**EXERCISE 1**

**1** Use the compound-angle formulae to write the following as surds.

**(i)** $\sin 75° = \sin(45° + 30°)$ **(ii)** $\cos 135° = \cos(90° + 45°)$

**(iii)** $\tan 15° = \tan(45° - 30°)$ **(iv)** $\tan 75° = \tan(45° + 30°)$

**2** Expand each of the following expressions.

**(i)** $\sin(\theta + 45°)$ **(ii)** $\cos(\theta - 30°)$

**(iii)** $\sin(60° - \theta)$ **(iv)** $\cos(2\theta + 45°)$

**(v)** $\tan(\theta + 45°)$ **(vi)** $\tan(\theta - 45°)$

**3** Simplify each of the following expressions.

**(i)** $\sin 2\theta \cos \theta - \cos 2\theta \sin \theta$

**(ii)** $\cos \phi \cos 3\phi - \sin \phi \sin 3\phi$

**(iii)** $\sin 120° \cos 60° + \cos 120° \sin 60°$

**(iv)** $\cos \theta \cos \theta - \sin \theta \sin \theta$

**4** Solve the following equations for values of $\theta$ in the range $0° \leqslant \theta \leqslant 180°$.

**(i)** $\cos(60° + \theta) = \sin \theta$

**(ii)** $\sin(45° - \theta) = \cos \theta$

**(iii)** $\tan(45° + \theta) = \tan(45° - \theta)$

**(iv)** $2\sin \theta = 3\cos(\theta - 60°)$

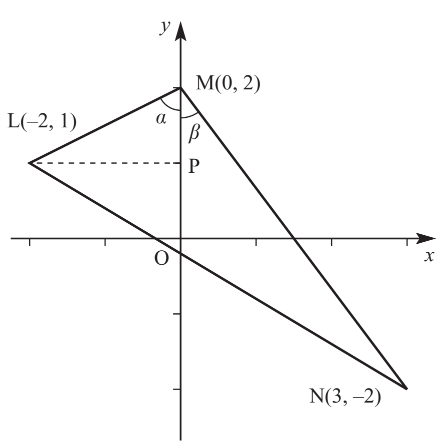
**(v)** $\sin \theta = \cos(\theta + 120°)$

**5** Solve the following equations for values of $\theta$ in the range $0 \leqslant \theta \leqslant \pi$.
(When the range is given in radians, the solutions should be in radians, using multiples of $\pi$ where appropriate.)

**(i)** $\sin\left(\theta + \frac{\pi}{4}\right) = \cos \theta$

**(ii)** $2\cos\left(\theta - \frac{\pi}{3}\right) = \cos\left(\theta + \frac{\pi}{2}\right)$

**6** Calculators are not to be used in this question.
The diagram shows three points L(–2, 1), M(0, 2) and N(3, –2) joined to form a triangle. The angles $\alpha$ and $\beta$ and the point P are shown in the diagram.



**(i)** Show that $\sin\alpha = \dfrac{2}{\sqrt{5}}$ and write down the value of $\cos\alpha$.

**(ii)** Find the values of $\sin\beta$ and $\cos\beta$.

**(iii)** Show that $\sin\angle\text{LMN} = \dfrac{11}{5\sqrt{5}}$.

**(iv)** Show that $\tan\angle\text{LNM} = \dfrac{11}{27}$.

**7** **(i)** Find $\int x\cos kx\,\mathrm{d}x$, where $k$ is a non-zero constant.

**(ii)** Show that

$$\cos(A - B) - \cos(A + B) = 2\sin A\sin B.$$

Hence express $2\sin 5x\sin 3x$ as the difference of two cosines.

**(iii)** Use the results in parts **(i)** and **(ii)** to show that

$$\int_0^{\frac{\pi}{4}} x\sin 5x\sin 3x\,\mathrm{d}x = \frac{\pi - 2}{16}.$$

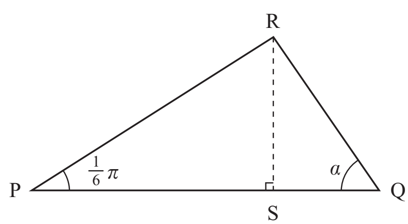
[MEI]

**8** **(i)** Use the formulae for $\cos(\theta + \phi)$ and $\cos(\theta - \phi)$ to prove that

$$\cos(\theta - \phi) - \cos(\theta + \phi) = 2\sin\theta\sin\phi. \qquad ⊛$$

Prove also that $\sin(\pi - \theta) = \sin\theta$.

In triangle PQR, angle P $= \frac{1}{6}\pi$ radians, angle Q $= \alpha$ radians, and QR $= 1$ unit. The point S is at the foot of the perpendicular from R to PQ, as shown in the diagram.




**(ii)** Show that PQ $= 2\sin\left(\alpha + \frac{1}{6}\pi\right)$.

By finding RS in terms of $\alpha$, deduce that the area $A$ of the triangle is given by

$$A = \sin\left(\alpha + \tfrac{1}{6}\pi\right)\sin\alpha.$$

Find the value of $\alpha$ for which the area $A$ is a maximum. [You may find the result ⊛ helpful.]

**(iii)** Expand $\sin\left(\alpha + \frac{1}{6}\pi\right)$, and hence show that, for small values of $\alpha$, $A \approx p\alpha + q\alpha^2$, where $p$ and $q$ are contants to be determined.

