HURWITZ EXISTENCE PROBLEM AND FIBER PRODUCTS

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Abstract. With each holomorphic map $f : R \to \mathbb{CP}^1$, where R is a compact Riemann surface, one can associate a combinatorial datum consisting of the genus g of R, the degree n of f, the number q of branching points of f, and the q partitions of n given by the local degrees of f at the preimages of the branching points. These quantities are related by the Riemann-Hurwitz formula, and the Hurwitz existence problem asks whether a combinatorial datum that fits this formula actually corresponds to some map f. In this paper, using results and techniques related to fiber products of holomorphic maps between compact Riemann surfaces, we prove a number of results that enable us to uniformly explain the non-realizability of many previously known non-realizable branch data, and to construct a large amount of new such data. We also deduce from our results the theorem of Halphen, proven in 1880, concerning polynomial solutions of the equation $A(z)^a + B(z)^b = C(z)^c$, where a, b, c are integers greater than one.

1. Introduction

Let R be a compact Riemann surface of genus $g = g(R)$ and $f : R \to \mathbb{CP}^1$ a holomorphic map of degree n. If $z_1, z_2, ..., z_q \in \mathbb{CP}^1$ are branching points of f, i.e. points $z \in \mathbb{CP}^1$ for which $f^{-1}\{z\}$ contains less than n points, then for each i, $1 \leq i \leq q$, the set $\Pi_i = (\pi_{i,1}, \pi_{i,2}, \dots, \pi_{i,p_i})$ of local degrees of f at points of $f^{-1}\{z_i\}$ is a partition of n. Furthermore, it follows from the Riemann-Hurwitz formula that the equality

$$\sum_{i=1}^{q} p_i = (q-2)n + 2 - 2g(R) \tag{1}$$

holds. We call a collection of the form $\Pi = (\Pi_1, \dots, \Pi_q, n, g)$, where $n \geq 1$, $g \geq 0$, and $q \geq 0$ are integers, and Π_i, $1 \leq i \leq q$, are partitions of n such that equality (1) holds, *a branch datum*. The Hurwitz existence problem is the following question: given a branch datum $\Pi = (\Pi_1, \dots, \Pi_q, n, g)$, determine whether there exists a compact Riemann surface of genus g and a holomorphic map $f : R \to \mathbb{CP}^1$ such that Π arises from f. If such a map exists, one says that Π is *realizable*; otherwise, that it is *non-realizable*.

Notice that the above problem is a special case of the broader problem of the existence of branched coverings maps between closed connected surfaces Σ_1 and Σ_2, which goes back to Hurwitz ([18]). However, except when Σ_2 is the sphere and Σ_1 is oriented, this problem is either solved or can be reduced to this specific case (see [9], [42]), in which, by the Riemann existence theorem, the existence of a branched covering map from Σ_1 to Σ_2 is equivalent to the existence of a holomorphic map $f : R \to \mathbb{CP}^1$ between compact Riemann surfaces. Since our results are easier to formulate in terms of holomorphic maps, we use the corresponding formulation.

Numerous papers employing various techniques have been devoted to the Hurwitz problem (see [1], [3]-[10], [12], [14], [18]-[22], [24]-[30], [35]-[44], [47]-[51]), but the problem is still far from being solved. A comprehensive introduction to the topic can be found in [42]. In this paper, using results and techniques related to fiber products of holomorphic maps between compact Riemann surfaces, we prove a number of results that enable us to uniformly explain the non-realizability of many previously known non-realizable branch data, and to construct a large amount of new such data.

Let $f : \mathbb{CP}^1 \to \mathbb{CP}^1$ be a rational function such that in the corresponding branch datum Π all entries in Π_1 and Π_2 are divisible by an integer $d \geq 2$. Then, assuming that critical values corresponding to Π_1 and Π_2 are 0 and ∞, and representing f as a quotient of two polynomials, we see that $f = z^d \circ q$ for some rational function q. For certain branch data, this straightforward statement permits to demonstrate their non-realizability, with the simplest example of this sort being $((2,2),(2,2),(1,3),4,0)$ (see [37], Section 5, and [48] for more details and examples). Roughly speaking, our first result provides a similar decomposability criterion for any finite group of rotations of the sphere. Since the most convenient way of formulating our results uses the notion of orbifold, we start by recalling the necessary definitions.

A Riemann surface orbifold is a pair $\mathcal{O} = (R, \nu)$ consisting of a Riemann surface R and a ramification function $\nu : R \to \mathbb{N}$ that takes the value $\nu(z) = 1$ except at isolated points. For an orbifold $\mathcal{O} = (R, \nu)$, *the Euler characteristic* of $\mathcal{O}$ is the number

$$\chi(\mathcal{O}) = \chi(R) + \sum_{z \in R} \left(\frac{1}{\nu(z)} - 1 \right),$$

the set of singular points of $\mathcal{O}$ is the set

$$c(\mathcal{O}) = \{z_1, z_2, \dots, z_s, \dots\} = \{z \in R \mid \nu(z) > 1\},$$

and *the signature* of $\mathcal{O}$ is the set

$$\nu(\mathcal{O}) = \{\nu(z_1), \nu(z_2), \dots, \nu(z_s), \dots\}.$$

Let $\mathcal{O}_1 = (R_1, \nu_1)$ and $\mathcal{O}_2 = (R_2, \nu_2)$ be orbifolds and let $f : R_1 \to R_2$ be a holomorphic branched covering map. We say that $f : \mathcal{O}_1 \to \mathcal{O}_2$ is *a covering map* between orbifolds if for any $z \in R_1$ the equality

$$\nu_2(f(z)) = \nu_1(z) \deg_z f \tag{2}$$

holds. For a holomorphic map $f : R' \to R$ between compact Riemann surfaces and an orbifold $\mathcal{O} = (R, \nu)$, we define an orbifold $f^*(\mathcal{O}) = (R', \nu')$ by the formula

$$\nu'(z) = \frac{\nu(f(z))}{\gcd(\deg_z f, \nu(f(z)))}, \qquad z \in R'. \tag{3}$$

Notice that if $f : \mathcal{O}_1 \to \mathcal{O}_2$ is a covering map between orbifolds, then $f^*(\mathcal{O}_2) = \mathcal{O}_1$.

A universal covering of an orbifold $\mathcal{O}$ is a covering map between orbifolds $\theta_{\mathcal{O}} : \widetilde{\mathcal{O}} \to \mathcal{O}$, where $\widetilde{\mathcal{O}}$ is an orbifold such that $\widetilde{R}$ is simply connected and $\widetilde{\mathcal{O}}$ is *non-ramified*, meaning $\widetilde{\nu}(z) \equiv 1$. If $\theta_{\mathcal{O}}$ is such a map, then there exists a group $\Gamma_{\mathcal{O}}$ of conformal automorphisms of $\widetilde{R}$ such that the equality $\theta_{\mathcal{O}}(z_1) = \theta_{\mathcal{O}}(z_2)$ holds for $z_1, z_2 \in \widetilde{R}$ if and only if $z_1 = \sigma(z_2)$ for some $\sigma \in \Gamma_{\mathcal{O}}$. A universal covering exists and is unique up to a conformal isomorphism of $\widetilde{R}$, unless $R = \mathbb{CP}^1$ with a single ramified point, or $R = \mathbb{CP}^1$ with two ramified points z_1, z_2 such that $\nu(z_1) \neq \nu(z_2)$. We say that an orbifold $\mathcal{O}$ is *good* if it has a universal covering, and *bad* otherwise.

Furthermore, if $\mathcal{O} = (R, \nu)$ is good, then the surface $\widetilde{R}$ is the unit disk $\mathbb{D}$ if and only if $\chi(\mathcal{O}) < 0$, the complex plane $\mathbb{C}$ if and only if $\chi(\mathcal{O}) = 0$, and the Riemann sphere $\mathbb{CP}^1$ if and only if $\chi(\mathcal{O}) > 0$. Finally, we recall that for an orbifold $\mathcal{O}$ on $\mathbb{CP}^1$ the inequality $\chi(\mathcal{O}) > 0$ holds if and only if $\nu(\mathcal{O})$ belongs to the list

$$\{d, d\}, \quad d \geq 1, \quad \{2, 2, d\}, \quad d \geq 2, \quad \{2, 3, 3\}, \quad \{2, 3, 4\}, \quad \{2, 3, 5\}.$$

The corresponding universal coverings $\theta_{\mathcal{O}}$ are well-known rational Galois coverings of $\mathbb{CP}^1$ by $\mathbb{CP}^1$ of degrees d, $d \geq 1$, $2d$, $d \geq 2$, 12, 24, 60, calculated by Klein ([23]). Most of the orbifolds considered in this paper are defined on $\mathbb{CP}^1$. For this reason, abusing notation, we will simply write "an orbifold $\mathcal{O}$" when the considered orbifold is defined on $\mathbb{CP}^1$.

In the above notation, our first result is the following statement.

Theorem 1.1. *Let $f : \mathbb{CP}^1 \to \mathbb{CP}^1$ be a rational function, and let $\mathcal{O}$ be a good orbifold such that $f^*(\mathcal{O})$ is non-ramified. Then $\chi(\mathcal{O}) > 0$, and $f = \theta_{\mathcal{O}} \circ q$ for some rational function q. In particular,* $\deg f$ *is divisible by* $\deg \theta_{\mathcal{O}}$.

Notice that the orbifold $f^*(\mathcal{O})$ is non-ramified if and only if $\nu(f(z))$ divides $\deg_z f$ for all $z \in \mathbb{CP}^1$. In particular, since the universal covering of the orbifold defined by the equalities

$$\nu(0) = d, \qquad \nu(\infty) = d, \qquad d \geq 2,$$

is $\theta_{\mathcal{O}} = z^d \circ \mu$, where μ is a Möbius transformation, for such orbifolds Theorem 1.1 reduces to the above mentioned statement.

Theorem 1.1 implies that a rational function such that $f^*(\mathcal{O})$ is non-ramified and $\deg f$ is not divisible by $\deg \theta_{\mathcal{O}}$ cannot exist. If $\mathcal{O}$ has the signature $\{d, d\}$, $d \geq 2$, this statement is not useful for proving non-realizability results, since if in a branch datum $\Pi = (\Pi_1, \dots, \Pi_q, n, 0)$ all entries in Π_1 and Π_2 are divisible by d, then n is also divisible by $d = \deg \theta_{\mathcal{O}}$. However, the situation changes when we consider other orbifolds, allowing us to obtain large families of non-realizable branch data by using the necessary condition $\deg \theta_{\mathcal{O}} \mid \deg f$ alone.

For example, for the signature $\{2, 3, 3\}$, non-realizable branch data obtained in this way with the minimum possible values of n and q equal to 18 and 3 are

$$((2^7, 4), (3^6), (3^6), 18, 0), \qquad ((2^9), (3^4, 6), (3^6), 18, 0),$$

while for the signature $\{2, 3, 5\}$ the first such data are

$$\big((2^{43}, 4), (3^{30}), (5^{18}), 90, 0\big), \qquad \big((2^{45}), (3^{28}, 6), (5^{18}), 90, 0\big),$$

and

$$\big((2^{45}), (3^{30}), (5^{16}, 10), 90, 0\big)$$

(hereinafter the symbol u^v appearing in a partition means a string consisting of the number u taken v times). For further examples of branch data whose non-realizability follows from Theorem 1.1 we refer the reader to Section 3.

It follows easily from the existence of a universal covering that if $f : \mathcal{O}_1 \to \mathcal{O}_2$ is a covering map between orbifolds, and $\mathcal{O}_2$ is good, then $\mathcal{O}_1$ is also good. The relationship between this property of covering maps and the Hurwitz problem, as well as its application in proving the non-realizability of certain branching data, was established in the paper [35]. For example, the branch data

$$((2, 2, 2, 2, 1), (3, 3, 3), (3, 3, 3), 9, 0)$$

is non-realizable, since if a rational function f with this branch data existed, it would be a covering map $f:\mathcal{O}_1\to\mathcal{O}_2$, with $\mathcal{O}_2$ defined by the equalities

$$\nu_2(z_1)=2,\quad \nu_2(z_2)=3,\quad \nu_3(z_3)=3,$$

where z_1,z_2,z_3 are critical values of f, and $\mathcal{O}_1$ defined by the equality $\nu_1(z_0)=2$, where z_0 is the unique point in $f^{-1}(z_1)$ that is not a critical point of f.

