

Integrability of matrices

S. Danielyan^b, A. Guterman^{a,b,c}, E. Kreines^{c,d,e}, F. Pakovich^e

^aBar-Ilan University, Ramat-Gan, 5290002, Israel

^bLomonosov Moscow State University, Moscow, 119991, Russia

^cMoscow Center of Fundamental and Applied Mathematics, Moscow, 119991, Russia

^dTel Aviv University, Tel Aviv 6997801, Israel

^eBen Gurion University of Negev, Beer-Sheva 8410501, Israel

Abstract

The concepts of differentiation and integration for matrices are known. As far as each matrix is differentiable, it is not clear a priori whether a given matrix is integrable or not. Recently some progress was obtained for diagonalizable matrices, however general problem remained open. In this paper, we present a full solution of the integrability problem. Namely, we provide necessary and sufficient conditions for a given matrix to be integrable in terms of its characteristic polynomial. Furthermore, we find necessary and sufficient conditions for the existence of integrable and non-integrable matrices with given geometric multiplicities of eigenvalues. Our approach relies on properties of some special classes of polynomials, namely, Shabat polynomials and conservative polynomials, arising in number theory and dynamics.

Keywords: polynomials, matrices, differentiators, integrators

1 Introduction

The concepts of differentiation and integration for matrices were introduced for studying zeros and critical points of complex polynomials. The notion of matrix differentiability was introduced by Davis in [6] and further investigated in [1, 2, 3, 9, 12]. The converse operation, namely, the operation of integration, is due to Bhat and Mukherjee ([1]). Notice that the integrability of matrices has applications to the nonnegative inverse eigenvalue problem and to inequalities like the dual Schoenberg type inequality (see [7], [5] and references therein).

Originally, differentiability and integrability of matrices were defined for matrices over \mathbb{C} , but in this paper we allow the main field to be an arbitrary algebraically closed field \mathbb{K} with $\text{char } \mathbb{K} = 0$. Accordingly, our main definition is following.

Definition 1.1. Let $B \in M_n(\mathbb{K})$. Then $A \in M_{n+1}(\mathbb{K})$ is called *an integral of B* if there exist $u, v \in \mathbb{K}^n$ and $b \in \mathbb{K}$ such that $A = \begin{bmatrix} B & u^\top \\ v & b \end{bmatrix}$ and $p_B(x) = \frac{1}{n+1}p'_A(x)$, where $p_M(x)$ denotes the characteristic polynomial of a matrix M . We say that B is integrable if there exists an integral of B .

Integrability of matrices was firstly investigated in [1]. In particular, it was proved in [1] that non-derogatory matrices, that is, matrices that have only one Jordan cell for each eigenvalue, are always integrable. Thus, the remaining problem is to determine if a matrix having several Jordan cells with the same eigenvalue

is integrable or not. In the subsequent paper [5], the integrability problem was solved for diagonalizable matrices by using methods of matrix analysis and linear algebra. However, the general case remained open.

In this paper, we provide a complete solution of the integrability problem for arbitrary matrices. Our first main result is the following statement.

Theorem 1.2. *Let $B \in M_n$ be a matrix, and $S = \{\lambda \mid \dim \ker(B - \lambda I) > 1\}$ the set of eigenvalues of B such that there exist more than one Jordan cells with the eigenvalue λ . Then B is integrable if and only if an integral $\int p_B(x)dx$ takes the same value on all elements of S .*

Solving the integrability problem for any given matrix, Theorem 1.2 also reduces finding necessary and sufficient conditions for the existence of integrable and non-integrable matrices with given geometric multiplicities of eigenvalues to solving the corresponding problem about integrals of polynomials. In more detail, let us introduce the concept of S -full integral of a polynomial by the following definition.

Definition 1.3. Let $f(x) \in \mathbb{K}[x]$ be a polynomial, $Z'(f)$ the set of zeros of f of multiplicity at least two, and S a subset of $Z'(f)$. A polynomial $F(x) \in \mathbb{K}[x]$ is called an S -full integral of $f(x)$ if

1. $F'(x) = f(x)$,
2. $F(t) = 0$ for any $t \in S$.

In view of Theorem 1.2, the existence problem for integrable and non-integrable matrices with given geometric multiplicities of eigenvalues is equivalent to the existence problem for S -full integrals of a polynomial with given multiplicities of zeros, and much of the paper is devoted to solving the last problem. A particular case of this problem with $S = Z'(f)$ was solved in the recent paper [4]. As a corollary, the relations between the spectrum of a diagonalizable matrix and its integrability were established (see [5, Theorem 3.13]).

In this paper, we solve the existence problem for S -full integral of a polynomial with given multiplicities of zeros, see Theorem 2.1 below. Our approach relies on some techniques having their origin in number theory and dynamics. Namely, to construct polynomials that possess S -full integrals we use Shabat polynomial arising in the Grothendieck “dessin d’enfant” theory, while to construct polynomials that do not possess S -full integrals we use conservative polynomials. A counterpart of Theorem 2.1 in the context of matrix integrability is the following statement.

Theorem 1.4. *Let $m, k \geq 0$ and $\alpha_1, \dots, \alpha_m \geq 2, \beta_1, \dots, \beta_k \geq 1$ be integers,*

$$n = \sum_{j=1}^m \alpha_j + \sum_{i=1}^k \beta_i, \quad M = \sum_{j=1}^m \alpha_j,$$

and \mathcal{M} the subset of M_n consisting of matrices B with pairwise different eigenvalues $(\lambda_1, \dots, \lambda_m, \mu_1, \dots, \mu_k)$ of the multiplicities $\alpha_1, \dots, \alpha_m, \beta_1, \dots, \beta_k$, correspondingly, satisfying the conditions

$$\dim(\text{Ker}(B - \lambda_i I)) > 1, 1 \leq i \leq m, \text{ and } \dim(\text{Ker}(B - \mu_j I)) = 1, 1 \leq j \leq k.$$

Then \mathcal{M} contains an integrable matrix if and only if $m \leq n - M + 1$, and contains a non-integrable matrix if and only if $m \geq 2$. Moreover,

1. *If $m \leq 1$, then all matrices in \mathcal{M} are integrable.*
2. *If $2 \leq m \leq n - M + 1$, then \mathcal{M} contains both integrable and non-integrable matrices.*
3. *If $m > n - M + 1$, then all matrices in \mathcal{M} are non-integrable.*

Let us mention that applying Theorem 1.4 for $m = 0$ we get the result of [1] that each non-derogatory matrix is integrable. On the other hand, for diagonalizable matrices Theorem 1.4 implies the main result of [5], see Theorem 3.11.

Our paper is organized as follows. In Section 2, after formulating Theorem 2.1, we recall some properties of Shabat polynomials and conservative polynomials used in the paper. In particular, we discuss the relations between these classes of polynomials and plane trees. Then, we prove Theorem 2.1. Specifically, we use Shabat polynomials to construct infinitely many polynomials having S -full integrals, while conservative polynomials are used for producing examples of polynomials without S -full integrals.

In Section 3, we prove Theorems 1.2 and 1.4 and some additional results. In particular, we prove that if a matrix is diagonalizable and integrable, then each matrix with the same spectrum is integrable.

2 S -full integrals of polynomials

2.1 Main result

We write the polynomials under consideration in the form

$$f(x) = (x - a_1)^{\alpha_1} \dots (x - a_m)^{\alpha_m} (x - b_1)^{\beta_1} \dots (x - b_k)^{\beta_k}, \quad (1)$$

where the zeros $a_1, \dots, a_m, b_1, \dots, b_k$ are pairwise different and the multiplicities satisfy the conditions $\alpha_1, \dots, \alpha_m \geq 2$, $\beta_1, \dots, \beta_k \geq 1$, so that the zeros a_1, \dots, a_m are multiple and the zeros b_1, \dots, b_k can be arbitrary. Thus, the set $S = \{a_1, \dots, a_m\}$ is always a subset of $Z'(f)$, and we can consider the question about S -full integrability of f .

Our main result concerning S -full integrals is the following statement.

Theorem 2.1. *Let $m, k \geq 0$, $\alpha_1, \dots, \alpha_m \geq 2$, $\beta_1, \dots, \beta_k \geq 1$ be integers, and*

$$n = \sum_{j=1}^m \alpha_j + \sum_{i=1}^k \beta_i, \quad M = \sum_{j=1}^m \alpha_j.$$

Then an S -integrable polynomial of the form (1) exists if and only if $m \leq n - M + 1$, and a non-integrable with respect to S polynomial of the form (1) exists if and only if $m \geq 2$. Moreover,

1) *If $m = 0$, then for all pairwise distinct $b_1, \dots, b_k \in \mathbb{K}$ the polynomial*

$$f(x) = (x - b_1)^{\beta_1} \dots (x - b_k)^{\beta_k}$$

has an S -full integral with respect to $S = \emptyset$.

2) *If $m = 1$, then for all pairwise distinct $a, b_1, \dots, b_k \in \mathbb{K}$ the polynomial*

$$f(x) = (x - a)^{\alpha_1} (x - b_1)^{\beta_1} \dots (x - b_k)^{\beta_k},$$

has an S -full integral with respect to $S = \{a\}$.

