Homework Set 3

Exercise 1 *In the risk neutral formulation a stock solves the Itô stochastic differential equation*

$$\begin{aligned} dS(t) =& rS(t)dt + \sigma S(t)dW(t), \\ S(0) =& S_0. \end{aligned} \tag{1}$$

As we have seen before,

$$S(T) = \exp((r - \sigma^2/2)T + \sigma W(T))\ S_0 \tag{2}$$

solves (1).
a) *Simulate the price*

$$f(0, S_0) = e^{-rT} E[\max(S(T) - K, 0)|S(0) = S_0],$$

of a European call option by a Monte Carlo method, where

$$S_0 = K = 35, \quad r = 0.04, \quad \sigma = 0.2, \quad T = \frac{1}{2}.$$

Use successively increasing number of samples and estimate the accuracy of your results by appealing to the Central Limit Theorem and computing a sample variance.
b) *Compute the corresponding sensitivity ("delta")*

$$\Delta \equiv \frac{\partial f(0, s)}{\partial s},$$

by approximating it with a finite difference quotient, for instance

$$\Delta \approx \frac{f(0, s + \Delta s) - f(0, s)}{\Delta s},$$

and determine a good choice of your Δs*. Estimate the accuracy of your results and suggest a better method to solve this problem.*

Exercise 2 *The following stochastic volatility model generalizes the well known Black-Scholes geometric Brownian motion model improving some aspects of option pricing. A simplified version of the model reads*

$$\begin{aligned} dS(t) &= rS(t)dt + e^{Y(t)}S(t)\ dW(t), & (3) \\ dY(t) &= \left(-\alpha(2 + Y(t)) + 0.4\sqrt{\alpha}\sqrt{1 - \rho^2}\right) dt + 0.4\sqrt{\alpha}\ d\hat{Z}(t), & (4) \end{aligned}$$

where W and Z are independent Wiener processes, $\alpha > 0$, and

$$\hat{Z}(t) \equiv \rho W(t) + \sqrt{1-\rho^2} Z(t).$$

Here the correlation coefficient is $\rho = -0.3$.
a) *Consider equation (4) alone and solve it in closed form in terms of an Itô integral. Compute $E[Y(t)]$, $Var[Y(t)]$ exactly and their limits as $t \to \infty$. Interpret the results.*
b) *For a stability analysis, consider the model equation*

$$dX(t) = -\alpha X(t)dt + \sqrt{\alpha} dW(t), \tag{5}$$

where W is a Wiener process.
Now consider the use of Forward Euler and Backward Euler to (5).

(i) *Compute expected value and variance of $X(t)$ and their corresponding limits as $t \to \infty$.*

(ii) *Compute expected value and variance of a Forward Euler approximation to $X(t)$ and their corresponding limits as $t \to \infty$.*

(iii) *Compute expected value and variance of a Backward Euler approximation to $X(t)$ and their corresponding limits as $t \to \infty$.*

(iv) *Interpret the results obtained in* **(i-iii)**.

c) *Use the Forward Euler method,*

$$\begin{aligned} S_{n+1} - S_n =& r S_n \Delta t + e^{Y_n} S_n \ \Delta W_n, \\ Y_{n+1} - Y_n =& \left(-\alpha(2+Y_n) + 0.4\sqrt{\alpha}\sqrt{1-\rho^2}\right) \Delta t + 0.4\sqrt{\alpha} \Delta \hat{Z}_n, \\ \hat{Z}_n =& \rho W_n + \sqrt{1-\rho^2} Z_n \end{aligned}$$

for the computation of the option value

$$e^{-rT} E[\max(S(T) - K, 0)].$$

Here use the parameter values $\alpha = 100$, $r = 0.04$, $T = \frac{3}{4}$, $Y_0 = -1$, and $S_0 = K = 100$.
d) *Use successively increasing number of samples and successively decreasing (uniform) time step size to estimate the accuracy of your results. Compute the option value to an estimated accuracy $TOL = 5 \times 10^{-2}$ with high confidence. Note your computational cost in terms of elapsed time and some computer independent measure; e.g. the total number of random variables sampled. Estimate the cost of computing the price to an estimated accuracy $TOL = 5 \times 10^{-3}$ with the same confidence.*