

HOLOMORPHIC MAPS SHARING PREIMAGES OVER FINITELY GENERATED FIELDS

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ABSTRACT. Let R be a compact Riemann surface, and let $P : R \rightarrow \mathbb{P}^1(\mathbb{C})$ and $Q : R \rightarrow \mathbb{P}^1(\mathbb{C})$ be holomorphic maps. In this paper, we investigate the following problem: under what conditions do the preimages $P^{-1}(K)$ and $Q^{-1}(K)$ coincide for some infinite set K contained in $\mathbb{P}^1(\mathbf{k})$, where \mathbf{k} is a finitely generated subfield of \mathbb{C} (e.g., a number field)? Equivalently, we study holomorphic correspondences that admit an infinite completely invariant set contained in $\mathbb{P}^1(\mathbf{k})$. We show that if such a set exists, then there is a holomorphic Galois covering $\Theta : R_0 \rightarrow \mathbb{P}^1(\mathbb{C})$, where R_0 has genus zero or one, such that P and Q are “compositional left factors” of Θ . We also consider a more general equation $P^{-1}(K_1) = Q^{-1}(K_2)$, where K_1 and K_2 are infinite subsets of $\mathbb{P}^1(\mathbf{k})$.

1. INTRODUCTION

The famous five-point-theorem of Nevanlinna [16] states that if f and g are two non-constant meromorphic functions on \mathbb{C} such that the preimages $f^{-1}(\{a\})$ and $g^{-1}(\{a\})$ coincide for five distinct points a of $\mathbb{P}^1(\mathbb{C})$, then $f \equiv g$. Nevanlinna obtained this result as an application of the deep theory of the distribution of values of meromorphic functions that he developed. On the other hand, if we assume that f and g are *rational* functions on $\mathbb{P}^1(\mathbb{C})$, then an easy application of the Riemann-Hurwitz formula implies that $f \equiv g$ whenever the above equalities hold for any four distinct points of $\mathbb{P}^1(\mathbb{C})$ (see [1], [29], [31]).

The problem of describing rational functions that share preimages becomes much more subtle when one considers the preimages of *sets* rather than those of individual points. This difficulty arises already in the case of polynomials. For example, the following problem, posed in [32], remained open for nearly twenty years. Let P and Q be non-constant polynomials of the same degree such that

$$P^{-1}(\{-1, 1\}) = Q^{-1}(\{-1, 1\}).$$

Does it follow that $P = \pm Q$? This problem was affirmatively solved in [17]. Further results regarding polynomials satisfying the condition

$$(1) \quad P^{-1}(K) = Q^{-1}(K)$$

for some *compact* set K in \mathbb{C} were obtained in [4], [5], [18], [19], [20].

A description of solutions to a more general equation

$$(2) \quad P^{-1}(K_1) = Q^{-1}(K_2),$$

where P and Q are non-constant polynomials and K_1 and K_2 are arbitrary compact subsets of \mathbb{C} , not necessarily equal, was obtained in [20]. Specifically, in [20],

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condition (2) was related to the functional equation

$$(3) \quad F \circ P = G \circ Q,$$

in polynomials. It is clear that for any polynomial solution of (3) and any compact set $K \subset \mathbb{C}$, one obtains a solution of (2) by setting

$$(4) \quad K_1 = F^{-1}(K), \quad K_2 = G^{-1}(K).$$

Somewhat unexpectedly, the main result of [20] states that *all* solutions to (2) can be constructed in this way, provided that the compact set defined by either side of (2) contains at least $\text{LCM}(\deg P, \deg Q)$ points. This condition, in particular, holds for any infinite K_1 and K_2 . Since Ritt's theory of polynomial decompositions [30] provides a quite precise description of solutions to (3), we thus obtain a precise description of solutions to (2). This description can then be applied to various related problems, including the classification of polynomials that share a Julia set, the description of commuting and semiconjugate polynomials, and the study of invariant curves for polynomial endomorphisms of $\mathbb{P}^1(\mathbb{C}) \times \mathbb{P}^1(\mathbb{C})$ (see [20], [22], [28] for further details).

The methods employed in the aforementioned papers are restricted to the polynomial case, and the problem of finding solutions to (1) and (2) when P and Q are arbitrary *rational* functions and K_1 and K_2 are compact subsets of \mathbb{C} or $\mathbb{P}^1(\mathbb{C})$ remains largely unresolved. With the exception of the note [25], where the solutions of the equation

$$\beta_1^{-1}[-1, 1] = \beta_2^{-1}[-1, 1]$$

were described in the case where β_1 and β_2 are rational Belyi functions, the only related paper known to us is [2], where the broader problem of describing invariant sets of correspondences was investigated.

We recall that a *correspondence*, defined by an algebraic curve

$$(5) \quad f(x, y) = 0,$$

over \mathbb{C} , is a multivalued map that assigns to a point x the set of all points y_i satisfying $f(x, y_i) = 0$. Along with this forward map, one also defines a backward map that assigns to a point y the set of all points x_j such that $f(x_j, y) = 0$. If R is a desingularization of the curve (5), and $z \mapsto (P(z), Q(z))$ is its parametrization given by a pair of holomorphic maps on R , then the forward and backward maps of the correspondence defined by (5) are described by the multivalued functions $Q(P^{-1}(z))$ and $P(Q^{-1}(z))$, respectively. Thus, a pair of holomorphic maps P and Q can be regarded as the primary object of investigation.

The correspondences can be naturally composed, and their dynamics have been extensively studied from various perspectives (see, for instance, the recent works [3], [6], [10], [12], and the references therein). In this context, describing the sets satisfying (1) for a pair of holomorphic maps P and Q on a compact Riemann surface R is clearly equivalent to describing the completely invariant sets of the associated correspondences. In particular, the case where the maps P and Q are rational functions corresponds to the situation in which the curve (5) has genus zero.