3) *If $2 \leq m \leq n - M + 1$, then there exists a subset $\{a_1, \dots, a_m, b_1, \dots, b_k\}$ of \mathbb{K} such that the polynomial $f_1(x) = (x - a_1)^{\alpha_1} \dots (x - a_m)^{\alpha_m} (x - b_1)^{\beta_1} \dots (x - b_k)^{\beta_k}$ has an S -full integral with respect to $S = \{a_1, \dots, a_m\}$, and there exists a subset $\{a'_1, \dots, a'_m, b'_1, \dots, b'_k\}$ of \mathbb{K} such that the polynomial $f_2(x) = (x - a'_1)^{\alpha_1} \dots (x - a'_m)^{\alpha_m} (x - b'_1)^{\beta_1} \dots (x - b'_k)^{\beta_k}$ does not have an S -full integral with respect to $S = \{a'_1, \dots, a'_m\}$.*

4) *If $m > n - M + 1$ then for all pairwise distinct $a_1, \dots, a_m, b_1, \dots, b_k \in \mathbb{K}$ the polynomial $f(x) = (x - a_1)^{\alpha_1} \dots (x - a_m)^{\alpha_m} (x - b_1)^{\beta_1} \dots (x - b_k)^{\beta_k}$ does not have an S -full integral with respect to $S = \{a_1, \dots, a_m\}$.*

Notice that if P is a polynomial, $S = \{a_1, \dots, a_m\}$ is a set, and $\mu = az + b$ is a non-constant linear map, then P has an S -full integral if and only if the polynomial $P \circ \mu$ has an \tilde{S} -full integral for $\tilde{S} = \{\mu^{-1}(a_1), \dots, \mu^{-1}(a_m)\}$. In particular, Theorem 2.1 implies that for all values of multiplicities satisfying $2 \leq m \leq n - M + 1$ there exist infinitely many polynomials that have S -full integrals and infinitely many polynomials that do not have S -full integrals.

We remark that the first and the second parts of the theorem are obvious, while the fourth part is obtained by a direct computation. The main difficulty, as well as the main interest, lies in proving the third part. To prove this part, we use the beautiful relations between plane trees and two types of complex polynomials: Shabat polynomials and conservative polynomials.

2.2 Plane trees

We recall that a *tree* is a connected graph without cycles, and a *plane tree* is a tree embedded into the plane. Two plane trees λ_1, λ_2 are called equivalent if there exists an orientation preserving homeomorphism μ of the plane such that $\mu(\lambda_1) = \lambda_2$. A trivial induction shows that a tree with n edges has $n + 1$ vertices. Let λ be a plane tree and $(\gamma_1, \gamma_2, \dots, \gamma_{n+1})$ the sequence of valencies of vertices of λ . Since λ has no loops, every edge of λ is adjacent exactly to two vertices of λ , implying that

$$\sum_{i=1}^{n+1} \gamma_i = 2n. \quad (2)$$

We will refer to this fact by saying that $(\gamma_1, \gamma_2, \dots, \gamma_{n+1})$ is a *partition* of $2n$.

The following two lemmas are known (see Section 1.5.2 and Section 1.6.1 of [10] for more details and generalizations). We provide full proofs of these results since their ideas are of importance for our proof of Theorem 2.1.

Lemma 2.2. *Let n be a positive integer. Then for any partition $(\gamma_1, \gamma_2, \dots, \gamma_{n+1})$ of $2n$ there exists a plane tree λ with n edges and the sequence of valencies of vertices $(\gamma_1, \gamma_2, \dots, \gamma_{n+1})$.*

Proof. The proof is by induction on n . For $n = 1$ the statement is clearly true. To prove the inductive step, we observe that for $n > 1$ equality (2) implies that at least one element of $(\gamma_1, \gamma_2, \dots, \gamma_{n+1})$ is equal to 1 and at least one does not. Assuming that $\gamma_1 = 1$, $\gamma_2 > 1$, let us consider the partition $(\gamma_2 - 1, \gamma_3, \dots, \gamma_{n+1})$ of the number $2(n - 1)$. By the induction assumption, there exists a tree λ' with $n - 1$ edges and the sequence of valencies of vertices $(\gamma_2 - 1, \gamma_3, \dots, \gamma_{n+1})$. To obtain now a required tree, λ it is enough to glue an extra edge to the vertex of valency $\gamma_2 - 1$ of λ' . \square

In this paper, instead of ordinary plane trees, we consider bicolored plane trees. By definition, a *bicolored plane tree* is a plane tree whose vertices are colored in black and white colors in such a way that any edge connects vertices of different colors. Two bicolored plane trees λ_1, λ_2 are called *equivalent* if they are equivalent as plane trees and the corresponding homeomorphism μ preserves the colors of vertices. Any plane tree λ can be “bicolored” by choosing one of two possible colorings for an arbitrary vertex of λ and expanding coloring to the remaining vertices.

One can easily see that if $(\alpha_1, \dots, \alpha_p)$ and $(\beta_1, \dots, \beta_q)$ are the sequences of valencies of white and black vertices of a bicolored plane tree with n edges, then the equalities

$$\sum_{i=1}^p \alpha_i = \sum_{i=1}^q \beta_i = n \quad (3)$$

and

$$p + q = n + 1 \quad (4)$$

hold. In turn, the analogue of Lemma 2.2 is the following statement.

Lemma 2.3. *Let n be a positive integer. Then for any partitions $(\alpha_1, \dots, \alpha_p)$ and $(\beta_1, \dots, \beta_q)$ of n such that $p + q = n + 1$ there exists a bicolored plane tree λ with n edges and the valency sequences of white and black vertices $(\alpha_1, \dots, \alpha_p)$ and $(\beta_1, \dots, \beta_q)$.*

Proof. As above, the proof is by induction on n . For $n = 1$ the statement is true. To prove the inductive step we observe that if $n > 1$, then (3) and (4) still imply that at least one of the numbers α_j, β_i is equal to one. Furthermore, if, say, $\alpha_1 = 1$, then it follows from (3) and (4) that at least one of the numbers β_i , say β_1 , is greater than one. By the induction assumption, there exists a bicolored plane tree λ' with $n - 1$ edges and the sets of valencies of white and black vertices $(\alpha_2, \dots, \alpha_p)$ and $(\beta_1 - 1, \dots, \beta_q)$. Gluing now an extra edge to the vertex of valency $\beta_1 - 1$ of λ' , we obtain a required tree λ . If instead of α_1 one of the numbers β_i is equal to one, then the proof is obtained by an obvious modification. \square

The main result of this section, which is used in the proof of Theorem 2.1, is the following variation of the above lemmas.

Lemma 2.4. *Let n be a positive integer and $(\gamma_1, \gamma_2, \dots, \gamma_{n+1})$ be a partition of $2n$. Assume that for an integer l , $1 \leq l \leq n$, the inequality*

$$\gamma_1 + \gamma_2 + \dots + \gamma_l \leq n \quad (5)$$

holds. Then there exist $p \geq l$ and a bicolored plane tree λ with n edges and the sequences of valencies of white and black vertices $(\alpha_1, \dots, \alpha_p)$ and $(\beta_1, \dots, \beta_q)$ such that:

- $(\gamma_1, \gamma_2, \dots, \gamma_{n+1}) = (\alpha_1, \dots, \alpha_p, \beta_1, \dots, \beta_q)$,
- $(\gamma_1, \dots, \gamma_l) = (\alpha_1, \dots, \alpha_l)$.

Proof. The proof is by induction on n . For $n = 1$ the statement is true. Furthermore, it follows from Lemma 2.2 that if $l = 1$, then it is true for any n , since any plane tree can be bicolored.

To prove the inductive step in case $l > 1$, let us observe first that conditions of the theorem imply that at least one of the following conditions holds:

- (a): the sequence $(\gamma_1, \gamma_2, \dots, \gamma_l)$ contains 1 and $(\gamma_{l+1}, \gamma_{l+2}, \dots, \gamma_{n+1})$ contains an element different from 1,
- (b): the sequence $(\gamma_{l+1}, \gamma_{l+2}, \dots, \gamma_{n+1})$ contains 1 and $(\gamma_1, \gamma_2, \dots, \gamma_l)$ contains an element different from 1.

Indeed, 1 belongs to the set $(\gamma_1, \gamma_2, \dots, \gamma_{n+1})$ and hence belongs to at least one of the sets $(\gamma_1, \gamma_2, \dots, \gamma_l)$ and $(\gamma_{l+1}, \gamma_{l+2}, \dots, \gamma_{n+1})$. If $1 \in (\gamma_1, \gamma_2, \dots, \gamma_l)$ and $(\gamma_{l+1}, \gamma_{l+2}, \dots, \gamma_{n+1})$ contains a non-unit, then (a) takes place. On the other hand, if all elements of the set $(\gamma_{l+1}, \gamma_{l+2}, \dots, \gamma_{n+1})$ are units, then necessarily at least one of the elements of the set $(\gamma_1, \gamma_2, \dots, \gamma_l)$ is not a unit, so that (b) takes place. In case $1 \in (\gamma_{l+1}, \gamma_{l+2}, \dots, \gamma_{n+1})$, the proof is similar.

