

QMS Journal Club

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A vector \vec{v} is an n -tuple of entries, which can be represented by a column vector,

$$\vec{v} = \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix}. \quad (1)$$

For our purposes, we focus only on complex vector space, which is the space of all complex vectors. The n -dimensional complex vector space is written as \mathbb{C}^n .

Vector space has two operations,

- 1 Vector addition, $\vec{v} = \vec{v}_1 + \vec{v}_2$
- 2 Scalar multiplication, $c\vec{v}$

A quantum bit (qubit) is a superposition (linear combination) of the basis states ($|0\rangle$, $|1\rangle$),

$$|\psi\rangle = \psi_0|0\rangle + \psi_1|1\rangle, \quad (2)$$

where ψ_0 and ψ_1 are called probability amplitudes, which are generally complex. The notation $|\psi\rangle$ is called a ket vector and can be represented as a column vector,

$$|\psi\rangle = \begin{pmatrix} \psi_0 \\ \psi_1 \end{pmatrix}, \quad (3)$$

where $|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $|1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. $|0\rangle$ and $|1\rangle$ are orthonormal to each other.

The dual to the ket vector is a bra vector,

$$\langle\psi| = \bar{\psi}_0\langle 0| + \bar{\psi}_1\langle 1| = (\bar{\psi}_0 \quad \bar{\psi}_1), \quad (4)$$

where the overhead bar is complex conjugate operation.

The state vectors of a qubit live in the Hilbert space, which is a two-dimensional complex vector space $\mathcal{H} = \mathbb{C}^2$. The Hilbert space is equipped with an inner product operation,

$$\begin{aligned}\langle\psi|\psi\rangle &= (\bar{\psi}_0 \quad \bar{\psi}_1) \begin{pmatrix} \psi_0 \\ \psi_1 \end{pmatrix} \\ &= |\psi_0|^2 + |\psi_1|^2.\end{aligned}\tag{5}$$

There is also an outer product (tensor product),

$$\begin{aligned}|\psi\rangle\langle\psi| &= \begin{pmatrix} \psi_0 \\ \psi_1 \end{pmatrix} (\bar{\psi}_0 \quad \bar{\psi}_1) \\ &= \begin{pmatrix} |\psi_0|^2 & \psi_0\bar{\psi}_1 \\ \psi_1\bar{\psi}_0 & |\psi_1|^2 \end{pmatrix}\end{aligned}\tag{6}$$

There exists four extremely important matrices, called the Pauli matrices:

$$\sigma_0 = I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad (7)$$

$$\sigma_1 = X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad (8)$$

$$\sigma_2 = Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad (9)$$

$$\sigma_3 = Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \quad (10)$$

It is important to take note that

$$XY = iZ; YX = -iZ, \quad (11)$$

$$YZ = iX; ZY = -iX, \quad (12)$$

$$ZX = iY; XZ = -iY, \quad (13)$$

$$XX = YY = ZZ = I. \quad (14)$$

The Pauli matrices are traceless, i.e. the sum of diagonal elements is zero, $\text{Tr}(\sigma_i) = 0$.

It is also Hermitian, i.e. its conjugate transpose is equivalent to itself, $\sigma_i^\dagger = \sigma_i$, where \dagger denotes conjugate transpose operation.

Let

$$|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}; |1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad (15)$$

$$|+\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}; |-\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \quad (16)$$

$$|+i\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ i \end{pmatrix}; |-i\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -i \end{pmatrix}. \quad (17)$$

The Pauli matrices have two eigenvalues ± 1 ,

$$X|+\rangle = |+\rangle; X|-\rangle = -|-\rangle, \quad (18)$$

$$Y|+i\rangle = |+i\rangle; Y|-i\rangle = -|-i\rangle, \quad (19)$$

$$Z|0\rangle = |0\rangle; Z|1\rangle = -|1\rangle. \quad (20)$$

The Pauli matrices can be formulated by outer products,

$$X = |+\rangle\langle+| - |-\rangle\langle-|, \quad (21)$$

$$Y = |+i\rangle\langle+i| - |-i\rangle\langle-i|, \quad (22)$$

$$Z = |0\rangle\langle 0| - |1\rangle\langle 1|. \quad (23)$$

Previously, we mentioned that $\langle\psi|$ is the dual to $|\psi\rangle$. By using conjugate transpose operation, they are related by

$$\langle\psi| = (|\psi\rangle)^\dagger. \quad (24)$$

If A and B are two matrices, then $(AB)^\dagger = B^\dagger A^\dagger$. Therefore, $(A|\psi\rangle)^\dagger = \langle\psi|A^\dagger$.

Also, $(A + B)^\dagger = A^\dagger + B^\dagger$. Using this fact, one can easily show that the Pauli matrices are Hermitian, $\sigma_i^\dagger = \sigma_i$.

A matrix is unitary if $U^\dagger = U^{-1}$. The Pauli matrices are Hermitian and unitary.