When applied to equation (1), the main result of [2] states that if $P : R \rightarrow \mathbb{P}^1(\mathbb{C})$ and $Q : R \rightarrow \mathbb{P}^1(\mathbb{C})$ are holomorphic maps on a compact Riemann surface R satisfying (1) for *infinitely many finite sets* K , then there exists a rational function

F such that

$$(6) \quad F \circ P = F \circ Q.$$

Observe that if (6) holds, then (1) is satisfied for every set of the form $K = F^{-1}(\widehat{K})$, where $\widehat{K} \subset \mathbb{P}^1(\mathbb{C})$, and, in particular, for every fiber of F . Consequently, all irreducible completely invariant sets of the associated correspondence are finite. Notice also the (6) obviously implies that

$$(7) \quad \deg P = \deg Q.$$

In this paper, we study equation (1), assuming that $P : R \rightarrow \mathbb{P}^1(\mathbb{C})$ and $Q : R \rightarrow \mathbb{P}^1(\mathbb{C})$ are holomorphic maps on a compact Riemann surface R , but replacing the condition that K is compact with the assumption that K is an infinite set contained in $\mathbb{P}^1(\mathbf{k})$, where \mathbf{k} is a finitely generated subfield of \mathbb{C} (e.g., a number field). In other words, we investigate correspondences that have an infinite, completely invariant set contained in $\mathbb{P}^1(\mathbf{k})$. We also study equation (2) under the assumption that K_1 and K_2 are infinite sets contained in $\mathbb{P}^1(\mathbf{k})$. Notice that (2) can also be interpreted in terms of correspondences: it expresses the condition that K_1 is mapped to K_2 by the forward map, while K_2 is mapped to K_1 by the backward map of the correspondence associated with the pair P, Q .

We remark that describing solutions of (2), where K_1 and K_2 are subsets of \mathbb{C} satisfying certain restrictions, reduces to describing solutions for which the maps P and Q have no non-trivial common compositional right factor in the following sense: the equalities

$$(8) \quad P = \tilde{P} \circ W, \quad Q = \tilde{Q} \circ W,$$

where

$$\tilde{P} : \tilde{R} \rightarrow \mathbb{P}^1(\mathbb{C}), \quad \tilde{Q} : \tilde{R} \rightarrow \mathbb{P}^1(\mathbb{C}), \quad \text{and} \quad W : R \rightarrow \tilde{R},$$

are holomorphic maps between compact Riemann surfaces, imply that $\deg W = 1$. Indeed, it follows from (2) and (8) that

$$W^{-1}(\tilde{P}^{-1}(K_1)) = W^{-1}(\tilde{Q}^{-1}(K_2)),$$

which implies

$$\tilde{P}^{-1}(K_1) = \tilde{Q}^{-1}(K_2).$$

Thus, any solution P, Q of (2) reduces to a solution \tilde{P}, \tilde{Q} , where \tilde{P} and \tilde{Q} have no non-trivial common compositional right factor, and we will primarily focus on such solutions.

In connection with the problem under consideration, we mention the recent note [27], where it was shown, using properties of height functions, that if P and Q are non-constant rational functions over \mathbb{C} , and K is an infinite subset of $\mathbb{P}^1(\mathbf{k})$, where \mathbf{k} a finitely generated subfield of \mathbb{C} , then the condition

$$P^{-1}(K) \subseteq Q^{-1}(K)$$

implies

$$\deg Q \geq \deg P.$$

In particular, equality (1) implies equality (7). Our approach is different; however, it does not immediately yield (7), even in the case when the holomorphic maps P and Q are rational functions on \mathbb{CP}^1 . Thus, the result of [27] is complementary to ours.

Our main result concerning equation (2) is the following statement.

Theorem 1.1. *Let R be a compact Riemann surface, and let $P : R \rightarrow \mathbb{P}^1(\mathbb{C})$ and $Q : R \rightarrow \mathbb{P}^1(\mathbb{C})$ be holomorphic maps having no non-trivial common compositional right factor. Assume that the equality*

$$P^{-1}(K_1) = Q^{-1}(K_2)$$

holds for some infinite sets K_1 and K_2 contained in $\mathbb{P}^1(\mathbf{k})$, where \mathbf{k} is a finitely generated subfield of \mathbb{C} . Then, there exist compact Riemann surfaces R_1 and R_2 of genus at most one and holomorphic Galois coverings $\Theta_1 : R_1 \rightarrow \mathbb{P}^1(\mathbb{C})$ and $\Theta_2 : R_2 \rightarrow \mathbb{P}^1(\mathbb{C})$ such that the equalities

$$\Theta_1 = P \circ U, \quad \Theta_2 = Q \circ V$$

hold for some holomorphic maps $U : R_1 \rightarrow R$ and $V : R_2 \rightarrow R$.

Theorem 1.1 implies, in particular, that $g(R) \leq 1$. More importantly, since Galois coverings $\Theta : R \rightarrow \mathbb{P}^1(\mathbb{C})$ with $g(R) \leq 1$ are easy to describe, Theorem 1.1 imposes strict constraints on P and Q . In particular, they can exhibit only very limited branching (see Lemma 2.1 below).

The simplest example of holomorphic maps P and Q satisfying the conclusion of Theorem 1.2 is any pair of rational Galois coverings. Moreover, we show that for any such pair, one can construct sets K_1 and K_2 contained in $\mathbb{P}^1(\mathbf{k})$, where \mathbf{k} is a finitely generated subfield of \mathbb{C} , such that (2) holds. These examples demonstrate the existence of pairs P, Q for which (2) holds, but (3) does not. Indeed, denoting the groups of covering transformations of P and Q by Γ_P and Γ_Q , we see that if (3) holds, then the rational function defined by any part of this equality must be invariant under the group $\Gamma = \langle \Gamma_P, \Gamma_Q \rangle$. This is possible only if Γ is finite, which is not necessarily the case. On the other hand, if Γ is finite, then such solutions are indeed obtained from the convenient formulas (3) and (4) (see Section 4).

Our second main result is the following specification of Theorem 1.2 in the case $K_1 = K_2$.

Theorem 1.2. *Let R be a compact Riemann surface, and let $P : R \rightarrow \mathbb{P}^1(\mathbb{C})$ and $Q : R \rightarrow \mathbb{P}^1(\mathbb{C})$ be holomorphic maps having no non-trivial common compositional right factor. Assume that the equality*

$$P^{-1}(K) = Q^{-1}(K)$$

holds for an infinite set K contained in $\mathbb{P}^1(\mathbf{k})$, where \mathbf{k} is a finitely generated subfield of \mathbb{C} . Then, there exist a compact Riemann surface R_0 of genus at most one and a holomorphic Galois covering $\Theta : R_0 \rightarrow \mathbb{P}^1(\mathbb{C})$ such that the equalities

$$\Theta = P \circ U, \quad \Theta = Q \circ V$$

hold for some holomorphic maps $U : R_0 \rightarrow R$ and $V : R_0 \rightarrow R$.