In case (a) holds, the inductive step goes as follows. Without loss of generality we may assume that $\gamma_1 = 1$ and $\gamma_{l+1} > 1$. Defining now a partition of the integer $2(n - 1)$ by the formula

$$(\gamma_2, \dots, \gamma_l, \gamma_{l+1} - 1, \gamma_{l+2}, \dots, \gamma_{n+1}) \quad (6)$$

and observing that (5) implies that

$$\gamma_2 + \gamma_3 + \dots + \gamma_l \leq n - 1,$$

we conclude by the induction assumption that there exists a bicolored plane tree λ' with $n - 1$ edges, whose sequence of valencies of vertices coincides with (6), and whose sequence of valencies of white vertices “contains” $(\gamma_2, \dots, \gamma_l)$. Therefore, gluing an extra edge to the vertex of valency $\gamma_{l+1} - 1$ of λ' we obtain a required tree λ . In case (b) holds, the proof is obtained by an obvious modification. \square

2.3 Shabat polynomials.

Let $P(z)$ be a complex polynomial. We recall that zeros w_1, \dots, w_n of $P'(x)$ are called (finite) *critical points* of $P(z)$ and the values $P(w_1), \dots, P(w_n)$ are called (finite) *critical values* of $P(z)$.

Definition 2.5. A complex polynomial $P(z)$ is called a *Shabat polynomial* if it has at most two (finite) critical values. Two Shabat polynomials $P_1(z), P_2(z)$ are called *equivalent* if there exist polynomials μ_1 and μ_2 of degree one such that $P_2 = \mu_1 \circ P_1 \circ \mu_2$.

Notice that by choosing an appropriate polynomial μ_1 it is always possible to assume that critical values of $P(z)$ are 0 and 1.

The following statement is a particular case of the correspondence between Belyi pairs and “dessins d’enfants” (see [10], [14], [13] for more detail).

Theorem 2.6. *There is a bijective correspondence between the equivalence classes of Shabat polynomials and the equivalence classes of bicolored plane trees.*

We briefly recall how this correspondence is constructed. Let $P(x)$ be a Shabat polynomial with critical values 0 and 1. Then the corresponding plane tree λ_P is defined as the preimage $\lambda_P = P^{-1}([0, 1])$ of the segment $[0, 1]$ with respect to the function $P(x) : \mathbb{C} \rightarrow \mathbb{C}$. By definition, white (resp., black) vertices of λ_P are preimages of the point 0 (resp., of the point 1) and edges of λ_P are preimages of the segment $[0, 1]$.

In the other direction, if λ is a bicolored tree with n edges and the sequences of valencies of white and of black vertices $\alpha_1, \dots, \alpha_p$ and β_1, \dots, β_q , correspondingly, then the corresponding Shabat polynomial $P(x) \in \mathbb{C}[x]$ with critical values 0 and 1 is defined by the conditions

$$\begin{cases} P(x) = c(x - a_1)^{\alpha_1} \dots (x - a_p)^{\alpha_p} \\ P(x) - 1 = c(x - b_1)^{\beta_1} \dots (x - b_q)^{\beta_q}, \end{cases}$$

where $a_1, \dots, a_p, b_1, \dots, b_q \in \mathbb{C}$ are pairwise distinct and $c \in \mathbb{C}$ is distinct from zero. Thus, a system that determines a Shabat polynomial of a tree is a system of polynomial equations with the unknowns $a_1, \dots, a_p, b_1, \dots, b_q, c$ obtained from equating coefficients of like terms in the equality

$$c(x - a_1)^{\alpha_1} \dots (x - a_p)^{\alpha_p} - 1 = c(x - b_1)^{\beta_1} \dots (x - b_q)^{\beta_q}. \quad (7)$$

Notice that there could be *several* trees with the same sequences $\alpha_1, \dots, \alpha_p$ and β_1, \dots, β_q . All corresponding Shabat polynomials satisfy the same system (7).

After fixing critical values of a Shabat polynomial, we still have a “degree of freedom” corresponding to a choice of μ_2 . Thus, we can impose some further restrictions on system (7). For example, we can assume that

$a_1 = 0$ and $b_1 = 1$. Theorem 2.6 implies that in this case the system (7) has only *finitely many* solutions. Since (7) provide us with equations in $a_1, \dots, a_p, b_1, \dots, b_q, c$ with *rational* and even integer coefficients, this implies that solutions are necessarily *algebraic numbers*. Thus, for any plane tree the corresponding equivalence class of Shabat polynomials contains polynomials with algebraic coefficients (see [10], [14]).

Theorem 2.6 combined with Lemma 2.4 allows us to prove the following statement, which is used for the proof of Theorem 2.1.

Corollary 2.7. *Let $m, k \geq 1$ and $\alpha_1, \dots, \alpha_m \geq 2, \beta_1, \dots, \beta_k \geq 1$ be integers, and*

$$n = \sum_{j=1}^m \alpha_j + \sum_{i=1}^k \beta_i, \quad M = \sum_{j=1}^m \alpha_j.$$

Assume that $m \leq n - M + 1$. Then there exist pairwise distinct $a_1, \dots, a_m, b_1, \dots, b_k \in \overline{\mathbb{Q}}$ and a Shabat polynomial $P(z) \in \overline{\mathbb{Q}}[z]$ of degree $n + 1$ such that

$$P'(x) = (x - a_1)^{\alpha_1} \dots (x - a_m)^{\alpha_m} (x - b_1)^{\beta_1} \dots (x - b_k)^{\beta_k} \quad (8)$$

and

$$P(a_1) = \dots = P(a_m) = 0. \quad (9)$$

Proof. Let us set

$$(\gamma_1, \gamma_2, \dots, \gamma_{n+2}) = (\alpha_1 + 1, \dots, \alpha_m + 1, \beta_1 + 1, \dots, \beta_k + 1, \underbrace{1, \dots, 1}_{n-(m+k)+2}). \quad (10)$$

It is easy to see that (10) is a partition of $2(n + 1)$. Indeed, the definition of n implies that $n \geq m + k$, so that (10) is well-defined. In addition,

$$\sum_{j=1}^m (\alpha_j + 1) + \sum_{i=1}^k (\beta_i + 1) + n - (m + k) + 2 = \sum_{j=1}^m \alpha_j + \sum_{i=1}^k \beta_i + n + 2 = 2n + 2.$$

Since (10) contains $n + 2$ elements and

$$\sum_{j=1}^m (\alpha_j + 1) = M + m \leq n + 1$$

by the condition, it follows from Lemma 2.4 that there exists a bicolored plane tree λ with $n + 1$ edges and sequences of white and black valencies (μ_1, \dots, μ_p) and (ν_1, \dots, ν_q) , where $p \geq m$, such that

$$(\gamma_1, \gamma_2, \dots, \gamma_{n+2}) = (\mu_1, \dots, \mu_p, \nu_1, \dots, \nu_q),$$

$$(\alpha_1 + 1, \dots, \alpha_m + 1) = (\mu_1, \dots, \mu_m),$$

and

$$(\beta_1 + 1, \dots, \beta_k + 1, \underbrace{1, \dots, 1}_{n-(m+k)+2}) = (\mu_{m+1}, \dots, \mu_p, \nu_1, \dots, \nu_q).$$

Applying Theorem 2.6, we see that there exists a Shabat polynomial $\tilde{P}(z) \in \overline{\mathbb{Q}}[z]$ of degree $n+1$ such that

$$\tilde{P}(z) = c(x - x_1)^{\mu_1} \dots (x - x_p)^{\mu_p} \quad (11)$$

and

$$\tilde{P}(z) - 1 = c(x - y_1)^{\nu_1} \dots (x - y_q)^{\nu_q}.$$

for some pairwise distinct $x_1, \dots, x_p, y_1, \dots, y_q \in \overline{\mathbb{Q}}$ and $0 \neq c \in \overline{\mathbb{Q}}$.

By construction, among the points $x_{m+1}, \dots, x_p, y_1, y_2, \dots, y_q$ there are exactly k points that are zeros of $P'(z)$. Denoting these points by z_1, z_2, \dots, z_k and setting

$$(a_1, a_2, \dots, a_m) = (x_1, x_2, \dots, x_m) \quad \text{and} \quad (b_1, b_2, \dots, b_k) = (z_1, z_2, \dots, z_k),$$

we see that $\tilde{P}'(z)$ is divisible by

$$(x - a_1)^{\alpha_1} \dots (x - a_m)^{\alpha_m} (x - b_1)^{\beta_1} \dots (x - b_k)^{\beta_k}.$$

Moreover, since the degree of the last polynomial is n , the equality

$$\tilde{P}'(z) = c(n+1)(x - a_1)^{\alpha_1} \dots (x - a_m)^{\alpha_m} (x - b_1)^{\beta_1} \dots (x - b_k)^{\beta_k} \quad (12)$$

holds. Finally, it follows from (11) and (12) that for the Shabat polynomial

$$P(z) = \frac{\tilde{P}(z)}{c(n+1)}$$

equalities (8) and (9) hold, and $P(z) \in \overline{\mathbb{Q}}[z]$. □

2.4 Conservative polynomials

In addition to a Shabat polynomial, with every plane tree one can associate a polynomial of a different type, described by the following definition.

