Dynamic Programming

Learning Objectives for Today:

- Understand Bellman Equations!
- Overview of a collection of classical solution methods for MDPs known as dynamic programming (DP)
- Using DP to compute value functions
- Taking Action: Computing optimal policies
- Algorithms
  - Policy Iteration
  - Value Iteration
- Discuss efficiency and utility of DP

Policy Evaluation (Prediction)

Policy Evaluation: for a given policy π, compute the state-value function v_π

Recall: **State-value function for policy π**

$$v_\pi(s) \doteq \mathbb{E}_\pi[G_t \mid S_t = s] = \mathbb{E}_\pi\left[\sum_{k=0}^{\infty} \gamma^k R_{t+k+1} \,\middle|\, S_t = s\right]$$

Recall: **Bellman equation for v_π**

$$v_\pi(s) = \sum_a \pi(a|s) \sum_{s',r} p(s', r|s, a)\left[r + \gamma v_\pi(s')\right]$$

—a system of $|\mathcal{S}|$ simultaneous equations

Iterative Policy Evaluation (Prediction)

$$v_0 \rightarrow v_1 \rightarrow \cdots \rightarrow v_k \rightarrow v_{k+1} \rightarrow \cdots \rightarrow v_\pi$$

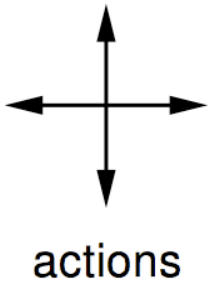
a "sweep"

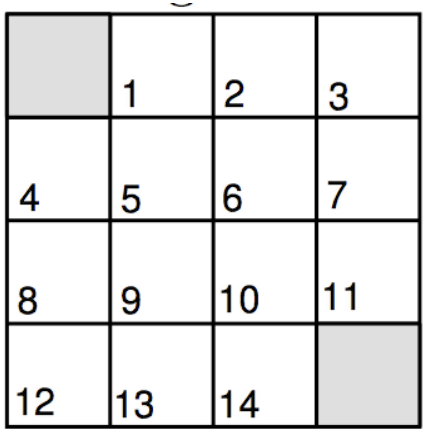
A sweep consists of applying a **backup operation** to each state.

A **full policy-evaluation backup**:

$$v_{k+1}(s) = \sum_a \pi(a|s) \sum_{s',r} p(s', r|s, a) \Big[r + \gamma v_k(s') \Big] \qquad \forall s \in \mathcal{S}$$

A Small Gridworld Example





$R = -1$
on all transitions

$\gamma = 1$

- An undiscounted episodic task
- Nonterminal states: 1, 2, . . ., 14;
- One terminal state (shown twice as shaded squares)
- Actions that would take agent off the grid leave state unchanged
- Reward is –1 until the terminal state is reached

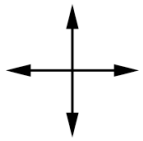
Iterative Policy Eval for the Small Gridworld

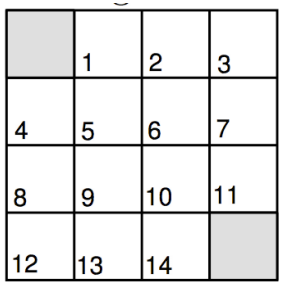
V_k for the Random Policy

$k = 0$

0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0

π = equiprobable random action choices





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$k = 1$

$k = 2$

$k = 3$

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$k = 10$

$$v_{k+1}(s) = \sum_a \pi(a|s) \sum_{s',r} p(s', r|s, a) \Big[r + \gamma v_k(s') \Big] \qquad \forall s \in \mathcal{S}$$

$k = \infty$

Iterative Policy Eval for the Small Gridworld

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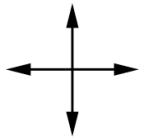

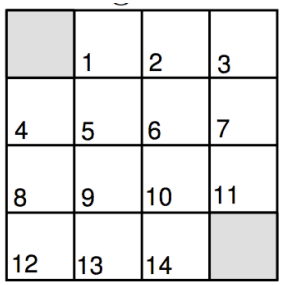

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V_k for the Random Policy

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0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0

$k = 1$

0.0	-1.0	-1.0	-1.0
-1.0	-1.0	-1.0	-1.0
-1.0	-1.0	-1.0	-1.0
-1.0	-1.0	-1.0	0.0

$k = 2$

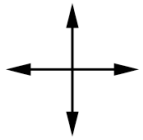
$k = 3$

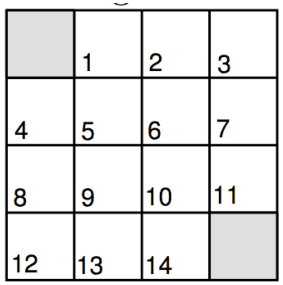
$k = 10$

$k = \infty$

Iterative Policy Eval for the Small Gridworld

π = equiprobable random action choices





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0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0

$k = 1$

0.0	-1.0	-1.0	-1.0
-1.0	-1.0	-1.0	-1.0
-1.0	-1.0	-1.0	-1.0
-1.0	-1.0	-1.0	0.0

$k = 2$

0.0	-1.7	-2.0	-2.0
-1.7	-2.0	-2.0	-2.0
-2.0	-2.0	-2.0	-1.7
-2.0	-2.0	-1.7	0.0

$k = 3$

$k = 10$

$$v_{k+1}(s) = \sum_a \pi(a|s) \sum_{s',r} p(s', r|s, a) \Big[r + \gamma v_k(s') \Big] \qquad \forall s \in \mathcal{S}$$

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Iterative Policy Eval for the Small Gridworld

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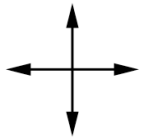

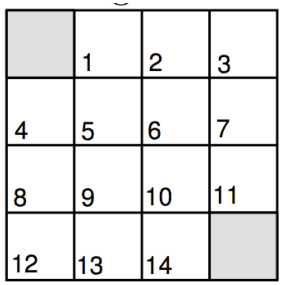

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0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0

