(F)$ there exists $\nu \in E_0(F)$ such that (33) holds. Let σ be an arbitrary element of $E_0(F)$. Then $F \circ \sigma \in E(F)$, implying by Lemma 3.4 and the theorem of Levin and Przytycki that

$$F^{\circ k_1} = F^{\circ k_2} \circ (F \circ \sigma)^{\circ l}$$

for some $k_1, l \ge 1, k_2 \ge 0$. Applying to the last equality recursively Theorem 2.4, we conclude that

(38)
$$F^{\circ(k_1-k_2)} = (F \circ \sigma)^{\circ l}.$$

Therefore, by Theorem 1.1, there exist Möbius transformations μ_i , $1 \le i \le l-1$, such that

$$F \circ \sigma = F \circ \mu_{r-1}, \quad F \circ \sigma = \mu_i^{-1} \circ F \circ \mu_{i-1}, \quad 1 < i < l, \quad \text{and} \quad F \circ \sigma = \mu_1^{-1} \circ F.$$

Thus, equality (33) holds for $\nu = \mu_1^{-1}$. Furthermore, since equality (38) implies that $k_1 - k_2 = l$, we have:

$$F^{\circ l} = (F \circ \sigma)^{\circ l} = (\mu_1^{-1} \circ F)^{\circ l},$$

implying by Lemma 3.3 that $\mu_1^{-1} \in \operatorname{Aut}_{\infty}(F) \subseteq E_0(F)$.

Notice that the equality $\operatorname{Aut}_{\infty}(F) = G_0(F)$ implies that every element of the semigroup $\langle \operatorname{Aut}_{\infty}(F), F \rangle$ can be represented in a unique way in the form $\alpha \circ F^{\circ s}$, where $\alpha \in \operatorname{Aut}_{\infty}(F)$ and $s \geq 1$.

Proof of Theorem 1.2. In view of Theorem 3.5, we only must show that

$$C_{\infty}(F) = E(F) = \langle \operatorname{Aut}_{\infty}(F), F \rangle.$$

Moreover, in view of (32), the first equality follows from the theorem of Levin and Przytycki and Theorem 2.4, since the latter implies that any G satisfying (2) satisfies (1) for $k = k_1 - k_2$. Since the semigroup $\langle \operatorname{Aut}_{\infty}(F), F \rangle$ is obviously a subsemigroup of $C_{\infty}(F)$, to finish the proof we only must show that if a rational function G satisfies (1), then it belongs to $\langle \operatorname{Aut}_{\infty}(F), F \rangle$.

Applying Corollary 2.10 to equality (1), we see that there exist Möbius transformations μ_i , $1 \le i \le l-1$, such that

$$G = F^{\circ s} \circ \mu_{l-1}, \quad G = \mu_i^{-1} \circ F^{\circ s} \circ \mu_{i-1}, \quad 1 < i < l, \quad \text{and} \quad G = \mu_1^{-1} \circ F^{\circ s},$$

where s = k/l. Moreover, since

$$F^{\circ sl}=G^{\circ l}=(\mu_1^{-1}\circ F^{\circ s})^{\circ l},$$

Lemma 3.3 implies that

$$\mu_1^{-1} \in \operatorname{Aut}(F^{\circ sl}) \subseteq \operatorname{Aut}_{\infty}(F).$$

Thus,

$$G = \mu_1^{-1} \circ F^{\circ s} \in \langle \operatorname{Aut}_{\infty}(F), F \rangle.$$

3.2. Groups and semigroups related to general rational functions. Let us recall that writing a rational function F = F(z) of degree m as F = P/Q, where

$$P(z) = a_m z^m + a_{m-1} z^{m-1} + \ldots + a_1 z + a_0, \quad Q(z) = b_m z^m + b_{m-1} z^{m-1} + \ldots + b_1 z + b_0$$

are polynomials of degree m without common roots, we can identify the space of rational functions of degree m with the algebraic variety

$$\operatorname{Rat}_m = \mathbb{CP}^{2m+1} \setminus \operatorname{Res}_{m,m,z}(P,Q),$$

where $\operatorname{Res}_{m,m,z}(P,Q)$ denotes the resultant of P and Q. We will say that some statement holds for general rational functions of degree m, if it holds for all $F \in \operatorname{Rat}_m$ with exception of some proper Zariski closed subset.