Definition 2.8. A complex polynomial $C(z)$ is called *conservative* if all its critical points are fixed, that is, if the equality $C'(\zeta) = 0$, $\zeta \in \mathbb{C}$, implies that $C(\zeta) = \zeta$. A conservative polynomial $C(z)$ is called *normalized* if $C(z)$ is monic and $C(0) = 0$. Two conservative polynomials $C_1(z)$ and $C_2(z)$ are called *equivalent* if there exists a complex polynomial μ of degree one such that $C_2 = \mu^{-1} \circ C_1 \circ \mu$.

Conservative polynomials were introduced by Smale [15] in connection with his “mean value conjecture”. Motivated by Smale’s conjecture Kostrikin proposed in [8] several conjectures concerning conservative polynomials. In particular, he conjectured that the number of normalized conservative polynomials of degree n is finite and is equal to C_{2n-2}^{n-1} . This conjecture was proved by Tischler in the paper [16]. In fact, he proved the following statement, implying the Kostrikin conjecture (see [16], Theorem 4.2).

Theorem 2.9. *There is a bijective correspondence between the equivalence classes of conservative polynomials of degree n and the equivalence classes of bicolored plane trees with $n-1$ edges.*

For a conservative polynomial C , the corresponding plane tree λ_C is constructed as follows (see [16] for more detail and [11] for some pictures). Let ζ be a critical point of $C(z)$ and $d \geq 2$ the local multiplicity of $C(z)$ at ζ . Then one can show that the immediate attractive basin B_ζ of ζ is a disk and that there is an analytic conjugation of $C(z)$ on B_ζ to $z \rightarrow z^d$ on the unit disk D such that the conjugating map $\phi_\zeta : \mathbb{D} \rightarrow B_\zeta$ extends continuously to the closed unit disk $\overline{\mathbb{D}}$. Let S be a union of $d - 1$ radial segments which are forward invariant under the map $z \rightarrow z^d$ on $\overline{\mathbb{D}}$, and S_ζ the image of S under the map ϕ_ζ , considered as a bicolored graph with a unique white vertex, which is the image of zero, and $d - 1$ black vertices, which are the images of end-points of S . In this notation, λ_C is defined as a union $\lambda_C = \cup_{i=1}^p S_{\zeta_i}$, where ζ_i , $1 \leq i \leq p$, are all finite critical points of $C(z)$. Note that by construction λ_C is a forward invariant of $C(z)$, and white (resp. black) vertices of λ_C are attractive (resp. repelling) fixed points of $C(z)$.

In the other direction, if λ is a bicolored plane tree with $n - 1$ edges and the sequence of valencies of white vertices $\alpha_1, \dots, \alpha_p$, then a corresponding conservative polynomial $P(z)$ satisfies the system

$$\begin{cases} C'(z) = c(z - c_1)^{\alpha_1} \dots (z - c_p)^{\alpha_p} \\ C(c_i) = c_i, \end{cases} \quad (13)$$

where $c_1, \dots, c_p \in \mathbb{C}$ are pairwise distinct and $c \in \mathbb{C}$ is distinct from zero.

Notice that in distinction with system (7) the valencies of black vertices do not appear in system (13). In addition, the number of edges of a tree corresponding to a conservative polynomial of degree n is $n - 1$ instead of n . Nevertheless, similar to system (7), system (13) reduces to a system of equations in c_1, \dots, c_p, c with rational coefficients. Furthermore, if $C(z)$ is normalized, then the number of solutions of (13) is finite and these solutions are algebraic numbers. Thus, for any plane tree the corresponding equivalence class of conservative polynomials contains polynomials with algebraic coefficients.

A counterpart of Corollary 2.7, which follows from Theorem 2.9 is the following statement.

Corollary 2.10. *Let $l \geq 1$ and $\gamma_1, \dots, \gamma_l \geq 1$ be integers, and $n = \sum_{i=1}^l \gamma_i$. Then there exist pairwise distinct $c_1, \dots, c_l \in \overline{\mathbb{Q}}$ and a conservative polynomial $C(z) \in \overline{\mathbb{Q}}[z]$ of degree $n + 1$ with algebraic coefficients such that*

$$C'(z) = (z - c_1)^{\gamma_1} \dots (z - c_l)^{\gamma_l} \quad (14)$$

and

$$C(c_i) = c_i, \quad 1 \leq i \leq l. \quad (15)$$

Proof. Let $(\delta_1, \dots, \delta_{n+1-l})$ be an arbitrary partition of the number n containing $n + 1 - l$ elements. For example, we can take

$$(\delta_1, \dots, \delta_{n+1-l}) = (l, \underbrace{1, \dots, 1}_{n-l}).$$

By Lemma 2.3, there exists a bicolored plane tree λ with n edges and the sequences of white and black valencies $(\gamma_1, \dots, \gamma_l)$ and $(\delta_1, \dots, \delta_{n+1-l})$. Therefore, by Theorem 2.9, there exist pairwise distinct $\tilde{c}_1, \dots, \tilde{c}_l \in \overline{\mathbb{Q}}$ and a conservative polynomial $\tilde{C}(z) \in \overline{\mathbb{Q}}[z]$ of degree $n + 1$ such that

$$\tilde{C}'(z) = c(z - \tilde{c}_1)^{\gamma_1} \dots (z - \tilde{c}_l)^{\gamma_l}$$

for some $0 \neq c \in \overline{\mathbb{Q}}$. Setting now $\mu = \varepsilon z$, where ε satisfies $\varepsilon^{n-1} = 1$, we see that the conservative polynomial $C = \mu^{-1} \circ \tilde{C} \circ \mu$ has algebraic coefficients and satisfies (14) and (15) for $c_i = \tilde{c}_i/\varepsilon$, $1 \leq i \leq l$. \square

2.5 Proof of Theorem 2.1

Since the condition $S = \emptyset$ provides no restrictions, the first part of the theorem is trivially true. The second part is also true since for any polynomial $F(x) \in \mathbb{K}[x]$ such that

$$F'(x) = f(x) = (x - a)^{\alpha_1} (x - b_1)^{\beta_1} \dots (x - b_k)^{\beta_k},$$

the polynomial $F(x) - F(a)$ obviously is an S -full integral of $f(x)$ for $S = \{a\}$.

To prove the fourth part, we observe that if $F(x)$ is an S -full integral of $f(x)$, then

$$(x - a_i)^{\alpha_i + 1} \mid F(x), \quad i = 1, \dots, m.$$

Therefore,

$$\deg F(x) \geq M + m,$$

implying that

$$m \leq \deg F(x) - M = n + 1 - M.$$

Let us prove now the third part. Notice that the condition

$$2 \leq m \leq n - M + 1 \tag{16}$$

implies that $k > 0$, for otherwise $n = M$ and (16) leads to a contradictory inequality $2 \leq m \leq 1$. Since $k > 0$ and $m \geq 2$, it follows from Corollary 2.7 that there exist pairwise distinct $a_1, \dots, a_m, b_1, \dots, b_k \in \overline{\mathbb{Q}}$ and a Shabat polynomial $P(z) \in \overline{\mathbb{Q}}[z]$ of degree $n + 1$ such that the equalities (8) and (9) hold. Thus, $P(z)$ is an S -full integral of $P'(z)$ for $S = \{a_1, \dots, a_m\}$, and hence the first statement of the third part is true in case $\mathbb{K} = \overline{\mathbb{Q}}$. Let now \mathbb{K} be an arbitrary algebraically closed field of characteristic zero. As it was mentioned above, system (7) for a Shabat polynomial reduces to a systems of equations in $a_1, \dots, a_p, b_1, \dots, b_q, c$ with rational coefficients. Thus, this system makes sense also over \mathbb{K} , since \mathbb{K} contains \mathbb{Q} . Furthermore, since \mathbb{K} is algebraically closed, it contains a “copy” of $\overline{\mathbb{Q}}$, implying that if the system (7) has solutions over $\overline{\mathbb{Q}}$, then it also has solutions over \mathbb{K} . Therefore, Corollary 2.7 remains true over \mathbb{K} , implying that the first statement of the third part also remains true over \mathbb{K} .

Further, since $k > 0$ and $m \geq 2$ imply that $k + m \geq 1$, we can apply Corollary 2.10 for

$$(\gamma_1, \dots, \gamma_l) = (\alpha_1, \dots, \alpha_m, \beta_1, \dots, \beta_k),$$

and find pairwise distinct $c_1, \dots, c_l \in \overline{\mathbb{Q}}$ and a conservative polynomial $C(z) \in \overline{\mathbb{Q}}[z]$ of degree $n + 1$ such that

$$C'(z) = (x - c_1)^{\alpha_1} \dots (x - c_m)^{\alpha_m} (x - c_{m+1})^{\beta_1} \dots (x - c_{m+k})^{\beta_k}$$

and

$$C(c_i) = c_i, \quad 1 \leq i \leq m + k. \tag{17}$$

Since any primitive $F(z)$ of $C'(z)$ has the form $F(z) = C(z) + c$, $c \in \mathbb{C}$, it follows from (17) that $C'(z)$ does not have an S -full integral for any subset S of $\{c_1, \dots, c_{m+k}\}$ that contains at least two elements. Thus, to prove the second statement of the third part of the theorem over $\overline{\mathbb{Q}}$, we can set for example

$$a_i = c_i, \quad 1 \leq i \leq m, \quad b_i = c_{i+m}, \quad 1 \leq i \leq k.$$

The general case can be proved as above, taking into account that (13) reduces to a systems of equations in c_1, \dots, c_p, c with rational coefficients. \square

3 S -full integrals and matrix integrability

In this section, we prove Theorem 1.2 and Theorem 1.4. Notice that our proof of Theorem 1.2 is effective: it shows how a matrix integral is constructed from an S -full integral of the characteristic polynomial. In this relation, we remark that some integrable matrices may have infinitely many integrals. For more detail, we refer the reader to [1].