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0.0	-1.0	-1.0	-1.0
-1.0	-1.0	-1.0	-1.0
-1.0	-1.0	-1.0	-1.0
-1.0	-1.0	-1.0	0.0

$k = 2$

0.0	-1.7	-2.0	-2.0
-1.7	-2.0	-2.0	-2.0
-2.0	-2.0	-2.0	-1.7
-2.0	-2.0	-1.7	0.0

$k = 3$

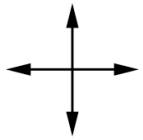
0.0	-2.4	-2.9	-3.0
-2.4	-2.9	-3.0	-2.9
-2.9	-3.0	-2.9	-2.4
-3.0	-2.9	-2.4	0.0

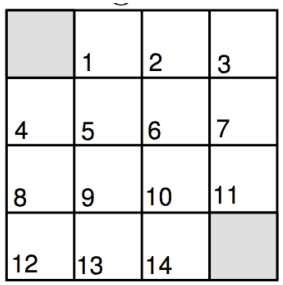
$k = 10$

$k = \infty$

Iterative Policy Eval for the Small Gridworld

π = equiprobable random action choices





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V_k for the Random Policy

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0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0

$k = 1$

0.0	-1.0	-1.0	-1.0
-1.0	-1.0	-1.0	-1.0
-1.0	-1.0	-1.0	-1.0
-1.0	-1.0	-1.0	0.0

$k = 2$

0.0	-1.7	-2.0	-2.0
-1.7	-2.0	-2.0	-2.0
-2.0	-2.0	-2.0	-1.7
-2.0	-2.0	-1.7	0.0

$k = 3$

0.0	-2.4	-2.9	-3.0
-2.4	-2.9	-3.0	-2.9
-2.9	-3.0	-2.9	-2.4
-3.0	-2.9	-2.4	0.0

$k = 10$

0.0	-6.1	-8.4	-9.0
-6.1	-7.7	-8.4	-8.4
-8.4	-8.4	-7.7	-6.1
-9.0	-8.4	-6.1	0.0

$k = \infty$

0.0	-14.	-20.	-22.
-14.	-18.	-20.	-20.
-20.	-20.	-18.	-14.
-22.	-20.	-14.	0.0

Algorithm: Iterative Policy Evaluation

Input π, the policy to be evaluated
Initialize an array $V(s) = 0$, for all $s \in \mathcal{S}^+$
Repeat
    $\Delta \leftarrow 0$
    For each $s \in \mathcal{S}$:
        $v \leftarrow V(s)$
        $V(s) \leftarrow \sum_a \pi(a|s) \sum_{s',r} p(s', r|s, a) \big[r + \gamma V(s')\big]$
        $\Delta \leftarrow \max(\Delta, |v - V(s)|)$
until $\Delta < \theta$ (a small positive number)
Output $V \approx v_\pi$

Note: this is a one-array version, V could be stored other ways.

Dynamic Programming

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- Overview of a collection of classical solution methods for MDPs known as dynamic programming (DP)
- Using DP to compute value functions
- **Taking Action: Computing optimal policies**
- Algorithms
  - Policy Iteration
  - Value Iteration
- Discuss efficiency and utility of DP

Recall: State-Value and Action-Value Functions

State - value function for policy π :

$$v_\pi(s) = E_\pi\left\{G_t \mid S_t = s\right\} = E_\pi\left\{\sum_{k=0}^{\infty} \gamma^k R_{t+k+1} \mid S_t = s\right\}$$

Action - value function for policy π :

$$q_\pi(s,a) = E_\pi\left\{G_t \mid S_t = s, A_t = a\right\} = E_\pi\left\{\sum_{k=0}^{\infty} \gamma^k R_{t+k+1} \mid S_t = s, A_t = a\right\}$$

Recall: State-Value and Action-Value Functions

State-Value Function:

$$\begin{aligned} v_*(s) &= \max_a q_{\pi_*}(s,a) \\ &= \max_a \mathbb{E}[R_{t+1} + \gamma v_*(S_{t+1}) \mid S_t = s, A_t = a] \\ &= \max_a \sum_{s',r} p(s',r|s,a)\big[r + \gamma v_*(s')\big]. \end{aligned}$$

Action-Value Function:

$$\begin{aligned} q_*(s,a) &= \mathbb{E}\Big[R_{t+1} + \gamma \max_{a'} q_*(S_{t+1},a') \;\Big|\; S_t = s, A_t = a\Big] \\ &= \sum_{s',r} p(s',r|s,a)\Big[r + \gamma \max_{a'} q_*(s',a')\Big]. \end{aligned}$$

Policy improvement theorem

- Given the value function for *any policy* π:

$$q_\pi(s, a) \qquad \text{for all } s, a$$

- It can always be greedified to obtain a *better policy*:

