QMS Journal Club

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QMS Journal Club Special Issue: Complex numbers and the formulation of quantum mechanics

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Postulates of quantum theory

1. To every physical system S, there corresponds a Hilbert space H_S and its states is represented by a normalized vector ψ in H_S, that is $\langle\psi|\psi\rangle = 1$.

2. The evolution of a closed quantum system is described by a unitary transformation. That is, the state $|\psi(t_1)\rangle$ of the system at time t_1 is related to the state $|\psi(t_2)\rangle$ of the system at time t_2 by a unitary operator U, $|\psi(t_2)\rangle = U|\psi(t_1)\rangle$. Explicitly, it can be described by the Schrodinger equation, $i\hbar\frac{d}{dt}|\psi\rangle = \hat{H}|\psi\rangle$.

3. A measurement Π in S corresponds to an ensemble $\{\Pi_r\}_r$ of projection operators acting on H_S, with $\sum_r \Pi_r = 1_S$. If we measure Π when system S is in state ψ, the probability of obtaining result r is given by $P(r) = \langle\psi|\Pi_r|\psi\rangle$.

4. The Hilbert space H_{ST} corresponding to the composition of two systems S and T is $H_S \otimes H_T$. Similarly, the state representing two independent preparations of the two systems is the tensor product of the two preparations.

Pure states vs mixed states

An individual polarized photon can be described as having a superposition of right or left circular polarization, $|R\rangle$ and $|L\rangle$, such that $\alpha|R\rangle + \beta|L\rangle$. Consider a vertical polarized photon, described by

$$|V\rangle = \frac{|R\rangle + |L\rangle}{\sqrt{2}}. \tag{1}$$

If we pass it through a circular polarizer that allows either $|R\rangle$ or $|L\rangle$, half of the photons are absorbed in both cases.

However, if we pass $|V\rangle$ through a linear polarizer, there is no absorption.

Unpolarized light cannot be described as any state of the form $\alpha|R\rangle + \beta|L\rangle$. It passes through a polarizer with 50% intensity loss regardless the orientation of the polarizer.

This suggests that unpolarized light should be described as a statistical ensemble, i.e. each photon having either $|R\rangle$ or $|L\rangle$ polarization with probability $\frac{1}{2}$.

Density matrix formalism

The density matrix formalism assigns each possible state a probability p_i, as an ensemble of pure states. The density operator is defined as

$$\rho = \sum_i p_i |\psi_i\rangle\langle\psi_i|, \tag{2}$$

where $\sum_i p_i = 1$.

Density matrix formalism is useful when

1. the preparation of a system can randomly produce different pure states, and thus one must deal with the statistics of the ensemble of possible preparations;

2. when one wants to describe a physical system that is entangled with another, without describing their combined state (i.e. finding the reduced density matrix of a subsystem).

The states at two consecutive time instances can be related by

$$\rho(t_2) = \sum_i p_i U|\psi_i\rangle\langle\psi_i|U^\dagger = U\rho(t_1)U^\dagger. \tag{3}$$

The probability of obtaining result r is given by

$$P(r) = \langle\psi|\Pi_r|\psi\rangle = Tr(\Pi_r|\psi_i\rangle\langle\psi_i|) = Tr(\Pi_r\rho). \tag{4}$$

1. To every physical system S, there corresponds a Hilbert space H_S and its states is represented by a density operator, which is a positive operator ρ with trace one. If a quantum system is in the state ρ_i with probability p_i, then the density operator for the system is given by $\rho = \sum_i p_i\rho_i$.

2. The evolution of a closed quantum system is described by a unitary transformation, $\rho(t_2) = U\rho(t_1)U^\dagger$.

3. If we measure Π when system S is in state ρ, the probability of obtaining result r is given by $P(r) = Tr(\Pi_r\rho)$.

4. The Hilbert space H_{ST} corresponding to the composition of two systems S and T is $H_S \otimes H_T$. The joint state of the total system is $\rho_1 \otimes \rho_2$.