Our second result offers a broad generalization of the aforementioned property of covering maps between orbifolds.

Theorem 1.2. *Let $f:\mathbb{CP}^1\to\mathbb{CP}^1$ be a rational function. Then for any good orbifold $\mathcal{O}$ on $\mathbb{CP}^1$ the orbifold $f^*(\mathcal{O})$ is good.*

As an example illustrating Theorem 1.2, let us consider the following well known series of non-realizable branch data

$$\big((2^k),(2^k),(l,2k-l),2k,0\big),\qquad l\neq k. \tag{4}$$

To see the non-realizability of (4) using Theorem 1.2, it is enough to observe that if a rational function f with this branch datum existed, then for a convenient orbifold $\mathcal{O}$ with $\nu(\mathcal{O})=\{2,2,t\}$, where $t=\max\{l,2k-l\}$, the set of singular points of the orbifold $f^*(\mathcal{O})$ would consist of a single point. As a more subtle corollary of Theorem 1.2 we mention the non-realizability of the branch data

$$\big((2^l),(3^k),(5^m,s),n,0\big),\qquad s\not\equiv 0 \pmod 5, \tag{5}$$

established in [1], [20].

Using some geometric objects, called “dessins d’enfants” or “constellations”, the realizability of a branch datum can be interpreted in geometric terms as the existence of a planar graph of a certain type (see e.g. [24], [25], [30], [37]). For instance, the non-realizability of branch data (5) is a particular case of the result in [20], which states that there is no triangulation of the sphere with the degrees of all vertices except one divisible by 5. Theorem 1.2 permits to extend the last statement from triangulations to graphs with the degrees of all faces divisible by three. More generally, Theorem 1.2 implies the following corollary concerning planar graphs.

Corollary 1.3. *There exists no connected planar graph with the degrees of all faces divisible by a number $k\geq 2$, and the degrees of all vertices except one divisible by a number $l\geq 2$.*

Another notable corollary of Theorem 1.1 is the following result.

Corollary 1.4. *Let G be a connected planar graph and let k, l be integers greater than one. Then the following holds:*

i) *If the degrees of all faces of G are divisible by k, and the degrees of all vertices of G are divisible by l except for two vertices with degrees u and v, then $\gcd(u,l)=\gcd(v,l)$.*
ii) *If the degrees of all faces of G are divisible by k except for one face with degree u, and the degrees of all vertices of G are divisible by l except for one vertex with degree v, then $k/\gcd(u,k)=l/\gcd(v,l)$.*

Notice that although Theorem 1.2 imposes no restrictions on $\chi(\mathcal{O})$, the orbifold $f^*(\mathcal{O})$ can be bad only if $\chi(\mathcal{O})>0$ (see Section 3). Accordingly, the above corollaries are primarily of interest when the pair $\{k,l\}$ is one of the following: $\{3,3\},\{3,4\},\{3,5\}$, or $\{2,d\}, d\geq 2$.

Theorems 1.1 and 1.2 address the situation where, for some orbifold $\mathcal{O}$, the orbifold $f^*(\mathcal{O})$ is either unbranched, or has signature $\{a\}$, or $\{a,b\}$ with $a \neq b$. In particular, $\chi(f^*(\mathcal{O})) > 0$. A natural question is whether the positivity of the Euler characteristic of $f^*(\mathcal{O})$ alone imposes constraints on f. The following theorem confirms that this is indeed the case and shows that Theorem 1.1 is, in fact, a special case of a more general result.

Theorem 1.5. *Let $f : \mathbb{CP}^1 \to \mathbb{CP}^1$ be a rational function, and let $\mathcal{O}$ be a good orbifold on $\mathbb{CP}^1$ such that $\chi(f^*(\mathcal{O})) > 0$. Then $\chi(\mathcal{O}) > 0$, and the following holds:*

i) *There exists a rational function q such that*

$$f \circ \theta_{f^*(\mathcal{O})} = \theta_{\mathcal{O}} \circ q.$$

In particular, the product $\deg \theta_{f^*(\mathcal{O})} \cdot \deg f$ is divisible by $\deg \theta_{\mathcal{O}}$.

ii) *There exist rational functions w and t such that*

$$f = w \circ t, \quad \theta_{\mathcal{O}} = w \circ \theta_{f^*(\mathcal{O})}.$$

Moreover, $w : f^*(\mathcal{O}) \to \mathcal{O}$ is a covering map between orbifolds. In particular, the equality $\deg w = \chi(f^*(\mathcal{O}))/\chi(\mathcal{O})$ holds.

In case the orbifold $f^*(\mathcal{O})$ is non-ramified, $\theta_{f^*(\mathcal{O})} = z$ and both conditions i) and ii) of Theorem 1.5 reduce to Theorem 1.2. In general case, condition ii) shows how large is "the common compositional left factor" of the functions f and $\theta_{\mathcal{O}}$, depending on $f^*(\mathcal{O})$, where by a compositional left factor of a rational function f we mean any rational function g such that $f = g \circ h$ for some rational function h. Note that the function w in condition (ii) may have degree one; this occurs when $\nu(f^*\mathcal{O}) = \nu(\mathcal{O})$. In this case, the theorem yields no direct corollaries relevant to the Hurwitz problem. However, condition (i) remains meaningful and essentially describes a semiconjugacy relation between the rational functions f and q (see Section 5).

Similar to Theorem 1.1, Theorem 1.5 allows us to establish the non-realizability of certain branch data simply by checking the divisibility of two numbers. In particular, if for a rational function f and an orbifold $\mathcal{O}$ the orbifold $f^*(\mathcal{O})$ has the signature $\{a,a\}$, then Theorem 1.5 yields that $a \deg f$ must be divisible by $\theta_{\mathcal{O}}$. This implies for example that the series of branch data

$$\big((2^{3l+3}), (3^{2l+2}), (5^{l+1}, 1, l), 6(l+1), 0\big), \qquad l \equiv 0 \,(\mathrm{mod}\, 2), \tag{6}$$

is non-realizable. Indeed, if f is a rational function realizing (6), then considering a convenient orbifold $\mathcal{O}$ with the signature $\{2,3,5\}$, we see that the orbifold $f^*(\mathcal{O})$ is either bad, or has the signature $\{5,5\}$. However, $5 \cdot 6(l+1)$ is not divisible by $\deg \theta_{\mathcal{O}} = 60$, in contradiction with Theorem 1.5. As an example when a more subtle condition ii) is used to prove non-realizability, we mention the known series

$$\big((2^k), (2^{k-2}, 1, 3), (k,k), 2k, 0\big), \quad k \geq 2,$$

(see Section 5).

Notice that Theorem 1.5 entails a rather unexpected consequence concerning functional decompositions of rational functions. Let us recall that a rational function f is called indecomposable if it cannot be represented as a composition of two rational functions of degree at least two. Otherwise, f is called decomposbale. In group theoretic terms, a rational function f is indecomposable if and only if the monodromy group of f is primitive. The decomposability is a rather subtle property that cannot typically be seen from the branch datum of f alone, as the same branch datum can correspond to different functions with distinct monodromy groups. Theorem 1.5 shows, however, that for certain branch data, nearly all functions with that branch data are necessarily decomposable. Specifically, we deduce from Theorem 1.5 the following general statement, which incorporates some earlier results obtained in [15] and [29] (see Section 5).

Corollary 1.6. *Let f be a rational function. Suppose that for some good orbifold $\mathcal{O}$ the conditions $\chi(f^*(\mathcal{O})) > 0$ and $\nu(f^*(\mathcal{O})) \neq \nu(\mathcal{O})$ hold. Then f is decomposable, unless it is an indecomposable compositional left factor of $\theta_{\mathcal{O}}$.*

In short, our approach to proving the theorems above is based on considering the fiber product of f and $\theta_{\mathcal{O}}$, which gives rise to the commutative diagram

$$\begin{array}{ccc} R & \xrightarrow{p} & \mathbb{CP}^1 \\ \Big\downarrow{\scriptstyle q} & & \Big\downarrow{\scriptstyle \theta_{\mathcal{O}}} \\ \mathbb{CP}^1 & \xrightarrow{f} & \mathbb{CP}^1 \end{array},$$

where p and q are holomorphic maps between compact Riemann surfaces. Under the assumptions of Theorem 1.1, further analysis shows that the map q is unramified, so that $\deg q = 1$. The assumptions of Theorem 1.2, on the other hand, imply that the orbifold $f^*(\mathcal{O})$ has a universal covering. Finally, Theorem 1.5 follows from Theorem 1.1 combined with general properties of fiber products and Galois coverings.

Let us stress that in the examples of non-realizable branch data obtained using Theorem 1.5 the obstacle to the realizability lies in the "forced decomposability" of the rational functions potentially realizing these branch data. In particular, examples obtained in this way, like other examples of non-realizabilty found so far, are consistent with the prime-degree conjecture proposed in [9], which posits that any branch datum with prime n is realizable. Moreover, the examples derived from Theorem 1.2 cannot contradict this conjecture either, as it is clear that the degree n in such examples is always divisible by at least one of the numbers a, b, c comprising the signature of the orbifold $\mathcal{O}$.

Our results show that a condition which may indicate the non-realizability of a branch datum Π with $g = 0$ is the existence of a good orbifold, necessarily with positive Euler characteristic, such that for any rational function f that might realize this datum, the orbifold $f^*(\mathcal{O})$ has positive Euler characteristic, or equivalently, the support of $f^*(\mathcal{O})$ contains at most three points. Moreover, this condition is quite general. Say, among the 59 non-realizable branch data for $g = 0$ and $n \leq 10$ found in [50], there is only seven data for which such an orbifold does not exist.

The findings in [50] demonstrate that a similar situation persists when passing from rational functions to holomorphic maps from a torus to a sphere. Specifically, for all 30 non-realizable branch data with $g = 1$ and $n \leq 20$ presented in [50], the following holds: there exists an orbifold $\mathcal{O}$ with $\chi(\mathcal{O}) \geq 0$ such that for a map f that might realize this datum, the orbifold $f^*(\mathcal{O})$ is either non-ramified or its set of singular points consists of one or two points. The approach used to prove Theorems 1.1 and 1.2 is not directly applicable to holomorphic maps from a torus to a sphere because non-ramified coverings of a torus with degree greater than one do exist, and every orbifold on a torus has a universal cover. Nonetheless, with a slight modification of the approach, we are able to prove the following result.

Theorem 1.7. *Let R be a compact Riemann surface of genus one, let $f : R \to \mathbb{CP}^1$ be a holomorphic map, and let $\mathcal{O}$ be a good orbifold on $\mathbb{CP}^1$ with $\chi(\mathcal{O}) > 0$ such that $f^*(\mathcal{O})$ is non-ramified. Then there exist a rational function $w : \mathbb{CP}^1 \to \mathbb{CP}^1$ and a holomorphic map $t : R \to \mathbb{CP}^1$ such that the equalities*

$$f = w \circ t, \quad \theta_{\mathcal{O}} = w \circ \theta_{w^*(\mathcal{O})}$$

hold, and the signature of $w^*(\mathcal{O})$ is either $\{d, d\}$ for some $d \geq 1$, or $\{2, 2, 2\}$.