$$\pi'(s) = \arg\max_a q_\pi(s, a) \qquad (\pi' \text{ is not unique})$$

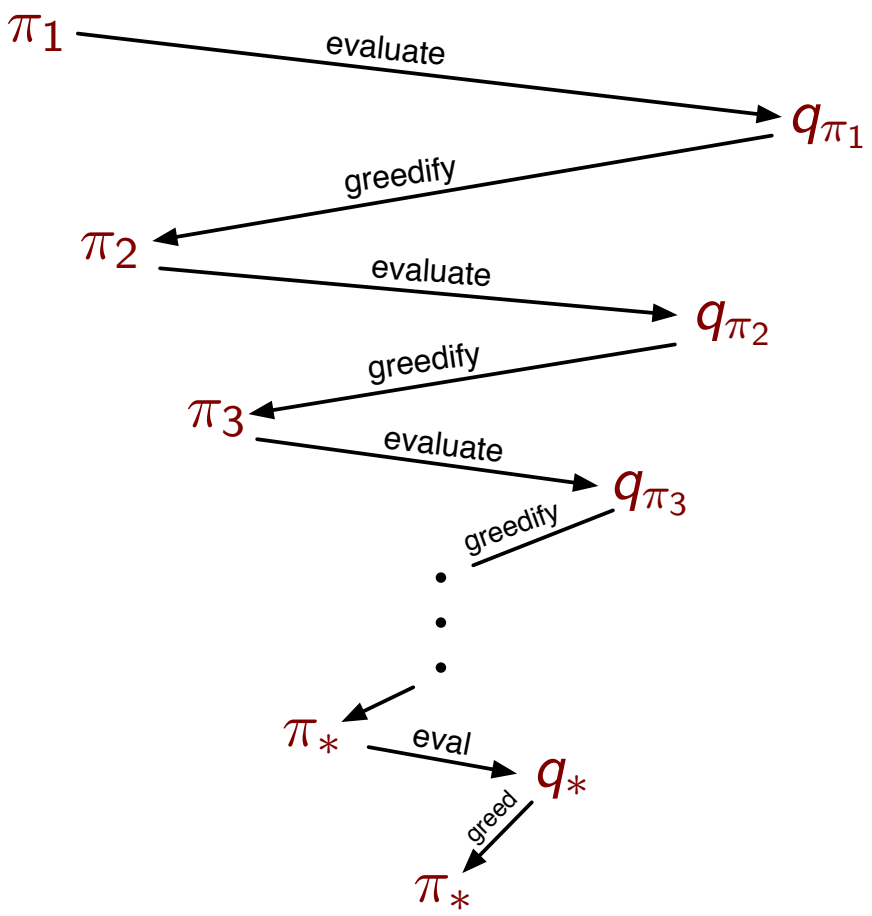
- where better means:

$$q_{\pi'}(s, a) \geq q_\pi(s, a) \qquad \text{for all } s, a$$

- with equality only if both policies are optimal

$$q_{\pi'}(s, a) = q_\pi(s, a) = q_*(s, a) \qquad (\text{if } \pi' \neq \pi)$$

The dance of policy and value (Policy Iteration)



Any policy evaluates to a unique value function (soon we will see how to learn it)

which can be greedified to produce a better policy

That in turn evaluates to a value function

which can in turn be greedified…

Each policy is *strictly better* than the previous, until *eventually both are optimal*

There are *no local optima*

The dance converges in a finite number of steps, usually very few

Policy Improvement

Suppose we have computed v_π for a deterministic policy π.

For a given state s,
would it be better to do an action $a \neq \pi(s)$?

It is better to switch to action a for state s if and only if

$$q_\pi(s,a) > v_\pi(s)$$

And, we can compute $q_\pi(s,a)$ from v_π by:

$$\begin{aligned} q_\pi(s,a) &= \mathbb{E}_\pi[R_{t+1} + \gamma v_\pi(S_{t+1}) \mid S_t = s, A_t = a] \\ &= \sum_{s',r} p(s',r|s,a)\Big[r + \gamma v_\pi(s')\Big]. \end{aligned}$$

Policy Improvement Cont.

Do this for all states to get a new policy $\pi' \geq \pi$ that is **greedy** with respect to v_π :

$$\begin{aligned}
\pi'(s) &= \arg\max_a q_\pi(s,a) \\
&= \arg\max_a \mathbb{E}[R_{t+1} + \gamma v_\pi(S_{t+1}) \mid S_t = s, A_t = a] \\
&= \arg\max_a \sum_{s',r} p(s', r|s, a)\Big[r + \gamma v_\pi(s')\Big],
\end{aligned}$$

What if the policy is unchanged by this?

Then the policy must be optimal!

Policy Improvement Cont.

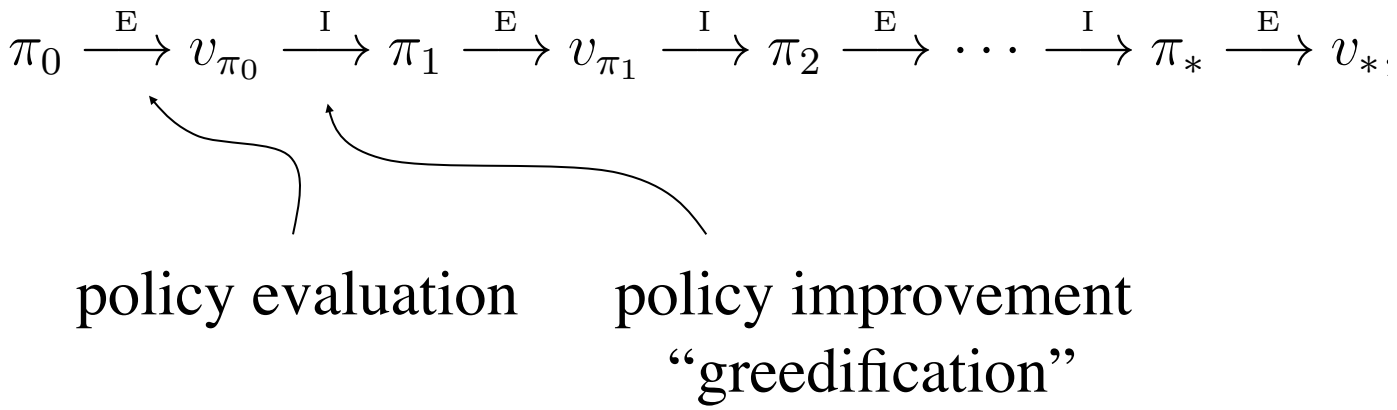
What if $v_{\pi'} = v_\pi$?

i.e., for all $s \in \mathcal{S}$, $v_\pi(s) = \max_a \sum_{s'} p(s' \mid s,a)\left[r(s,a,s') + \gamma\, v_\pi(s')\right]$?

