

# QMS Journal Club

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## QMS Journal Club 2nd meeting: Postulates of quantum mechanics and Bloch sphere

Postulate 1: Associated to any isolated physical system is a complex vector space with inner product, i.e. a Hilbert space, known as the state space of the system. The system is completely described by its state vector, which is a unit vector in the systems' state space.

Postulate 2: The evolution of a closed quantum system is described by a unitary transformation. That is, the state  $|\psi\rangle$  of the system at time  $t_1$  is related to the state  $|\psi'\rangle$  of the system at time  $t_2$  by a unitary operator  $U$  which depends only on the times  $t_1$  and  $t_2$ ,

$$|\psi'\rangle = U|\psi\rangle. \quad (1)$$

Explicitly, it can be described by the Schrodinger equation,

$$i\hbar \frac{d}{dt} |\psi\rangle = \hat{H} |\psi\rangle. \quad (2)$$

Or,

$$|\psi(t_2)\rangle = e^{\frac{-i\hat{H}(t_2-t_1)}{\hbar}}|\psi(t_1)\rangle. \quad (3)$$

Question: Suppose  $A$  and  $B$  are commuting Hermitian operators. Show that  $e^A e^B = e^{(A+B)}$  by expanding the left and right hand sides until  $n = 3$ .

Question: Show that  $e^{iK}$  is unitary for some Hermitian matrix  $K$ . You only need to expand until  $n = 6$ .

Note that the matrix exponential is given by

$$e^A = \sum_{n=0}^{\infty} \frac{A^n}{n!}. \quad (4)$$

Postulate 3: Quantum measurements are described by a collection  $\{M_m\}$  of measurement operators. These are operators acting on the state space of the system being measured. The index  $m$  refers to the measurement outcomes that may occur in the experiment. If the state of the quantum system is  $|\psi\rangle$  immediately before the measurement then the probability that result  $m$  occurs is given by

$$p(m) = \langle \psi | M_m^\dagger M_m | \psi \rangle, \quad (5)$$

and the state of the system after the measurement is

$$\frac{M_m |\psi\rangle}{\sqrt{\langle \psi | M_m^\dagger M_m | \psi \rangle}}. \quad (6)$$

The measurement operators satisfies the completeness equation,

$$\sum_m M_m^\dagger M_m = I. \quad (7)$$

A projective measurement is described by an observable,  $M$ , a Hermitian operator on the state space of the system being observed. The observable can be written as

$$M = \sum_m m |m\rangle \langle m|, \quad (8)$$

where  $P_m = |m\rangle \langle m|$  is the projector onto the eigenspace of  $M$  with eigenvalue  $m$ , satisfying  $P_m P_m^\dagger = \delta_{mm'} P_m$ .

The expected value of the measurement is

$$\begin{aligned} E(M) &= \langle M \rangle = \sum_m m p(m) \\ &= \sum_m m \langle \psi | P_m | \psi \rangle \\ &= \langle \psi | \sum_m m |m\rangle \langle m| | \psi \rangle \\ &= \langle \psi | M | \psi \rangle \end{aligned} \quad (9)$$

Question: The variance associated to observations of  $M$ ,  $[\Delta(M)]^2$  is given by

$$[\Delta(M)]^2 = \langle (M - \langle M \rangle)^2 \rangle. \quad (10)$$

Show that it is equivalent to  $[\Delta(M)]^2 = \langle M^2 \rangle - \langle M \rangle^2$ .

Question: Suppose  $A$  and  $B$  are two Hermitian operators, and  $\langle \psi | AB | \psi \rangle = x + iy$ . Show that  $\langle \psi | [A, B] | \psi \rangle = 2iy$ ,  $\langle \psi | \{A, B\} | \psi \rangle = 2x$ .

The above implies that

$$|\langle \psi | [A, B] | \psi \rangle|^2 + |\langle \psi | \{A, B\} | \psi \rangle|^2 = 4|\langle \psi | AB | \psi \rangle|^2. \quad (11)$$

By the Cauchy-Schwarz inequality,

$$|\langle \psi | AB | \psi \rangle|^2 \leq \langle \psi | A^2 | \psi \rangle \langle \psi | B^2 | \psi \rangle. \quad (12)$$

Hence,

$$|\langle \psi | [A, B] | \psi \rangle|^2 \leq 4 \langle \psi | A^2 | \psi \rangle \langle \psi | B^2 | \psi \rangle. \quad (13)$$



Question: Suppose  $C$  and  $D$  are two observables, and  $A = C - \langle C \rangle$ ,  $B = D - \langle D \rangle$ . Obtain the Heisenberg's uncertainty principle,

$$\Delta(C)\Delta(D) \geq \frac{|\langle \psi | [C, D] | \psi \rangle|}{2}. \quad (14)$$