Lemma 3.6. A general rational function F of degree $m \geq 2$ is simple.

Proof. First, let us observe that the Wronskian

$$W(z) = P'(z)Q(z) - P(z)Q'(z)$$

of $F \in \text{Rat}_m$ has degree 2m-2, unless F belongs to the projective hypersurface

$$U: a_m b_{m-1} - b_m a_{m-1}$$

in \mathbb{CP}^{2m+1} . Defining now a polynomial R(t) by the formula

$$R(t) = \operatorname{Res}_{2m-2,m,z}(W(z), P(z) - Q(z)t)$$

and using the well-known property of the resultant, we see that for $F \in \text{Rat}_m \setminus U$ the equality

(39)
$$R(t) = c \prod_{\zeta, W(\zeta) = 0} (P(\zeta) - Q(\zeta)t)$$

holds for some $c \in \mathbb{C}^*$. Since the set of zeroes of W(z) coincides with the set of finite critical points of F, it follows from (39) that the set of finite critical values of F coincides with the set of zeroes of R(t), and the inequality $\deg R(t) < 2m - 2$ holds if and only if infinity is a critical value of F. Therefore, if Z is the projective hypersurface Z in \mathbb{CP}^{2m+1} defined by the equality

$$Z = \text{Res}_{2m-2,2m-3,t}(R(t), R'(t)),$$

then $F \in \operatorname{Rat}_m \setminus U$ belongs to Z if and only if either some of finite critical values of F coincide, or infinity is a critical value of F. Thus, every $F \in \operatorname{Rat}_m \setminus Z \cup U$ is simple.

Lemma 3.7. For a general rational function F of degree $m \geq 3$ the group G(F) is trivial.

Proof. Let us recall that the group G(F) is non-trivial if and only if there exist $\alpha, \beta \in \operatorname{Aut}(\mathbb{CP}^1)$ such that either equality (34) holds (in which case the group G(P) is infinite), or

(40)
$$\alpha \circ F \circ \beta = z^r R(z^d),$$

for some $R \in \mathbb{C}(z) \setminus \mathbb{C}$ and integers r and d satisfying $2 \leq d \leq m$, $0 \leq r \leq d-1$ (see e.g. [39], Section 4 or [46], Section 2).

Equality (40) is equivalent to the vanishing of certain groups of coefficients in its left part, implying easily that (40) holds if and only if the coefficients of α , β , and F belong to some projective algebraic variety

$$W \subseteq \mathbb{CP}^3 \times \mathbb{CP}^{2m+1} \times \mathbb{CP}^3$$
.

hold.

The same is true for equality (34). Considering now a union of such varieties for all possible r and d and taking into account that the projection

$$p_k: (\mathbb{CP}^1)^l \times (\mathbb{CP}^1)^k \to (\mathbb{CP}^1)^k, \qquad k, l > 1,$$

is a closed map (see e.g. [31]), we conclude that the set of $F \in \operatorname{Rat}_m$ with $G(F) \neq id$ is contained in some projective algebraic variety $Z \subseteq \mathbb{CP}^{2m+1}$.

To finish the proof, we only must show that Z does not contain Rat_m . For this purpose, it is enough to show that for every $m \geq 3$ there exists a polynomial F of degree m such that G(F) = id. Let us recall that for any polynomial F of degree $m \geq 2$ such that condition (34) does not hold the group G(F) is a finite cyclic group generated by a polynomial (see e.g. [46], Section 2). On the other hand, it is easy to see that if F has the form

(41)
$$F = z^m + a_{m-2}z^{m-2} + a_{m-3}z^{m-3} + \dots + a_0,$$

then (33) holds for polynomials $\sigma = az + b$, $\mu = cz + d$ only if b = 0 and a is a root of unity. Therefore, for any polynomial of the form (41) with $a_{m-2} \neq 0$, $a_{m-3} \neq 0$ the group G(F) is trivial.

Notice that Lemma 3.7 is not true for m=2. Indeed, for every rational function F of degree two there exist $\alpha, \beta \in \operatorname{Aut}(\mathbb{CP}^1)$ such that equality (34) holds, implying that the group G(P) is non-trivial.

Since $G_0(F)$ is a subgroup of G(F), Theorem 1.2 combined with Lemma 3.6 and Lemma 3.7 implies the following result.

