FSF3581: Homework 3

Problem 1

Given the following Ito's SDE of a risk neutral stock

$$dS_t = rS_t dt + \sigma S_t dW_t$$
$$S(0) = S_0$$

with the following solution,

$$S_T = exp((r - \frac{\sigma^2}{2})T + \sigma W_T)S_0$$

Part 1a

Simulate the price of a European option call,

$$f(0,S_0) = e^{-rT}\,\mathbb{E}\left[max(S_T - K, 0)|S_0\right]$$

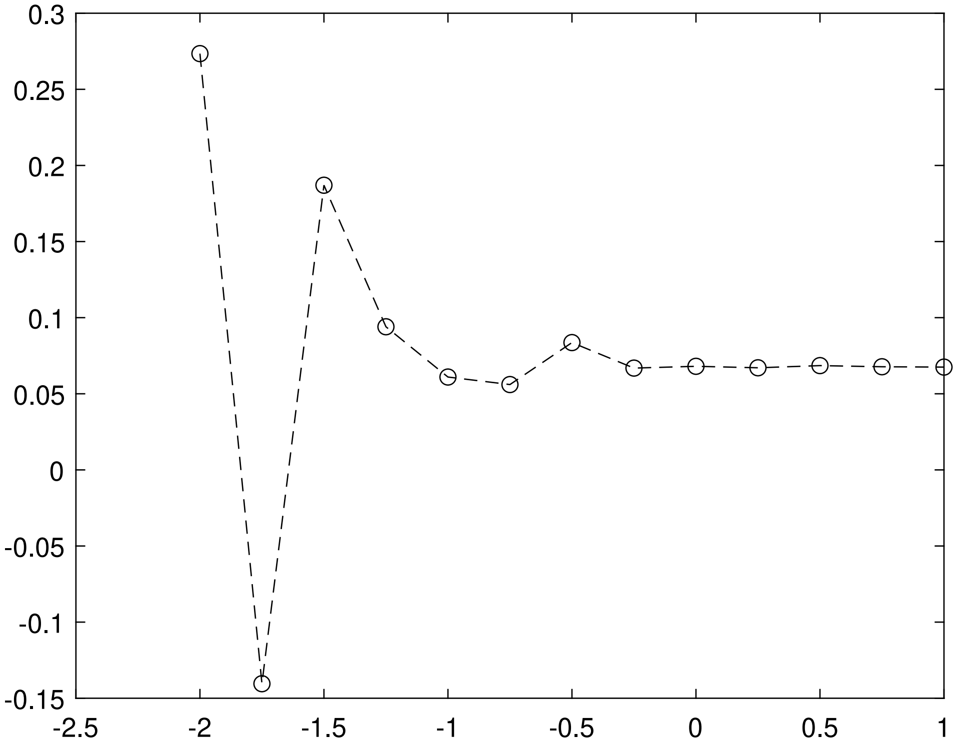
using a Monte Carlo method where $S_0 = K = 35, r = 0.04, \sigma = 0.2, T = \frac{1}{2}$.
Use a successively larger sample and estimate the accuracy of your results by appealing to the the Central Limit Theorem and computing a sample variance.

Answer: We run a Monte Carlo simulation with a fixed step length of $\Delta t = \frac{1}{2^7} = \frac{1}{128}$ and 5 different sample sizes of $N \in [10^3, 10^4, 10^5, 10^6, 10^7]$, from $t = 0$ to $t = 0.5 = T$. For each case (of sample size) we noted the sample variance and estimation error (as a deviation from the analytic solution). The results are given in the table below. (*Note:* Monte Carlo's sample variance is calculated as the variance of the difference between the F-E approximation and analytical solution at every time step.)

Good.

N	$\hat{\sigma}_S^2$ (10^{-7})	$\hat{\sigma}_f^2$ (10^{-7})	$\mathbb{E}[\epsilon_S]$ (10^{-4})	$\mathbb{E}[\epsilon_f]$ (10^{-4})	$(f_T - f_N)$ (10^{-3})
1,000	2.2329	4.3847	-8.647	1.614	-1.521
10,000	0.2889	0.3428	1.410	0.769	-0.644
100,000	0.0425	0.0383	-0.098	-0.041	0.139
1,000,000	0.0041	0.0034	-0.252	-0.147	0.029
10,000,000	0.0032	0.0006	-0.320	-0.211	0.030

Accordingly to the Central Limit Theorem, Monte Carlo's sample variance scaled down proportionally with the increase in sample size, for both stock price (S_t) and option price (f_t) (2nd and 3rd columns, respectively, in table). A similar observation can be made about the deviation between the the analytical solution and the F-E approximation of option price at the call time ($t = T$) (last column in the table). MATLAB code for this simulation is given in Appendix.



?

number of sampled?

Figur 1: Δ (y-axis) as a function of $\log(\Delta s)$ (x-axis)

Part 1b

Compute the corresponding sensitivity parameter ("delta"),

$$\Delta \equiv \frac{\partial f(0,s)}{\partial s}$$

using a finite difference approximation,

$$\Delta \approx \frac{f(0,s+\Delta s) - f(0,s)}{\Delta s}$$

and determine a good choice of Δs. Estimate also accuracy of the results.

Answer: We looked at two points to the left and right of $S_0 = 35$, thus $\mathcal{N} = [(S_0 - \Delta s), S_0, (S_0 + \Delta s)]$ at various distances $\Delta s \in [10^{-2}, 10^{-1.75}, ..., 10^{0.75}, 10^{1}]$; in other words, we systematically searched at 13 different distances from 0.01 to 10 around S_0, computed $f(0,s+\Delta s)$ at each of the points on the left and right of S_0 and calculated the mean Δ, to see how it would change as a function of Δs. The results of this iterative search are plotted against the log(int) in Figure 1.