But this is the Bellman Optimality Equation.

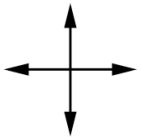
So $v_\pi = v_*$ and both π and π' are optimal policies.

Policy Iteration

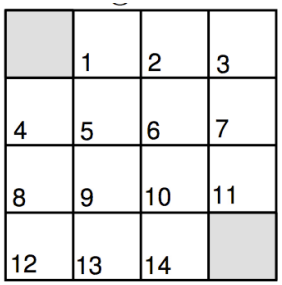


Iterative Policy Eval for the Small Gridworld

π = equiprobable random action choices



actions



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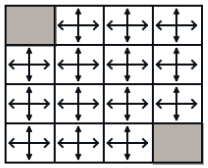
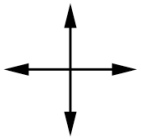
for all $s \in \mathcal{S}$

	V_k for the Random Policy	Greedy Policy w.r.t. V_k
$k = 0$	0.0 0.0 0.0 0.0 0.0 0.0 0.0 0.0 0.0 0.0 0.0 0.0 0.0 0.0 0.0 0.0	 ← random policy
$k = 1$	0.0 -1.0 -1.0 -1.0 -1.0 -1.0 -1.0 -1.0 -1.0 -1.0 -1.0 -1.0 -1.0 -1.0 -1.0 0.0	
$k = 2$	0.0 -1.7 -2.0 -2.0 -1.7 -2.0 -2.0 -2.0 -2.0 -2.0 -2.0 -1.7 -2.0 -2.0 -1.7 0.0	
$k = 3$	0.0 -2.4 -2.9 -3.0 -2.4 -2.9 -3.0 -2.9 -2.9 -3.0 -2.9 -2.4 -3.0 -2.9 -2.4 0.0	
$k = 10$	0.0 -6.1 -8.4 -9.0 -6.1 -7.7 -8.4 -8.4 -8.4 -8.4 -7.7 -6.1 -9.0 -8.4 -6.1 0.0	
$k = \infty$	0.0 -14. -20. -22. -14. -18. -20. -20. -20. -20. -18. -14. -22. -20. -14. 0.0	

Iterative Policy Eval for the Small Gridworld

π = equiprobable random action choices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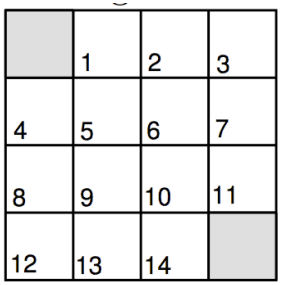

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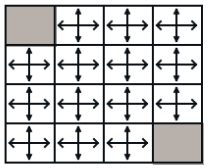
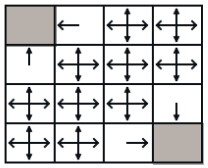
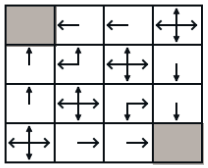
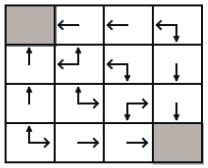
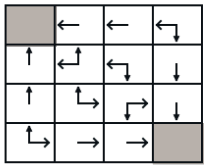
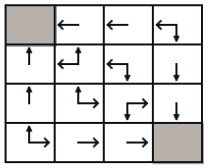
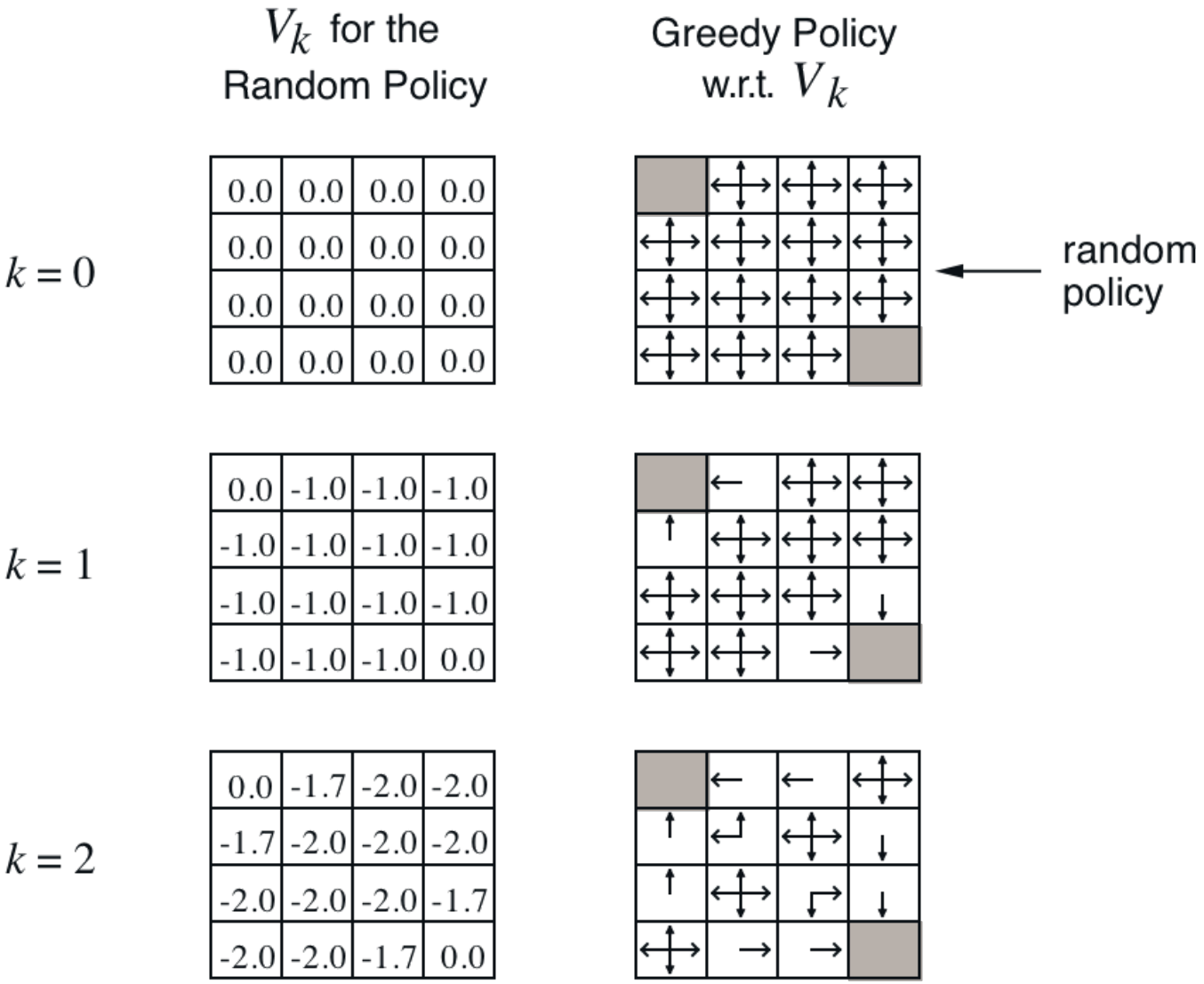
for all $s \in \mathcal{S}$