Theorem 3.8. For a general rational function F of degree $m \geq 4$, the equalities

$$E_0(F) = \operatorname{Aut}_{\infty}(F) = G_0(F) = id$$
 and $E(F) = C_{\infty}(F) = \langle F \rangle$

4. Proof of Theorem 1.3 and Theorem 1.4

4.1. Generalized Lattès maps, semiconjugate rational functions, and invariant curves. In this section, we recall basic definitions and results related to descriptions of semiconjugate rational functions and invariant curves for endomorphisms of $(\mathbb{CP}^1)^2$.

An orbifold \mathbb{C} on \mathbb{CP}^1 is a ramification function $\nu: \mathbb{CP}^1 \to \mathbb{N}$ which takes the value $\nu(z) = 1$ except at a finite set of points. For an orbifold \mathbb{O} , the Euler characteristic of \mathbb{O} is the number

$$\chi(0) = 2 + \sum_{z \in R} \left(\frac{1}{\nu(z)} - 1 \right),\,$$

the set of singular points of O is the set

$$c(0) = \{z_1, z_2, \dots, z_s, \dots\} = \{z \in \mathbb{CP}^1 \mid \nu(z) > 1\},\$$

and the signature of O is the set

$$\nu(0) = {\{\nu(z_1), \nu(z_2), \dots, \nu(z_s), \dots\}}.$$

Let A be a rational function, and \mathcal{O}_1 , \mathcal{O}_2 orbifolds with ramifications functions ν_1 and ν_2 . We say that $A: \mathcal{O}_1 \to \mathcal{O}_2$ is a covering map between orbifolds if for any $z \in \mathbb{CP}^1$ the equality

$$\nu_2(A(z)) = \nu_1(z) \deg_z A$$

holds. In case the weaker condition

$$\nu_2(A(z)) = \nu_1(z) GCD(\deg_z A, \nu_2(A(z)))$$

is satisfied, we say that $A: \mathcal{O}_1 \to \mathcal{O}_2$ is a minimal holomorphic map between orbifolds.

In the above terms, a Lattès map can be defined as a rational function A such that $A: \mathcal{O} \to \mathcal{O}$ is a covering self-map for some orbifold \mathcal{O} (see [29], [40]). Following [40], we say that a rational function A of degree at least two is a generalized Lattès map if there exists an orbifold \mathcal{O} , distinct from the non-ramified sphere, such that $A: \mathcal{O} \to \mathcal{O}$ is a minimal holomorphic map. Thus, A is a Lattès map if there exists an orbifold \mathcal{O} such that

(42)
$$\nu(A(z)) = \nu(z)\deg_z A, \quad z \in \mathbb{CP}^1,$$

and A is a generalized Lattès map if there exists an orbifold ${\mathcal O}$ such that

(43)
$$\nu(A(z)) = \nu(z) GCD(\deg_z A, \nu(A(z))), \quad z \in \mathbb{CP}^1.$$

Since (42) implies (43), any Lattès map is a generalized Lattès map. More generally, any special function is a generalized Lattès map (see [40]). Notice that a rational function A is a generalized Lattès map if and only if some iterate $A^{\circ d}$, $d \geq 1$, is a generalized Lattès map (see [44], Section 2.3). Notice also that if \mathcal{O} is an orbifold such that (43) holds for some rational function A, then the Euler characteristic of \mathcal{O} is necessarily non-negative. This condition is quite restrictive, and signatures of orbifolds satisfying $\chi(\mathcal{O}) \geq 0$ can be described explicitly (see e.g. [40] for more detail).

Our proof of Theorem 1.3 relies on the following result, which is a particular case of the classifications of semiconjugate rational functions (see [40], Theorem 3.2 or [44], Proposition 3.3).

Theorem 4.1. If A, X, B is a solution of (4) and A is not a generalized Lattès map, then there exists a rational function Y such that the diagram

$$\begin{array}{ccc} \mathbb{CP}^1 & \stackrel{B}{\longrightarrow} & \mathbb{CP}^1 \\ x \Big\downarrow & & \Big\downarrow x \\ \mathbb{CP}^1 & \stackrel{A}{\longrightarrow} & \mathbb{CP}^1 \\ y \Big\downarrow & & \Big\downarrow y \\ \mathbb{CP}^1 & \stackrel{B}{\longrightarrow} & \mathbb{CP}^1, \end{array}$$

commutes, and the equalities

$$Y \circ X = B^{\circ d}, \qquad X \circ Y = A^{\circ d}$$

hold.

