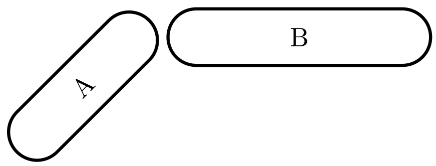

The velocity of a point with position $\mathbf{r}$ relative to the center of mass of a cell with its main axis along the unit vector $\hat{\mathbf{a}}$ is:

$$
\begin{aligned}
\mathbf{v}_{\mathbf{r}} &= \mathbf{v}_{\text{linear}} + \mathbf{v}_{\text{angular}} + \mathbf{v}_{\text{growth}} \\
&= \mathbf{v} + \boldsymbol{\omega} \times \mathbf{r} + \frac{1}{2}\frac{\hat{\mathbf{a}} \cdot \mathbf{r}}{l} v_{growth} \\
&= \mathbf{v} - \mathbf{r} \times \boldsymbol{\omega} + \frac{1}{2}\frac{\hat{\mathbf{a}} \cdot \mathbf{r}}{l} v_{growth} \\
&= \frac{1}{m}\mathbf{p} - \mathbf{r} \times \mathbf{I}^{-1}\mathbf{L} + \frac{1}{2}\frac{\hat{\mathbf{a}} \cdot \mathbf{r}}{l}\frac{g}{G} \\
&= \frac{1}{m}\mathbf{p} - [\mathbf{r}]_{\times}\mathbf{I}^{-1}\mathbf{L} + \frac{1}{2}\frac{\hat{\mathbf{a}} \cdot \mathbf{r}}{l}\frac{g}{G}
\end{aligned}
$$

This expression is linear in $\mathbf{p}$, $\mathbf{L}$, and g, so the velocity of the point after an impulse $\Delta\mathbf{p}$ is:

$$
\begin{aligned}
\mathbf{v}_{\mathbf{r}} &= \mathbf{v}_{\mathbf{r}0} + \Delta\mathbf{v} \\
&= \mathbf{v}_{\mathbf{r}0} + \frac{1}{m}\Delta\mathbf{p} - [\mathbf{r}]_{\times}\mathbf{I}^{-1}\Delta\mathbf{L} \times \mathbf{r} + \frac{1}{2}\frac{\hat{\mathbf{x}} \cdot \mathbf{r}}{l}\frac{\Delta g}{G} \\
&= \mathbf{v}_{\mathbf{r}0} + \mathbf{M}_{\hat{\mathbf{a}},\mathbf{r}}\Delta\mathbf{p}
\end{aligned}
$$

...where:

$$
\mathbf{M}_{\hat{\mathbf{a}},\mathbf{r}} = \begin{bmatrix} \frac{1}{m}I & [\mathbf{r}]_{\times}\mathbf{I}^{-1} & \frac{\hat{\mathbf{a}} \cdot \mathbf{r}}{2lG}\hat{\mathbf{a}} \end{bmatrix}
$$

...and:

$$
\Delta\mathbf{p} = \begin{bmatrix} \Delta\mathbf{p} \\ \Delta\mathbf{L} \\ \Delta g \end{bmatrix} = \begin{bmatrix} \Delta p_x \\ \Delta p_y \\ \Delta p_z \\ \Delta L_x \\ \Delta L_y \\ \Delta L_z \\ \Delta g \end{bmatrix}
$$

In the above, I is the identity matrix and $\mathbf{I}^{-1}$ is the inverse inertia tensor. No confusion should arise from the use of $\Delta\mathbf{p}$ to signify both general momentum and specifically linear momentum.