	V_k for the Random Policy	Greedy Policy w.r.t. V_k
$k = 0$	0.0 0.0 0.0 0.0 0.0 0.0 0.0 0.0 0.0 0.0 0.0 0.0 0.0 0.0 0.0 0.0	 ← random policy
$k = 1$	0.0 -1.0 -1.0 -1.0 -1.0 -1.0 -1.0 -1.0 -1.0 -1.0 -1.0 -1.0 -1.0 -1.0 -1.0 0.0	
$k = 2$	0.0 -1.7 -2.0 -2.0 -1.7 -2.0 -2.0 -2.0 -2.0 -2.0 -2.0 -1.7 -2.0 -2.0 -1.7 0.0	
$k = 3$	0.0 -2.4 -2.9 -3.0 -2.4 -2.9 -3.0 -2.9 -2.9 -3.0 -2.9 -2.4 -3.0 -2.9 -2.4 0.0	
$k = 10$	0.0 -6.1 -8.4 -9.0 -6.1 -7.7 -8.4 -8.4 -8.4 -8.4 -7.7 -6.1 -9.0 -8.4 -6.1 0.0	 ← optimal policy
$k = \infty$	0.0 -14. -20. -22. -14. -18. -20. -20. -20. -20. -18. -14. -22. -20. -14. 0.0	

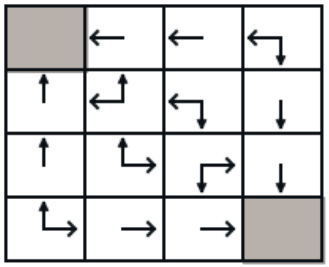
Iterative Policy Eval for the Small Gridworld



Iterative Policy Eval for the Small Gridworld

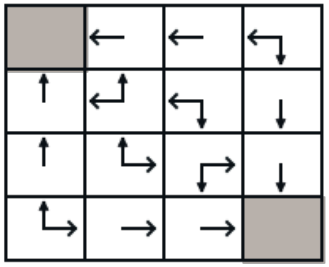
$k = 3$

0.0	-2.4	-2.9	-3.0
-2.4	-2.9	-3.0	-2.9
-2.9	-3.0	-2.9	-2.4
-3.0	-2.9	-2.4	0.0



$k = 10$

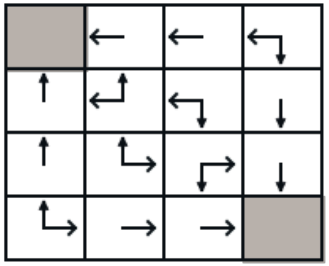
0.0	-6.1	-8.4	-9.0
-6.1	-7.7	-8.4	-8.4
-8.4	-8.4	-7.7	-6.1
-9.0	-8.4	-6.1	0.0



optimal policy

$k = \infty$

0.0	-14.	-20.	-22.
-14.	-18.	-20.	-20.
-20.	-20.	-18.	-14.
-22.	-20.	-14.	0.0



Jack's Car Rental

- $10 for each car rented (must be available when request received)
- Two locations, maximum of 20 cars at each
- Cars returned and requested **randomly**
  - *n* returns/requests with prob $\frac{\lambda^n}{n!}e^{-\lambda}$ (Poisson distribution)
  - 1st location: average requests = 3, average returns = 3
  - 2nd location: average requests = 4, average returns = 2
- Can move up to 5 cars between locations overnight
  - at a cost of $2/car

- What are the States, Actions, Rewards?
- Transition probabilities? Discounting?