In turn, our proof of Theorem 1.4 uses the following description of invariant curves for for endomorphisms

$$(z_1, z_2) \to (A_1(z_1), A_2(z_2)),$$

where A_1 , A_2 are rational functions of degree at least two that are not generalized Lattès maps (see [44], Theorem 1.1).

Theorem 4.2. Let A_1 , A_2 be rational functions of degree at least two that are not generalized Lattès maps, and C an irreducible algebraic curve in $(\mathbb{CP}^1)^2$ that is not a vertical or horizontal line. Then C is (A_1, A_2) -invariant if and only if there exist rational functions X_1 , X_2 , Y_1 , Y_2 , B such that:

1. The diagram

$$(\mathbb{CP}^{1})^{2} \xrightarrow{(B,B)} (\mathbb{CP}^{1})^{2}$$

$$(X_{1},X_{2}) \downarrow \qquad \qquad \downarrow (X_{1},X_{2})$$

$$(\mathbb{CP}^{1})^{2} \xrightarrow{(A_{1},A_{2})} (\mathbb{CP}^{1})^{2}$$

$$(Y_{1},Y_{2}) \downarrow \qquad \qquad \downarrow (Y_{1},Y_{2})$$

$$(\mathbb{CP}^{1})^{2} \xrightarrow{(B,B)} (\mathbb{CP}^{1})^{2}$$

commutes,

2. The equalities

$$X_1 \circ Y_1 = A_1^{\circ d}, \qquad X_2 \circ Y_2 = A_2^{\circ d},$$

$$Y_1 \circ X_1 = Y_2 \circ X_2 = B^{\circ d}$$

hold for some $d \geq 0$,

- 3. The map $t \to (X_1(t), X_2(t))$ is a parametrization of C.
- 4.2. **Proof of Theorem 1.3 and Theorem 1.4.** We start by proving the following lemma.

Lemma 4.3. Let F be a simple rational function of degree $m \ge 4$. Then F is not a generalized Lattès map.

Proof. If F is a simple rational function of degree $m \geq 4$, then the preimage of any k distinct points of \mathbb{CP}^1 under F contains at least $k(m-2) \geq 2k$ distinct points z such that $\deg_z F = 1$. In turn, this implies that if the equality

$$\nu(F(z)) = \nu(z) \text{GCD}(\deg_z F, \nu(F(z))), \quad z \in \mathbb{CP}^1$$

holds for some orbifold \mathcal{O} distinct from the non-ramified sphere, then the preimage $F^{-1}\{c(\mathcal{O})\}$ contains at least $2|c(\mathcal{O})|$ points where $\nu(z) > 1$. However, this is impossible since any such a point belongs to $c(\mathcal{O})$.

Proof of Theorem 1.3. Since F is not a generalized Lattès map by Lemma 4.3, it follows from Theorem 4.1 that there exists a rational function Y such that the equality

$$X \circ Y = F^{\circ rd}$$

holds for some $d \geq 1$. By Corollary 2.10, this implies that

$$X = F^{\circ l} \circ \mu$$

for some Möbius transformation μ and $l \geq 1$. Thus, (5) reduces to the equality

$$F^{\circ r} \circ F^{\circ l} \circ \mu = F^{\circ l} \circ \mu \circ G$$
,

and applying to this equality Theorem 2.4, we conclude that

$$G = \mu^{-1} \circ F^{\circ r} \circ \mu.$$

Lemma 4.4. Let F be a simple rational function of degree $m \geq 4$. Then $\gamma(\operatorname{Aut}(F^{\circ k})) = \operatorname{Aut}(F^{\circ k})$ for every $k \geq 1$.

Proof. Since $\operatorname{Aut}_{\infty}(F) = G_0(F)$ by Theorem 3.5, for every $\nu \in \operatorname{Aut}(F^{\circ k})$ there exists $\nu' \in \operatorname{Aut}_{\infty}(F)$ such that

$$F \circ \nu = \nu' \circ F$$
.