The inertia tensor for a rod of mass m, radius r, and length l centered at the origin and lying along the x axis is:

$$\mathbf{I} = \frac{1}{12} m \begin{bmatrix} 6r^2 & 0 & 0 \\ 0 & 3r^2 + l^2 & 0 \\ 0 & 0 & 3r^2 + l^2 \end{bmatrix}$$

Therefore the inverse inertia tensor for the same cylinder is:

$$\mathbf{I}^{-1} = \frac{12}{m} \begin{bmatrix} \frac{1}{6r^2} & 0 & 0 \\ 0 & \frac{1}{3r^2+l^2} & 0 \\ 0 & 0 & \frac{1}{3r^2+l^2} \end{bmatrix}$$

Suppose $\mathbf{T}_i$ transforms from world coordinates to local coordinates for cell i — i.e. it transforms the x axis to the main axis of that cell. Then the inverse inertia tensor for cell i in world coordinates is given by:

$$\mathbf{I}_i^{-1} = \mathbf{T}_i \mathbf{I}_{\text{cyl}}^{-1} \mathbf{T}_i^{-1}$$

So we now have:

$$\mathbf{M}_{\hat{\mathbf{a}},\mathbf{r}} = \begin{bmatrix} \frac{1}{m} I & [\mathbf{r}]_\times \mathbf{T}_i \mathbf{I}_{\text{cyl}}^{-1} \mathbf{T}_i^{-1} & \frac{\hat{\mathbf{a}} \cdot \mathbf{r}}{2lG} \hat{\mathbf{a}} \end{bmatrix}$$

We assume that colliding capsules — those that overlap and are moving toward each other (the dot product of the velocity vectors of the contact points on the two capsules is negative) — collide inelastically. In this case, the relative velocity of the two contact points along the normal $\hat{\mathbf{n}}$ after the impulse provided by the collision must be zero. So:

$$\begin{aligned}
\hat{\mathbf{n}}^{\mathrm{T}} \left(\mathbf{v}_{\mathbf{r}_a} - \mathbf{v}_{\mathbf{r}_b} \right) \cdot &= 0 \\
\hat{\mathbf{n}}^{\mathrm{T}} \mathbf{M}_{\mathbf{r}_a} \mathbf{p}_a - \hat{\mathbf{n}}^{\mathrm{T}} \mathbf{M}_{\mathbf{r}_b} \mathbf{p}_b &= 0 \\
\hat{\mathbf{n}}^{\mathrm{T}} \mathbf{M}_{\mathbf{r}_a} \left(\mathbf{p}_{a_0} + \Delta \mathbf{p}_a \right) - \hat{\mathbf{n}}^{\mathrm{T}} \mathbf{M}_{\mathbf{r}_b} \left(\mathbf{p}_{b_0} + \Delta \mathbf{p}_b \right) &= 0 \\
\hat{\mathbf{n}}^{\mathrm{T}} \mathbf{M}_{\mathbf{r}_a} \Delta \mathbf{p}_a - \hat{\mathbf{n}}^{\mathrm{T}} \mathbf{M}_{\mathbf{r}_b} \Delta \mathbf{p}_b &= \hat{\mathbf{n}}^{\mathrm{T}} \mathbf{M}_{\mathbf{r}_b} \mathbf{p}_{b_0} - \hat{\mathbf{n}}^{\mathrm{T}} \mathbf{M}_{\mathbf{r}_a} \mathbf{p}_{a_0} \\
\hat{\mathbf{n}}^{\mathrm{T}} \mathbf{M}_{\mathbf{r}_a} \Delta \mathbf{p}_a - \hat{\mathbf{n}}^{\mathrm{T}} \mathbf{M}_{\mathbf{r}_b} \Delta \mathbf{p}_b &= \hat{\mathbf{n}}^{\mathrm{T}} \mathbf{v}_{b_0} - \hat{\mathbf{n}}^{\mathrm{T}} \mathbf{v}_{a_0} \\
\hat{\mathbf{n}}^{\mathrm{T}} \mathbf{M}_{\mathbf{r}_a} \Delta \mathbf{p}_a - \hat{\mathbf{n}}^{\mathrm{T}} \mathbf{M}_{\mathbf{r}_b} \Delta \mathbf{p}_b &= \hat{\mathbf{n}}^{\mathrm{T}} \mathbf{v}_{\text{rel}_{a,b}}
\end{aligned}$$