Let \mathbb{K} be an arbitrary algebraically closed field of characteristic 0. We denote the set of $n \times n$ matrices over \mathbb{K} by $M_n := M_n(\mathbb{K})$, and the set of row vectors of the length n over \mathbb{K} by \mathbb{K}^n . Let $E_{ij} \in M_k$ be (i, j) -th matrix unit, i.e., the matrix with 1 on the (i, j) -th position and 0 elsewhere, $J_k = E_{12} + E_{23} + \dots + E_{k-1k} \in M_k$ the Jordan matrix of the size k if $k \geq 2$, and $J_1 = 0 \in M_1$. We denote by X^\top the transposed matrix and by $p_X(x) = \det(xI - X)$ the characteristic polynomial of $X \in M_n$. Finally, we denote by $\text{diag}(X_1, \dots, X_n)$ the block-diagonal matrix with the blocks X_1, \dots, X_n .

3.1 Proofs of Theorems 1.2 and 1.4

The following lemma is proved in [1, Lemma 7].

Lemma 3.1. *If $B \in M_n$ has an integral A , then $\begin{pmatrix} X & 0^\top \\ 0 & 1 \end{pmatrix} A \begin{pmatrix} X^{-1} & 0^\top \\ 0 & 1 \end{pmatrix}$ is an integral of XBX^{-1} for any invertible $X \in M_n$.*

The following corollary is straightforward:

Corollary 3.2. *1. If $B \in M_n$ is integrable, then for any invertible $X \in M_n$ the matrix XBX^{-1} is integrable.*

2. If B has an integral A and the matrix X commutes with B , then $\begin{pmatrix} X & 0^\top \\ 0 & 1 \end{pmatrix} A \begin{pmatrix} X^{-1} & 0^\top \\ 0 & 1 \end{pmatrix}$ is also an integral of B .

Lemma 3.3. *For any non-zero vector $v \in \mathbb{K}^n$, let k be the smallest positive integer such that $v_k \neq 0$, i.e., $v = (\underbrace{0, \dots, 0}_{k-1}, v_k, v_{k+1}, \dots, v_n)$, where $1 \leq k \leq n$, $v_k \neq 0$. Then there exists a polynomial $h(x) \in \mathbb{K}[x]$ of the degree at most $(n - k)$ such that $h(J_n)$ is invertible and $v \cdot h(J_n) = (\underbrace{0, \dots, 0}_{k-1}, 1, \underbrace{0, \dots, 0}_{n-k})$.*

Proof. It is well-known that

$$h(J_n) = \begin{pmatrix} h(0) & \frac{h'(0)}{1!} & \frac{h''(0)}{2!} & \frac{h'''(0)}{3!} & \dots & \frac{h^{(n-2)}(0)}{(n-2)!} & \frac{h^{(n-1)}(0)}{(n-1)!} \\ 0 & h(0) & \frac{h'(0)}{1!} & \frac{h''(0)}{2!} & \dots & \frac{h^{(n-3)}(0)}{(n-3)!} & \frac{h^{(n-2)}(0)}{(n-2)!} \\ 0 & 0 & h(0) & \frac{h'(0)}{1!} & \dots & \frac{h^{(n-4)}(0)}{(n-4)!} & \frac{h^{(n-3)}(0)}{(n-3)!} \\ \dots & \dots & \ddots & \ddots & & \dots & \dots \\ \dots & \dots & & \ddots & \ddots & \dots & \dots \\ 0 & 0 & \dots & \dots & 0 & h(0) & \frac{h'(0)}{1!} \\ 0 & 0 & \dots & \dots & 0 & 0 & h(0) \end{pmatrix}$$

and a direct computation shows that

$$v \cdot h(J_n) = \left(\underbrace{0, \dots, 0}_{k-1}, v_k h(0), v_k \frac{h'(0)}{1!} + v_{k+1} h(0), \dots, \sum_{i=k}^n v_i \frac{h^{(n-i)}(0)}{(n-i)!} \right).$$

Thus, the equality $v \cdot h(J_n) = (\underbrace{0, \dots, 0}_{k-1}, \underbrace{1, 0, \dots, 0}_{n-k})$ is equivalent to the system

$$\begin{cases} h(0) = \frac{1}{v_k} \\ h'(0) = -\frac{1!}{v_k} v_{k+1} h(0) \\ h''(0) = -\frac{2!}{v_k} \left(v_{k+2} h(0) + v_{k+1} \frac{h'(0)}{1!} \right) \\ \vdots \\ h^{n-k}(0) = -\frac{(n-k)!}{v_k} \sum_{i=k+1}^n v_i \frac{h^{(n-i)}(0)}{(n-i)!} \end{cases} \quad (18)$$

Writing $h(x)$ as $h(x) = c_0 + c_1 x + \dots + c_{n-k} x^{n-k}$ with unknown coefficients c_0, \dots, c_{n-k} , we see that (18) transforms into the triangular linear system for c_0, \dots, c_{n-k} , which is solvable since the determinant of this system is equal to $\prod_{l=0}^{n-k} l! \neq 0$. Moreover $\det(h(J_n)) = (h(0))^n = \frac{1}{v_k^n} \neq 0$ and hence $h(J_n)$ is invertible as desired. \square

Definition 3.4. Let $B \in M_n$. Any matrix $A \in M_{n+1}$ of the form $A = \begin{pmatrix} B & u^\top \\ v & b \end{pmatrix}$ is called an *integral extension* of the matrix B .

Notice that an integral extension is not necessarily an integral of the matrix B , but any integral of B is an integral extension by the definition. Note also that if a matrix A is an integral of B then necessarily $b = \frac{\text{tr}(B)}{n}$. Indeed, $b = \text{tr}(A) - \text{tr}(B)$. On the other hand, $p'_A(x) = (n+1)p_B(x)$ implies $\text{tr}(A) \cdot n = (n+1) \text{tr}(B)$.

By Corollary 3.2(1), if a matrix B is integrable, then all matrices similar to B are integrable as well. Thus, we may assume that the matrix $B \in M_n$ is in the Jordan normal form. Namely,

$$B = \text{diag}(B_1, \dots, B_m), \quad (19)$$

where B_i is the Jordan block for b_i , i.e. the union of all Jordan cells of B with the eigenvalue b_i ordered in non-increasing order of block sizes. We denote the number of Jordan cells in B_i by β_i , and the sizes of the Jordan cells of B_i by $k_{i,1} \geq \dots \geq k_{i,\beta_i}$. Then

$$B_i = \text{diag}(b_i I_{k_{i,1}} + J_{k_{i,1}}, b_i I_{k_{i,2}} + J_{k_{i,2}}, \dots, b_i I_{k_{i,\beta_i}} + J_{k_{i,\beta_i}}) \in M_{\alpha_i} \quad (20)$$

where

$$\alpha_i = k_{i,1} + \dots + k_{i,\beta_i}, \quad i = 1, \dots, m. \quad (21)$$

Note that in the introduced notations the characteristic polynomial of B is

$$p_B(x) = (x - b_1)^{\alpha_1} \dots (x - b_m)^{\alpha_m},$$

where b_1, \dots, b_m are pairwise distinct.

Let $A = \begin{pmatrix} B & u^\top \\ v & b \end{pmatrix} \in M_{n+1}$ be an integral extension of B . Then in accordance with the introduced notations

$$A = \begin{pmatrix} b_1 I_{k_{1,1}} + J_{k_{1,1}} & & & & & & (u^{1,1})^\top \\ & \ddots & & & & & \vdots \\ & & b_1 I_{k_{1,\beta_1}} + J_{k_{1,\beta_1}} & & & & (u^{1,\beta_1})^\top \\ & & & \ddots & & & \vdots \\ & & & & b_m I_{k_{m,1}} + J_{k_{m,1}} & & (u^{m,1})^\top \\ & & & & & \ddots & \vdots \\ & & & & & & b_m I_{k_{m,\beta_m}} + J_{k_{m,\beta_m}} & (u^{m,\beta_m})^\top \\ v^{1,1} & \dots & v^{1,\beta_1} & \dots & v^{m,1} & \dots & v^{m,\beta_m} & b \end{pmatrix} \in M_{n+1}, \quad (22)$$

where $v^{i,j}, u^{i,j} \in \mathbb{K}^{k_{i,j}}$.

Definition 3.5. In the above notations if

$$v^{i,j} = (\underbrace{0, \dots, 0}_{r_{i,j}}, 1, 0, \dots, 0) \in \mathbb{K}^{k_{i,j}} \quad (23)$$

for some $r_{i,j}$, $0 \leq r_{i,j} \leq k_{i,j}$, then the matrix A is called a *normalized* integral extension of B .

Notice that in case $r_{i,j} = k_{i,j}$ we have $v^{i,j} = 0$.