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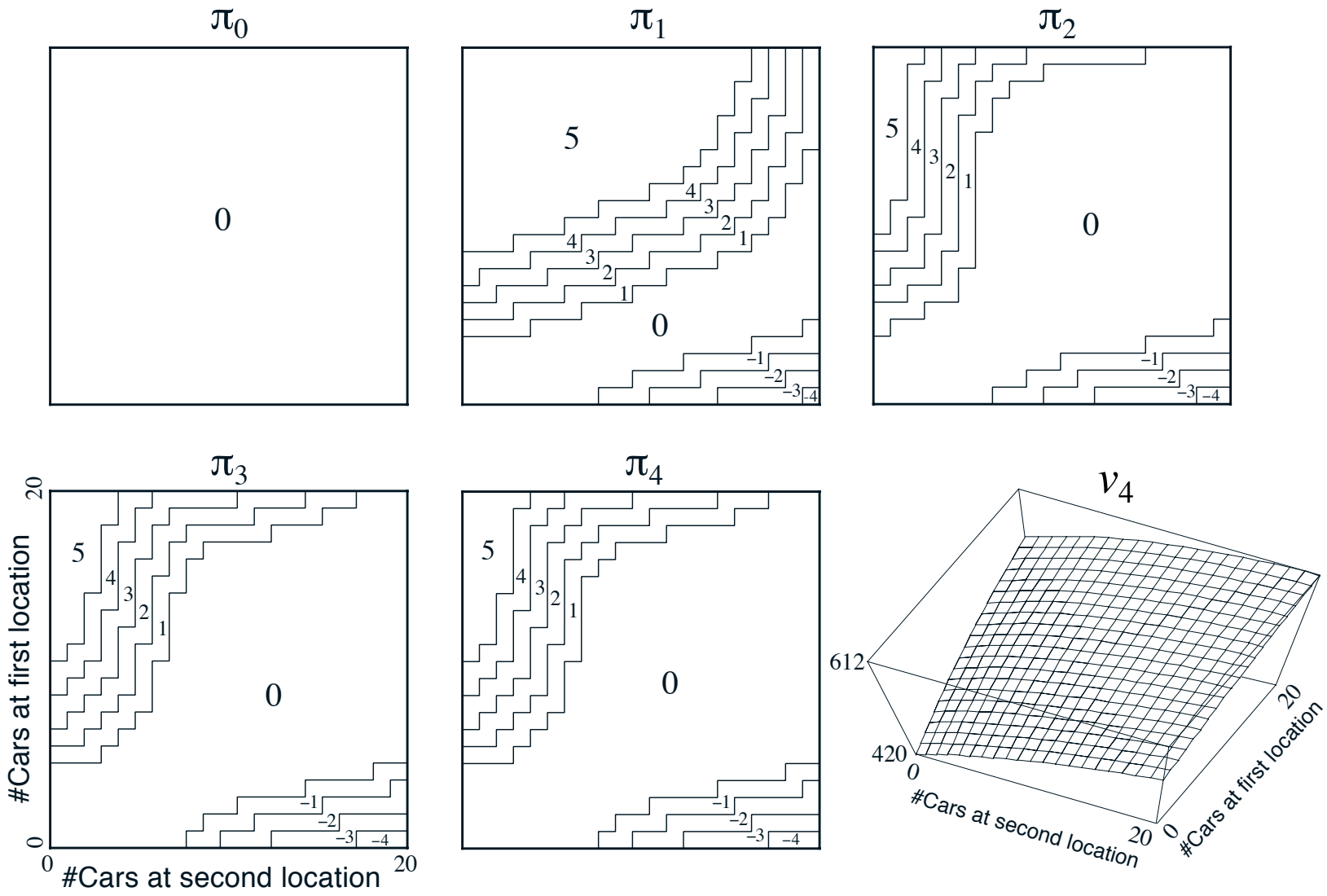
Jack's Car Rental



Dynamic Programming

Learning Objectives for Today:

- Understand Bellman Equations!
- Overview of a collection of classical solution methods for MDPs known as dynamic programming (DP)
- Using DP to compute value functions
- Taking Action: Computing optimal policies
- **Algorithms**
  - Policy Iteration
  - Value Iteration
- Discuss efficiency and utility of DP

Policy Iteration – One array version (+ policy)

1. Initialization
   $V(s) \in \mathbb{R}$ and $\pi(s) \in \mathcal{A}(s)$ arbitrarily for all $s \in \mathcal{S}$

2. Policy Evaluation
   Repeat
   $\Delta \leftarrow 0$
   For each $s \in \mathcal{S}$:
   $v \leftarrow V(s)$
   $V(s) \leftarrow \sum_{s',r} p(s', r|s, \pi(s))\big[r + \gamma V(s')\big]$
   $\Delta \leftarrow \max(\Delta, |v - V(s)|)$
   until $\Delta < \theta$ (a small positive number)

3. Policy Improvement
   policy-stable $\leftarrow$ *true*
   For each $s \in \mathcal{S}$:
   $a \leftarrow \pi(s)$
   $\pi(s) \leftarrow \arg\max_a \sum_{s',r} p(s', r|s, a)\big[r + \gamma V(s')\big]$
   If $a \neq \pi(s)$, then *policy-stable* $\leftarrow$ $false$
   If *policy-stable*, then stop and return V and π; else go to 2

Value Iteration

Recall the **full policy-evaluation backup**:

$$v_{k+1}(s) = \sum_a \pi(a|s) \sum_{s',r} p(s', r|s, a) \Big[r + \gamma v_k(s') \Big] \qquad \forall s \in \mathcal{S}$$

Here is the **full value-iteration backup**:

$$v_{k+1}(s) = \max_a \sum_{s',r} p(s', r|s, a) \Big[r + \gamma v_k(s') \Big] \qquad \forall s \in \mathcal{S}$$

Value Iteration – One array version

Initialize array V arbitrarily (e.g., $V(s) = 0$ for all $s \in \mathcal{S}^+$)

Repeat
  $\Delta \leftarrow 0$
  For each $s \in \mathcal{S}$:
    $v \leftarrow V(s)$
    $V(s) \leftarrow \max_a \sum_{s',r} p(s', r|s, a)\big[r + \gamma V(s')\big]$
    $\Delta \leftarrow \max(\Delta, |v - V(s)|)$
until $\Delta < \theta$ (a small positive number)