Write:

$$\begin{aligned}
\mathbf{N}_{\hat{\mathbf{a}},\mathbf{r}} &= \begin{bmatrix} \hat{\mathbf{n}}^{\mathrm{T}} \frac{1}{m} I & \hat{\mathbf{n}}^{\mathrm{T}} [\mathbf{r}]_\times \mathbf{T}_i \mathbf{I}_{\text{cyl}}^{-1} \mathbf{T}_i^{-1} & \hat{\mathbf{n}}^{\mathrm{T}} \frac{\hat{\mathbf{a}} \cdot \mathbf{r}}{2lG} \hat{\mathbf{a}} \end{bmatrix} \\
&= \begin{bmatrix} \frac{1}{m} \hat{\mathbf{n}} & \hat{\mathbf{n}}^{\mathrm{T}} [\mathbf{r}]_\times \mathbf{T}_i \mathbf{I}_{\text{cyl}}^{-1} \mathbf{T}_i^{-1} & \frac{1}{2lG} \left(\hat{\mathbf{a}} \cdot \mathbf{r} \right) \left(\hat{\mathbf{a}} \cdot \hat{\mathbf{n}} \right) \end{bmatrix}
\end{aligned}$$

This is a 1×7 matrix which when multiplied by the 7×1 matrix $\Delta\mathbf{p}$ gives a change in velocity along the normal $\hat{\mathbf{n}}$.

In block matrix form, we can write:

$$\begin{bmatrix} \hat{\mathbf{n}}^{\mathrm{T}}\mathbf{M}_{\mathbf{r}_a} & -\hat{\mathbf{n}}^{\mathrm{T}}\mathbf{M}_{\mathbf{r}_b} \end{bmatrix} \begin{bmatrix} \Delta\mathbf{p}_a \\ \Delta\mathbf{p}_b \end{bmatrix} = \hat{\mathbf{n}}^{\mathrm{T}}\mathbf{v}_{\mathrm{rel}_{a,b}}$$

More generally, we will have a system of equations with one such equation for each pair of colliding cells. We build a larger block matrix with a row for each constraint and a column for each cell in the system. If the ith constraint involves cells a and b (in that order), then:

$$\mathbf{M}_{ij} = \left\{ \begin{array}{ll} \hat{\mathbf{n}}^{\mathrm{T}}\mathbf{M}_{\mathbf{r}_a} & \text{if } j = a \\ -\hat{\mathbf{n}}^{\mathrm{T}}\mathbf{M}_{\mathbf{r}_b} & \text{if } j = b \\ 0 & \text{otherwise} \end{array} \right\}$$

A sytem with four cells and four collisions might have the equation:

$$\begin{bmatrix} \mathbf{M}_{\mathbf{r}_{1a}} & -\mathbf{M}_{\mathbf{r}_{1b}} & 0 & 0 \\ 0 & \mathbf{M}_{\mathbf{r}_{2b}} & -\mathbf{M}_{\mathbf{r}_{2c}} & 0 \\ \mathbf{M}_{\mathbf{r}_{3a}} & 0 & 0 & -\mathbf{M}_{\mathbf{r}_{3d}} \\ 0 & 0 & \mathbf{M}_{\mathbf{r}_{4c}} & -\mathbf{M}_{\mathbf{r}_{4d}} \end{bmatrix} \begin{bmatrix} \Delta\mathbf{p}_a \\ \Delta\mathbf{p}_b \\ \Delta\mathbf{p}_c \\ \Delta\mathbf{p}_d \end{bmatrix} = \begin{bmatrix} \mathbf{v}_{\mathrm{rel}_{a,b}} \\ \mathbf{v}_{\mathrm{rel}_{b,c}} \\ \mathbf{v}_{\mathrm{rel}_{a,d}} \\ \mathbf{v}_{\mathrm{rel}_{c,d}} \end{bmatrix}$$

We can transform this into a square matrix by multiplying by $\mathbf{M}^{\mathrm{T}}$.

$$\mathbf{M}^{\mathrm{T}}\mathbf{M}\Delta\mathbf{p} = \mathbf{M}^{\mathrm{T}}\mathbf{v}_{\mathrm{rel}}$$

Consider $\mathbf{A} = \mathbf{M}^{\mathrm{T}}\mathbf{M}$. We have:

$$\mathbf{A}_{ij} = \sum_k \mathbf{M}^{\mathrm{T}}_{ik}\mathbf{M}_{kj}$$

. . . where $k \in$ constraints.