Notice that if in Theorem 1.7 the orbifold $\mathcal{O}$ itself has signature $\{d, d\}$, $d \geq 1$, or $\{2, 2, 2\}$, then the conclusion holds trivially for $w = id$. However, if $\mathcal{O}$ has a different signature, then Theorem 1.7 implies that f is decomposable, and this fact can be used in the study of the Hurwitz problem. For example, using Theorem 1.7 one can establish the non-realizability of branch data

$$\big((2^k, k+3), (3^{k+1}), (3^{k+1}), 3k+3, 0\big), \quad k \equiv 1 \,(\mathrm{mod}\, 4),$$

and

$$\big((2^{3k+6}), (3^{2k+4}), (3, 9, 6^k), 6k+12, 0\big), \quad k \equiv 1 \,(\mathrm{mod}\, 2),$$

(see Section 5).

This paper is organized as follows. In Section 2, we collect the necessary definitions and results concerning orbifolds and fiber products used in the subsequent sections. In Section 3, we prove Theorem 1.1 and explain how it gives rise to large families of non-realizable branch data. In Section 4, we prove Theorem 1.2, present several examples, and provide applications to planar graphs and permutation groups. In Section 5, we prove Theorems 1.5 and 1.7 along with some of their corollaries.

Finally, in Section 6, we deduce from Theorems 1.1 and 1.2 a result of Halphen [16] concerning polynomial solutions of the equation

$$A(z)^a + B(z)^b = C(z)^c,$$

where a, b, and c are integers greater than one.

2. Orbifolds and fiber products

2.1. **Minimal holomorphic maps between orbifolds.** In addition to the notion of a covering map between orbifolds defined in the introduction, we will also use the notion of a holomorphic map between orbifolds. Let $\mathcal{O}_1 = (R_1, \nu_1)$ and $\mathcal{O}_2 = (R_2, \nu_2)$ be orbifolds and let $f : R_1 \to R_2$ be a holomorphic branched covering map. We say that $f : \mathcal{O}_1 \to \mathcal{O}_2$ is a holomorphic map between orbifolds, if for any $z \in R_1$ instead of equality (2) a weaker condition

$$\nu_2(f(z)) \mid \nu_1(z) \deg_z f \tag{7}$$

holds.

Holomorphic maps between orbifolds lift to holomorphic maps between their universal covers. Specifically, the following proposition is true (see [32], Proposition 3.1, for a more detailed formulation and a proof).

Proposition 2.1. *For any holomorphic map between orbifolds $f:\mathcal{O}_1\to\mathcal{O}_2$, there exists a holomorphic map $F:\widetilde{\mathcal{O}_1}\to\widetilde{\mathcal{O}_2}$ such that the diagram*

$$\begin{array}{ccc} \widetilde{\mathcal{O}_1} & \xrightarrow{F} & \widetilde{\mathcal{O}_2} \\ \Big\downarrow{\scriptstyle\theta_{\mathcal{O}_1}} & & \Big\downarrow{\scriptstyle\theta_{\mathcal{O}_2}} \\ \mathcal{O}_1 & \xrightarrow{f} & \mathcal{O}_2 \end{array}$$

is commutative. The holomorphic map F is an isomorphism if and only if f is a covering map between orbifolds. □

If $f:\mathcal{O}_1\to\mathcal{O}_2$ is a covering map between orbifolds with compact R_1 and R_2, then the Riemann-Hurwitz formula implies that

$$\chi(\mathcal{O}_1)=d\chi(\mathcal{O}_2), \tag{8}$$

where $d=\deg f$. For holomorphic maps the following statement is true (see [32], Proposition 3.2).

Proposition 2.2. *Let $f:\mathcal{O}_1\to\mathcal{O}_2$ be a holomorphic map between orbifolds with compact R_1 and R_2. Then*

$$\chi(\mathcal{O}_1)\leq\chi(\mathcal{O}_2)\deg f, \tag{9}$$

and the equality holds if and only if $f:\mathcal{O}_1\to\mathcal{O}_2$ is a covering map between orbifolds. □

Let R_1, R_2 be Riemann surfaces and $f:R_1\to R_2$ a holomorphic branched covering map. Assume that R_2 is provided with ramification function ν_2. In order to define a ramification function ν_1 on R_1 so that f would be a holomorphic map between orbifolds $\mathcal{O}_1=(R_1,\nu_1)$ and $\mathcal{O}_2=(R_2,\nu_2)$ we must satisfy condition (7), and it is easy to see that for any $z\in R_1$ a minimal possible value for $\nu_1(z)$ is defined by the equality

$$\nu_2(f(z))=\nu_1(z)\mathrm{GCD}(\deg{}_z f,\nu_2(f(z))). \tag{10}$$

In case if (10) is satisfied for any $z\in R_1$ we say that f is a minimal holomorphic map between orbifolds $\mathcal{O}_1=(R_1,\nu_1)$ and $\mathcal{O}_2=(R_2,\nu_2)$.

It follows from the definition that for any orbifold $\mathcal{O}=(R,\nu)$ and any holomorphic branched covering map $f\colon R'\to R$, there exists a unique orbifold structure ν' on R' such that f becomes a minimal holomorphic map between the orbifolds $\mathcal{O}'=(R',\nu')$ and $\mathcal{O}=(R,\nu)$. We denote the corresponding orbifold $\mathcal{O}'$ by $f^*\mathcal{O}$, consistent with the notation and formula (3) from the introduction. Let us emphasize that for any minimal holomorphic map $f\colon\mathcal{O}_1\to\mathcal{O}_2$, the equality

$$\mathcal{O}_1=f^*(\mathcal{O}_2) \tag{11}$$

holds simply by definition. Notice also that any covering map between orbifolds $f\colon\mathcal{O}_1\to\mathcal{O}_2$ is a minimal holomorphic map. In particular, equality (11) holds.

Minimal holomorphic maps between orbifolds possess the following fundamental property (see [32], Theorem 4.1).

Theorem 2.3. *Let $f:R''\to R'$ and $g:R'\to R$ be holomorphic branched covering maps, and $\mathcal{O}=(R,\nu)$ an orbifold. Then*

$$(g\circ f)^*(\mathcal{O})=f^*(g^*(\mathcal{O})).$$ □

Theorem 2.3 implies in particular the following corollaries (see [32], Corollary 4.1 and Corollary 4.2).

Corollary 2.4. *Let $f : \mathcal{O}_1 \to \mathcal{O}'$ and $g : \mathcal{O}' \to \mathcal{O}_2$ be minimal holomorphic maps (resp. covering maps) between orbifolds. Then $g \circ f : \mathcal{O}_1 \to \mathcal{O}_2$ is a minimal holomorphic map (resp. covering map).* □

Corollary 2.5. *Let $f : R_1 \to R'$ and $g : R' \to R_2$ be holomorphic branched covering maps, and $\mathcal{O}_1 = (R_1, \nu_1)$ and $\mathcal{O}_2 = (R_2, \nu_2)$ orbifolds. Assume that $g \circ f : \mathcal{O}_1 \to \mathcal{O}_2$ is a minimal holomorphic map between orbifolds (resp. a covering map). Then $g : g^*(\mathcal{O}_2) \to \mathcal{O}_2$ and $f : \mathcal{O}_1 \to g^*(\mathcal{O}_2)$ are minimal holomorphic maps (resp. covering maps).* □

For orbifolds $\mathcal{O}_1 = (R_1, \nu_1)$ and $\mathcal{O}_2 = (R_2, \nu_2)$, we write

$$\mathcal{O}_1 \preceq \mathcal{O}_2$$

if $R_1 = R_2$, and for any $z \in R_1$ the condition

$$\nu_1(z) \mid \nu_2(z)$$

holds. Abusing notation we use the symbol $\mathbb{CP}^1$ both for the Riemann sphere and for the non-ramified orbifold defined on $\mathbb{CP}^1$.

2.2. **Fiber products.** Let $f : C_1 \to \mathbb{CP}^1$ and $g : C_2 \to \mathbb{CP}^1$ be holomorphic maps between compact Riemann surfaces. The collection

$$(C_1, f) \times (C_2, g) = \bigcup_{j=1}^{n(f,g)} \{R_j, p_j, q_j\}, \tag{12}$$

where $n(f, g)$ is an integer positive number and R_j are compact Riemann surfaces provided with holomorphic maps

$$p_j : R_j \to C_1, \quad q_j : R_j \to C_2, \quad 1 \le j \le n(f, g),$$

is called the *fiber product* of f and g if

$$f \circ p_j = g \circ q_j, \quad 1 \le j \le n(f, g), \tag{13}$$

and for any holomorphic maps $p : R \to C_1$, $q : R \to C_2$ between compact Riemann surfaces satisfying

$$f \circ p = g \circ q \tag{14}$$

there exist a uniquely defined index j and a holomorphic map $w : R \to R_j$ such that

$$p = p_j \circ w, \quad q = q_j \circ w.$$

The fiber product exists and is defined in a unique way up to natural isomorphisms.

The fiber product can be described by the following algebro-geometric construction. Let us consider the algebraic curve

$$L = \{(x, y) \in C_1 \times C_2 \mid f(x) = g(y)\}. \tag{15}$$

Let us denote by L_j, $1 \le j \le n(f, g)$, irreducible components of L, by R_j, $1 \le j \le n(f, g)$, their desingularizations, and by

$$\pi_j : R_j \to L_j, \quad 1 \le j \le n(f, g),$$

the desingularization maps. Then the compositions

$$x \circ \pi_j : L_j \to C_1, \quad y \circ \pi_j : L_j \to C_2, \quad 1 \le j \le n(f,g),$$

extend to holomorphic maps

$$p_j : R_j \to C_1, \quad q_j : R_j \to C_2, \quad 1 \le j \le n(f,g),$$

and the collection $\bigcup_{j=1}^{n(f,g)} \{R_j, p_j, q_j\}$ is the fiber product of f and g. Abusing notation, we refer to the Riemann surfaces R_j, $1 \le j \le n(f,g)$, as the irreducible components of the fiber product of f and g, and say that the fiber product is irreducible if $n(f,g) = 1$.

It follows from the definition that for every j, $1 \le j \le n(f,g)$, the functions p_j, q_j have no *non-trivial common compositional right factor* in the following sense: the equalities

$$p_j = \widetilde{p} \circ t, \quad q_j = \widetilde{q} \circ t,$$

where

$$t : R_j \to \widetilde{R}, \quad \widetilde{p} : \widetilde{R} \to C_1, \quad \widetilde{q} : \widetilde{R} \to C_2$$

are holomorphic maps between compact Riemann surfaces, imply that $\deg t = 1$. Denoting by $\mathcal{M}(R)$ the field of meromorphic functions on a Riemann surface R, we can restate this condition as the equality

$$p_j^*\mathcal{M}(C_1) \cdot q_j^*\mathcal{M}(C_2) = \mathcal{M}(R_j),$$

meaning that the field $\mathcal{M}(R_j)$ is the compositum of its subfields $p_j^*\mathcal{M}(C_1)$ and $q_j^*\mathcal{M}(C_2)$. In the other direction, if q and p satisfy (13) and have no non-trivial common compositional right factor, then

$$p = p_j \circ t, \quad q = q_j \circ t$$

for some j, $1 \le j \le n(f,g)$, and an isomorphism $t : R_j \to R_j$.

Notice that since p_i, q_i, $1 \le j \le n(f,g)$, parametrize components of (15), the equalities

$$\sum_j \deg p_j = \deg g, \quad \sum_j \deg q_j = \deg f \tag{16}$$

hold. In particular, if $(C_1, f) \times (C_2, g)$ consists of a unique component $\{R, p, q\}$, then

$$\deg p = \deg g, \quad \deg q = \deg f. \tag{17}$$

Vice versa, if holomorphic maps q and p satisfy (13) and (17), and have no non-trivial common compositional right factor, then $(C_1, f) \times (C_2, g)$ is irreducible.