For Theorem 1.2, the simplest examples of maps P and Q that satisfy its conclusion are any rational Galois coverings P and Q such that

$$Q = P \circ \mu$$

for some $\mu \in \text{Aut}(\mathbb{P}^1(\mathbb{C}))$. Moreover, for such P and Q , one can construct a set K contained in a finitely generated field such that (1) holds, and these examples generally do not reduce to (6) (see Section 4).

For illustration, we consider the following simple example. Let

$$P = z^2, \quad Q = (z+1)^2 = P \circ (z+1),$$

and let K be the set of squares of integers. Then we have

$$P^{-1}(K) = Q^{-1}(K) = \mathbb{Z}.$$

However, the equality (6) is impossible, because, if it held, the rational function defined by any part of this equality would be invariant under the transformation $z \mapsto z+1$.

This paper is organized as follows. In the second section, we begin by recalling several definitions and results related to fiber products and normalizations, and then prove two results concerning equation (1), extending Theorem 1.1. In the third section, we prove several results related to equation (2) and establish Theorem 1.2, again in an extended form. Finally, in the fourth section, we construct examples of solutions to (1) and (2), illustrating Theorems 1.1 and 1.2.

2. HOLOMORPHIC MAPS SHARING PREIMAGES OF DIFFERENT SETS

2.1. Fiber products and normalizations. In this subsection, we review several definitions and results concerning fiber products and normalizations.

Let $V_i : E \rightarrow R_i$, $1 \leq i \leq k$, where $k \geq 2$, be holomorphic maps between compact Riemann surfaces. We say that the maps V_i , $1 \leq i \leq k$, have no non-trivial common compositional right factor if the equalities

$$V_i = \tilde{V}_i \circ W, \quad 1 \leq i \leq k,$$

where $W : E \rightarrow \tilde{E}$ and $\tilde{V}_i : \tilde{E} \rightarrow R_i$, $1 \leq i \leq k$, are holomorphic maps between compact Riemann surfaces imply that $\deg W = 1$. Denoting by $\mathcal{M}(S)$ the field of meromorphic functions on a compact Riemann surface S , this condition can be restated as the requirement

$$\mathcal{M}(E) = V_1^*(\mathcal{M}(R_1)) \cdot V_2^*(\mathcal{M}(R_2)) \cdot \dots \cdot V_k^*(\mathcal{M}(R_k)),$$

meaning that the field $\mathcal{M}(R)$ is the compositum of its subfields $V_i^*(\mathcal{M}(E_i))$, $1 \leq i \leq k$.

Let us recall that if $P_i : R_i \rightarrow \mathbb{P}^1(\mathbb{C})$, $1 \leq i \leq k$, are holomorphic maps between compact Riemann surfaces, then the fiber product of P_i , $1 \leq i \leq k$, is a collection

$$P_1 \times P_2 \times \dots \times P_k = \bigcup_{j=1}^n \{E_j, V_{j1}, V_{j2}, \dots, V_{jk}\},$$

where $n = n(P_1, P_2, \dots, P_k)$ is an integer positive number and E_j , $1 \leq j \leq n$, are compact Riemann surfaces provided with holomorphic maps

$$V_{ji} : E_j \rightarrow R_i, \quad 1 \leq i \leq k,$$

such that

$$P_1 \circ V_{j1} = P_2 \circ V_{j2} = \dots = P_k \circ V_{jk}, \quad 1 \leq j \leq n,$$

and for any holomorphic maps $T_i : E \rightarrow R_i$, $1 \leq i \leq k$, between compact Riemann surfaces satisfying

$$(9) \quad P_1 \circ T_1 = P_2 \circ T_2 = \dots = P_k \circ T_k$$

there exist a uniquely defined index j , $1 \leq j \leq n$, and a holomorphic map $W : E \rightarrow E_j$ such that

$$(10) \quad T_i = V_{ji} \circ W, \quad 1 \leq i \leq k.$$

Notice that the definition implies that for every j , $1 \leq j \leq n$, the maps $V_{ji} : E_j \rightarrow R_i$, $1 \leq i \leq n$, have no non-trivial common compositional right factor. In the other direction, if $T_i : E \rightarrow R_i$, $1 \leq i \leq k$, are holomorphic maps between compact Riemann surfaces satisfying (9) and having no non-trivial common compositional right factor, then (10) holds for some j , $1 \leq j \leq n$, and isomorphism $W : E \rightarrow E_j$. We will call each collection $\{E_j, V_{j1}, V_{j2}, \dots, V_{jk}\}$, $1 \leq j \leq n$, a component of the fiber product $P_1 \times P_2 \times \dots \times P_k$. We call the genus of a component the genus of E_j , $1 \leq j \leq n$.

The fiber product is defined in a unique way up to natural isomorphisms, and can be described by the following algebro-geometric construction. Let us consider the algebraic variety

$$L = \{(x_1, x_2, \dots, x_k) \in R_1 \times R_2 \times \dots \times R_k \mid P_1(x_1) = P_k(x_2) = \dots = P_k(x_k)\}.$$

Let us denote by L_j , $1 \leq j \leq n$, irreducible components of L , by E_j , $1 \leq j \leq n$, their desingularizations, and by

$$\pi_j : E_j \rightarrow L_j, \quad 1 \leq j \leq n,$$

the desingularization maps. Then the compositions

$$\pi_i \circ \pi_j : E_j \rightarrow R_i, \quad 1 \leq i \leq k, \quad 1 \leq j \leq n,$$

extend to holomorphic maps

$$V_j : E_j \rightarrow R_i, \quad 1 \leq i \leq k, \quad 1 \leq j \leq n,$$

and the collection $\bigcup_{j=1}^n \{E_j, V_{j1}, V_{j2}, \dots, V_{jk}\}$ is the fiber product $P_1 \times P_2 \times \dots \times P_k$.