Lemma 3.6. Let $A = \begin{pmatrix} B & u^\top \\ v & b \end{pmatrix} \in M_{n+1}$ be an integral extension of $B \in M_n$. Then there exists a normalized integral extension $\tilde{A} \in M_{n+1}$ of B such that the matrices A and \tilde{A} are similar and the similarity matrix $C = \text{diag}(C_1, 1)$ satisfies $C_1 B = B C_1$.

Proof. Let B and A be determined by the equalities (19) and (22), correspondingly. For each pair (i, j) , $1 \leq i \leq m$, $1 \leq j \leq \beta_i$, we consider the vector $v^{i,j} \in \mathbb{K}^{k_{i,j}}$ and define a matrix $C_{i,j} \in M_{k_{i,j}}$ as follows. If $v^{i,j} = 0$, then we set $C_{i,j} = I_{k_{i,j}}$. If $v^{i,j} \neq 0$, then by Lemma 3.3 there exists $h_{i,j} \in \mathbb{K}_{k_{i,j}}[x]$ such that $h_{i,j}(J_{k_{i,j}})$ is invertible and $v_{i,j} \cdot h_{i,j}(J_{k_{i,j}})$ has only one nonzero entry. In this case, we set $C_{i,j} = h_{i,j}(J_{k_{i,j}})$. Note that $C_{i,j}$ commutes with $b_i I_{k_{i,j}} + J_{k_{i,j}}$ since the matrix $J_{k_{i,j}}$ commutes with a polynomial of itself. Hence,

$$B \cdot \text{diag}(C_{1,1}, \dots, C_{m,\beta_m}) = \text{diag}(C_{1,1}, \dots, C_{m,\beta_m}) \cdot B.$$

Let us consider the matrix

$$C = \text{diag}(C_{1,1}, \dots, C_{m,\beta_m}, 1) \in M_{n+1}$$

and the matrix $\tilde{A} = C A C^{-1}$. By the choice of the blocks $C_{i,j}$, we have:

$$\tilde{A} = \begin{pmatrix} b_1 I_{k_{1,1}} + J_{k_{1,1}} & & & & & & (\tilde{u}^{1,1})^\top \\ & \ddots & & & & & \vdots \\ & & b_1 I_{k_{1,\beta_1}} + J_{k_{1,\beta_1}} & & & & (\tilde{u}^{1,\beta_1})^\top \\ & & & \ddots & & & \vdots \\ & & & & b_m I_{k_{m,1}} + J_{k_{m,1}} & & (\tilde{u}^{m,1})^\top \\ & & & & & \ddots & \vdots \\ & & & & & & b_m I_{k_{m,\beta_m}} + J_{k_{m,\beta_m}} & (\tilde{u}^{m,\beta_m})^\top \\ \tilde{v}^{1,1} & \dots & \tilde{v}^{1,\beta_1} & \dots & \tilde{v}^{m,1} & \dots & \tilde{v}^{m,\beta_m} & b \end{pmatrix} \in M_{n+1},$$

where $\tilde{v}^{i,j} = (\underbrace{0, \dots, 0}_{r_{i,j}}, 1, 0, \dots, 0) \in \mathbb{K}^{k_{i,j}}$, as required. \square

Corollary 3.7. Assume that $B \in M_n$ is integrable, and let $A \in M_{n+1}$ be its integral. Then there exists an integral $\tilde{A} \in M_{n+1}$ of B such that \tilde{A} is a normalized integral extension of B and $p_{\tilde{A}}(x) = p_A(x)$.

Proof. Let $A \in M_{n+1}$ be an integral of B . By Lemma 3.6, there exists a normalized integral extension \tilde{A} of B . By Corollary 3.2(2), the matrix \tilde{A} is an integral of B since $\text{diag}(C_{1,1}, \dots, C_{m,\beta_m})$ commutes with B . Finally, $p_A(x) = p_{\tilde{A}}(x)$ since the matrices A and \tilde{A} are similar. \square

We denote by $\mathbb{K}(y)$ the field of formal rational functions in the variable y over the field \mathbb{K} .

Lemma 3.8. Let $k \geq 1$, $0 \leq r \leq k$ be integers, and $X = (yI_k - J_k) \in M_k(\mathbb{K}(y))$. Then for the vector $v = (\underbrace{0, \dots, 0}_r, 1, \underbrace{0, \dots, 0}_{k-r-1}) \in \mathbb{K}^k$ and an arbitrary vector $u = (u_1, \dots, u_k) \in \mathbb{K}^k$ it holds that

$$vX^{-1}u^\top = \sum_{t=1}^{k-r} u_{t+r}y^{-t}. \quad (24)$$

Proof. We use the notation $J_k^0 = I_k$. Note that in case $k = r$ the vector v is the zero vector. Otherwise the direct multiplication

$$(yI_k - J_k) \sum_{t=1}^k y^{-t} J_k^{t-1} = \sum_{t=1}^k (y^{1-t} J_k^{t-1} - y^{-t} J_k^t) = I_k - y^{-k} J_k^k = I_k$$

shows that $X^{-1} = \sum_{t=1}^k y^{-t} J_k^{t-1}$.

Therefore,

$$vX^{-1} = \left(\underbrace{0, \dots, 0}_r, y^{-1}, y^{-2}, \dots, y^{-(k-r)} \right)$$

since it is the $(r+1)$ -th row of X^{-1} . Hence, we obtain:

$$(vX^{-1})u^\top = \sum_{t=1}^{k-r} u_{t+r}y^{-t},$$

as desired. \square

Lemma 3.9. Assume $B \in M_n$ is in the Jordan normal form, and let $A = \begin{pmatrix} B & u^\top \\ v & b \end{pmatrix} \in M_{n+1}$ be its normalized integral extension. Then in the notation (19) – (23) the characteristic polynomial of A is

$$p_A(x) = (x - b)p_B(x) - \sum_{i=1}^m \sum_{j=1}^{\beta_i} \sum_{t=1}^{k_{i,j}-r_{i,j}} u_{t+r_{i,j}}^{i,j} \frac{p_B(x)}{(x - b_i)^t}. \quad (25)$$

Proof. Using the formula for the determinant of a block matrix with invertible block X_1

$$\det \begin{pmatrix} X_1 & X_2 \\ X_3 & X_4 \end{pmatrix} = \det(X_1) \det(X_4 - X_3 X_1^{-1} X_2),$$

we obtain that

$$p_A(x) = \det(xI_{n+1} - A) = \det \begin{pmatrix} xI_n - B & -u^\top \\ -v & x - b \end{pmatrix} = \det(xI_n - B) \det((x - b) - v(xI_n - B)^{-1}u^\top),$$

and therefore

$$p_A(x) = p_B(x) \left((x - b) - v(xI_n - B)^{-1}u^\top \right). \quad (26)$$

Considering $(xI_n - B)$ as an element of $M_n(\mathbb{K}(x))$, in the notation (19) – (23) we obtain:

$$\begin{aligned} (xI_n - B)^{-1} &= [xI_n - \text{diag}(b_1I_{k_{1,1}} + J_{k_{1,1}}, \dots, b_mI_{k_{m,\beta_m}} + J_{k_{m,\beta_m}})]^{-1}, \\ (xI_n - B)^{-1} &= [\text{diag}((x - b_1)I_{k_{1,1}} - J_{k_{1,1}}, \dots, (x - b_m)I_{k_{m,\beta_m}} - J_{k_{m,\beta_m}})]^{-1}, \\ (xI_n - B)^{-1} &= \text{diag}([(x - b_1)I_{k_{1,1}} - J_{k_{1,1}}]^{-1}, \dots, [(x - b_m)I_{k_{m,\beta_m}} - J_{k_{m,\beta_m}}]^{-1}). \end{aligned}$$

Hence,

$$v(xI_n - B)^{-1}u^\top = \sum_{i=1}^m \sum_{j=1}^{\beta_i} v^{i,i} [(x - b_i)I_{k_{i,j}} - J_{k_{i,j}}]^{-1} (u^{i,j})^\top. \quad (27)$$

Now we apply the formula (24) with $k = k_{i,j}$, $r = r_{i,j}$, $y = (x - b_i)$ to each summand and obtain that

$$v^{i,i} [(x - b_i)I_{k_{i,j}} - J_{k_{i,j}}]^{-1} (u^{i,j})^\top = \sum_{t=1}^{k_{i,j}-r_{i,j}} u_{t+r_{i,j}}^{i,j} (x - b_i)^{-t}. \quad (28)$$

It remains to substitute the expressions (27) and (28) into the equality (26), and the result follows. \square

Lemma 3.10. *Let $f(x), g(x) \in \mathbb{K}[x]$ be polynomials, and $\deg g(x) > \deg f(x)$. Then for any $t \in \mathbb{K}$ satisfying $f(t) \neq 0$ and for any k , $0 \leq k \leq \deg g(x) - \deg f(x)$, there exists a polynomial $h(x) \in \mathbb{K}[x]$, $\deg(h(x)) \leq k$, such that*

$$(fh)^{(i)}(t) = g^{(i)}(t), \quad i = 0, \dots, k. \quad (29)$$

Proof. Let us write $h(x) = a_0 + a_1(x - t) + \dots + a_k(x - t)^k$ with unknown coefficients $a_0, \dots, a_k \in \mathbb{K}$. Then $h^{(i)}(t) = i!a_i, i = 0, \dots, k$. Further,

$$(fh)^{(s)}(t) = \sum_{i=0}^s f^{(s-i)} h^{(i)}(t) = \sum_{i=0}^s i!a_i f^{(s-i)}(t).$$