2.3. **Functional equations and orbifolds.** With each holomorphic map $h : R_1 \to R_2$ between compact Riemann surfaces, one can associate two orbifolds $\mathcal{O}_1^h = (R_1, \nu_1^h)$ and $\mathcal{O}_2^h = (R_2, \nu_2^h)$ in a natural way, setting $\nu_2^h(z)$ equal to the least common multiple of local degrees of h at the points of the preimage $h^{-1}\{z\}$, and

$$\nu_1^h(z) = \frac{\nu_2^h(h(z))}{\deg_z h}.$$

By construction,

$$h : \mathcal{O}_1^h \to \mathcal{O}_2^h$$

is a covering map between orbifolds. It is easy to see that this covering map is minimal in the following sense. For any covering map $h : \mathcal{O}_1 \to \mathcal{O}_2$, we have:

$$\mathcal{O}_1^h \preceq \mathcal{O}_1, \quad \mathcal{O}_2^h \preceq \mathcal{O}_2. \tag{18}$$

Notice that for the universal covering $\theta_{\mathcal{O}}$ of an orbifold $\mathcal{O}$ of positive Euelr characteristic the equalities

$$\mathcal{O}_2^{\theta_{\mathcal{O}}} = \mathcal{O}, \qquad \mathcal{O}_1^{\theta_{\mathcal{O}}} = \mathbb{CP}^1$$

hold.

We will widely use the following fact (see [32], Lemma 4.2).

Lemma 2.6. *For any holomorphic map $h : R_1 \to R_2$ between compact Riemann surfaces, the orbifolds $\mathcal{O}_1^h$ and $\mathcal{O}_2^h$ are good.* □

The orbifolds defined above are useful for the study of the functional equation (14), where

$$p : R \to C_1, \quad f : C_1 \to \mathbb{CP}^1, \quad q : R \to C_2, \quad g : C_2 \to \mathbb{CP}^1$$

are holomorphic maps between compact Riemann surface. Usually, we will write this equation in the form of a commutative diagram

$$\begin{array}{ccc} R & \xrightarrow{q} & C_2 \\ \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle g} \\ C_1 & \xrightarrow{f} & \mathbb{CP}^1 . \end{array} \tag{19}$$

The main result we use for dealing with equation (19) is the following statement (see [32], Theorem 4.2).

Theorem 2.7. *Let f, p, g, q be holomorphic maps between compact Riemann surface such that diagram (19) commutes, the fiber product of f and g has a unique component, and p and q have no non-trivial common compositional right factor. Then the commutative diagram*

$$\begin{array}{ccc} \mathcal{O}_1^p & \xrightarrow{q} & \mathcal{O}_1^g \\ \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle g} \\ \mathcal{O}_2^p & \xrightarrow{f} & \mathcal{O}_2^g \end{array}$$

consists of minimal holomorphic maps between orbifolds. □

Notice that since vertical arrows in the above diagram are covering maps between orbifolds, they are automatically minimal holomorpic maps. The nontrivial part of the theorem that will be used is the equalities

$$\mathcal{O}_2^p = f^*(\mathcal{O}_2^g), \qquad \mathcal{O}_1^p = q^*(\mathcal{O}_1^g),$$

which describe the orbifolds $\mathcal{O}_2^p$ and $\mathcal{O}_1^p$ as pullbacks.

2.4. **Normalizations and the Fried theorem.** Let $p: R \to C$ be a holomorphic map between compact Riemann surfaces. Let us recall that p is called a *Galois covering* if its automorphism group

$$\mathrm{Aut}(R,p) = \{\sigma \in \mathrm{Aut}(R) : p \circ \sigma = p\}$$

acts transitively on fibers of p. Equivalently, p is a Galois covering if the field extension $\mathcal{M}(R)/p^*\mathcal{M}(C)$ is a Galois extension. In case p is a Galois covering, for the corresponding Galois group the isomorphism

$$\mathrm{Aut}(R,p) \cong \mathrm{Gal}\left(\mathcal{M}(R)/p^*\mathcal{M}(C)\right)$$

holds. Notice that since the action of $\mathrm{Aut}(C,p)$ on fibers of p has no fixed points, p is a Galois covering if and only if the equality

$$|\mathrm{Aut}(R,p)| = \deg p \tag{20}$$

holds.

Let $f : C \to \mathbb{CP}^1$ be an arbitrary holomorphic map between compact Riemann surfaces. Then the *normalization* of f is defined as a compact Riemann surface N_f together with a holomorphic Galois covering of the lowest possible degree $\widehat{f} : N_f \to \mathbb{CP}^1$ such that

$$\widehat{f} = f \circ h$$

for some holomorphic map $h : N_f \to C$. The map $\widehat{f}$ is defined up to the change $\widehat{f} \to \widehat{f} \circ \alpha$, where $\alpha \in \mathrm{Aut}(N_f)$, and is characterized by the property that the field extension $\mathcal{M}(N_f)/\widehat{f}^*\mathcal{M}(\mathbb{CP}^1)$ is isomorphic to the Galois closure $\widetilde{\mathcal{M}(C)}/f^*\mathcal{M}(\mathbb{CP}^1)$ of the extension $\mathcal{M}(C)/f^*\mathcal{M}(\mathbb{CP}^1)$.

The main technical tool for working with reducible fiber products is the following result of Fried (see [13], Proposition 2, or [31], Theorem 3.5).

Theorem 2.8. *Assume that the fiber product of holomorphic maps between compact Riemann surfaces $f : C_1 \to \mathbb{CP}^1$ and $g : C_2 \to \mathbb{CP}^1$ is reducible. Then there exist holomorphic maps between compact Riemann surfaces $f_1 : R_1 \to \mathbb{CP}^1$, $g_1 : R_2 \to \mathbb{CP}^1$, and $f_2 : C_1 \to R_1$, $g_2 : C_2 \to R_2$ such that*

$$f = f_1 \circ f_2, \quad g = g_1 \circ g_2, \tag{21}$$

$n(f,g) = n(f_1,g_1)$, and $\widehat{f_1} = \widehat{g}_1$. □

Notice that both f_1 and g_1 must have degree at least two; otherwise $n(f_1,g_1) = 1$, which contradicts the assumption $n(f,g) > 1$. Notice also that the fiber product of f and g is always reducible if equalities (21) hold for *equal* f_1 and g_1 of degree at least two. In this case the condition $\widehat{f_1} = \widehat{g}_1$ is trivially satisfied. In general, the reducibility of the fiber product of f and g does not imply that f and g have a common compositional left factor of degree at least two. Nonetheless, Fried's theorem states that f and g at least have compositional left factors with the same normalization.

3. Proof of Theorem 1.1

Let us begin by noting that if, for a rational function f and an orbifold $\mathcal{O}$ on $\mathbb{CP}^1$, the inequality $f^*(\mathcal{O}) > 0$ holds (including the cases where $f^*(\mathcal{O})$ is non-ramified or bad), then $\chi(\mathcal{O}) > 0$. Indeed, it follows from (9) that

$$\chi(\mathcal{O}) \deg f \geq \chi(f^*(\mathcal{O})) > 0,$$

whence $\chi(\mathcal{O}) > 0$. We will use this fact below without explicitly mentioning it.

The simplest way to prove Theorem 1.1 is by using Proposition 2.1 as follows.

The first proof of Theorem 1.1. Since the sphere is simply connected, the universal covering of the non-ramified orbifold $\mathbb{CP}^1 = f^*(\mathcal{O})$ is the pair $(\mathbb{CP}^1, id)$. Therefore, Proposition 2.1 implies that $f = \theta_{\mathcal{O}} \circ q$ for some rational function q. □

Notice that in the above proof we used only that $f : \mathbb{CP}^1 \to \mathcal{O}$ is a holomorphic map between orbifolds, without the minimality assumption. However, it is easy to see that if $\mathcal{O}_1$ is non-ramified, then both conditions (7) and (10) reduce to the same condition that $\nu_2(f(z))$ divides $\deg_z f$.

Let $h : R \to C$ be a holomorphic map between compact Riemann surfaces. We say that h is *uniform*, if the orbifold $\mathcal{O}_1^h$ is non-ramified. Notice that every Galois covering is uniform, but the inverse is not true in general.

The second way to prove Theorem 1.1 is by using the following statement.

Theorem 3.1. *Let f, g, p, q be holomorphic maps between compact Riemann surfaces such that the diagram*

$$\begin{array}{ccc} R & \xrightarrow{\ q\ } & C_2 \\ \Big\downarrow{\scriptstyle p} & & \Big\downarrow{\scriptstyle g} \\ C_1 & \xrightarrow{\ f\ } & \mathbb{CP}^1 \end{array}$$

commutes, and p and q have no non-trivial common compositional right factor. Assume that g is uniform and $f^*(\mathcal{O}_2^g)$ is non-ramified. Then p is unbranched.

Proof. Let us set

$$F = f \circ p = g \circ q.$$

It follows from the description of the fiber product of f and g in terms of the monodromy groups of f and g (see [31], Section 2), or, in the more algebraic setting, from the Abhyankar lemma (see e. g. [46], Theorem 3.9.1) that for every point t_0 of R the equality

$$e_F(t_0) = \operatorname{lcm}\big(e_f(p(t_0)), e_g(q(t_0))\big) \tag{22}$$

holds, where $e_h(t)$ denotes the local multiplicity of a holomorphic map h at a point t. Since g is uniform,

$$e_g(q(t_0)) = \nu_2^g\big(g(q(t_0))\big) = \nu_2^g\big(f(p(t_0))\big). \tag{23}$$

On the other hand, since $f^*(\mathcal{O}_2^g)$ is non-ramified,

$$\nu_2^g(f(p(t_0)) \mid e_f(p(t_0)).$$

It follows now from (22) and (23) that

$$e_F(t_0) = e_f(p(t_0)), \qquad t_0 \in R,$$

implying by the chain rule that $e_p(t_0) = 1$, for all $t_0 \in R$. □

The second proof of Theorem 1.1. It follows from the Riemann-Hurwitz formula that a holomorphic map between compact Riemann surfaces $p : R \to \mathbb{CP}^1$ is unramified if and only if $\deg p = 1$. Therefore, applying Theorem 3.1 to a component of the fiber product of f and $g = \theta_{\mathcal{O}}$, we conclude that $f = \theta_{\mathcal{O}} \circ q$ for some rational function q. □

To formulate corollaries of Theorem 1.1 concerning non-realizable coverings, it is convenient to associate to each triple of integers (a, b, c) greater than one certain characteristics derived from those of orbifolds with the corresponding signatures. Namely, for a triple (a, b, c), we define $\chi(a, b, c) = \chi(\mathcal{O})$, where $\mathcal{O}$ is an orbifold with signature $\{a, b, c\}$. Furthermore, for triples with $\chi(a, b, c) > 0$, we introduce additional parameters:

$$n(a, b, c) = \deg \theta_{\mathcal{O}}, \quad l(a, b, c) = \operatorname{lcm}(a, b, c).$$

Notice that since the Euler characteristic of a non-ramified orbifold on $\mathbb{CP}^1$ equals two, we have

$$\chi(a, b, c) = \frac{2}{n(a, b, c)},$$

by (8). In addition, it is easy to see that

$$l(a, b, c) = \frac{n(a, b, c)}{2}, \tag{24}$$

unless $(a, b, c) = (2, 2, d)$, where d is odd, in which case $l(a, b, c) = n(a, b, c)$.

In this notation, the following statement holds.

Corollary 3.2. *Let $\Pi = (\Pi_1, \Pi_2, \ldots, \Pi_q, n, 0)$ be a branch datum. Assume that there exists a triple of integers greater than one (a, b, c) such that all entries of Π_1 are divisible by a, all entries of Π_2 are divisible by b, and all entries of Π_3 are divisible by c. Then $\chi(a, b, c) > 0$ and Π is non-realizable, unless n is divisible by $n(a, b, c)$.*

Proof. Let f be a rational function whose branch datum satisfies the conditions of the corollary. Then for the orbifold $\mathcal{O}$ defined by the equalities

$$\nu(z_1) = a, \quad \nu(z_2) = b, \quad \nu(z_3) = c,$$

where z_1, z_2, z_3 are critical values of f corresponding to the partitions Π_1, Π_2, Π_3, the orbifold $f^*(\mathcal{O})$ is non-ramified. Therefore, $n(a, b, c)$ must divide $\deg f$ by Theorem 1.1. $\square$

For a partition $\Pi = (\pi_1, \pi_2, \ldots, \pi_p)$ of a number n, we define the number $d(\Pi)$ by the formula

$$d(\Pi) = n - p.$$

The following proposition demonstrates the existence of branch data whose non-realizability results from Corollary 3.2 and provides a method for the practical construction of such data.