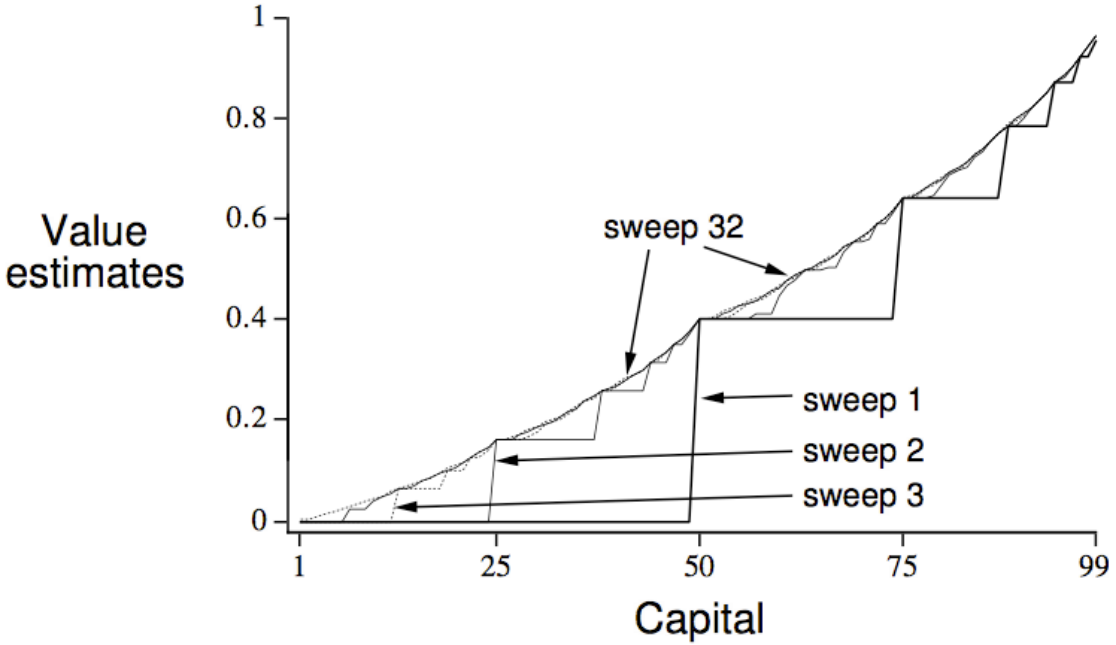
Output a deterministic policy, π, such that
  $\pi(s) = \arg\max_a \sum_{s',r} p(s', r|s, a)\big[r + \gamma V(s')\big]$

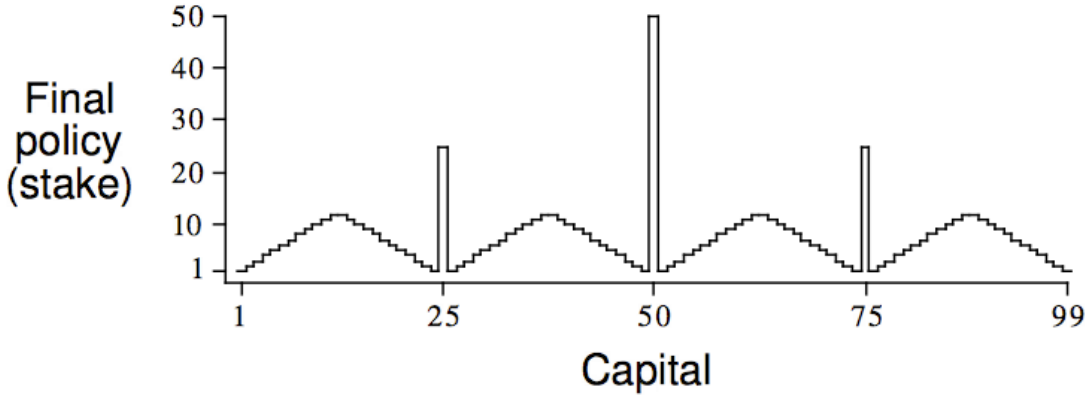
Gambler's Problem

- Gambler can repeatedly bet $ on a coin flip
- Heads he wins his stake, tails he loses it
- Initial capital $\in \{\$1, \$2, \ldots \$99\}$
- Gambler wins if his capital becomes $100
  loses if it becomes $0
- Coin is unfair
  - Heads (gambler wins) with probability $p = .4$

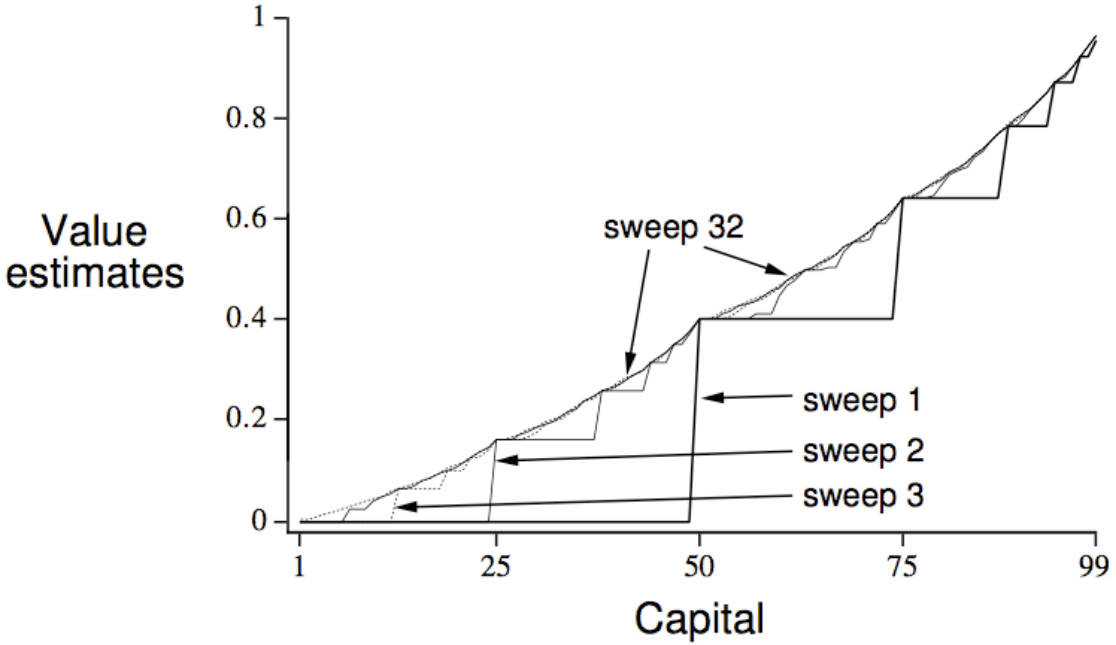
- States, Actions, Rewards? Discounting?