Thus, the condition (29) is equivalent to the equality

$$\begin{pmatrix} 0!f(t) & 0 & 0 & \dots & 0 \\ 0!f'(t) & 1!f(t) & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0!f^{(k)}(t) & 1!f^{(k-1)}(t) & 2!f^{(k-2)}(t) & \dots & k!f(t) \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ \vdots \\ a_k \end{pmatrix} = \begin{pmatrix} g(t) \\ g'(t) \\ \vdots \\ g^{(k)}(t) \end{pmatrix}$$

The determinant of this linear system is equal to $0!f(t) \cdot 1!f(t) \cdot \dots \cdot k!f(t)$. Hence the matrix of the system is invertible, since $f(t) \neq 0$. Then (29) is satisfied for

$$\begin{pmatrix} a_0 \\ a_1 \\ \vdots \\ a_k \end{pmatrix} = \begin{pmatrix} 0!f(t) & 0 & 0 & \dots & 0 \\ 0!f'(t) & 1!f(t) & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0!f^{(k)}(t) & 1!f^{(k-1)}(t) & 2!f^{(k-2)}(t) & \dots & k!f(t) \end{pmatrix}^{-1} \begin{pmatrix} g(t) \\ g'(t) \\ \vdots \\ g^{(k)}(t) \end{pmatrix}.$$

\square

Proof of Theorem 1.2. In the notation (19) – (23), we set $S = \{b_i \mid \beta_i > 1\}$.

1. First, we show that if A is an integral of B , then $p_A(x)$ is an S -full integral of $(n+1)p_B(x)$. By Corollary 3.7, we can assume that A is normalized.

Since A is an integral of B , the equality $p'_A(x) = (n+1)p_B(x)$ follows from the definition. It remains to show that $p_A(x)$ is an S -full integral, i.e., that $p_A(b_i) = 0$ for all $b_i \in S$. Observe that $\beta_i > 1$ since $b_i \in S$. Hence, there are at least two summands in the equality (21) for α_i . Since $k_{i,j} > 0$, we obtain

$$\alpha_i = \sum_{j=1}^{\beta_i} k_{i,j} > \max_{1 \leq j \leq \beta_i} (k_{i,j}).$$

Since t ranges from 1 till $(k_{i,j} - r_{i,j})$ in the decomposition (25), it follows that

$$t \leq \max_{1 \leq j \leq \beta_i} (k_{i,j} - r_{i,j}) \leq \max_{1 \leq j \leq \beta_i} (k_{i,j}) < \alpha_i.$$

Since b_i is the zero of the polynomial $p_B(x)$ of multiplicity α_i , we have

$$(x - b_i) \mid \frac{p_B(x)}{(x - b_i)^t} \in \mathbb{K}[x]$$

for each $t = 1, \dots, (k_{i,j} - r_{i,j})$. Thus, every summand in the decomposition (25) of $p_A(x)$ is divisible by $(x - b_i)$, and hence $p_A(b_i) = 0$. Therefore, $p_A(x)$ is an S -full integral of $(n+1)p_B(x)$ as desired.

2. Now let us assume that $F(x)$ is an S -full integral of $(n+1)p_B(x)$ and prove that there exist vectors $u, v \in \mathbb{K}^n$ such that for $A = \begin{pmatrix} B & u^T \\ v & b \end{pmatrix} \in M_{n+1}$, where $b = \frac{\text{tr}(B)}{n}$, the equality $p_A(x) = F(x)$ holds. Let us observe first that to prove the theorem it is enough to find u, v such that

$$p_A^{(j)}(b_i) = F^{(j)}(b_i), \quad i = 1, \dots, m; \quad j = 0, \dots, \alpha_i - 1. \quad (30)$$

Indeed, (30) implies that the polynomial $(p_A - F)(x)$ has $\sum_{i=1}^m \alpha_i = n$ zeros counting with the multiplicities. Notice that the coefficient at x^{n-1} of $p_B(x)$ is equal to $-\text{tr}(B)$. Hence, since $F'(x) = (n+1)p_B(x)$, the coefficient at x^n of $F(x)$ is equal to $-\frac{n+1}{n}\text{tr}(B)$. On the other hand, the coefficient at x^n of $p_A(x)$ is equal to

$$-\text{tr}(A) = -(\text{tr}(B) + b) = -\frac{n+1}{n}\text{tr}(B).$$

Thus, since both $p_A(x)$ and $F(x)$ are monic and the coefficient at x^n of both polynomials is equal to $\frac{n+1}{n}\text{tr}(B)$, we obtain that $\deg((p_A - F)(x)) \leq n + 1 - 2 = n - 1$. Therefore, $(p_A - F)(x) \equiv 0$, and so $p_A(x) = F(x)$ as desired.

Now, let us prove (30).

2.1. At first, we consider $b_i \in S$. Let us set $v^{i,j} = u^{i,j} = 0 \in \mathbb{K}^{k_{i,j}}$ for each $j = 1, \dots, \beta_i$ and each $i = 1, \dots, m$ satisfying $b_i \in S$. Then by the formula (25) we obtain that $(x - b_i)^{\alpha_i} \mid p_A(x)$ for any values of the other coordinates of the vectors u, v . Observe that by definition of $F(x)$ and properties of zeros of derivatives for any $b_i \in S$ it holds that $(x - b_i)^{\alpha_i} \mid F(x)$ or, equivalently, $F^{(j)}(b_i) = 0$, $0 \leq j < \alpha_i$. Hence, for any $b_i \in S$ we obtain

$$p_A^{(j)}(b_i) = F^{(j)}(b_i) = 0, \quad 0 \leq j < \alpha_i. \quad (31)$$

2.2. Now let us consider an eigenvalue b_i of B such that $b_i \notin S$. According to the notation (19) — (23) this means that $\beta_i = 1$ and $k_{i,1} = \alpha_i$. Let us set $f(x) = \frac{p_B(x)}{(x-b_i)^{\alpha_i}}$. Then $f(x) \in \mathbb{K}^{n-\alpha_i}[x]$ is a polynomial with the property $f(b_i) \neq 0$. Since

$$\deg(F(x)) - \deg(f(x)) = n + 1 - (n - \alpha_i) = \alpha_i + 1 > \alpha_i - 1 \geq 0,$$

it follows from Lemma 3.10 that the system of equations

$$(f(x)h(x))^{(l)}(b_i) = F^{(l)}(b_i), \quad 0 \leq l < \alpha_i,$$

on the coefficients of a polynomial $h(x)$ has a solution $h_0(x) \in \mathbb{K}_{\alpha_i-1}[x]$ of degree $\deg h_0(x) = q$ with $0 \leq q \leq \alpha_i - 1$.

Now we define the elements $w_t \in \mathbb{K}$, $t = 0, \dots, \alpha_i - 1$, as follows. Let us expand the polynomial $h_0(x)$ on the degrees of $x - b_i$, i.e., $h_0(x) = \sum_{t=0}^q w_t(x - b_i)^t$. This expansion defines w_0, \dots, w_q . For $q + 1 \leq t \leq \alpha_i - 1$, we set $w_t = 0$. Let us show that for $v^{i,1} = (1, 0, \dots, 0)$ and $u_{t+1}^{i,1} = w_{\alpha_i-t}$ it holds that

$$p_A^{(l)}(b_i) = (f(x)h_0(x))^{(l)}(b_i) = F^{(l)}(b_i), \quad 0 \leq l < \alpha_i. \quad (32)$$

Indeed, let us consider

$$\tilde{p}_A(x) = \sum_{j=1}^{\beta_i} \sum_{t=1}^{k_{i,j}-r_{i,j}} u_{t+r_{i,j}}^{i,j} \frac{p_B(x)}{(x-b_i)^t}. \quad (33)$$

Then by the decomposition (25)

$$p_A(x) + \tilde{p}_A(x) = (x-b)p_B(x) - \sum_{\substack{\varphi=1,\dots,m \\ \varphi \neq i}} \sum_{j=1}^{\beta_\varphi} \sum_{t=1}^{k_{\varphi,j}-r_{\varphi,j}} u_{t+r_{\varphi,j}}^{\varphi,j} \frac{p_B(x)}{(x-b_\varphi)^t},$$

and using the fact that $(x-b_i)^{\alpha_i} \mid p_B(x)$, we obtain

$$(x-b_i)^{\alpha_i} \mid (p_A(x) + \tilde{p}_A(x)). \quad (34)$$

Since $\beta_i = 1$ and $k_{i,1} = \alpha_i$, we can remove the first summation in (33) and simplify the second as follows:

$$\tilde{p}_A(x) = \sum_{t=1}^{k_{i,1}-r_{i,1}} u_{t+r_{i,1}}^{i,1} \frac{p_B(x)}{(x-b_i)^t} = \sum_{t=1}^{\alpha_i-r_{i,1}} u_{t+r_{i,1}}^{i,1} \frac{p_B(x)}{(x-b_i)^t}.$$