Let R be a compact Riemann surface and $Q : R \rightarrow \mathbb{P}^1(\mathbb{C})$ a holomorphic map. A *normalization* of Q is defined as a compact Riemann surface N_Q together with a holomorphic Galois covering of the minimal degree $\tilde{Q} : N_Q \rightarrow \mathbb{P}^1(\mathbb{C})$ such that $\tilde{Q} = Q \circ H$ for some holomorphic map $H : N_Q \rightarrow R$. The normalization is characterized by the property that the field extension

$$\mathcal{M}(N_Q)/\tilde{Q}^*(\mathcal{M}(\mathbb{P}^1(\mathbb{C})))$$

is isomorphic to the Galois closure of the extension

$$\mathcal{M}(R)/Q^*(\mathcal{M}(\mathbb{P}^1(\mathbb{C}))).$$

We recall that an *orbifold* \mathcal{O} on $\mathbb{P}^1(\mathbb{C})$ is a ramification function $\nu : \mathbb{P}^1(\mathbb{C}) \rightarrow \mathbb{N}$, which takes the value $\nu(z) = 1$ except at finite points, and the *Euler characteristic* of \mathcal{O} is the number

$$\chi(\mathcal{O}) = 2 + \sum_{z \in \mathbb{P}^1(\mathbb{C})} \left(\frac{1}{\nu(z)} - 1 \right).$$

The *signature* $\nu(\mathcal{O})$ of \mathcal{O} is defined as the list of all values $\nu(z)$, where z ranges over the points of \mathcal{O} with $\nu(z) > 1$, and each value is included as many times as it occurs. With each holomorphic map $Q : R \rightarrow \mathbb{P}^1(\mathbb{C})$ between compact Riemann surfaces, one can associate its *ramification orbifold* \mathcal{O}^Q by setting $\nu^Q(z)$ equal to the least common multiple of the local degrees of Q at the points of the preimage $Q^{-1}\{z\}$.

The following statement provides different characterizations of the condition appeared in Theorem 1.1 and Theorem 1.2.

Lemma 2.1. *Let R be a compact Riemann surface, and $P : R \rightarrow \mathbb{P}^1(\mathbb{C})$ a holomorphic map of degree at least two. Then the following conditions are equivalent.*

- (i) *There exist a compact Riemann surfaces R_0 of genus at most one and a holomorphic Galois covering $\Theta : R_0 \rightarrow \mathbb{P}^1(\mathbb{C})$ such that $\Theta = P \circ V$ for some holomorphic map $V : R_0 \rightarrow R$.*
- (ii) *The inequality $g(N_P) \leq 1$ holds.*
- (iii) *The inequality $\chi(\mathcal{O}^P) \geq 0$ holds.*
- (iv) *The signature $\nu(\mathcal{O}^P)$ belongs to the list*

$$\{2, 2, 2, 2\}, \quad \{3, 3, 3\}, \quad \{2, 4, 4\}, \quad \{2, 3, 6\},$$

$$\{l, l\}, \quad l \geq 2, \quad \{2, 2, l\}, \quad l \geq 2, \quad \{2, 3, 3\}, \quad \{2, 3, 4\}, \quad \{2, 3, 5\}.$$

Proof. The equivalency $i) \Leftrightarrow ii)$ follows from the minimality of the Galois covering \tilde{Q} and the fact that the genus does not increase under holomorphic maps. The equivalency $ii) \Leftrightarrow iii)$ follows from the Riemann-Hurwitz formula (see e.g. [24], Lemma 3.1). Finally, the equivalency $iii) \Leftrightarrow iv)$ is obtained by a direct calculation (see e.g. [7], IV.9.3, IV.9.5). \square

The normalization of a holomorphic map $P : R \rightarrow \mathbb{P}^1(\mathbb{C})$ of degree d can be described in terms of the fiber product of P with itself d times as follows (see [9], §I.G, or [24], Section 2.2). Consider the algebraic variety

$$L^P = \{(x_1, x_2, \dots, x_k) \in R \times R \times \cdots \times R \mid P(x_1) = P(x_2) = \cdots = P(x_k)\}.$$

Let \hat{L}^P be a variety obtained from L^P by removing the components contained in varieties of the form

$$x_{j_1} = x_{j_2}, \quad 1 \leq j_1, j_2 \leq d, \quad j_1 \neq j_2,$$

and let N be an irreducible component of \hat{L}^P . Further, let $\pi' : N_P \rightarrow N$ be the desingularization map, and $\tilde{P} : N_P \rightarrow \mathbb{P}^1(\mathbb{C})$ a holomorphic map induced by the composition

$$N_P \xrightarrow{\pi'} N \xrightarrow{\pi_i} \mathbb{P}^1(\mathbb{C}) \xrightarrow{X} \mathbb{P}^1(\mathbb{C}),$$

where π_i is the projection to any coordinate.

In this notation, the following statement holds.

Theorem 2.2. *The map $\tilde{P} : N_P \rightarrow R$ is the normalization of P .* \square

2.2. Proof of Theorem 1.1 and its extensions. We deduce Theorem 1.1 from the following result, which follows from the Faltings theorem.

Theorem 2.3. *Let R be a compact Riemann surface, and let $P_i : R \rightarrow \mathbb{P}^1(\mathbb{C})$, $1 \leq i \leq k$, where $k \geq 2$, be holomorphic maps having no non-trivial common compositional right factor. Assume that for some infinite set K contained in $\mathbb{P}^1(\mathbf{k})$, where \mathbf{k} is a finitely generated subfield of \mathbb{C} , the inclusions*

$$P_i(P_k^{-1}(K)) \subset \mathbb{P}^1(\mathbf{k}), \quad 1 \leq i \leq k - 1,$$

hold. Then $g(N_{P_k}) \leq 1$.

Proof. The image of R under the map

$$(11) \quad \theta : z \rightarrow (P_1(z), P_2(z), \dots, P_k(z))$$

defines an irreducible algebraic curve X in the space $(\mathbb{P}^1(\mathbb{C}))^k$ with coordinates (x_1, x_2, \dots, x_k) . Moreover, since P_i , $1 \leq i \leq k$, have no non-trivial common compositional right factor, R is the desingularization of X .

Setting $d = \deg P_k$, let us consider the product X^d in the space $(\mathbb{P}^1(\mathbb{C}))^{kd}$ with coordinates

$$(x_1^1, x_2^1, \dots, x_k^1, x_1^2, x_2^2, \dots, x_k^2, \dots, x_1^d, x_2^d, \dots, x_k^d)$$

and define the algebraic variety E as the intersection of X^d and the algebraic variety

$$x_k^1 = x_k^2 = \dots = x_k^d$$

in $(\mathbb{P}^1(\mathbb{C}))^{kd}$. By adjoining, if necessary, finitely many coefficients of the equations defining E in $(\mathbb{P}^1(\mathbb{C}))^{kd}$ to \mathbf{k} , we may assume that E is defined over \mathbf{k} .