[For small $\theta$, $\sin\theta \approx \theta$ and $\cos\theta \approx 1$.]

Find the value of this expression when $\alpha = 0.1$, and find also the corresponding value of $A$ given by the expression in part **(ii)**. [MEI]

## EXERCISE 2

**1** Solve the following equations for $0° \leqslant \theta \leqslant 360°$.

**(i)** $2\sin 2\theta = \cos\theta$ **(ii)** $\tan 2\theta = 4\tan\theta$ **(iii)** $\cos 2\theta + \sin\theta = 0$

**(iv)** $\tan\theta\tan 2\theta = 1$ **(v)** $2\cos 2\theta = 1 + \cos\theta$

**2** Solve the following equations for $-\pi \leqslant \theta \leqslant \pi$.

**(i)** $\sin 2\theta = 2\sin\theta$ **(ii)** $\tan 2\theta = 2\tan\theta$ **(iii)** $\cos 2\theta - \cos\theta = 0$

**(iv)** $1 + \cos 2\theta = 2\sin^2\theta$ **(v)** $\sin 4\theta = \cos 2\theta$

(**Hint:** Write the expression in part **(v)** as an equation in $2\theta$.)

**3** By first writing $\sin 3\theta$ as $\sin(2\theta + \theta)$, express $\sin 3\theta$ in terms of $\sin\theta$. Hence solve the equation $\sin 3\theta = \sin\theta$ for $0 \leqslant \theta \leqslant 2\pi$.

**4** Solve $\cos 3\theta = 1 - 3\cos\theta$ for $0° \leqslant \theta \leqslant 360°$.

**5** Simplify $\dfrac{1 + \cos 2\theta}{\sin 2\theta}$.

**6** Express $\tan 3\theta$ in terms of $\tan\theta$.

**7** Show that $\dfrac{1 - \tan^2\theta}{1 + \tan^2\theta} = \cos 2\theta$.

**8** **(i)** Show that $\tan\left(\dfrac{\pi}{4} + \theta\right)\tan\left(\dfrac{\pi}{4} - \theta\right) = 1$.

**(ii)** Given that $\tan 26.6° = 0.5$, solve $\tan\theta = 2$ without using your calculator. Give $\theta$ to 1 decimal place, where $0° < \theta < 90°$.

**9** **(i)** Sketch on the same axes the graphs of

$$y = \cos 2x \quad \text{and} \quad y = 3\sin x - 1 \quad \text{for} \quad 0 \leqslant x \leqslant 2\pi.$$

**(ii)** Show that these curves meet at points whose $x$ co-ordinates are solutions of the equation $2\sin^2 x + 3\sin x - 2 = 0$.

**(iii)** Solve this equation to find the values of $x$ in terms of $\pi$ for $0 \leqslant x \leqslant 2\pi$.

**[MEI]**

**EXAMPLE**

Solve $\sin 3\theta + \sin\theta = 0$ for $0° \leqslant \theta \leqslant 360°$.

**SOLUTION**

Using

$$\sin\alpha + \sin\beta = 2\sin\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right)$$

and putting $\alpha = 3\theta$ and $\beta = \theta$ gives

$$\sin 3\theta + \sin\theta = 2\sin 2\theta\cos\theta$$

so the equation becomes

$$2\sin 2\theta\cos\theta = 0$$

$\Rightarrow \quad \cos\theta = 0 \quad$ or $\quad \sin 2\theta = 0$.

From the graphs for $y = \cos\theta$ and $y = \sin\theta$

$\cos\theta = 0$ gives $\theta = 90°$ or $270°$

$\sin 2\theta = 0$ gives $2\theta = 0°, 180°, 360°, 540°$ or $720°$

so $\theta = 0°, 90°, 180°, 270°$ or $360°$.

You should only list each root once in the final answer.

The complete set of roots in the range given is $\theta = 0°, 90°, 180°, 270°, 360°$.

**EXERCISE 3**

*The questions in this exercise relate to enrichment material.*

**1** Factorise the following expressions.

**(i)** $\sin 4\theta - \sin 2\theta$

**(ii)** $\cos 5\theta + \cos\theta$

**(iii)** $\cos 7\theta - \cos 3\theta$

**(iv)** $\cos(\theta + 60°) + \cos(\theta - 60°)$

**(v)** $\sin(3\theta + 45°) + \sin(3\theta - 45°)$

**2** Factorise $\cos 4\theta + \cos 2\theta$. Hence, for $0° < \theta < 180°$, solve

$$\cos 4\theta + \cos 2\theta = \cos\theta.$$

**3** Simplify $\dfrac{\sin 5\theta + \sin 3\theta}{\sin 5\theta - \sin 3\theta}$.

**4** Solve the equation $\sin 3\theta - \sin\theta = 0$ for $0 \leqslant \theta \leqslant 2\pi$.

**5** Factorise $\sin(\theta + 73°) - \sin(\theta + 13°)$ and use your result to sketch the graph of $y = \sin(\theta + 73°) - \sin(\theta + 13°)$.

**6** Prove that $\sin^2 A - \sin^2 B = \sin(A - B)\sin(A + B)$.

**7** **(i)** Use a suitable factor formula to show that $\sin 3\theta + \sin\theta = 4\sin\theta\cos^2\theta$.

**(ii)** Hence show that $\sin 3\theta = 3\sin\theta - 4\sin^3\theta$.