Proposition 3.3. *Let (a, b, c) be a triple of integers greater than one with $\chi(a, b, c) > 0$, distinct from $(2, 2, d)$, where d is odd. Then for any integer of the form*

$$n = \frac{n(a, b, c)}{2} k, \quad k \geq 2, \tag{25}$$

where k is odd, there exists a non-realizable branch datum $\Pi = (\Pi_1, \Pi_2, \ldots, \Pi_q, n, 0)$ such that all entries of Π_1 are divisible by a, all entries of Π_2 are divisible by b, and all entries of Π_3 are divisible by c.

Proof. If $\Pi = (\Pi_1, \Pi_2, \dots, \Pi_q, n, 0)$, where n has the form (25), is a branch datum such that

$$\Pi_1 = (au_1, au_2, \dots, au_{p_1}), \quad \Pi_2 = (bv_1, bv_2, \dots, bv_{p_2}), \quad \Pi_3 = (cw_1, \dots, cw_{p_3}), \tag{26}$$

then

$$\sum_{i=1}^{p_1} u_i = \frac{n}{a}, \qquad \sum_{j=1}^{p_2} v_j = \frac{n}{b}, \qquad \sum_{e=1}^{p_3} w_e = \frac{n}{c}, \tag{27}$$

implying that

$$\begin{aligned}\sum_{i=1}^{p_1}(u_i-1)+\sum_{j=1}^{p_2}(v_j-1)+\sum_{e=1}^{p_3}(w_e-1) &= \frac{n}{a}+\frac{n}{b}+\frac{n}{c}-(p_1+p_2+p_3) = \\ = \frac{n}{a}+\frac{n}{b}+\frac{n}{c}-n-(p_1+p_2+p_3-n) &= n\chi(a,b,c)-(p_1+p_2+p_3-n).\end{aligned}$$

Furthermore, by (1),

$$p_1+p_2+p_3 = (q-2)n+2-\sum_{i=4}^{q} p_i = n+2+\sum_{i=4}^{q}(n-p_i) = n+(2+\sum_{i=4}^{q} d(\Pi_i)), \tag{28}$$

and hence

$$\begin{aligned}\sum_{i=1}^{p_1}(u_i-1)+\sum_{j=1}^{p_2}(v_j-1)+\sum_{e=1}^{p_3}(w_e-1) &= n\chi(a,b,c)-(2+\sum_{i=4}^{q} d(\Pi_i)) = \\ &= \frac{n}{\frac{n(a,b,c)}{2}}-(2+\sum_{i=4}^{q} d(\Pi_i)) = k-(2+\sum_{i=4}^{q} d(\Pi_i)).\end{aligned} \tag{29}$$

In the other direction, if Π_1, Π_2, Π_3 and $\Pi_4, \dots \Pi_q$ are partitions of a number n given by (25) such that Π_1, Π_2, Π_3 have the form (26) and equality (29) holds, then the collection $\Pi = (\Pi_1, \Pi_2, \dots, \Pi_q, n, 0)$ satisfies (28) and therefore is a branch datum. Moreover, since k is odd, all such data are non-realizable by Corollary 3.2.

Finally, it is easy to see that if $q = 3$, or, more generally, if $q \geq 3$ and $\Pi_4, \dots \Pi_q$ are arbitrary partitions of n given by (25) satisfying the additional condition

$$\sum_{i=4}^{q} d(\Pi_i) \leq k-2, \tag{30}$$

then the system (27), (29) has solutions in u_i, v_j, w_e. Indeed, condition (30) implies that the right-hand side of (29) is non-negative. Therefore, setting

$$d = \sum_{i=4}^{q} d(\Pi_i)$$

to simplify notation, one obtains a solution, for example, by taking $u_1 = k-d-1$ and assigning all other u_i, v_j, and w_e the value one, in such a way that (27) is satisfied. The last requirement can always be fulfilled since (24) implies that the right-hand side of the equalities in (27) are integers and the number n/a satisfies

$$\frac{n}{a} = \frac{n(a,b,c)}{2a}k = \frac{l(a,b,c)}{a}k \geq k > k-d-1.$$

The branch datum corresponding to this specific solution is

$$\Pi(k) = \big((a(k-d-1), a^{n/a-(k-d-1)}), (b^{n/b}), (c^{n/c}), \Pi_4, \dots \Pi_q, n, 0\big). \tag{31}$$

$\square$

Concrete examples of non-realizable branch data of the form (31) for the triples $(2,3,3)$ and $(2,3,5)$ with $q=k=3$ are given in the introduction. In addition to these, let us present a similar example for the triple $(2,3,3)$ with $q=4$, assuming, for instance, that $\Pi_4=(2,1^{n-1})$. In this case, $d=1$, and (30) is satisfied for $k\geq 3$. In particular, for $k=3$ we obtain the following non-realizable branch datum:

$$\big((2^9),(3^6),(3^6),(2,1^{16}),18,0\big).$$

In conclusion, note that the set of data for which non-realizability follows from Theorem 1.1 is certainly not restricted to the case where $\deg f$ is not divisible by $\deg\theta_{\mathcal{O}}$, as in Proposition 3.3. Let us consider, for instance, the branch datum (31) supposing that

$$d\leq \frac{k}{2}-2, \tag{32}$$

where k is even, and $a\geq b\geq c$. By Theorem 1.1, a rational function f with such a branch datum has the form $f=\theta_{\mathcal{O}}\circ q$ for some rational function q of degree $k/2$. On the other hand, it follows easily from $a\geq b\geq c$ by the chain rule that such q must have a critical point of order at least $k-d-1$. Since $k-d-1>k/2=\deg q$ whenever (32) holds, we obtain a contradiction. Thus, under the above conditions (31) is non-realizable. The simplest example of this sort is the branch datum

$$\big((9,3^5),(3^8),(2^{12}),24,0\big),$$

where $a=b=3$, $c=2$, and $k=4$.

4. Proof of Theorem 1.2

The proof of Theorem 1.2 relies on a combination of Fried's theorem and a classical result of complex analysis regarding existence of a universal cover, which was mentioned in the introduction. We recall that this result states that a universal covering for an orbifold $\mathcal{O}=(R,\nu)$ exists and is unique up to a conformal isomorphism of $\widetilde{R}$ if and only if $\mathcal{O}$ is not bad (see e. g. [11], Section IV.9.12). In fact, we only need the following corollary of this result also mentioned in the introduction.

Lemma 4.1. *Let $\mathcal{O}_1=(\mathbb{CP}^1,\nu_1)$ and $\mathcal{O}_2=(\mathbb{CP}^1,\nu_2)$ be orbifolds, and f a rational function. Assume that $f:\mathcal{O}_1\rightarrow\mathcal{O}_2$ is a covering map between orbifolds, and $\mathcal{O}_2$ is good. Then $\mathcal{O}_1$ is also good.*

Proof. Let $\theta_2:\widetilde{\mathcal{O}}_2\rightarrow\mathcal{O}_2$ be a universal covering of $\mathcal{O}_2$, and f^{-1} a germ of the algebraic function inverse to f. Let us define θ_1 as a complete analytic continuation of the germ $f^{-1}\circ\theta_2$. Since $f:\mathcal{O}_1\rightarrow\mathcal{O}_2$ is a covering map,

$$\nu_2^f(z)\mid\nu_2(z),\quad z\in\mathbb{CP}^1,$$

by (18). On the other hand,

$$\nu_2(\theta_2(z))=\deg{}_z\theta_2,\quad z\in\widetilde{\mathcal{O}}_2,$$

since θ_2 is uniform. Thus,

$$\nu_2^f(\theta_2(z))\mid\nu_2(\theta_2(z))=\deg{}_z\theta_2,\quad z\in\widetilde{\mathcal{O}}_2.$$

By the definition of $\mathcal{O}_2^f$ and θ_1, this implies that θ_1 has no local branching. Therefore, since $\widetilde{\mathcal{O}}_2$ is simply connected, θ_1 is single valued, and it is clear that the equality $f\circ\theta_1=\theta_2$ holds.

Since $\theta_2 : \widetilde{\mathcal{O}_2} \to \mathcal{O}_2$ is a covering map between orbifolds, it follows from the last equality by Corollary 2.5 that

$$f : f^*(\mathcal{O}_2) \to \mathcal{O}_2, \quad \theta_1 : \widetilde{\mathcal{O}}_2 \to f^*(\mathcal{O}_2)$$

are covering map between orbifolds. Since $f^*(\mathcal{O}_2) = \mathcal{O}_1$ by (11), and the orbifold $\widetilde{\mathcal{O}}_2$ is non-ramified, we conclude that θ_1 is a universal covering of $\mathcal{O}_1$. Thus, $\mathcal{O}_1$ is good. □

Lemma 4.2. *Assume that the fiber product of holomorphic maps between compact Riemann surfaces $f : C_1 \to \mathbb{CP}^1$ and $g : C_2 \to \mathbb{CP}^1$, where g is a Galois covering, is reducible. Then there exist holomorphic maps between compact Riemann surfaces $u : C_1 \to R$, $v : C_2 \to R$, and $h : R \to \mathbb{CP}^1$ such that:*

(1) The inequality $\deg h \geq 2$ holds,
(2) The equalities $f = h \circ u$, $g = h \circ v$ hold,
(3) The minimal holomorphic map between orbifolds $h : h^*(\mathcal{O}_2^g) \to \mathcal{O}_2^g$ is a covering map.

Proof. By Theorem 2.8, there exist holomorphic maps between compact Riemann surfaces $f_1 : R_1 \to \mathbb{CP}^1$, $g_1 : R_2 \to \mathbb{CP}^1$, and $f_2 : C_1 \to R_1$, $g_2 : C_2 \to R_2$ such that equalities (21) hold, $n(f_1, g_1) = n(f, g)$, and $\widehat{f_1} = \widehat{g}_1$. Moreover, f_1 and g_1 have degree at least two.

Let us show that f_1 is a compositional left factor of g. Clearly, f_1 is a compositional left factor of $\widehat{f_1}$ and hence of $\widehat{g}_1$ since $\widehat{f_1} = \widehat{g}_1$. On the other hand, since g_1 is a compositional left factor of a Galois covering g, and $\widehat{g}_1$ is a minimal Galois covering that factors through g_1, $\widehat{g}_1$ is a compositional left factor of g. Thus, f_1 is a compositional left factor of g.

The above implies that the first two conclusions of the lemma are satisfied for $R = R_1$, $h = f_1$, $u = f_2$, and a convenient holomorphic map $v : C_2 \to R$. Finally, the last conclusion follows from the equality $g = h \circ v$ by Corollary 2.5 since $g : \mathcal{O}_1^g \to \mathcal{O}_2^g$ is a covering map between orbifolds. □

Below, we will frequently use the fact that for a map f and orbifolds $\mathcal{O}_1$, $\mathcal{O}_2$ the condition $f^*(\mathcal{O}_2) = \mathcal{O}_1$ is simply a reformulation of the condition that $f : \mathcal{O}_1 \to \mathcal{O}_2$ is a minimal holomorphic map between orbifolds. In particular, Theorem 1.2 can be considered as a generalization of Lemma 4.1, stating that its conclusion remains true when the condition that f is a covering map between orbifolds is replaced by the weaker condition that f is a minimal holomorphic map between orbifolds.

For a holomorphic map f between compact Riemann surfaces, we denote by $r(f)$ the largest integer r such that f can be written as a composition of r holomorphic maps between compact Riemann surfaces, each of degree at least two.