Since the map (11) is an isomorphism off a finite set, the map

$$(12) \quad (\theta, \theta, \dots, \theta) : R^d \rightarrow X^d \subset (\mathbb{P}^1(\mathbb{C}))^{kd}$$

induces an isomorphism between components of the curve $\hat{L}^{P_k} \subset R^d$ defined above and components of the curve $E \subset X^n$. On the other hand, it is easy to see that the condition of the theorem implies that the image of \hat{L}^{P_k} under (12) has infinitely many points over \mathbf{k} . Therefore, at least one of irreducible components of this image also has infinitely many points over \mathbf{k} . Now the statement of the theorem follows from Theorem 2.2 and the Faltings theorem, which states that if an irreducible algebraic curve C defined over a finitely generated field \mathbf{k} of characteristic zero has infinitely many \mathbf{k} -points, then $g(C) \leq 1$ ([8]). \square

The proof of Theorem 2.3 given above is a modification of the proof of one of the implications of the main result in [26], which characterizes algebraic curves $X : F(x, y) = 0$ defined over $\overline{\mathbb{Q}}$ that satisfy the following property: there exist a number field \mathbf{k} and an infinite set $S \subset \mathbf{k}$ such that, for every $y \in S$, the roots of the polynomial $F(x, y)$ belong to \mathbf{k} . The connection between these problems becomes evident upon noting that if a curve X as above is parametrized by holomorphic maps $\varphi : R \rightarrow \mathbb{P}^1(\mathbb{C})$ and $\psi : R \rightarrow \mathbb{P}^1(\mathbb{C})$, then $\varphi(\psi^{-1}(S)) \subset \mathbf{k}$.

Theorem 2.3 implies the following result, which extends Theorem 1.1.

Theorem 2.4. *Let R be a compact Riemann surface, and let $P_i : R \rightarrow \mathbb{P}^1(\mathbb{C})$, $1 \leq i \leq k$, where $k \geq 2$, be holomorphic maps having no non-trivial common compositional right factor. Assume that the equality*

$$(13) \quad P_1^{-1}(K_1) = P_2^{-1}(K_2) = \dots = P_k^{-1}(K_k)$$

holds for some infinite sets K_i , $1 \leq i \leq l$, contained in $\mathbb{P}^1(\mathbf{k})$, where \mathbf{k} is a finitely generated subfield of \mathbb{C} . Then $g(N_{P_i}) \leq 1$, $1 \leq i \leq k$.

Proof. Since equality (13) implies that for every j , $1 \leq j \leq k$, the inclusion

$$P_i(P_j^{-1}(K_j)) \subset \mathbb{P}^1(\mathbf{k}), \quad 1 \leq i \leq k, \quad i \neq j,$$

holds, the statement of the theorem follows from Theorem 2.3. \square

3. HOLOMORPHIC MAPS SHARING PREIMAGES OF THE SAME SET

In this section, using fiber products, we derive refined versions of Theorem 2.4 under the additional assumption that the equality

$$(14) \quad K_1 = K_2 = \cdots = K_k$$

holds in (13).

Lemma 3.1. *Let R be a compact Riemann surface, and let $P_i : R \rightarrow \mathbb{P}^1(\mathbb{C})$, $1 \leq i \leq k$, where $k \geq 2$, be holomorphic maps having no non-trivial common compositional right factor. Assume that the equality*

$$(15) \quad K' = P_1^{-1}(K) = P_2^{-1}(K) = \cdots = P_k^{-1}(K)$$

holds for some subsets $K \subset \mathbb{P}^1(\mathbb{C})$ and $K' \subset R$. Then for any holomorphic maps $V_i : E \rightarrow R$, $1 \leq i \leq k$, between compact Riemann surfaces satisfying

$$(16) \quad P_1 \circ V_1 = P_2 \circ V_2 = \cdots = P_k \circ V_k$$

the equalities

$$(17) \quad V_1^{-1}(K') = V_2^{-1}(K') = \cdots = V_k^{-1}(K')$$

and

$$(18) \quad (P_{i_1} \circ V_{j_1})^{-1}(K) = (P_{i_2} \circ V_{j_2})^{-1}(K), \quad 1 \leq i_1, i_2, j_1, j_2 \leq k,$$

hold.

Proof. Setting

$$F = P_1 \circ V_1 = P_2 \circ V_2 = \cdots = P_k \circ V_k,$$

we see that

$$V_i^{-1}(K') = (P_i \circ V_i)^{-1}(K) = F^{-1}(K), \quad 1 \leq i \leq k.$$

Equality (18) follows now from (15) and (17). \square

Lemma 3.2. *Let R_1 , R_2 , R_3 be compact Riemann surfaces, $V_i : R_1 \rightarrow R_2$, $1 \leq i \leq n$, holomorphic maps having no non-trivial common compositional right factor, and $U_j : R_2 \rightarrow R_3$, $1 \leq j \leq m$, holomorphic maps having no non-trivial common compositional right factor. Then the holomorphic maps $U_j \circ V_i : R_1 \rightarrow R_3$, $1 \leq i \leq n$, $1 \leq j \leq m$, also have no non-trivial common compositional right factor.*

Proof. Let K be the compositum of the fields

$$(U_j \circ V_i)^*(\mathcal{M}(R_3)), \quad 1 \leq i \leq n, \quad 1 \leq j \leq m,$$

and K_j , $1 \leq j \leq k$, the compositum

$$K_j = (U_1 \circ V_j)^*(\mathcal{M}(R_3)) \cdot (U_2 \circ V_j)^*(\mathcal{M}(R_3)) \cdot \dots \cdot (U_m \circ V_j)^*(\mathcal{M}(R_3)), \quad 1 \leq j \leq n.$$

By the condition,

$$U_1^*(\mathcal{M}(R_3)) \cdot U_2^*(\mathcal{M}(R_3)) \cdot \dots \cdot U_m^*(\mathcal{M}(R_3)) = \mathcal{M}(R_2),$$

implying that

$$K_j = V_j^*(\mathcal{M}(R_2)), \quad 1 \leq j \leq n.$$

Since K contains K_j , $1 \leq j \leq n$, and

$$V_1^*(\mathcal{M}(R_2)) \cdot V_2^*(\mathcal{M}(R_2)) \cdot \dots \cdot V_n^*(\mathcal{M}(R_2)) = \mathcal{M}(R_3)$$

by the condition, this implies that $K = \mathcal{M}(R_3)$. \square

Since for any holomorphic maps $P : R \rightarrow \mathbb{P}^1(\mathbb{C})$ and $V : \tilde{R} \rightarrow R$ between compact Riemann surfaces the inequality $g(N_P) \leq g(N_{P \circ V})$ holds, the following result may be viewed as a stronger version of Theorem 2.4 in the case where (14) holds.