Moreover,

$$F^{\circ k} \circ \nu' \circ F = F^{\circ k} \circ F \circ \nu = F \circ F^{\circ k} \circ \nu = F \circ \nu \circ F^{\circ k} = \nu' \circ F \circ F^{\circ k} = \nu' \circ F^{\circ k} \circ F,$$
 implying that $\nu' \in \operatorname{Aut}(F^{\circ k})$.

Proof of Theorem 1.4. Assume that

$$(44) (F_1, F_2)^{\circ d}(C) = C, d \ge 1.$$

Then Theorem 4.2 and Theorem 1.3 imply that C is parametrized by the functions

$$X_1 = (F_1^{\circ d_1} \circ \beta)(t), \qquad X_2 = (F_2^{\circ d_2} \circ \alpha)(t),$$

where β and α are Möbius transformations such that

$$\beta^{-1} \circ F_1^{\circ d} \circ \beta = \alpha^{-1} \circ F_2^{\circ d} \circ \alpha$$

and d_1 and d_2 are non-negative integers. It is clear that without loss of generality we may assume that $\beta = z$, implying that

$$(45) F_1^{\circ d} = \alpha^{-1} \circ F_2^{\circ d} \circ \alpha = (\alpha^{-1} \circ F_2 \circ \alpha)^{\circ d}.$$

Thus, $\alpha^{-1} \circ F_2^{\circ d} \circ \alpha \in C_{\infty}(F_1)$ and hence

(46)
$$\alpha^{-1} \circ F_2 \circ \alpha = \mu \circ F_1$$

for some $\mu \in \operatorname{Aut}_{\infty}(F_1)$, by Theorem 1.2. Further, equalities (45) and (46) imply by Lemma 3.3 that $\mu \in \operatorname{Aut}(F_1^{\circ d})$. Therefore,

$$F_2 = \alpha \circ \mu \circ F_1 \circ \alpha^{-1}$$

for some $\mu \in \operatorname{Aut}(F_1^{\circ d})$, and C is parametrized by the functions

(47)
$$X_1 = F_1^{\circ d_1}(t), \qquad X_2 = \alpha \circ (\mu \circ F_1)^{\circ d_2}(t).$$

Moreover, it follows from (47) by Lemma 4.4 that there exists $\mu' \in \operatorname{Aut}(F_1^{\circ d})$ such that

(48)
$$X_1 = F_1^{\circ d_1}(t), \qquad X_2 = \alpha \circ \mu' \circ F_1^{\circ d_2}(t).$$

If $d_1 \leq d_2$, then (48) implies that C is parametrized by the functions

$$X_1 = t, \quad X_2 = (\alpha \circ \mu' \circ F_1^{\circ (d_2 - d_1)})(t), \quad \mu' \in \operatorname{Aut}(F_1^{\circ d}).$$

On the other hand, if $d_1 > d_2$, then C is parametrized by the functions

$$X_1 = F_1^{\circ (d_1 - d_2)}(t), \qquad X_2 = (\alpha \circ \mu')(t).$$

Taking into account that by Lemma 4.4,

$$F_1^{\circ (d_1 - d_2)} \circ \mu'^{-1} \circ \alpha^{-1} = \mu'' \circ F_1^{\circ (d_1 - d_2)} \circ \alpha^{-1}$$

for some $\mu'' \in Aut(F_1^{\circ d})$, we see that in this case C is also parametrized by the functions

$$X_1 = (\mu'' \circ F_1^{\circ (d_1 - d_2)} \circ \alpha^{-1})(t), \quad X_2 = t, \quad \mu'' \in \operatorname{Aut}(F_1^{\circ d}).$$

In the other direction, assume that (45) holds and C is a curve parametrized by

$$X_1 = t, \quad X_2 = (\alpha \circ \nu \circ F_1^{\circ s})(t)$$

for some $\alpha \in \operatorname{Aut}(\mathbb{CP}^1)$, $\nu \in \operatorname{Aut}(F_1^{\circ d})$, and $s \geq 1$. Then

$$F_1^{\circ d} \circ X_1(t) = X_1 \circ F_1^{\circ d}(t)$$

and

$$F_2^{\circ d}\circ X_2(t)=\alpha\circ F_1^{\circ d}\circ \nu\circ F_1^{\circ s}(t)=\alpha\circ \nu\circ F_1^{\circ d}\circ F_1^{\circ s}(t)=X_2\circ F_1^{\circ d}(t),$$