EPR paradox

In a complete physical theory, there is an element corresponding to each element of reality. A sufficient condition for the reality of a physical quantity is **the possibility of predicting it with certainty, without disturbing the system**.

In quantum mechanics, in the case of two physical quantities described by non-commuting operators, the knowledge of one precludes the knowledge of the other.

Then, either

1. The description of reality given by the wavefunction in quantum mechanics is not complete;
2. These two quantities cannot have simultaneous reality.

Bell's theorem

Bell's theorem assumes a local hidden variable model, whereby

1. "local" refers to the principle of locality, i.e. an object is influenced directly only by its immediate surroundings. In order for a cause at one point to have an effect at another point, something in the space between those points must mediate the action.

2. "Hidden variable" is a deterministic model to explain the probabilistic nature of quantum theory by introducing additional, possibly inaccessible, variables.

The correlations between the outcomes must obey a specific mathematical constraint, called Bell inequality.

Consider the following setup:

1. Alice and Bob stand in widely separated locations. A pair of particles is prepared. One particle is sent to Alice, the other to Bob.
2. Alice and Bob are free to choose two possible measurements, A_0 and A_1, B_0 and B_1, respectively. Each of these measurements produce ± 1.
3. Suppose that each measurement reveals a property that the particle already possessed. For example, if Alice chooses to measure A_0 and gets $+1$, then the particle she received has $+1$ for a_0.
4. Consider the combination for all possible properties a_0, a_1, b_0, b_1:

$$a_0 b_0 + a_0 b_1 + a_1 b_0 - a_1 b_1 = (a_0 + a_1) b_0 + (a_0 - a_1) b_1. \quad (5)$$

5. Because a_0 and a_1 takes the values ± 1, therefore it is either $a_0 = a_1$ or $a_0 = -a_1$. In the former case, $(a_0 - a_1) b_1 = 0$, while in the latter case, $(a_0 + a_1) b_0 = 0$. One of the terms on the RHS will vanish, and the remaining term will give ± 2.
6. If the experiment repeats many times, the average will be given by

$$|\langle A_0 B_0 \rangle + \langle A_0 B_1 \rangle + \langle A_1 B_0 \rangle - \langle A_1 B_1 \rangle| \leq 2. \quad (6)$$

Quantum entanglement violates this formulation of Bell inequality, called Clauser-Horne-Shimony-Holt (CHSH) inequality.
Consider the bell state,

$$|\psi\rangle = \frac{|01\rangle - |10\rangle}{\sqrt{2}}. \tag{7}$$

The first qubit is passed to Alice, while the second to Bob. Alice and Bob measure the following observables,

$$A_0 = \sigma_z, \ A_1 = \sigma_x; \ B_0 = -\frac{\sigma_x + \sigma_z}{\sqrt{2}}, \ B_1 = \frac{\sigma_x - \sigma_z}{\sqrt{2}}.$$

The expectation values are

$$\langle A_0 \otimes B_0 \rangle = \langle A_0 \otimes B_1 \rangle = \langle A_1 \otimes B_0 \rangle = \frac{1}{\sqrt{2}}, \ \langle A_1 \otimes B_1 \rangle = -\frac{1}{\sqrt{2}}.$$

Therefore, the CHSH inequality is violated,
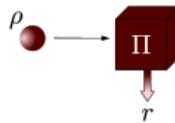
$$\langle A_0 B_0 \rangle + \langle A_0 B_1 \rangle + \langle A_1 B_0 \rangle - \langle A_1 B_1 \rangle = 2\sqrt{2}. \tag{8}$$

A reading on "Quantum theory based on real numbers can be experimentally falsified"

Going back to the first postulate of quantum theory. Can we replace complex by real numbers in the Hilbert space formulation of quantum theory without limiting its predictions?
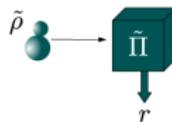
The description of a complex density matrix ρ can be replaced by

$$\tilde{\rho} = Re(\rho) \otimes \frac{I}{2} + Im(\rho) \otimes \frac{1}{2} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

$$= \frac{1}{2}(\rho \otimes |+i\rangle\langle+i| + \bar{\rho} \otimes |-i\rangle\langle-i|). \tag{9}$$

Complex

Real

Consider a single-site quantum experiment, where a system ρ is probed by the measurement $\{\Pi_r\}$.