Proof of Theorem 1.2. We will prove the theorem by induction on $r(f)$. First, let us assume that $r(f) = 1$, meaning f is indecomposable. Let us consider the fiber product f and $g = \theta_{\mathcal{O}}$. Since $\mathcal{O}_2^{\theta_{\mathcal{O}}} = \mathcal{O}$, if this product is irreducible, then, by Theorem 2.7,

$$f^*(\mathcal{O}) = \mathcal{O}_2^q$$

for some holomorphic map between compact Riemann surfaces $q : R \to \mathbb{CP}^1$, implying that $f^*(\mathcal{O})$ is good by Lemma 2.6. On the other hand, if the fiber product of f and $\theta_{\mathcal{O}}$ is reducible, then it follows from Lemma 4.2, taking into account that f is indecomposable, that

$$f : f^*(\mathcal{O}) \to \mathcal{O}$$

is a covering map. In this case $f^*(\mathcal{O})$ is good by Lemma 4.1.

Let us now assume that $r(f) > 1$. Applying Theorem 2.7 and Lemma 4.2 in the same manner as before, we deduce that either the fiber product of f and $\theta_{\mathcal{O}}$ is irreducible and $f^*(\mathcal{O})$ is good, or there exist rational functions h and u, with $\deg h \geq 2$, such that

$$f = h \circ u, \tag{33}$$

and

$$h : h^*(\mathcal{O}) \to \mathcal{O}$$

is a covering map between orbifolds. By Lemma 4.1, the orbifold $h^*(\mathcal{O})$ is good. Moreover, it follows from (33) by Corollary 2.5 that

$$u : f^*(\mathcal{O}) \to h^*(\mathcal{O})$$

is a minimal holomorphic map between orbifolds. As $r(u) < r(f)$ by construction, the induction assumption implies that $f^*(\mathcal{O})$ is good. $\square$

Theorem 1.2 implies the following corollary.

Corollary 4.3. *A branch datum $\Pi = (\Pi_1, \Pi_2, \dots, \Pi_q, n, 0)$ is non-realizable whenever there exist integers a, b, c, each greater than one, such that one of the following conditions holds:*

Condition 1:

(1) All entries of Π_1 are divisible by a.
(2) All entries of Π_2 are divisible by b.
(3) All entries of Π_3 except one are divisible by c.

Condition 2:

(1) All entries of Π_1 are divisible by a.
(2) All entries of Π_2 are divisible by b.
(3) All entries of Π_3 are divisible by c except for two entries u and v.
(4) $\gcd(u, c) \neq \gcd(v, c)$.

Condition 3:

(1) All entries of Π_1 are divisible by a.
(2) All entries of Π_2 are divisible by b except for one entry u.
(3) All entries of Π_3 are divisible by c except for one entry v.
(4) $\frac{b}{\gcd(u,b)} \neq \frac{c}{\gcd(v,c)}$.

Proof. Assume that a rational function f with a branch datum satisfying to one of the above conditions exists. Then for the orbifold $\mathcal{O}$ defined by the equalities

$$\nu(z_1) = a, \quad \nu(z_2) = b, \quad \nu(z_3) = c,$$

where z_1, z_2, z_3 are critical values of f corresponding to the partitions Π_1, Π_2, Π_2, the orbifold $f^*(\mathcal{O})$ is bad in contradiction with Theorem 1.2. $\square$

Corollary 4.3 allows us to construct many non-realizable branch data. For instance, by slightly modifying the series (4) and the argument from the introduction, we obtain the non-realizable series

$$\big((2^k), (2^{k-2}, 4), (k - s, k - s, 2s), 2k, 0\big), \quad k > 3s.$$

Similarly, we see that, along with the series (5), the series

$$\big((2^l), (3^k, s), (5^m), n, 0\big), \quad s \not\equiv 0 \ (\mathrm{mod}\ 3), \tag{34}$$

$$\big((2^l, s), (3^k), (5^m), n, 0\big), \quad s \not\equiv 0 \pmod 2, \tag{35}$$

or, for instance, the series

$$\big((2^l), (3^k, d), (5^m, s), n, 0\big), \tag{36}$$

where at least one of the conditions $d \not\equiv 0 \pmod 3$, $s \not\equiv 0 \pmod 5$ holds, are also non-realizable. Many other examples can be obtained through calculations similar to those performed in Section 3, by considering branch data such that in the first three partitions all entries but one or two are divisible by suitable numbers a, b, c, though not necessarily equal to those numbers as in (5), (34), (35), (36).

To prove Corollary 1.3 and Corollary 1.4, we will use the link between Hurwitz existence problem and "dessins d'enfants" theory. Below we briefly list necessary definitions and results referring for more detail and proofs to Chapter 2 of [25].

A rational function f is called a Belyi function if it does not have critical values outside the set $\{0, 1, \infty\}$. Let us take the segment $[0, 1]$, color the point 0 in black and the point 1 in white, and consider the preimage $D = f^{-1}([0, 1])$; we will call this preimage a dessin. The dessin $D = f^{-1}([0, 1])$ is a connected planar graph, which has a bipartite structure: black vertices are preimages of 0, and white vertices are preimages of 1. The degrees of the black vertices are local degrees of f at points of $f^{-1}\{0\}$, and the degrees of the white ones are local degrees of f at points of $f^{-1}\{1\}$. Thus, the sum of the degrees in both cases is equal to $n = \deg f$, which is also the number of edges. Since the graph D is bipartite, the number of edges surrounding each face is even; it is convenient to define the face degree as this number divided by two. Then the sum of the face degrees is also equal to $n = \deg f$. In more detail, inside each face there is a single pole of f, and the multiplicity of this pole is equal to the degree of the face.

The above construction works in the opposite direction as well. Specifically, for every bicolored connected planar graph M, there exists a Belyi function f, unique up to a change $f \to f \circ \mu$, where μ is a Möbius transformation, such that the dessin $D = f^{-1}([0, 1])$ is isomorphic to M in the following sense: there exists an orientation-preserving homeomorphism of the sphere that transforms M into D, respecting the colors of the vertices.

The correspondence between dessins and Belyi functions implies that a rational function with a branch datum $\Pi = (\Pi_1, \Pi_2, \Pi_3, n, 0)$ exists if and only if there exists a bicolored graph with black vertices having valencies Π_1, white vertices having valencies Π_2, and faces having valencies Π_3. This fact is of fundamental importance and is widely used in works addressing the Hurwitz problem.

Proof of Corollary 1.3 and Corollary 1.4. Any connected planar graph G with n edges can be transformed into a bipartite graph G' with $2n$ edges by labeling all vertices of G as "black" and introducing new "white" vertices at the midpoints of the edges of G. Moreover, if $\Pi = (\Pi_1, \Pi_2, \Pi_3, n, 0)$ is the branch data of the corresponding Belyi function, then Π_1 represents the list of vertex degrees of G, Π_3 represents the list of face degrees of G, and $\Pi_2 = (2^n)$. Therefore, both statements follow from Theorem 1.2 applied to the Belyi function for G'. □

Notice that the graphs considered in Corollary 1.3 and Corollary 1.4 may include loops, with the convention that a loop is counted as contributing two units to the degree of its endpoint. Additionally, these corollaries hold true if we switch the roles of vertices and faces in the formulations, as we can pass to the dual graph.

Let us recall that, compared to dessins d'enfants, a more classical approach to studying the Hurwitz problem is through the consideration of special permutation groups. For example, the existence of a rational function with the branch datum $\Pi = (\Pi_1, \dots, \Pi_q, n, 0)$ is equivalent to the existence of permutations $\alpha_1, \alpha_2, \dots, \alpha_q$ in S_n satisfying the following three conditions (see e.g. [9]):

(i) The group generated by α_i, $1 \le i \le q$, in S_n is transitive.
(ii) The total number of cycles in α_i, $1 \le i \le q$, is $(q-2)n+2$.
(iii) The lengths of the cycles of α_i, $1 \le i \le q$, form the partition Π_i, $1 \le i \le q$.

Thus, Theorem 1.2 implies the following statement.

Corollary 4.4. *Let $\alpha_1, \alpha_2, \dots, \alpha_q \in S_n$ be permutations satisfying conditions (i) and (ii). Then for any integers a, b, c greater than one none of the following conditions can be true:*

CONDITION 1:

(1) Lengths of all cycles of α_1 are divisible by a.
(2) Lengths of all cycles of α_2 are divisible by b.
(3) Lengths of all cycles of α_3 except one are divisible by c.

CONDITION 2:

(1) Lengths of all cycles of α_1 are divisible by a.
(2) Lengths of all cycles of α_2 are divisible by b.
(3) Lengths of all cycles of α_3 are divisible by c except for two cycles of lengths u and v.
(4) $\gcd(u, l) \neq \gcd(v, l)$.

CONDITION 3:

(1) Lengths of all cycles of α_1 are divisible by a.
(2) Lengths of all cycles of α_2 are divisible by b except for one cycle of length u.
(3) Lengths of all cycles of α_3 are divisible by c except for one cycle of length v.
(4) $b/\gcd(b, u) \neq c/\gcd(c, v)$.

We remind that the proof of Theorem 1.2 critically relies on the analytical theorem regarding the existence of a universal covering of an orbifold. On the other hand, Corollaries 1.3, 1.4, and 4.4 are formulated in discrete terms and seemingly have no connection to analysis. In this context, the following question appears interesting: is there a purely geometric proof for Corollaries 1.3 and 1.4, and a purely algebraic proof for Corollary 4.4?

5. Proof of Theorem 1.5 and Theorem 1.7

We start by proving the following statement of independent interest.

Theorem 5.1. *Let f, p, g, q be holomorphic maps between compact Riemann surface such that the diagram*

$$\begin{array}{ccc} R & \xrightarrow{q} & C_2 \\ \Big\downarrow{\scriptstyle p} & & \Big\downarrow{\scriptstyle g} \\ C_1 & \xrightarrow{f} & \mathbb{CP}^1 \end{array} \tag{37}$$

commutes, the fiber product of f and g is irreducible, and p and q have no non-trivial common compositional right factor. Assume that g and p are Galois coverings. Then $\mathrm{Aut}(R, p) \cong \mathrm{Aut}(C_2, g)$.

Proof. We construct the isomorphism $\mathrm{Aut}(R,p)$ and $\mathrm{Aut}(C_2,g)$ explicitly by modifying the proof of Theorem 5.1 in [32]. Specifically, we show that for every $\sigma \in \mathrm{Aut}(R,p)$ the equality

$$q \circ \sigma = \varphi(\sigma) \circ q \tag{38}$$

holds for some $\varphi(\sigma) \in \mathrm{Aut}(C_2,g)$, and the correspondence $\sigma \to \varphi(\sigma)$ is an isomorphism of groups.

Clearly, the commutativity of (37) implies that for every $\sigma \in \mathrm{Aut}(R,p)$ the equality

$$g \circ (q \circ \sigma) = g \circ q$$

holds. On the other hand, for the fiber product of g with itself, the functions p_j, q_j in (12) are

$$p_j = \mu_j, \quad q_i = id, \quad \mu_j \in \mathrm{Aut}(C_2,g).$$

Indeed, clearly, p_j, q_j defined in this way satisfy (13) and have no non-trivial common compositional right factor. Moreover, since

$$\sum_j \deg \mu_i = |\mathrm{Aut}(C_2,g)| = \deg g,$$

by (20), these functions exhaust all p_j, q_j in (12) by (16). Thus, the universality property

$$q \circ \sigma = p_j \circ w, \quad q = q_i \circ w$$

reduces to equality (38) for some $\varphi(\sigma) \in \mathrm{Aut}(C_2,g)$. Finally, one can easily see that the correspondence $\sigma \mapsto \varphi(\sigma)$ defines a homomorphism of groups.