If $i \neq j$ and there is no constraint between cells i and j then $\mathbf{M}^{\mathrm{T}}{}_{ik}\mathbf{M}_{kj}$ will be zero. If there is a constraint between i and j, then:

$$\mathbf{A}_{ij} = -\mathbf{M}^{\mathrm{T}}_{\mathbf{r}_{ik}}\mathbf{M}_{\mathbf{r}_{jk}}$$

. . . where $\mathbf{r}_{ik}$ is the vector from the center of cell i to the point on the surface of that cell associated with contact k.

If $i = j$, then:

$$\mathbf{A}_{ii} = \sum_k \mathbf{M}^{\mathrm{T}}_{\mathbf{r}_{ik}}\mathbf{M}_{\mathbf{r}_{ik}}$$

. . . where k ranges over all constraints involving cell i.

In general:

$$
\mathbf{M}_{\mathbf{s}}^{\mathrm{T}}\mathbf{M}_{\mathbf{r}} = \begin{bmatrix}
\frac{1}{M_1 M_2} & 0 & 0 & 0 & \frac{\mathbf{s}_z}{I_2} & -\frac{\mathbf{s}_y}{I_2} & \frac{\mathbf{a_s}_x}{2l_2G_2} \\
0 & \frac{1}{M_1 M_2} & 0 & -\frac{\mathbf{s}_z}{I_2} & 0 & \frac{\mathbf{s}_x}{I_2} & \frac{\mathbf{a_s}_y}{2l_2G_2} \\
0 & 0 & \frac{1}{M_1 M_2} & \frac{\mathbf{s}_y}{I_2} & -\frac{\mathbf{s}_x}{I_2} & 0 & \frac{\mathbf{a_s}_z}{2l_2G_2} \\
0 & -\frac{\mathbf{r}_z}{I_1} & \frac{\mathbf{r}_y}{I_1} & \frac{\mathbf{r}_y\mathbf{s}_y+\mathbf{r}_z\mathbf{s}_z}{I_1I_2} & -\frac{\mathbf{r}_y\mathbf{s}_x}{I_1I_2} & -\frac{\mathbf{r}_z\mathbf{s}_x}{I_1I_2} & \frac{\mathbf{r}_y\mathbf{a_s}_z}{2I_1l_2G_2} \\
\frac{\mathbf{r}_z}{I_1} & 0 & -\frac{\mathbf{r}_x}{I_1} & -\frac{\mathbf{r}_x\mathbf{s}_y}{I_1I_2} & \frac{\mathbf{r}_x\mathbf{s}_x+\mathbf{r}_z\mathbf{s}_z}{I_1I_2} & -\frac{\mathbf{r}_z\mathbf{s}_y}{I_1I_2} & \frac{\mathbf{r}_z\mathbf{a_s}_x}{2I_1l_2G_2} \\
-\frac{\mathbf{r}_y}{I_1} & \frac{\mathbf{r}_x}{I_1} & 0 & -\frac{\mathbf{r}_x\mathbf{s}_z}{I_1I_2} & -\frac{\mathbf{r}_y\mathbf{s}_z}{I_1I_2} & \frac{\mathbf{r}_y\mathbf{s}_y+\mathbf{r}_z\mathbf{s}_z}{I_1I_2} & \frac{\mathbf{r}_x\mathbf{a_s}_y}{2I_1l_2G_2} \\
\frac{\mathbf{a_r}_x}{2l_1G_1} & \frac{\mathbf{a_r}_y}{2l_1G_1} & \frac{\mathbf{a_r}_z}{2l_1G_1} & \frac{\mathbf{s}_y\mathbf{a_r}_z-\mathbf{s}_z\mathbf{a_r}_y}{2I_2l_1G_1} & \frac{\mathbf{s}_z\mathbf{a_r}_x-\mathbf{s}_x\mathbf{a_r}_z}{2I_2l_1G_1} & \frac{\mathbf{s}_x\mathbf{a_r}_y-\mathbf{s}_y\mathbf{a_r}_x}{2I_2l_1G_1} & \frac{\mathbf{a_r}\cdot\mathbf{a_s}}{4l_1l_2G_1G_2}
\end{bmatrix}
$$

