## 8.2 Tests About a Population Mean



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## Tests About a Population Mean

- Confidence intervals for a population mean $\mu$ focused on three different cases.

- We now develop test procedures for these cases.

Case I : Normal Population with Known σ
Case II : Large-Sample Tests ตัวอย่างใหญ่
Case III : Normal Population Distribution กระจายข้อมูลปกติ
→ t-test

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# Case I: A Normal Population with Known $\sigma$



## Case I: A Normal Population with Known $\sigma$

- Although assumption that value of $\sigma$ is known is rarely met in practice, this case provides a good starting point because of the ease with which general procedures and their properties can be developed.

$H_0: \mu = \mu_0$

null value

- Null hypothesis in all three cases will state that $\mu$ has a particular numerical value, the ***null value***, which we will denote by $\mu_0$ .

ค่าตัวเลขจำเพาะ

ค่าตัวเลขจำเพาะ

- Let $X_1, \ldots, X_n$ represent random sample of size $n$ from normal population.

## Case I: A Normal Population with Known $\sigma$

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- Then sample mean $\bar{X}$ has a normal distribution with expected value $\mu_{\bar{X}} = \mu$ and standard deviation $\sigma_{\bar{X}} = \frac{\sigma}{\sqrt{n}}$

null Value

- When $H_0$ is true, $\mu_{\bar{X}} = \mu_0$

- Consider now the statistic $Z$ obtained by standardizing $\bar{X}$ under assumption that $H_0$ is true:

$$Z = \frac{\bar{X} - \mu_0}{\sigma / \sqrt{n}}$$

- Substitution of computed sample mean $\bar{x}$ gives $z$, distance between $\bar{x}$ and $\mu_0$ expressed in "standard deviation units."

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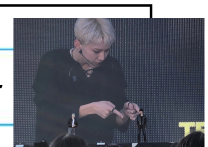

## Case I: A Normal Population with Known $\sigma$

- For example, if null hypothesis is

$\mu_0$

$$H_0 : \mu = 100 \qquad \sigma_{\bar{X}} = \frac{\sigma}{\sqrt{n}} = \frac{10}{\sqrt{25}} = 2.0 \qquad \text{and} \qquad \bar{x} = 103$$

then the test statistic value is

$$z = \frac{\bar{x} - \mu}{\sigma_{\bar{X}}} \Rightarrow z = \frac{\bar{x} - \mu}{\frac{\sigma}{\sqrt{n}}} \Rightarrow z = \frac{103 - 100}{\frac{10}{\sqrt{25}}} = 1.5$$

- That is, the observed value of $\bar{x}$ is 1.5 standard Deviations (of $\bar{X}$ ) larger than what we expect it to be when $H_0$ is true.

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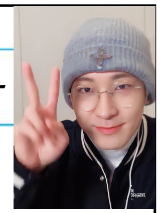
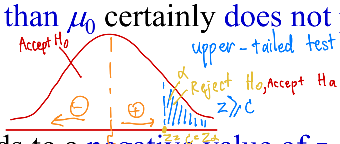
## Case I: A Normal Population with Known $\sigma$



- Statistic $Z$ is natural measure of distance between $\overline{X}$, estimator of $\mu$, and its expected value when $H_0$ is true.
- If this distance is too great in a direction consistent with $H_a$, null hypothesis should be rejected.
- Suppose first that the alternative hypothesis has the form

$$\boxed{H_a : \mu > \mu_0.}$$

- Then an $\bar{x}$ value less than $\mu_0$ certainly does not provide support for $H_a$.

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$$\boxed{z = \frac{\bar{x} - \mu_0}{\frac{\sigma}{\sqrt{n}}}}$$

- Such an $\bar{x}$ corresponds to a negative value of $z$ (since $\bar{x} - \mu_0$ is negative and the divisor $\sigma / \sqrt{n}$ is positive).

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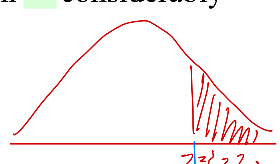
## Case I: A Normal Population with Known $\sigma$

- Similarly, an $\bar{x}$ value that exceeds $\mu_0$ by only a small amount (corresponding to $z$, which is positive but small) does not suggest that $H_0$ should be rejected in favor of $H_a$.