By the choice of $v^{i,1}, u^{i,1}$, we have $r_{i,1} = 0$ and $u_t^{i,1} = w_{\alpha_i-t}$. Therefore,

$$\begin{aligned} \tilde{p}_A(x) &= \sum_{t=1}^{\alpha_i} u_t^{i,1} \frac{p_B(x)}{(x-b_i)^t} = \sum_{t=1}^{\alpha_i} w_{\alpha_i-t} \frac{p_B(x)}{(x-b_i)^t} = \sum_{s=0}^{\alpha_i-1} w_s \frac{p_B(x)}{(x-b_i)^{\alpha_i-s}} \\ &= \sum_{s=0}^{\alpha_i-1} w_s (x-b_i)^s \frac{p_B(x)}{(x-b_i)^{\alpha_i}} = (f \cdot h_0)(x), \end{aligned}$$

where the last equality in the first row is due to the substitution $s = \alpha_i - t$. Now (34) implies that

$$p_A^{(l)}(b_i) = (f \cdot h_0)^{(l)}(b_i), \quad 0 \leq l < \alpha_l.$$

Therefore, by the choice of $h_0(x)$ we get

$$p_A^{(l)}(b_i) = (f \cdot h_0)^{(l)}(b_i) = F^{(l)}(b_i), \quad 0 \leq l < \alpha_l.$$

Combining the equalities (31) and (32) we obtain (30), and the result follows. \square

Proof of Theorem 1.4. By Theorem 1.2, the matrix B is integrable if and only if an integral $\int p_B(x)dx$ takes the same value on all elements of $S = \{\lambda_1, \dots, \lambda_m\}$, which means that $p_B(x)$ has an S -full integral. It remains to apply Theorem 2.1 to get the desired statement. \square

Notice that Theorem 1.4 implies the following result proved in [5, Theorem 3.13].

Theorem 3.11. *Let $m, k \geq 0$ and $\alpha_1, \dots, \alpha_m \geq 2, \beta_1, \dots, \beta_k \geq 1$ be integers, $n = k + \sum_{j=1}^m \alpha_j$, and \mathcal{M} the subset of M_n consisting of diagonal matrices B with pairwise different eigenvalues $(\lambda_1, \dots, \lambda_m, \mu_1, \dots, \mu_k)$ of the multiplicities $\alpha_1, \dots, \alpha_m, 1, \dots, 1$, correspondingly.*

Then \mathcal{M} contains an integrable matrix if and only if $m \leq k + 1$, and contains a non-integrable matrix if and only if $m \geq 2$. Moreover,

1. *If $m \leq 1$, then all matrices in \mathcal{M} are integrable.*
2. *If $2 \leq m \leq k + 1$, then \mathcal{M} contains both integrable and non-integrable matrices.*
3. *If $m > k + 1$, then all matrices in \mathcal{M} are non-integrable.*

Proof. If B is diagonalizable, then $\beta_1 = \dots = \beta_k = 1$ and hence $n - M + 1 = k + 1$. Thus, Theorem 1.4 provides the desired result. \square

3.2 Supplements

The main aim of this section is to prove the following theorem:

Theorem 3.12. *Let $B \in M_n$ be diagonalizable, and $B' \in M_n$ have the same eigenvalues as B counting the multiplicities, i.e., $p_{B'}(x) = p_B(x)$. If B is integrable, then B' is integrable.*

Since by [5, Corollary 5.2] an integrable diagonalizable matrix has a diagonalizable integral, Theorem 3.12 follows from the lemma below.

Lemma 3.13. *Let $B \in M_n$ be diagonalizable, and let $B' \in M_n$ have the same eigenvalues as B counting the multiplicities, i.e., $p_{B'}(x) = p_B(x)$. If B is integrable and $A = \begin{pmatrix} B & u^\top \\ v & b \end{pmatrix}$ is a diagonalizable integral of B , then there exists $X \in GL_n(\mathbb{K})$ and there exists an integral A' of $XB'X^{-1}$ such that $A' = \begin{pmatrix} XB'X^{-1} & u^\top \\ v & b \end{pmatrix}$ with the same vectors (u, v) and an element b as A has.*

Proof. Without loss of generality, we assume that

$$B = \text{diag}(\underbrace{b_1, \dots, b_1}_{\alpha_1}, \dots, \underbrace{b_m, \dots, b_m}_{\alpha_m}).$$

By Lemma 3.6, without loss of generality we may assume that A is a normalized integral extension of B . Since B is diagonalizable, in the notations of the formula (22) we have that $k_{i,j} = 1$ for all i, j . Thus $\beta_i = \alpha_i$, and $v^{i,j}, u^{i,j} \in \mathbb{K}$. By the criterion of diagonalizability of an integral [5, Theorem 4.1], we obtain that if $\alpha_i > 1$, then $v^{i,j} = u^{i,j} = 0$, $j = 1, \dots, \alpha_i$. By Lemma 3.9, these equalities imply that

$$p_A(x) = (x - b)p_B(x) - \sum_{\substack{s=1; \\ \alpha_s=1}}^m u_{s,1} \frac{p_B(x)}{x - b_s}.$$

Let B'' be the Jordan normal form of B' such that the order of the diagonal elements of B'' coincides with the order of the diagonal elements of B . Let us consider $A' = \begin{pmatrix} B'' & u^T \\ v & b \end{pmatrix}$. Since A is a normalized extension of B and $v^{i,j} = u^{i,j} = 0$, $j = 2, \dots, \alpha_i$, it follows that A' is a normalized extension of B'' by definition. Hence, by Lemma 3.9, we obtain

$$p_{A'}(x) = (x - b)p_{B''}(x) - \sum_{\substack{s=1; \\ \alpha_s=1}}^m u_{s,1} \frac{p_{B''}(x)}{x - b_s}.$$

Since $p_{B''}(x) = p_{B'}(x) = p_B(x)$, this implies $p_{A'}(x) = p_A(x)$, and therefore

$$p'_{A'}(x) = (n + 1)p_B(x) = (n + 1)p_{B''}(x).$$

Thus, A' is an integral of B'' . □

Remark 3.14. Even if $A = \begin{pmatrix} B & u^T \\ v & b \end{pmatrix}$ is a diagonalizable integral of B , the integral $A' = \begin{pmatrix} B' & u^T \\ v & b \end{pmatrix}$ of B' can be non-diagonalizable. Consider, for example, $B = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}$, $B' = \begin{pmatrix} a & 1 \\ 0 & a \end{pmatrix}$. Then $p_B(x) = p_{B'}(x)$ and the vectors $u = v = (0, 0)$ provide a diagonal integral $\begin{pmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & a \end{pmatrix}$ of B . However, $\begin{pmatrix} a & 1 & 0 \\ 0 & a & 0 \\ 0 & 0 & a \end{pmatrix}$ is not diagonalizable.

Corollary 3.15. Consider an arbitrary (not necessarily diagonalizable) $B \in M_n$. If $p_B(x)$ has an S -full integral, where S is the set of all multiple zeros of $p_B(x)$, then B is integrable.

Proof. If $p_B(x)$ has a full integral, then a diagonal matrix B' with the same spectrum as B is integrable by Theorem 1.2. Hence, the matrix B is integrable by Theorem 3.12. □

Remark 3.16. The assertion opposite to Corollary 3.15 is not true. Consider the non-diagonalizable matrix

$$B = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

Since $p_B(x) = (x - 1)^2(x + 1)^2$, it follows from [5, Theorem 2.14] that the matrix $B' = \text{diag}(1, 1, -1, -1)$ is not integrable. Nevertheless, the matrix

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & -1 & \frac{4}{3} \\ 0 & 0 & 1 & 1 & 0 \end{pmatrix}$$

is an integral of B , since $p_A(x) = (x - 1)^2(x^3 + 2x^2 - \frac{1}{3}x + \frac{8}{3}) = x^5 - \frac{10}{3}x^3 + 5x - \frac{8}{3}$ and

$$p'_A(x) = 5(x^4 - 2x^2 + 1) = 5(x - 1)^2(x + 1)^2 = 5p_B(x).$$

As a conclusion, we made the following observation:

Remark 3.17. Consider an ordered set of matrices $\{B_1, \dots, B_m\} \subset M_n$ having the same eigenvalues counting the multiplicities. Let us assume that the following conditions are satisfied:

$$\dim(\text{Ker}(B_i - \lambda I)) \geq \dim(\text{Ker}(B_{i+1} - \lambda I))$$

for any λ , and

$$\dim(\text{Ker}(B_i - \lambda I)) > \dim(\text{Ker}(B_{i+1} - \lambda I))$$

for at least one value of λ , i.e., the number of Jordan cells in a Jordan block does not increase with the growth of i , and decreases for at least one block for each i . In this case, similarly to the proof of Theorem 1.2, it can be shown that if B_i is integrable for a certain index i , then B_{i+1} is integrable. Since the number of cells in a certain block decreases, we obtain just 1 cell on a certain step. Therefore, that if m is big enough, then B_m is non-derogatory, and hence integrable. Thus, if B_1 is not integrable, then there exists a positive integer k such that the matrices B_1, \dots, B_k are not integrable, but the matrices B_{k+1}, \dots, B_m are integrable.

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