Since $\deg p = \deg g$ by (17), it follows from (20) that $|\mathrm{Aut}(R,p)| = |\mathrm{Aut}(C_2,g)|$. Thus, to complete the proof, we only need to show that the group $\mathrm{Ker}\,\psi$ is trivial. To this end, observe that if $\mathrm{Ker}\,\psi$ is non-trivial, then the set of meromorphic functions h on R satisfying the condition

$$h \circ \sigma = h, \qquad \sigma \in \mathrm{Ker}\,\psi,$$

form a subfield k of $\mathcal{M}(R)$ distinct from $\mathcal{M}(R)$. Clearly, q belongs to k. Moreover, p also belongs to k, since $\mathrm{Ker}\,\psi$ is a subgroup of $\mathrm{Aut}(R,p)$. Thus, if $\mathrm{Ker}\,\psi$ is non-trivial,

$$p^*\mathcal{M}(C_1) \cdot q^*\mathcal{M}(C_2) \neq \mathcal{M}(R),$$

in contradiction with the assumption that p and q have no non-trivial common compositional right factor. □

Lemma 5.2. *Let $\mathcal{O}_1$ and $\mathcal{O}_2$ be orbifolds of positive Euler characteristic. Then $\nu(\mathcal{O}_1) = \nu(\mathcal{O}_2)$ if and only if there exists a Möbius transformation w such that $w : \mathcal{O}_1 \to \mathcal{O}_2$ is a covering map between orbifolds.*

Proof. The proof follows easily from the definition of a covering map, combined with the following facts: orbifolds of positive Euler characteristic are ramified at most at three points, and any triple of points on the sphere can be transformed into any other triple by a suitable Möbius transformation.

Proof of Theorem 1.5. To prove the first part of the theorem it is enough to observe that the composition

$$f \circ \theta_{f^*(\mathcal{O})} : \mathbb{CP}^1 \to \mathcal{O} \tag{39}$$

is a minimal holomorphic map between orbifolds by Corollary 2.4. Thus,

$$f\circ\theta_{f^*(\mathcal{O})}=\theta_{\mathcal{O}}\circ q \tag{40}$$

for some rational function q by Theorem 1.1.

The proof of the second part proceeds by induction on $r(f)$. Let us assume first that $r(f)=1$, and consider the fiber product of f and $g=\theta_{\mathcal{O}}$. If this fiber product is reducible, then by Lemma 4.2, taking into account that f is indecomposable and $\mathcal{O}_2^{\theta_{\mathcal{O}}}=\mathcal{O}$, we conclude that $f:f^*(\mathcal{O})\to\mathcal{O}$ is a covering map between orbifolds. By Corollary 2.4 this implies that (39) is also a covering map, and hence $f\circ\theta_{f^*(\mathcal{O})}=\theta_{\mathcal{O}}$ by the uniqueness of a universal covering. Thus, in this case, the equalities

$$f=w\circ t,\quad \theta_{\mathcal{O}}=w\circ\theta_{f^*(\mathcal{O})}. \tag{41}$$

hold for $w=f$, $t=id$. On the other hand, if the fiber product of f and $g=\theta_{\mathcal{O}}$ is irreducible, then applying Theorem 5.1 to equality (40), we conclude that

$$\mathrm{Aut}(\mathbb{CP}^1,\theta_{f^*(\mathcal{O})})\cong\mathrm{Aut}(\mathbb{CP}^1,\theta_{\mathcal{O}}).$$

Since for a rational Galois covering $\theta_{\mathcal{O}}$, the signature of $\mathcal{O}$ is defined by the group $\mathrm{Aut}(\mathbb{CP}^1,\theta_{\mathcal{O}})$, this implies by Lemma 5.2 that there exists a Möbius transformation w such that $w:f^*(\mathcal{O})\to\mathcal{O}$ is a covering map between orbifolds. Thus, in this case equalities (41) hold for this w and $t=w^{-1}\circ f$.

Assume now that $r(f)>1$. If the fiber product of f and g is irreducible, we conclude as above that equalities (41) hold for some rational function w of degree one. Now, let us suppose that the fiber product of f and g is reducible, and consider the rational functions h, $\deg h\geq 2$, and u, provided by Lemma 4.2, such that

$$f=h\circ u \tag{42}$$

and

$$h:h^*(\mathcal{O})\to\mathcal{O} \tag{43}$$

is a covering map. The equality (42) implies that

$$u:f^*(\mathcal{O})\to h^*(\mathcal{O}) \tag{44}$$

is a minimal holomorphic map between orbifolds by Corollary 2.5. Thus, since

$$u^*(h^*(\mathcal{O}))=f^*(\mathcal{O})$$

by Theorem 2.3, and $r(u)<r(f)$ by construction, it follows from the induction assumption applied to minimal holomorphic map (44) and the good orbifold $h^*(\mathcal{O})$ that there exist rational functions w' and t such that the equalities

$$u=w'\circ t,\qquad \theta_{h^*(\mathcal{O})}=w'\circ\theta_{f^*(\mathcal{O})}$$

hold, and

$$w':f^*(\mathcal{O})\to h^*(\mathcal{O}) \tag{45}$$

is a covering map between orbifolds.

Since (43) is a covering map, it follows from Corollary 2.4 that

$$h\circ\theta_{h^*(\mathcal{O})}:\mathbb{CP}^1\to\mathcal{O}$$

is also a covering map, implying by the uniqueness of a universal covering that

$$\theta_{\mathcal{O}}=h\circ\theta_{h^*(\mathcal{O})}=h\circ w'\circ\theta_{f^*(\mathcal{O})}.$$

Moreover, we have:

$$f = h \circ u = h \circ w' \circ t.$$

Thus, equalities (41) hold for

$$w = h \circ w'.$$

Finally, since (43) and (45) are covering maps,

$$w : f^*(\mathcal{O}) \to \mathcal{O}$$

is also covering map by Corollary 2.4. □

To illustrate how Theorem 1.5 can be used for proving non-realizability, we consider the series of branch data

$$\big((2^k), (2^{k-2}, 1, 3), (k, k), 2k, 0\big), \quad k \geq 2, \tag{46}$$

mentioned in the introduction, whose non-realizability is known (see [30], [37]). Assume that a rational function f realizing (46) exists, and let z_1, z_2, z_3 be critical values of f corresponding to the partitions (2^k), $(2^{k-2}, 1, 3)$, and (k, k). Then for the orbifold $\mathcal{O}$ on $\mathbb{CP}^1$ defined by the equalities

$$\nu(z_1) = 2, \quad \nu(z_2) = 2, \quad \nu(z_3) = k,$$

the orbifold $f^*(\mathcal{O})$ has the signature $\{2, 2\}$, implying by Theorem 1.5 that equalities (41) hold. Moreover, it follows from

$$\deg \theta_{f^*(\mathcal{O})} = 2, \quad \deg \theta_{\mathcal{O}} = 2k, \quad \deg f = 2k$$

that

$$\deg w = k, \quad \deg t = 2.$$

It is well known (see, e.g., [34], Section 4.2) that the branch datum of any compositional left factor w of $\theta_{\mathcal{O}}$ of degree k for even k has one of the following forms:

$$\left((2^{\frac{k}{2}}), (2^{\frac{k}{2}-1}, 1, 1), (k)\right), \quad \left((2^{\frac{k}{2}}), (2^{\frac{k}{2}}), (k/2, k/2)\right),$$

while for odd k, it has the form

$$\left((2^{\frac{k-1}{2}}, 1), (2^{\frac{k-1}{2}}, 1), (k)\right).$$

Now, using the chain rule, it is easy to see that by composing such w with a rational function t of degree two, it is impossible to obtain a rational function with the branch datum given in (46), since (46) contains the entry 3, and only once.

Notice that if the function w in Theorem 1.5 has degree one, then it follows from the fact that $w : f^*(\mathcal{O}) \to \mathcal{O}$ is a covering map between orbifolds by Lemma 5.2 that the rational function $\widehat{f} = f \circ w^{-1}$ is a minimal *self-holomorphic map* between orbifolds, $\widehat{f} : \mathcal{O} \to \mathcal{O}$. Such rational functions are called *generalized Lattès maps*, by analogy with ordinary Lattès maps, which can be defined as *self-covering* maps between orbifolds. Generalized Lattès maps play a crucial role in the study of semiconjugate rational functions (see [32], [33]). In particular, equation (40) can be written as

$$\widehat{f} \circ \theta_{\mathcal{O}} = \theta_{\mathcal{O}} \circ q.$$

It would be interesting to understand which branch data potentially corresponding to generalized Lattès maps are realizable. For ordinary Lattès maps, this question was resolved in [35].

Proof of Corollary 1.6. If f is indecomposable, the first equality in (41) implies that either $\deg w = 1$ or $\deg t = 1$. In the first case however $\nu(f^*(\mathcal{O})) = \nu(\mathcal{O})$ by Lemma 5.2, contradicting the assumption. Therefore, $\deg t = 1$, which implies that f is a compositional left factor of $\theta_{\mathcal{O}}$. □

Notice that results equivalent to Corollary 1.6 in certain special cases were previously obtained in the context of describing possible monodromy groups of indecomposable rational functions (see [15], Proposition 2.0.16–Corollary 2.0.18, and [29], Lemma 9.1). For example, the three items of Lemma 9.1 in [29] follow from Corollary 1.6 applied in the following cases:

- $\nu(\mathcal{O}_2) = (p, p)$, where p is a prime, and $\mathcal{O}_1$ is non-ramified.
- $\nu(\mathcal{O}_2) = (p, 2, 2)$, where p is a prime, and $\nu(\mathcal{O}_1) = (2, 2)$.
- $\nu(\mathcal{O}_2) = (2, 3, 3)$, and $\nu(\mathcal{O}_1) = (3, 3)$.

Proof of Theorem 1.7. The proof is similar to the proof of Theorem 1.5, and uses the induction on $r(f)$. Let us assume that $r(f) = 1$, and consider the fiber product of f and $g = \theta_{\mathcal{O}}$. First, we observe that this product cannot be reducible. Indeed, if it were, then by Lemma 4.2, considering that f is indecomposable, it would follow that f, which is defined on a torus, is a compositional left factor of $\theta_{\mathcal{O}}$, which is defined on the sphere. Thus, the fiber product of f and $g = \theta_{\mathcal{O}}$ is irreducible. Considering now the corresponding commutative diagram (19), where p and q have no non-trivial common compositional factor, and applying Theorem 2.7, we see that $f : \mathcal{O}_2^p \to \mathcal{O}$ is a minimal holomorphic map. Thus, the orbifold $\mathcal{O}_2^p = f^*(\mathcal{O})$ is non-ramified, implying that the holomorphic map p has no branching. As $g(C_1) = 1$, this condition implies by the Riemann-Hurwitz formula that $g(R) = 1$.

Let us recall now that any holomorphic map between compact Riemann surfaces of genus one is a Galois covering with an Abelian automorphism group (see [45], Theorem 4.10). On the other hand, the automorphism groups D_{2n}, $n > 2$, A_4, S_4, A_5 of $\theta_{\mathcal{O}}$ corresponding to orbifolds $\mathcal{O}$ with the signatures $\{2, 2, d\}$, $d > 2$, $\{2, 3, 3\}$, $\{2, 3, 4\}$, $\{2, 3, 5\}$ are non-Abelian. Since the automorphism groups of p and g are isomorphic by Theorem 5.1, we conclude that the signature of $\mathcal{O}$ is $\{d, d\}$, $d \geq 1$, or $\{2, 2, 2\}$. Hence, equalities

$$f = w \circ t, \quad \theta_{\mathcal{O}} = w \circ \theta_{w^*(\mathcal{O})} \tag{47}$$

trivially hold for $w = id$ and $t = f$.

Assume now that $r(f) > 1$. If the fiber product of f and $g = \theta_{\mathcal{O}}$ is irreducible, we conclude as above that equalities (47) trivially hold. On the other hand, if the fiber product of f and $g = \theta_{\mathcal{O}}$ is reducible, then considering the holomorphic maps h, $\deg h \geq 2$, and u, provided by Lemma 4.2, such that $f = h \circ u$ and

$$h : h^*(\mathcal{O}) \to \mathcal{O} \tag{48}$$

is a covering map, we see that

$$u : f^*(\mathcal{O}) \to h^*(\mathcal{O}) \tag{49}$$

is a minimal holomorphic map between orbifolds. Moreover, since (48) is a covering map, it follows from (8) that $\chi(h^*(\mathcal{O})) > 0$, implying that u is a map from a torus to the sphere, as no orbifolds on a torus have positive Euler characteristic.