เกินกว่า

- Rejection of $H_0$ is appropriate only when $\bar{x}$ considerably exceeds $\mu_0$—that is, when the $z$ **value** is positive and **large**.




- In summary, appropriate rejection region, based on the test statistic $Z$ rather than $\overline{X}$, has the form $z \geq c$.

Rejection Region
$z \geq c$
$z \geq z_\alpha$ นั่นเอง

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## Case I: A Normal Population with Known $\sigma$

เลือกค่า $\alpha$

- As we have discussed earlier, **cutoff value $c$** should be chosen to control probability of **type I error** at the **desired level $\alpha$.**
- The required cutoff $c$ is **$z$ critical value** that captures upper-tail area $\alpha$ under the $z$ curve.



## Case I: A Normal Population with Known $\sigma$

- As an example,
- let $c = 1.645$, value that captures tail area 0.05($z_{0.05} = 1.645$).

Table A.3 Standard Normal Curve Areas *(cont.)* $\Phi(z) = P(Z \le z)$

| $z$ | .00 | .01 | .02 | .03 | .04 | .05 | .06 | .07 | .08 | .09 |
|---|---|---|---|---|---|---|---|---|---|---|
| 1.6 | .9452 | .9463 | .9474 | .9484 | .9495 + | .9505 | .9515 | .9525 | .9535 | .9545 |

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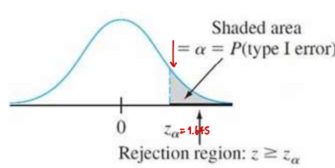
$c, z_\alpha$ ?

Then, $\alpha = P(\text{type I error})$

$= P(H_0 \text{ is } \boxed{\text{rejected}} \text{ when } H_0 \text{ is true})$

$1-P(Z \le 1.645)$ $= P(Z \ge 1.645 \text{ when } Z \sim N(0,1))$

$= 1 - \Phi(1.645) = 1 - 0.95 = 0.05$

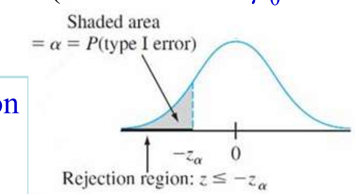



- More generally, rejection region $z \ge z_\alpha$ has type I error probability $\alpha$.
- The test procedure is *upper-tailed* because rejection region consists only of large values of test statistic.

## Case I: A Normal Population with Known $\sigma$

Lower – tailed test

- Analogous reasoning for alternative hypothesis $\boxed{H_a: \mu < \mu_0}$ suggests a rejection region of the form $z \le c$, where $c$ is a suitably chosen negative number ($\bar{x}$ is far below $\mu_0$ if and only if $z$ is quite negative).




- Because $Z$ has standard normal distribution when $H_0$ is true, taking $c = -z_\alpha$ yields $P$(type I error) $= \alpha$.

This is a ***lower-tailed* test.**

- For example, $z_{0.10} = 1.28$ implies that the rejection region $z \le -1.28$ specifies a test with significance level 0.10.

| $z$ | .00 | .01 | .02 | .03 | .04 | .05 | .06 | .07 | .08 | .09 |
|---|---|---|---|---|---|---|---|---|---|---|
| −1.2 | .1151 | .1131 | .1112 | .1093 | .1075 | .1056 | .1038 | .1020 | .1003 | .0985 |

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## Case I: A Normal Population with Known $\sigma$

$H_0: \mu = \mu_0$

$\boxed{H_a: \mu < \mu_0}$ } Lower – tailed test

↳ Rejection Region : $z \le c$

$\therefore c = -z_\alpha$

Ex ให้ $\alpha = 0.10$

$\alpha = P(\text{Type I error})$

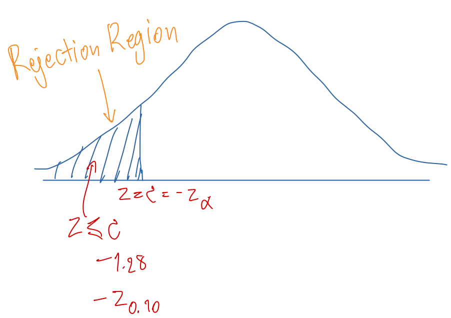
$= P($ปฏิเสธ $H_0$ เมื่อ $H_0$ เป็นจริง$)$