implying that (44) holds. Similarly, if C is parametrized by

$$X_1 = (\nu \circ F_1^{\circ s} \circ \alpha^{-1})(t), \quad X_2 = t,$$

then

$$F_2^{\circ d} \circ X_2(t) = X_2 \circ F_2^{\circ d}(t)$$

and

$$F_1^{\circ d} \circ X_1(t) = F_1^{\circ d} \circ \nu \circ F_1^{\circ s} \circ \alpha^{-1}(t) = \nu \circ F_1^{\circ d} \circ F^{\circ s} \circ \alpha^{-1}(t) =$$

$$= \nu \circ F_1^{\circ s} \circ \alpha^{-1} \circ \alpha \circ F_1^{\circ d}(t) \circ \alpha^{-1} = X_1 \circ F_2^{\circ d}(t).$$

4.3. Invariant curves for general pairs of rational functions. Identifying a pair of rational functions F_1, F_2 of degree m with a point of $\operatorname{Rat}_m \times \operatorname{Rat}_m$, we will say that some statement holds for general pairs of rational functions of degree m, if it holds for all $F \in \operatorname{Rat}_m \times \operatorname{Rat}_m$ with exception of some proper Zariski closed subset.

Let us recall that a variant of a conjecture of Zhang ([57]) on the existence of Zariski dense orbits for endomorphisms of varieties states that if K is a subfield of \mathbb{C} and $F_1, F_2 \in K(z)$ are rational functions of degree $m \geq 2$, then there is a point in $(\mathbb{P}^1(K))^2$ whose (F_1, F_2) -forward orbit is Zariski dense in $(\mathbb{P}^1(K))^2$. For polynomials this conjecture was proved in [28], and for rational functions in [56] and [44].

The Zhang conjecture is closely related to the problem of describing periodic curves. Indeed, if the (F_1, F_2) -orbit \mathcal{O} of a point (x_0, y_0) is not dense, then it is easy to see (see e.g. [28], Lemma 7.20) that all but finitely many elements of \mathcal{O} are contained in some (F_1, F_2) -invariant algebraic set $Z \subset (\mathbb{P}^1(\mathbb{C}))^2$. Moreover, if x_0 and y_0 are not preperiodic points of A, then Z is a finite union of curves that are not vertical or horizontal lines.

Theorem 1.4 combined with Lemma 3.6 and Lemma 3.7 implies that for general pairs of rational functions the following statement holds.

Theorem 4.5. For a general pair of rational functions F_1, F_2 of degree $m \geq 4$ the endomorphism $(z_1, z_2) \to (F_1(z_1), F_2(z_2))$ has no periodic curves distinct from vertical or horizontal lines. In particular, the (F_1, F_2) -orbit of a point (x, y) in $(\mathbb{P}^1(\mathbb{C}))^2$ is Zariski dense in $(\mathbb{P}^1(\mathbb{C}))^2$, unless x is a preperiodic point of F_1 , or y is a preperiodic point of F_2 .

Proof. By Lemma 3.6 and Lemma 3.7, a general rational function of degree $m \geq 4$ is simple with the trivial group G(F). For such F, equality (45) is equivalent to the equality

(49)
$$F_2 = \alpha \circ F_1 \circ \alpha^{-1}, \quad \alpha \in \operatorname{Aut}(\mathbb{CP}^1),$$

by Theorem 1.2. Therefore, by Theorem 1.4, for a general pair of rational functions F_1, F_2 of degree m the endomorphism $(z_1, z_2) \to (F_1(z_1), F_2(z_2))$ has no periodic

curves distinct from vertical or horizontal lines, unless F_1 and F_2 are conjugated. Since equality (49) holds if and only if the coefficients of F_1 , F_2 , and α belong to some proper algebraic variety in

$$\mathbb{CP}^{2m+1} \times \mathbb{CP}^{2m+1} \times \mathbb{CP}^3$$
.

arguing as in Lemma 3.7 we conclude that for general pairs of rational functions F_1 , F_2 the endomorphism $(z_1, z_2) \to (F_1(z_1), F_2(z_2))$ has no periodic curves distinct from vertical or horizontal lines.

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