Since $r(u) < r(f)$, it follows from the induction assumption applied to minimal holomorphic map (49) that there exist a rational function $w' : \mathbb{CP}^1 \to \mathbb{CP}^1$ and a holomorphic map $t : R \to \mathbb{CP}^1$ such that the equalities

$$u = w' \circ t, \qquad \theta_{h^*(\mathcal{O})} = w' \circ \theta_{w'^*(h^*(\mathcal{O}))}$$

hold and the signature of $w'^*(h^*(\mathcal{O}))$ is $\{d, d\}$, $d \geq 1$, or $\{2, 2, 2\}$. Moreover, since (48) is a covering map, the equality

$$\theta_{\mathcal{O}} = h \circ \theta_{h^*(\mathcal{O})}$$

holds, implying that

$$\theta_{\mathcal{O}} = h \circ w' \circ \theta_{w'^*(h^*(\mathcal{O}))}, \qquad f = h \circ w' \circ t.$$

Hence, equalities (47) hold for $w = h \circ w'$. Finally, since

$$w'^*(h^*(\mathcal{O})) = w^*(\mathcal{O})$$

by Theorem 2.3, the signature of $w^*(\mathcal{O})$ is $\{d, d\}$, $d \geq 1$, or $\{2, 2, 2\}$. □

As an example illustrating Theorem 1.7, we show that the seris of branch data

$$\big((2^k, k + 3), (3^{k+1}), (3^{k+1}), 3k + 3, 0\big), \qquad k \equiv 1 \,(\mathrm{mod}\, 4), \tag{50}$$

is non-realizable. Assume that f is a holomorphic map realizing (50), and let z_1, z_2, z_3 be critical values of f corresponding to the partitions $(2^k, k+3)$, (3^{k+1}), and (3^{k+1}). Then for the orbifold $\mathcal{O}$ on $\mathbb{CP}^1$ defined by the equalities

$$\nu(z_1) = 2, \qquad \nu(z_2) = 3, \qquad \nu(z_3) = 3,$$

the orbifold $f^*(\mathcal{O})$ is non-ramified, implying by Theorem 1.7 that equalities (47) hold. On the other hand, any decomposition of $\theta_{\mathcal{O}}$ into a composition of indecomposable rational functions of degree at least two has either the form

$$\theta_{\mathcal{O}} = w_1 \circ w_2 \circ w_3,$$

where $\deg w_1 = 3$, $\deg w_2 = 2$, $\deg w_3 = 2$, or the form

$$\theta_{\mathcal{O}} = s_1 \circ s_2,$$

where $\deg s_1 = 4$, $\deg s_2 = 3$. Moreover, the branch datum of w_1 is $\big((3), (3), 3, 0\big)$ (see e.g. [34], Section 4.3).

Thus, since the degree of f is not divisible by 4 by the condition $k \equiv 1 \,(\mathrm{mod}\, 4)$, we conlcude that $f = w \circ t$, where w is a rational function of degree 3 with the branch datum $\big((3), (3), 3, 0\big)$ and t is a holomorphic map of degree $k + 1$. Let us observe now that the chain rule implies that z_1 is not a critical value of w, since 2 is not divisible by 3. Therefore, again by the chain rule, the map t must have a critical point of order $k + 3$, and this is impossible since $k + 3 > k + 1 = \deg t$.

As another example of using Theorem 1.7 for proving non-realizability, we consider the series of branch data

$$\big((2^{3k+6}), (3^{2k+4}), (3, 9, 6^k), 6k + 12, 0\big), \qquad k \equiv 1 \,(\mathrm{mod}\, 2). \tag{51}$$

Arguing as above one can see that if f is a holomorphic map realizing (51), then $f = w \circ t$, where w is a rational function of degree 3 with the branch datum $\big((3), (3), 3, 0\big)$ and t is a holomorphic map of degree $2k + 4$. Moreover, if z_1 is a critical value of f corresponding to the partition (2^{3k+6}), then z_1 is not a critical value of w. This implies easily that the branch data of t has the form

$$\big((2^{k+2}), (2^{k+2}), (2^{k+2}), (1, 3, 2^k), 2k + 4, 0\big).$$

However, it is known that the last branch data are non-realizable (see [24], Section 5). Therefore, the branch data (51) are non-realizable as well.

Extending the definition of decomposable rational functions on holomorphic maps between compact Riemann surfaces in the obvious way, we obtain the following corollary of Theorem 1.7, similar to Corollary 1.6.

Corollary 5.3. *Let R be a compact Riemann surface of genus one and $f : R \to \mathbb{CP}^1$ a holomorphic map. Assume that for some orbifold $\mathcal{O}$ on $\mathbb{CP}^1$ with the signature $\{2, 2, d\}$, $d > 2$, $\{2, 3, 3\}$, $\{2, 3, 4\}$, or $\{2, 3, 5\}$ the orbifold $f^*(\mathcal{O})$ is non-ramified. Then f is decomposable.*

Proof. If f is indecomposable, then in the first equality of (47), either $\deg w = 1$ or $\deg t = 1$. Both of these assumptions lead to a contradiction. Indeed, if $\deg w = 1$, then the second equality in (47) implies, by Lemma 5.2, that the signature of $\mathcal{O}$ is $\{d, d\}$, with $d \geq 1$, or $\{2, 2, 2\}$, which contradicts the assumption. The equality $\deg t = 1$ is also impossible, since w is defined on the Riemann sphere, while f is defined on a torus. □

6. The Halphen theorem

In this section, we deduce from Theorem 1.1 and Theorem 1.2 the Halphen theorem (see [16] or [2]) concerning polynomial solutions of the generalized Fermat equation

$$X^a + Y^b = Z^c, \tag{52}$$

where (a, b, c) is a triple of integers grater than one.

We start from constructing solutions of (52) from rational Galois coverings. Assume that (a, b, c) satisfies $\chi(a, b, c) > 0$, and let $\theta_{\mathcal{O}}$ be a universal covering of an orbifold $\mathcal{O}$ defined by the equalities

$$\nu(1) = a, \quad \nu(\infty) = b, \quad \nu(0) = c. \tag{53}$$

Since $\theta_{\mathcal{O}}$ is uniform, there exist coprime polynomials P, Q, R such that

$$\theta_{\mathcal{O}} = \frac{R^c}{P^b}, \quad \theta_{\mathcal{O}} - 1 = \frac{Q^a}{P^b}, \tag{54}$$

and changing if necessary $\theta_{\mathcal{O}}$ to $\theta_{\mathcal{O}} \circ \mu$, where μ is a convenient Möbius transfromation, we can assume that ∞ is not a critical point of $\theta_{\mathcal{O}}$, implying that

$$c \deg R = b \deg P = a \deg Q = n, \tag{55}$$

where $n = \deg \theta_{\mathcal{O}}$.

It is clear that

$$Q^a(z) + P^b(z) = R^c(z).$$

Moreover, this equality remains true after the substitution $z = U/V$, where U and V are coprime polynomials. Taking into account (55), this implies that for any non-zero complex numbers α, β, γ such that

$$\alpha^n = \beta^n = \gamma^n,$$

the rational functions

$$X = \alpha V^{n/a} Q\left(\frac{U}{V}\right), \quad Y = \beta V^{n/b} P\left(\frac{U}{V}\right), \quad Z = \gamma V^{n/c} R\left(\frac{U}{V}\right) \tag{56}$$

are also coprime polynomials satisfying (52).

In the above notation, the Halphen theorem is the following statement.

Theorem 6.1. *Let (a,b,c) be a triple of integers greater than one. Then equation* (52) *has no solutions in coprime non-constant polynomials X, Y, Z unless $\chi(a,b,c) > 0$. On the other hand, if $\chi(a,b,c) > 0$, then any solution of* (52) *has the form* (56)*, where P, Q, R is any triple of coprime polynomials satisfying* (54)*,* (55) *for the orbifold $\mathcal{O}$ defined by* (53)*.*

Proof. Assume that X, Y, Z satisfy (52). Then at least two of the numbers $a \deg X$, $b \deg Y$, $c \deg Z$ are equal. Assume say that

$$b \deg Y = c \deg Z, \tag{57}$$

for other cases the proof is similar. Let us set

$$F = \frac{Z^c}{Y^b}, \tag{58}$$

and let Π be the branch datum of F. If Π_1, Π_2, Π_3 are partitions of Π corresponding to the critical values $1, \infty, 0$ of F, then equality (57) implies that all entries in Π_3 are divisible by c and all entries in Π_2 are divisible by b. Furthermore, it follows from

$$F - 1 = \frac{Z^c}{Y^b} - 1 = \frac{X^a}{Y^b} \tag{59}$$

that all entries in Π_1 are divisible by a with a single possible exception corresponding to the point ∞.

The above implies that either $F^*(\mathcal{O})$ is non-ramified, or the set of singular points of $F^*(\mathcal{O})$ consists of a single point. The last case is impossible by Theorem 1.2. Therefore,

$$a \deg X = b \deg Y = c \deg Z,$$

and $F = \theta_{\mathcal{O}} \circ q$ for some rational function q by Theorem 1.1. Representing now q as a quotient of two coprime polynomials U and V, we have:

$$\frac{Z^c}{Y^b} = F = \theta_{\mathcal{O}}\left(\frac{U}{V}\right) = \frac{R^c\left(\frac{U}{V}\right)}{P^b\left(\frac{U}{V}\right)} = \frac{R^c\left(\frac{U}{V}\right)V^n}{P^b\left(\frac{U}{V}\right)V^n} = \frac{\left(R\left(\frac{U}{V}\right)V^{n/c}\right)^c}{\left(P\left(\frac{U}{V}\right)V^{n/b}\right)^b}.$$

Since P, Q, R and X, Y, Z are triples of coprime polynomials, this implies that

$$Y = \beta V^{n/b} P\left(\frac{U}{V}\right), \quad Z = \gamma V^{n/c} R\left(\frac{U}{V}\right), \tag{60}$$

where $\beta^n = \gamma^n$. Similarly,

$$\frac{X^a}{Y^b} = 1 - \frac{Z^c}{Y^b}\left(\frac{U}{V}\right) = \frac{Q^a\left(\frac{U}{V}\right)}{P^b\left(\frac{U}{V}\right)} = \frac{Q^a\left(\frac{U}{V}\right)V^n}{P^b\left(\frac{U}{V}\right)V^n} = \frac{\left(Q\left(\frac{U}{V}\right)V^{n/a}\right)^a}{\left(P\left(\frac{U}{V}\right)V^{n/b}\right)^b},$$

implying that

$$Y = \widetilde{\beta} V^{n/b} P\left(\frac{U}{V}\right), \quad X = \alpha V^{n/c} R\left(\frac{U}{V}\right), \tag{61}$$

where $\widetilde{\beta}^n = \alpha^n$. Finally, the first equalities in (60) and (61) imply that $\widetilde{\beta} = \beta$. $\square$

Notice that the Halphen theorem implies in turn Theorem 1.1. Indeed, if f satisfies the conditions of Theorem 1.1, then assuming without loss of generality that the orbifold $\mathcal{O}$ is defined by the equalities (53) and that ∞ is not a critical point of f, we see that there exist coprime non-constant polynomials X, Y, Z such that equalities (58) and (59) hold. Thus, X, Y, Z is a solution of (52), implying by the Halphen theorem that

$$F = Z^c/Y^b = \frac{\left(\gamma V^{n/c} R\left(\frac{U}{V}\right)\right)^c}{\left(\beta V^{n/b} P\left(\frac{U}{V}\right)\right)^b} = \frac{R^c\left(\frac{U}{V}\right)}{P^b\left(\frac{U}{V}\right)} = \theta_{\mathcal{O}}\left(\frac{U}{V}\right).$$

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