$= P(z \le c)$

$0.1 = P(z \le c)$

$= \phi(-1.28)$

$\therefore c = -1.28 \longrightarrow c = -z_\alpha = -z_{0.10}$



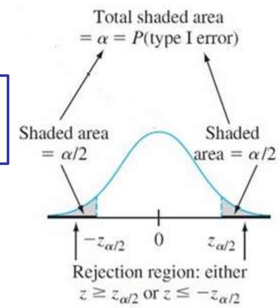

| $z$ | .00 | .01 | .02 | .03 | .04 | .05 | .06 | .07 | .08 | .09 |
|---|---|---|---|---|---|---|---|---|---|---|
| −1.2 | .1151 | .1131 | .1112 | .1093 | .1075 | .1056 | .1038 | .1020 | .1003 | .0985 |

Two - tail test

## Case I: A Normal Population with Known $\sigma$

- Finally, when alternative hypothesis is $\boxed{H_a: \mu \neq \mu_0}$,
  $H_0$ should be rejected if $\bar{x}$ is too far to **either side of $\mu_0$**.

- This is equivalent to rejecting $H_0$ either if $z \geq c$ or if $z \leq -c$.
- Suppose we desire $\alpha = 0.05$. Then,




$$\boxed{\begin{array}{l}0.05 = P(Z \geq c \text{ or } Z \leq -c \\ \qquad \text{when } Z \text{ has a standard normal distribution})\end{array}}$$

$$= \Phi(-c) + (1 - \Phi(c)) = 2[1 - \Phi(c)]$$

- Thus $c$ is such that $1 - \Phi(c)$ ,
  area under the $z$ curve to the right of $c$, is 0.025 (and not 0.05!).



## Case I: A Normal Population with Known $\sigma$

- From Appendix Table A.3, $c = 1.96$ , and the rejection region is
  $z \geq 1.96$ *or* $z \leq -1.96$.

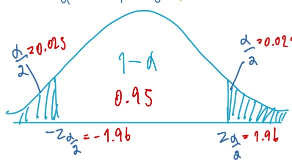
| z | .00 | .01 | .02 | .03 | .04 | .05 | .06 | .07 | .08 | .09 |
|---|---|---|---|---|---|---|---|---|---|---|
| −1.9 | .0287 | .0281 | .0274 | .0268 | .0262 | .0256 | .0250 | .0244 | .0239 | .0233 |
| 1.9 | .9713 | .9719 | .9726 | .9732 | .9738 | .9744 | .9750 | .9756 | .9761 | .9767 |

- For any $\alpha$, the *two-tailed* rejection region $z \geq z_{\alpha/2}$ *or* $z \leq -z_{\alpha/2}$
  has type I error probability $\alpha$
  (since area $\alpha/2$ is captured under each of two tails of the $z$ curve).

- Again, key reason for using standardized test statistic $Z$ is that
  because $Z$ has a known distribution when
  $H_0$ is true (standard normal), rejection region with desired type I
  error probability is easily obtained by using appropriate critical value.

## Case I: A Normal Population with Known $\sigma$

$H_0: \mu = \mu_0$
$H_a: \mu \neq \mu_0$ } two-tailed test



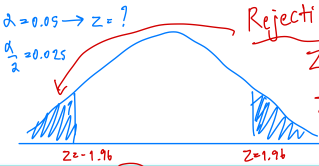

Rejection Region $z \geq c$ หรือ $z \leq -c$

$\alpha = P(\text{Type I error})$
$= P(\text{ปฏิเสธ } H_0 \text{ เมื่อ } H_0 \text{ เป็นจริง})$
$= P(Z \leq -c) + P(Z \geq c)$
$= P(Z \geq c) + P(Z \geq c)$
$= 2P(Z \geq c)$
$= 2(1 - P(Z \leq c))$
$\alpha = 2(1 - \Phi(c))$