Tensor product manifests itself in matrices as Kronecker product. Given two matrices A and B ,

$$\begin{aligned} A \otimes B &= \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \otimes \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} \\ &= \begin{pmatrix} a_{11} \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} & a_{12} \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} \\ a_{21} \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} & a_{22} \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} \end{pmatrix} \\ &= \begin{pmatrix} a_{11}b_{11} & a_{11}b_{12} & a_{12}b_{11} & a_{12}b_{12} \\ a_{11}b_{21} & a_{11}b_{22} & a_{12}b_{21} & a_{12}b_{22} \\ a_{21}b_{11} & a_{21}b_{12} & a_{22}b_{11} & a_{22}b_{12} \\ a_{21}b_{21} & a_{21}b_{22} & a_{22}b_{21} & a_{22}b_{22} \end{pmatrix}. \end{aligned} \tag{25}$$

In general, $A \otimes B \neq B \otimes A$.

If we have three vectors $|v_1\rangle, |v_2\rangle \in V$ and $|w\rangle \in W$,

$$(|v_1\rangle + |v_2\rangle) \otimes |w\rangle = |v_1\rangle \otimes |w\rangle + |v_2\rangle \otimes |w\rangle = |v_1 w\rangle + |v_2 w\rangle, \quad (26)$$

$$|w\rangle \otimes (|v_1\rangle + |v_2\rangle) = |w\rangle \otimes |v_1\rangle + |w\rangle \otimes |v_2\rangle = |wv_1\rangle + |wv_2\rangle. \quad (27)$$

If A acts on the vector space V and B acts on the vector space W , then

$$(A \otimes B)|v_1 w\rangle = A|v_1\rangle \otimes B|w\rangle. \quad (28)$$

Question: The Hadamard operator on one-qubit can be written as

$$H = \frac{1}{\sqrt{2}} [(|0\rangle + |1\rangle)\langle 0| + (|0\rangle - |1\rangle)\langle 1|]. \quad (29)$$

Show that the Hadamard transform on two qubits can be written as

$$H^{\otimes 2} = H \otimes H = \frac{1}{2} \sum_{x_1, x_2, y_1, y_2 \in \{0, 1\}} (-1)^{x_1 y_1 + x_2 y_2} |x_1 x_2\rangle \langle y_1 y_2|. \quad (30)$$

$$\begin{aligned}
H \otimes H &= \frac{1}{\sqrt{2}} [(|0\rangle + |1\rangle)\langle 0| + (|0\rangle - |1\rangle)\langle 1|] \otimes \\
&\quad \frac{1}{\sqrt{2}} [(|0\rangle + |1\rangle)\langle 0| + (|0\rangle - |1\rangle)\langle 1|] \\
&= \frac{1}{2} [(|00\rangle + |01\rangle + |10\rangle + |11\rangle)\langle 00| \\
&\quad + (|00\rangle - |01\rangle + |10\rangle - |11\rangle)\langle 01| \\
&\quad + (|00\rangle + |01\rangle - |10\rangle - |11\rangle)\langle 10| \\
&\quad + (|00\rangle - |01\rangle - |10\rangle + |11\rangle)\langle 11|] \\
&= \frac{1}{2} \sum_{x_1, x_2, y_1, y_2 \in \{0, 1\}} (-1)^{x_1 y_1 + x_2 y_2} |x_1 x_2\rangle \langle y_1 y_2|.
\end{aligned}$$

The trace of a matrix A is defined as the sum of its diagonal elements,

$$\begin{aligned} \text{Tr}(A) &= \text{Tr}(|A|) = \text{Tr} \left(\sum_{jk} |j\rangle \langle j| A |k\rangle \langle k| \right) \\ &= \text{Tr} \left(\sum_{jk} A_{jk} |j\rangle \langle k| \right) = \sum_{ijk} A_{jk} \langle i|j\rangle \langle k|i\rangle = \sum_{ijk} A_{jk} \langle k|i\rangle \langle i|j\rangle \\ &= \sum_{jk} A_{jk} \langle k|j\rangle = \sum_{jk} A_{jk} \delta_{kj} = \sum_j A_{jj}. \end{aligned} \quad (31)$$

The trace of a matrix satisfies a few properties:

- ① $\text{Tr}(AB) = \text{Tr}(BA)$
- ② $\text{Tr}(ABC) = \text{Tr}(BCA) = \text{Tr}(CAB)$
- ③ $\text{Tr}(A + B) = \text{Tr}(A) + \text{Tr}(B)$
- ④ $\text{Tr}(cA) = c\text{Tr}(A)$
- ⑤ $\text{Tr}(A^T) = \text{Tr}(A)$
- ⑥ $\text{Tr}(A^T B) = \text{Tr}(AB^T) = \text{Tr}(B^T A) = \text{Tr}(BA^T)$
- ⑦ $\text{Tr}(A \otimes B) = \text{Tr}(A)\text{Tr}(B)$

Suppose $|\psi\rangle$ is a unit vector and A is an arbitrary operator.

$$\begin{aligned} \text{Tr}(A|\psi\rangle\langle\psi|) &= \sum_i \langle i|A|\psi\rangle\langle\psi|i\rangle \\ &= \sum_i \langle\psi|i\rangle\langle i|A|\psi\rangle \\ &= \langle\psi|A|\psi\rangle. \end{aligned} \tag{32}$$