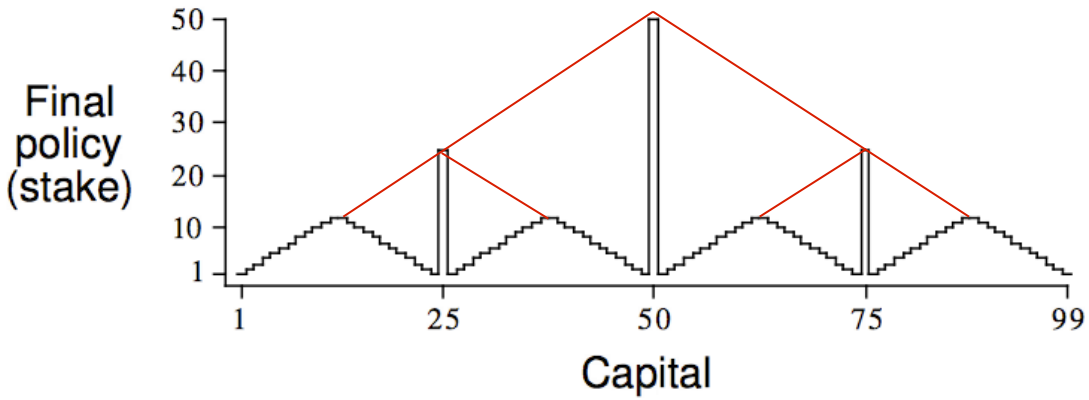
Gambler's Problem Solution





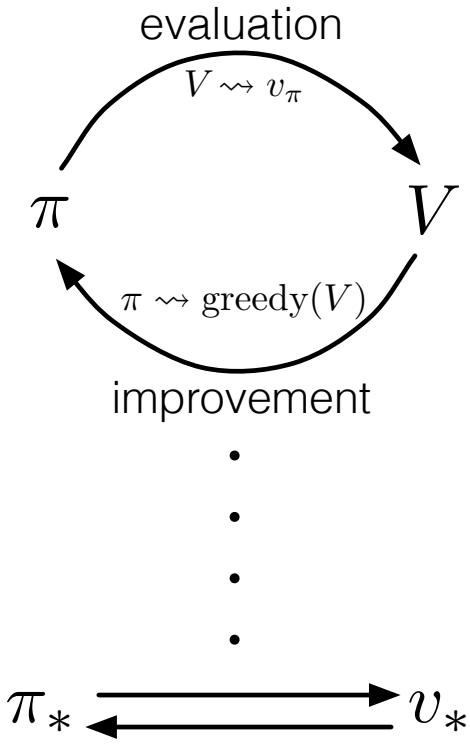
Gambler's Problem Solution



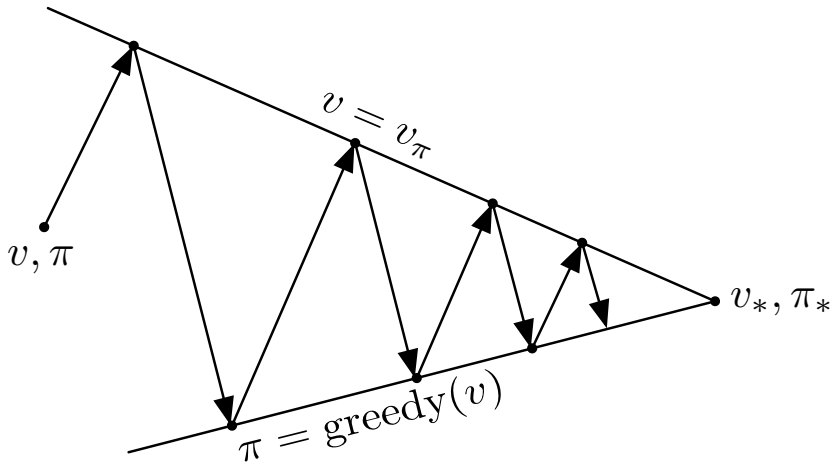


Generalized Policy Iteration

Generalized Policy Iteration (GPI):
any interaction of policy evaluation and policy improvement, independent of their granularity.



A geometric metaphor for convergence of GPI:



Alternative Algorithm: Asynchronous DP

- All the DP methods described so far require exhaustive sweeps of the entire state set.
- Asynchronous DP does not use sweeps. Instead it works like this:
  - Repeat until convergence criterion is met:
    - Pick a state at random and apply the appropriate backup
- Still need lots of computation, but does not get locked into hopelessly long sweeps
- Can you select states to backup intelligently? YES: an agent's experience can act as a guide.

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Efficiency of DP

- To find an optimal policy is polynomial in the number of states…
- BUT, the number of states is often astronomical, e.g., often growing exponentially with the number of state variables (what Bellman called "the curse of dimensionality").
- In practice, classical DP can be applied to problems with a few millions of states.
- Asynchronous DP can be applied to larger problems, and is appropriate for parallel computation.
- It is surprisingly easy to come up with MDPs for which DP methods are not practical.

Summary

- Policy evaluation: backups without a max (prediction)
- Policy improvement: form a greedy policy, if only locally
- Policy iteration: alternate the above two processes (control)
- Value iteration: backups with a max (control)

- Full backups (to be contrasted later with sample backups)
- Generalized Policy Iteration (GPI)
- Asynchronous DP: a way to avoid exhaustive sweeps
- **Bootstrapping**: updating estimates based on other estimates
- Biggest limitation of DP is that it requires a *probability model* (as opposed to a generative or simulation model)