$1 - \Phi(c) = \frac{\alpha}{2}$

Ex $\alpha = 0.05 \rightarrow z = ?$
$\frac{\alpha}{2} = 0.025$

Rejection Region
$Z \leq -1.96$
$Z \geq 1.96$




| z | .00 | .01 | .02 | .03 | .04 | .05 | .06 | .07 | .08 | .09 |
|---|---|---|---|---|---|---|---|---|---|---|
| −1.9 | .0287 | .0281 | .0274 | .0268 | .0262 | .0256 | .0250 | .0244 | .0239 | .0233 |
| 1.9 | .9713 | .9719 | .9726 | .9732 | .9738 | .9744 | .9750 | .9756 | .9761 | .9767 |



## Case I: A Normal Population with Known $\sigma$

- **Test procedure** for **case I** is summarized in accompanying box, and corresponding rejection regions are illustrated in Figure 8.2.

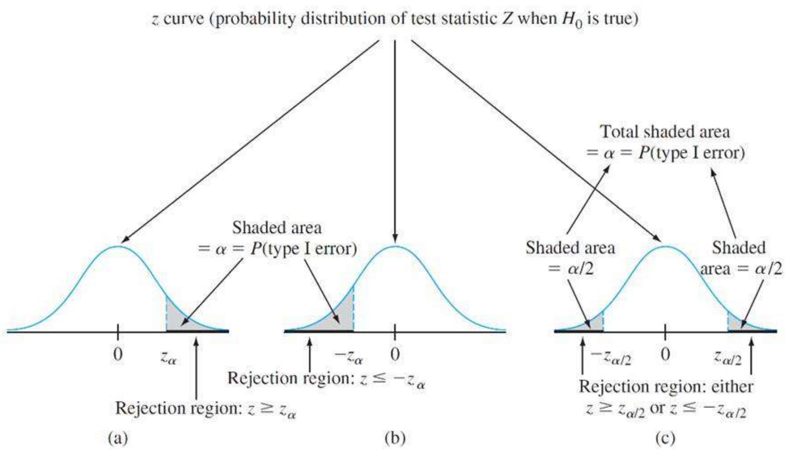



**Figure 8.2** Rejection regions for *z* **tests**: (a) **upper-tailed test**; (b) **lower-tailed test**; (c) **two-tailed test**

## Case I: A Normal Population with Known $\sigma$

Null hypothesis: $H_0: \mu = \mu_0$

Test statistic value : $z = \dfrac{\bar{x} - \mu_0}{\sigma/\sqrt{n}}$

**Alternative Hypothesis** **Rejection Region for Level $\alpha$ Test**

$H_a: \mu > \mu_0$ $z \geq z_\alpha$ (upper-tailed test) $z_\alpha$

$H_a: \mu < \mu_0$ $z \leq -z_\alpha$ (lower-tailed test) $-z_\alpha$

$H_a: \mu \neq \mu_0$ either $z \geq z_{\alpha/2}$ or $z \leq -z_{\alpha/2}$ (two-tailed test)

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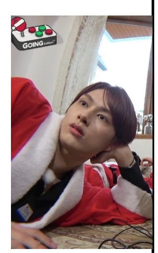
## Case I: A Normal Population with Known $\sigma$

- Use of the following sequence of steps is recommended when testing hypotheses about a parameter.



1. Identify the parameter of interest and describe it in the context of the problem situation.
2. Determine the null value and state the null hypothesis. $\mu_0$ $H_0: \mu = \mu_0$
3. State the appropriate alternative hypothesis. $H_a$
4. Give the formula for the computed value of the test statistic (substituting the null value and the known values of any other parameters, but *not* those of any sample-based quantities). $z = \dfrac{\bar{x} - \mu_0}{\sigma/\sqrt{n}}$

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## Case I: A Normal Population with Known $\sigma$

5. State the rejection region for the selected significance level $\alpha$.

6. Compute any necessary sample quantities, substitute into the formula for the test statistic value, and compute that value.