Theorem 3.3. *Let R be a compact Riemann surface, and let $P_i : R \rightarrow \mathbb{P}^1(\mathbb{C})$, $1 \leq i \leq k$, where $k \geq 2$, be holomorphic maps having no non-trivial common compositional right factor. Assume that the equality*

$$P_1^{-1}(K) = P_2^{-1}(K) = \cdots = P_k^{-1}(K)$$

holds for some infinite set K contained in $\mathbb{P}^1(\mathbf{k})$, where \mathbf{k} is a finitely generated subfield of \mathbb{C} . Then for every component $\{E, V_1, V_2, \dots, V_k\}$ of the fiber product $P_1 \times P_2 \times \cdots \times P_k$ the inequalities

$$g(N_{P_i \circ V_j}) \leq 1, \quad 1 \leq i, j \leq k,$$

hold.

Proof. Since P_i , $1 \leq i \leq k$, have no non-trivial common compositional right factor by assumption and the same holds for the maps V_i , $1 \leq i \leq k$, as they form a component of a fiber product, it follows from Lemma 3.2 that the holomorphic maps $P_i \circ V_j$, $1 \leq i, j \leq k$, also have no non-trivial common compositional right factor. Moreover, by Lemma 3.1, the equalities (16) imply the equalities (18). Applying now Theorem 2.3 to the maps $P_i \circ V_j$, $1 \leq i, j \leq k$, we conclude that $g(N_{P_i \circ V_j}) \leq 1$, $1 \leq i, j \leq k$. \square

For holomorphic maps P_i , $1 \leq i \leq k$, of degree d_i , $1 \leq i \leq k$, let us consider a component $\{E, V_{i,j}, 1 \leq i \leq k, 1 \leq j \leq d_i\}$ of the fiber product

$$\Pi = P_1^{\times d_1} \times P_2^{\times d_2} \times \cdots \times P_k^{\times d_k}$$

such that the corresponding irreducible component in the variety

$$\begin{aligned} P_1(x_{11}) &= P_1(x_{12}) = \cdots = P_1(x_{1d_1}) = P_2(x_{21}) = \cdots = P_2(x_{2d_2}) = \cdots \\ &\cdots = P_k(x_{k1}) = \cdots = P_k(x_{kd_k}) \end{aligned}$$

is not contained in a variety of the form

$$x_{i,j_1} = x_{i,j_2}, \quad 1 \leq i \leq k, \quad 1 \leq j_1, j_2 \leq d_i, \quad j_1 \neq j_2,$$

and set

$$(19) \quad F = P_i \circ V_{i,j}, \quad 1 \leq i \leq k, \quad 1 \leq j \leq d_i.$$

Theorem 3.4. *Let R be a compact Riemann surface, and let $P_i : R \rightarrow \mathbb{P}^1(\mathbb{C})$, $1 \leq i \leq k$, where $k \geq 2$, be holomorphic maps having no non-trivial common compositional right factor. Assume that the equality*

$$P_1^{-1}(K) = P_2^{-1}(K) = \cdots = P_k^{-1}(K)$$

holds for some infinite set K contained in $\mathbb{P}^1(\mathbf{k})$, where \mathbf{k} is a finitely generated subfield of \mathbb{C} . Then $g(E) \leq 1$ and F is a Galois covering.

Proof. Since the functions P_i , $1 \leq i \leq k$, where each function is taken d_i times still have no non-trivial common compositional right factor, $g(E)$ is one or zero by Theorem 3.3. Thus, we only must prove that F is a Galois covering. Let

$\{\tilde{E}, U_1, U_2, \dots, U_k\}$ be a component of the fiber product $\tilde{P}_1 \times \tilde{P}_2 \times \dots \times \tilde{P}_k$, where \tilde{P}_i is a normalization of P_i , $1 \leq i \leq k$, and

$$\tilde{F} = \tilde{P}_1 \circ U_1 = \tilde{P}_2 \circ U_2 = \dots = \tilde{P}_k \circ U_k.$$

Notice that since \tilde{P}_i , $1 \leq i \leq k$, are Galois coverings, all such components are isomorphic and \tilde{F} is a Galois covering. Thus, to prove the theorem, it is enough to prove that E and \tilde{E} are isomorphic and the equality $F = \tilde{F} \circ \mu$ holds for some isomorphism $\mu : E \rightarrow \tilde{E}$.

For each fixed i , $1 \leq i \leq k$, the equalities (19) imply, by Theorem 2.2, that

$$(20) \quad V_{i,j} = \tilde{U}_{i,j} \circ W_i, \quad 1 \leq j \leq d_i,$$

where $\{\tilde{E}_i, \tilde{U}_{i,j}, 1 \leq j \leq d_i\}$ is a component of $P_i^{\times d_i}$, and $W_i : \tilde{E} \rightarrow \tilde{E}_i$ is a holomorphic map. Substituting (20) into (19) for all i , $1 \leq i \leq k$, we obtain

$$F = \tilde{P}_1 \circ W_1 = \tilde{P}_2 \circ W_2 = \dots = \tilde{P}_k \circ W_k.$$

Moreover, W_1, W_2, \dots, W_k have no nontrivial common compositional right factor; otherwise, (20) would imply that $V_{i,j}$, $1 \leq i \leq k$, $1 \leq j \leq d_i$, have such a factor. Thus, $\{E, W_1, W_2, \dots, W_k\}$ is a component of the fiber product $\tilde{P}_1 \times \tilde{P}_2 \times \dots \times \tilde{P}_k$ and $F = \tilde{F} \circ \mu$ for some isomorphism $\mu : E \rightarrow \tilde{E}$. \square

Theorem 3.4 implies the following statement extending Theorem 1.2 from the introduction.