In order to reproduce the measurement statistics using real quantum theory, we need to add an extra real qubit. The real measurement operator is replaced with

$$\tilde{\Pi}_r = \Pi_r \otimes |+i\rangle\langle+i| + \bar{\Pi}_r \otimes |-i\rangle\langle-i|. \qquad (10)$$

Since probabilities are real, $P(r) = \overline{P(r)} = Tr(\bar{\rho}\bar{\Pi}_r)$. Hence, $P(r) = Tr(\tilde{\rho}\tilde{\Pi}_r)$.

This construction doubles the Hilbert space dimension of the original complex quantum system.

A second test can be done with CHSH inequality. Previously, the maximum violation by complex quantum system can be attained using real measurements on a real two-qubit state.

A more complicated setup, with a combination of three CHSH inequalities is used:

$$CHSH_1 = \langle A_0 B_0 \rangle + \langle A_0 B_1 \rangle + \langle A_1 B_0 \rangle - \langle A_1 B_1 \rangle, \quad (11)$$

$$CHSH_2 = \langle A_0 B_2 \rangle + \langle A_0 B_3 \rangle + \langle A_2 B_2 \rangle - \langle A_2 B_3 \rangle, \quad (12)$$

$$CHSH_3 = \langle A_1 B_4 \rangle + \langle A_1 B_5 \rangle + \langle A_2 B_4 \rangle - \langle A_2 B_5 \rangle, \quad (13)$$

$$CHSH_{total} = CHSH_1 + CHSH_2 + CHSH_3 \leq 6. \quad (14)$$
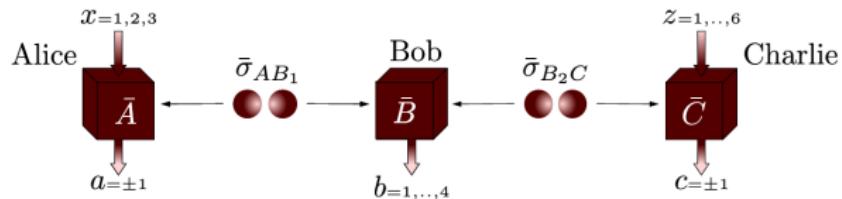
The maximal violation of $CHSH_{total}$ is $6\sqrt{2}$.

Such a setup, however, can be reproduced by real quantum theory, as long as an extra qubit can be allocated for each observer.
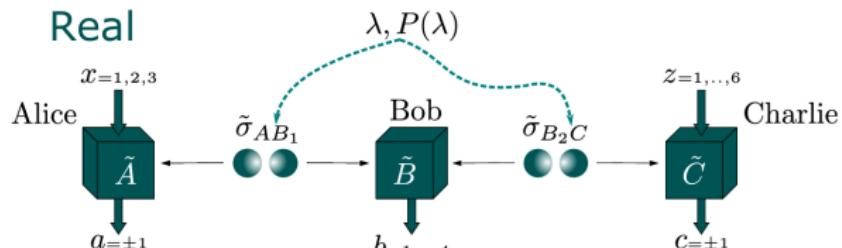
Consider a network scenario, involving three parties Alice, Bob, and Charlie, in an entanglement swapping protocol.

The entanglement swapping protocol is as follow:

1. Bob prepares two-qubit states $\tilde{\sigma}_{AB_1}$, $\tilde{\sigma}_{B_2C}$ and send qubit A to Alice, qubit C to Charlie;

2. Bob performs a Bell-state projection between B_1B_2. In the case of obtaining the Bell state $|\Phi^+\rangle = \frac{|00\rangle + |11\rangle}{\sqrt{2}}$, Alice's and Charlie's qubits A and C are entangled accordingly with the same Bell state $|\Phi^+\rangle$.