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7. Decide whether $H_0$ should be rejected, and state this conclusion in the problem context.

- The formulation of hypotheses (Steps 2 and 3) should be done before examining the data.

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## Example 6


ระบบหัวกระจายน้ำดับเพลิงอัตโนมัติ

- A manufacturer of sprinkler systems used for fire protection in office buildings claims that true average system-activation temperature is 130°. → $\mu_0$
- A sample of $n = 9$ systems, when tested, yields a sample average activation temperature of 131.08°F. — $\bar{x}$
- If the distribution of activation times is normal with standard deviation 1.5°F, → $\sigma = 1.5$
  does the data contradict the manufacturer's claim at significance level $\alpha = 0.01$?

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$\frac{\alpha}{2} = 0.005$

# Example 6

cont'd

1. Parameter of interest: $\mu$ = true average activation temperature.

$H_0: \mu = \mu_0$

2. Null hypothesis: $H_0$: $\mu = 130$ (null value = $\mu_0 = 130$).

two - tailed test

3. Alternative hypothesis: $H_a$: $\boxed{\mu \neq 130}$ (a departure from the claimed value in *either* direction is of concern).

4. Test statistic value:

$\mu_0 = 130$

$\sigma = 1.5$

$n = 9$

$$z = \frac{\bar{x} - \mu_0}{\sigma/\sqrt{n}} = \frac{\bar{x} - 130}{1.5/\sqrt{n}}$$

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# Example 6

$H_a$: $\mu \neq 130$ either $z \geq z_{\alpha/2}$ or $z \leq -z_{\alpha/2}$ (two-tailed test)

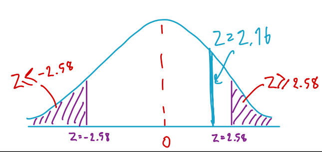
cont'd

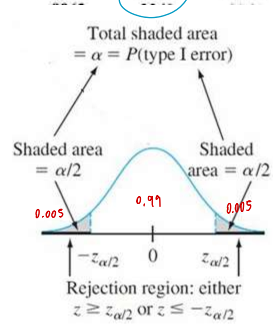
$\alpha = 0.01$ $\frac{\alpha}{2} = 0.005$

5. Rejection region: The form of $H_a$ implies use of two-tailed test with rejection region *either* $z \geq z_{0.005}$ *or* $z \leq -z_{0.005}$ .

| $z$ | .00 | .01 | .02 | .03 | .04 | .05 | .06 | .07 | .08 | .09 |
|---|---|---|---|---|---|---|---|---|---|---|
| −2.5 | .0062 | .0060 | .0059 | .0057 | .0055 | .0054 | .0052 | .0051 | .0049 | .0038 |
| 2.5 | .9938 | .9940 | .9941 | .9943 | .9945 | .9946 | .9948 | .9949 | .9951 | .9952 |

- From Appendix Table A.3, $z_{0.005} = 2.58$ ,
  so we reject $H_0$
  if either $z \geq 2.58$ or $z \leq -2.58$



## Example 6

$H_a$: $\mu \neq 130$ either $z \geq z_{\alpha/2}$ or $z \leq -z_{\alpha/2}$ (two-tailed test)

cont'd

6. Substituting $n = 9$ and $\bar{x} = 131.08$ ,

$$z = \frac{\bar{x} - \mu_0}{\sigma/\sqrt{n}} = \frac{\bar{x} - 130}{1.5/\sqrt{n}} \Rightarrow z = \frac{131.08 - 130}{1.5/\sqrt{9}} = \frac{1.08}{0.5} = 2.16$$

- That is, the observed sample mean is a bit more than 2 standard deviations above what would have been expected were $H_0$ true.

7. The computed value $z = 2.16$ does not fall in rejection region ($-2.58 < 2.16 < 2.58$), so $H_0$ cannot be rejected at significance level 0.01.

Data does not give strong support to the claim that the true average differs from the design value of 130.

# End of Section 8.2

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