Theorem 3.5. *Let R be a compact Riemann surface, and let $P_i : R \rightarrow \mathbb{P}^1(\mathbb{C})$, $1 \leq i \leq k$, where $k \geq 2$, be holomorphic maps having no non-trivial common compositional right factor. Assume that the equality*

$$P_1^{-1}(K) = P_2^{-1}(K) = \dots = P_k^{-1}(K)$$

holds for some infinite set K contained in $\mathbb{P}^1(\mathbf{k})$, where \mathbf{k} is a finitely generated subfield of \mathbb{C} . Then there exist a compact Riemann surface R_0 and a holomorphic Galois covering $\Theta : R_0 \rightarrow \mathbb{P}^1(\mathbb{C})$ such that the equalities

$$\Theta = P_i \circ U_i, \quad 1 \leq i \leq k,$$

hold for some holomorphic maps $U_i : R_0 \rightarrow R$, $1 \leq i \leq k$.

Proof. Since, by Theorem 3.4, the function F in (19) is a Galois covering, we can set $\Theta = F$ and $U_i = V_{i,1}$ for $1 \leq i \leq k$. \square

4. EXAMPLES OF HOLOMORPHIC MAPS SHARING PREIMAGES

In this section, we construct examples of solutions to equations (1) and (2) illustrating Theorems 1.1 and 1.2.

Theorem 4.1. *Let R be a compact Riemann surface of genus zero or one, and let $P_i : R \rightarrow \mathbb{P}^1(\mathbb{C})$, $1 \leq i \leq k$, where $k \geq 2$, be holomorphic Galois coverings. Then there exist a finitely generated subfield \mathbf{k} of \mathbb{C} and infinite sets $K_i \subset \mathbb{P}^1(\mathbf{k})$, $1 \leq i \leq k$, such that the equality*

$$P_1^{-1}(K_1) = P_2^{-1}(K_2) = \dots = P_k^{-1}(K_k)$$

holds.

Proof. Assume first that $g(R) = 0$, that is, $R = \mathbb{P}^1(\mathbb{C})$. Let us denote by Γ_i , $1 \leq i \leq k$, the group of covering transformations of P_i , $1 \leq i \leq k$, and consider the group Γ generated by Γ_i , $1 \leq i \leq k$, and some Möbius transformation of infinite order μ , for example, $\mu(z) = z + 1$. Let us take now an arbitrary point $z_0 \in \mathbb{P}^1(\mathbb{C})$, and consider its orbit

$$(21) \quad S = \bigcup_{\sigma \in \Gamma} \sigma(z_0)$$

under the action of Γ . Notice that since $\mu \in \Gamma$, the set S is infinite.

Since for every $z \in S$ the set S contains the orbit of z under the action of each group Γ_i , $1 \leq i \leq k$, there exist infinite sets $K_i \subset \mathbb{P}^1(\mathbb{C})$, $1 \leq i \leq k$, such that $S = P_i^{-1}(K_i)$, $1 \leq i \leq k$. Moreover, it is clear that $S \subset \mathbb{P}^1(\mathbf{k}')$, where \mathbf{k}' is generated over \mathbb{Q} by the coefficients of elements of Γ , and $K_i \subset \mathbb{P}^1(\mathbf{k})$, $1 \leq i \leq k$, where \mathbf{k} is generated over \mathbf{k}' by the coefficients of the rational functions P_i , $1 \leq i \leq k$. Thus, \mathbf{k} is finitely generated. Notice that since any finitely generated algebraic extension of \mathbb{Q} is a number field, if z_0 and the coefficients of P_i , $1 \leq i \leq k$, are algebraic numbers, then the field \mathbf{k} is a number field.

In case $g(R) = 1$, the proof is modified as follows. Let us consider some holomorphic maps φ and ψ on R having no non-trivial common compositional right factor, and a plane curve $X : f(x, y) = 0$ parametrized by $\theta : z \rightarrow (\varphi(z), \psi(z))$. Since φ and ψ have no non-trivial common compositional right factor, R is the desingularization of X , implying that for every holomorphic map $F : R \rightarrow \mathbb{P}^1(\mathbb{C})$ there exists a rational function $\widehat{F}(x, y) \in \mathbb{C}(x, y)$ such that

$$\widehat{F} \circ \theta = F$$

for all but finitely many points $z \in R$. Similarly, for every $\sigma \in \text{Aut}(R)$ there exists a rational function $\widehat{\sigma}(x, y) \in \mathbb{C}(x, y)$ such that

$$\widehat{\sigma} \circ \theta = \theta \circ \sigma$$

for all but finitely many points $z \in R$.

Let $\Gamma \subset \text{Aut}(R)$ be the group generated by the groups Γ_{P_i} , $1 \leq i \leq k$, of covering transformations of P_i , together with a shift μ of infinite order on R , and let $\widehat{\Gamma} \subset \text{Aut}(X)$ be the group consisting of all elements $\widehat{\sigma}$, where $\sigma \in \Gamma$. Let us take an arbitrary point $(x_0, y_0) \in X(\mathbb{C})$, and set

$$(22) \quad \widehat{S} = \bigcup_{\widehat{\sigma} \in \widehat{\Gamma}} \widehat{\sigma}(x_0, y_0).$$

Since the set \widehat{S} is countable, we may, if necessary, change (x_0, y_0) so that the equalities

$$\widehat{P}_i \circ \theta = P_i, \quad 1 \leq i \leq k,$$

hold for every $z \in S$. Thus, to prove the theorem it is enough to show that there exist a finitely generated subfield \mathbf{k} of \mathbb{C} and infinite sets $K_i \subset \mathbb{P}^1(\mathbf{k})$, $1 \leq i \leq k$, such that the equality

$$\widehat{P}_1^{-1}(K_1) = \widehat{P}_2^{-1}(K_2) = \cdots = \widehat{P}_k^{-1}(K_k)$$

holds. Taking into account that the addition operation on X is defined over the field of definition of X , this can be done as in the first part of the proof. \square

The formulation of Theorem 4.1 does not require the maps P_i , $1 \leq i \leq k$, to have no non-trivial common compositional right factor. On the other hand, since these maps are Galois coverings, they have such a factor if and only if the intersection

$$(23) \quad \Gamma_1 \cap \Gamma_2 \cap \cdots \cap \Gamma_k$$

is nontrivial. Furthermore, if

$$P_i = \tilde{P}_i \circ W, \quad 1 \leq i \leq k,$$

where $W : R \rightarrow \tilde{R}_i$ and $\tilde{P}_i : \tilde{R}_i \rightarrow E_i$, $1 \leq i \leq k$, are holomorphic maps between compact Riemann surfaces such that $\deg W > 1$ and the maps \tilde{P}_i , $1 \leq i \leq k$, have no nontrivial common compositional right factor, then the group (23) is the group of covering transformations of W .