... and:

$$
\mathbf{M}_{\mathbf{r}}^{\mathrm{T}}\mathbf{M}_{\mathbf{r}} = \begin{bmatrix}
\frac{1}{M_1^2} & 0 & 0 & 0 & \frac{\mathbf{r}_z}{I_1} & -\frac{\mathbf{r}_y}{I_1} & \frac{\mathbf{a_r}_x}{2l_1G_1} \\
0 & \frac{1}{M_1^2} & 0 & -\frac{\mathbf{r}_z}{I_1} & 0 & \frac{\mathbf{r}_x}{I_1} & \frac{\mathbf{a_r}_y}{2l_1G_1} \\
0 & 0 & \frac{1}{M_1^2} & \frac{\mathbf{r}_y}{I_1} & -\frac{\mathbf{r}_x}{I_1} & 0 & \frac{\mathbf{a_r}_z}{2l_1G_1} \\
0 & -\frac{\mathbf{r}_z}{I_1} & \frac{\mathbf{r}_y}{I_1} & \frac{\mathbf{r}_y^2+\mathbf{r}_z^2}{I_1^2} & -\frac{\mathbf{r}_y\mathbf{r}_x}{I_1^2} & -\frac{\mathbf{r}_z\mathbf{r}_x}{I_1^2} & \frac{\mathbf{r}_y\mathbf{a_r}_z}{2I_1l_1G_1} \\
\frac{\mathbf{r}_z}{I_1} & 0 & -\frac{\mathbf{r}_x}{I_1} & -\frac{\mathbf{r}_x\mathbf{r}_y}{I_1^2} & \frac{\mathbf{r}_x^2+\mathbf{r}_z^2}{I_1^2} & -\frac{\mathbf{r}_z\mathbf{r}_y}{I_1^2} & \frac{\mathbf{r}_z\mathbf{a_s}_x}{2I_1l_1G_1} \\
-\frac{\mathbf{r}_y}{I_1} & \frac{\mathbf{r}_x}{I_1} & 0 & -\frac{\mathbf{r}_x\mathbf{r}_z}{I_1^2} & -\frac{\mathbf{r}_y\mathbf{r}_z}{I_1^2} & \frac{\mathbf{r}_y^2+\mathbf{r}_z^2}{I_1^2} & \frac{\mathbf{r}_x\mathbf{a_r}_y}{2I_1l_1G_1} \\
\frac{\mathbf{a_r}_x}{2l_1G_1} & \frac{\mathbf{a_r}_y}{2l_1G_1} & \frac{\mathbf{a_r}_z}{2l_1G_1} & \frac{\mathbf{r}_y\mathbf{a_r}_z-\mathbf{r}_z\mathbf{a_r}_y}{2I_1l_1G_1} & \frac{\mathbf{r}_z\mathbf{a_r}_x-\mathbf{r}_x\mathbf{a_r}_z}{2I_1l_1G_1} & \frac{\mathbf{r}_x\mathbf{a_r}_y-\mathbf{r}_y\mathbf{a_r}_x}{2I_1l_1G_1} & \frac{\mathbf{a_r}\cdot\mathbf{a_r}}{4l_1^2G_1^2}
\end{bmatrix}
$$

... where:

$$
\begin{aligned}
\mathbf{a_r} &= (\hat{\mathbf{x}}_1 \cdot \mathbf{r})\,\mathbf{r} \\
\mathbf{a_s} &= (\hat{\mathbf{x}}_2 \cdot \mathbf{s})\,\mathbf{s}
\end{aligned}
$$

So we are left with the equation:

$$
\mathbf{A}\Delta\mathbf{p} = \Delta\mathbf{v}
$$

... where:

$$
\mathbf{A}_{ij} = \left\{ \begin{array}{ll} \mathbf{M}_i^{\mathrm{T}}\mathbf{M}_i & \text{if } i = j \\ -\mathbf{M}_i^{\mathrm{T}}\mathbf{M}_j & \text{if cell } i \text{ and cell } j \text{ are colliding} \end{array} \right\}
$$