Alice can perform three measurements,

$$A_0 = \sigma_z, \ A_1 = \sigma_x, \ A_2 = \sigma_y. \tag{15}$$

Meanwhile, Charlie can perform six measurements,

$$C_0 = D_{zx}, \ C_1 = E_{zx}, \ C_2 = D_{zy}, \ C_3 = E_{zy}, \ C_4 = D_{xy}, \ C_5 = E_{xy}, \tag{16}$$

where $D_{ij} = \frac{\sigma_i + \sigma_j}{\sqrt{2}}$ and $E_{ij} = \frac{\sigma_i - \sigma_j}{\sqrt{2}}$.
Bob can obtain four Bell states,

$$b = b_1 b_2 = 00 = |\Phi^+\rangle = \frac{|00\rangle + |11\rangle}{\sqrt{2}}, \tag{17}$$

$$b = b_1 b_2 = 01 = |\Psi^+\rangle = \frac{|10\rangle + |01\rangle}{\sqrt{2}}, \tag{18}$$

$$b = b_1 b_2 = 10 = |\Phi^-\rangle = \frac{|00\rangle - |11\rangle}{\sqrt{2}}, \tag{19}$$

$$b = b_1 b_2 = 11 = |\Psi^-\rangle = \frac{|10\rangle - |01\rangle}{\sqrt{2}}. \tag{20}$$

The Bell-type functional

$$\mathcal{I}_b(P) = (-1)^{b_2}(S_{11}^b + S_{12}^b) + (-1)^{b_1}(S_{21}^b - S_{22}^b) + (-1)^{b_2}(S_{13}^b + S_{14}^b)$$
$$- (-1)^{b_1+b_2}(S_{33}^b - S_{34}^b) + (-1)^{b_1}(S_{25}^b + S_{26}^b)$$
$$- (-1)^{b_1+b_2}(S_{35}^b - S_{36}^b) \tag{21}$$

gives us the conditional expectation value based on the outcome that Bob receives, where $S_{11}^b = \langle A_1 C_1 \rangle$. For $b = 00$, $\mathcal{I}_{00}(P)$ gives us the $CHSH_{total}$ for Alice and Charlie,

$$\langle A_0 B_0 \rangle + \langle A_0 B_1 \rangle + \langle A_1 B_0 \rangle - \langle A_1 B_1 \rangle + \langle A_0 B_2 \rangle + \langle A_0 B_3 \rangle + \langle A_2 B_2 \rangle$$
$$- \langle A_2 B_3 \rangle + \langle A_1 B_4 \rangle + \langle A_1 B_5 \rangle + \langle A_2 B_4 \rangle - \langle A_2 B_5 \rangle,$$

while other results of b gives us other variants of CHSH inequality for three inequalities.

It holds that $\mathcal{I}_b(P) = 6\sqrt{2}P(b) = \frac{6\sqrt{2}}{4}$, where $P(b) = \frac{1}{4}$. Hence, $\mathcal{I}(P) = \sum_b \mathcal{I}_b(P) = 6\sqrt{2}$.

Some remarks

According to quantum physics, the observed correlation is given by

$$P(abc|xz) = Tr[(\sigma_{AB_1} \otimes \sigma_{B_2C})(A \otimes B \otimes C)]. \qquad (22)$$

For real quantum theory, it is given by

$$P(abc|xz) = \sum_{\lambda} P(\lambda) Tr[(\sigma_{AB_1} \otimes \sigma_{B_2C})(A \otimes B \otimes C)], \qquad (23)$$

where λ is a shared randomness source that controls two spacelike-seperated sources.

Having this causal structure is more general than without such source.

The maximum violation with real quantum theory is $\mathcal{I}(P) \leq 7.6605$.