Notice also that if the group

$$(24) \quad G = \langle \Gamma_1, \Gamma_2, \dots, \Gamma_k \rangle$$

is finite, then there exists a holomorphic Galois covering $A : R \rightarrow \mathbb{P}^1(\mathbb{C})$ such that G is its group of covering transformations. The inclusions $\Gamma_i \subseteq G$, $1 \leq i \leq k$, then imply that

$$(25) \quad A = F_1 \circ P_1 = F_2 \circ P_2 = \cdots = F_k \circ P_k$$

for some rational functions F_i , $1 \leq i \leq k$. Moreover, since S is a union of G -orbits, there exists a set K such that $S = A^{-1}(K)$. Thus,

$$S = P_i^{-1}(K_i) = P_i^{-1}(F_i^{-1}(K)), \quad 1 \leq i \leq k,$$

implying that

$$K_i = F_i^{-1}(K), \quad 1 \leq i \leq k.$$

In the general case, for the solutions constructed in Theorem 4.1, rational functions F_i , $1 \leq i \leq k$, satisfying (25) do not exist, since (25) would imply that G is finite, which need not hold.

Theorem 4.2. *Let R be a compact Riemann surface of genus zero or one, and let $P_i : R \rightarrow \mathbb{P}^1(\mathbb{C})$, $1 \leq i \leq k$, where $k \geq 2$, be holomorphic Galois coverings of the form*

$$P_i = P_k \circ \mu_i, \quad 1 \leq i \leq k-1,$$

where $\mu_i \in \text{Aut}(R)$, $1 \leq i \leq k-1$. Then there exist a finitely generated subfield \mathbf{k} of \mathbb{C} and an infinite set $K \subset \mathbb{P}^1(\mathbf{k})$ such that the equality

$$P_1^{-1}(K) = P_2^{-1}(K) = \cdots = P_k^{-1}(K)$$

holds.

Proof. The proof is obtained by a modification of the proof of Theorem 4.1 as follows. Assuming first that $R = \mathbb{P}^1(\mathbb{C})$, let us define a group $\Gamma \subset \text{Aut}(\mathbb{P}^1(\mathbb{C}))$ as the group generated by the group Γ_k of covering transformation of P_k , the Möbius transformations μ_i , $1 \leq i \leq k-1$, and some Möbius transformation of infinite order μ . Let us take now a point $z_0 \in \mathbb{P}^1(\mathbb{C})$ and define a subset S of $\mathbb{P}^1(\mathbb{C})$ by the formula (21). Since the group Γ_i , $1 \leq i \leq k-1$, of covering transformations of P_i , $1 \leq i \leq k-1$, satisfies

$$\Gamma_i = \mu_i^{-1} \circ \Gamma_k \circ \mu_i, \quad 1 \leq i \leq k-1,$$

the group Γ contains all the groups Γ_i , $1 \leq i \leq k$, implying as in the proof of Theorem 4.1 that there exist a finitely generated subfield \mathbf{k} of \mathbb{C} and infinite sets $K_i \subset \mathbb{P}^1(\mathbf{k})$, $1 \leq i \leq k$, such that the equality

$$(26) \quad (P_k \circ \mu_1)^{-1}(K_1) = \cdots = (P_k \circ \mu_{k-1})^{-1}(K_{k-1}) = P_k^{-1}(K_k) = S$$

holds.

Setting

$$(27) \quad \tilde{K}_i = P_k^{-1}(K_i), \quad 1 \leq i \leq k,$$

we see that (26) implies the equality

$$(28) \quad \mu_1^{-1}(\tilde{K}_1) = \mu_2^{-1}(\tilde{K}_2) = \cdots = \mu_{k-1}^{-1}(\tilde{K}_{k-1}) = \tilde{K}_k = S.$$

On the other hand, since the Möbius transformations μ_i , $1 \leq i \leq k-1$, belong to Γ , the set S is invariant with respect to these transformations. Thus, (28) implies the equality

$$\tilde{K}_1 = \tilde{K}_2 = \cdots = \tilde{K}_{k-1} = \tilde{K}_k = S.$$

It follows now from (27) that

$$K_1 = K_2 = \cdots = K_{k-1} = K_k = P_k(S).$$

The case $g(R) = 1$ can be treated using the same approach as in the proof of Theorem 4.1, with appropriate modifications. Namely, we consider the algebraic curve X defined in the proof of Theorem 4.1, and the group $\Gamma \subset \text{Aut}(R)$ generated by the group Γ_k , the automorphisms μ_i , $1 \leq i \leq k-1$, and an arbitrary shift μ of infinite order on R . We then define $\widehat{\Gamma} \subset \text{Aut}(X)$ as the group consisting of all elements $\widehat{\sigma}$ with $\sigma \in \Gamma$, and consider the set (22). \square

Notice that, as in the examples given by Theorem 4.1, the holomorphic maps P_i , $1 \leq i \leq k$, provided by Theorem 4.2 may or may not have a nontrivial common compositional right factor, depending on whether the group (23) is trivial. Furthermore, if the group (24) is finite, then there exist a holomorphic Galois covering $A : R \rightarrow \mathbb{P}^1(\mathbb{C})$ and a rational function F such that the following holds: G is the group of covering transformations of A , the equality

$$A = F \circ P_1 = F \circ P_2 = \cdots = F \circ P_k$$

holds, and $K = F^{-1}(\widehat{K})$ for some set \widehat{K} , contained in $\mathbb{P}^1(\mathbf{k})$ for some finitely generated field \mathbf{k} .

Indeed, in the considered case, equality (25) reduces to the following:

$$A = F_1 \circ P_k \circ \mu_1 = F_2 \circ P_k \circ \mu_2 = \cdots = F_k \circ P_k.$$

Since μ_i , $1 \leq i \leq k-1$, belong to Γ , we have

$$(29) \quad A = A \circ \mu_i, \quad 1 \leq i \leq k-1.$$

Substituting $F_i \circ P_k \circ \mu_i$ for A in the left-hand side and $F_k \circ P_k$ for A in the right-hand side of (29), we obtain

$$F_i \circ P_k \circ \mu_i = F_k \circ P_k \circ \mu_i, \quad 1 \leq i \leq k-1,$$

which implies that

$$F_1 = F_2 = \cdots = F_k.$$

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