From the figure, we can see that starting from $\Delta s \approx 10^{-0.25} = 0.56$, the change in $f(0,s)$ per unit change of s stabilizes to $\Delta \approx 0.1$. Thus, this is arguably a good choice for a cut-off value, namely $\Delta s \geq 0.56$. MATLAB code for this simulation is given in Appendix.

Problem 2

Given a (simplified) stochastic volatility model as follow,

$$dS_t = rS_t dt + e^{Y_t} S_t dW_t \tag{1}$$

$$dY_t = (-\alpha(2 + Y_t) + 0.4\sqrt{\alpha}\sqrt{1-\rho^2})dt + 0.4\sqrt{\alpha} d\hat{Z}_t \tag{2}$$

$$\hat{Z}_t = \rho W_t + \sqrt{1-\rho^2} Z_t \tag{3}$$

where W and Z are independent Wiener processes, $\alpha > 0$, and $\rho = -0.3$

Part 2a

Derive the analytical solution of Y_t from (2) above, and calculate $\mathbb{E}[Y_t]$ and $Var[Y_t]$ exactly and their limit as $t \to \infty$.

Answer: First, we re-arrange dY_t as follow (substituting $\rho = -0.3$),

$$\begin{aligned} dY_t &= \left(-2\alpha + 0.4\sqrt{0.91\alpha} - \alpha Y_t\right) dt + 0.4\sqrt{\alpha} d\hat{Z}_t \\ &= (a + bY_t)\, dt + c d\hat{Z}_t \end{aligned}$$

where,

$$\begin{aligned} a &= -2\alpha + 0.4\sqrt{0.91\alpha} \\ b &= -\alpha \\ c &= 0.4\sqrt{\alpha} \end{aligned}$$

Re-arranging dY_t further to,

$$dY_t = -b\left(-\frac{a}{b} - Y_t\right) dt + c d\hat{Z}_t$$

we can infer,

- Steady-state (asymptotic) mean of Y_t is $\theta = -\frac{a}{b} = -2 + 0.4\sqrt{\frac{0.91}{\alpha}}$
- Time constant of decay in Y_t is $-b = \alpha$ (thus, decay term is $e^{-\alpha t}$)
- Coefficient of stochastic (Wiener) component in Y_t is $c = 0.4\sqrt{\alpha}$

Then, we can derive the solution of Y_t using the following steps:

(i) Mean-centered process: $M_t = Y_t - \theta$,

$$\begin{aligned} dM_t = dY_t &= -b(-M_t)dt + c d\hat{Z}_t \\ &= bM_t dt + c d\hat{Z}_t \end{aligned}$$

(ii) Transient-free process: $N_t = e^{-bt} M_t$,

$$\begin{aligned} dN_t &= -be^{-bt} M_t dt + e^{-bt} dM_t \\ &= -be^{-bt} M_t dt + e^{-bt}(bM_t dt + c d\hat{Z}_t) \\ &= ce^{-bt} d\hat{Z}_t \end{aligned}$$

(iii) Ito's integral of both sides of (ii),

$$\begin{aligned}
\int_s^t dN_u &= \int_s^t ce^{-bu} d\hat{Z}_u \\
N_t - N_s &= \int_s^t ce^{-bu} d\hat{Z}_u \\
N_t &= N_s + \int_s^t ce^{-bu} d\hat{Z}_u
\end{aligned}$$

(iv) Returning back to decaying process,

$$\begin{aligned}
e^{-bt} M_t &= e^{-bs} M_s + \int_s^t ce^{-bu} d\hat{Z}_u \\
M_t &= e^{b(t-s)} M_s + \int_s^t ce^{b(t-u)} d\hat{Z}_u
\end{aligned}$$

(v) Returning back the steady-state mean,

$$\begin{aligned}
Y_t - \theta &= e^{b(t-s)}(Y_s - \theta) + \int_s^t ce^{b(t-u)} d\hat{Z}_u \\
Y_t &= \theta + e^{b(t-s)}(Y_s - \theta) + \int_s^t ce^{b(t-u)} d\hat{Z}_u
\end{aligned}$$

(vi) Finally, obtaining the differential stochastic term from (3),

$$d\hat{Z}_t = d\left(-0.3W_t + \sqrt{0.91}Z_t\right) = -0.3dW_t + \sqrt{0.91}dZ_t$$

We obtain the final form of the solution,

$$Y_t = -2 + 0.4\sqrt{\frac{0.91}{\alpha}} + e^{-\alpha(t-s)}\left(Y_s + 2 - 0.4\sqrt{\frac{0.91}{\alpha}}\right) + 0.4\sqrt{\alpha}\int_s^t e^{-\alpha(t-u)}(-0.3dW_u + \sqrt{0.91}dZ_u)$$

Then, the mean of Y_t is,

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$$\begin{aligned}
\mathbb{E}\left[Y_t\right] &= \theta + e^{-\alpha(t-s)}\left(Y_s - \theta\right) + 0.4\sqrt{\alpha}\,\underbrace{\mathbb{E}\left[\int_s^t e^{-\alpha(t-u)}(-0.3dW_u + \sqrt{0.91}dZ_u)\right]}_{=0} \\
&= \theta + e^{-\alpha(t-s)}\left(Y_s - \theta\right) \\
&= -2 + 0.4\sqrt{\frac{0.91}{\alpha}} + e^{-\alpha(t-s)}\left(Y_s + 2 - 0.4\sqrt{\frac{0.91}{\alpha}}\right)
\end{aligned}$$

and the asymptotic ($t \to \infty$) mean is,

$$\mathbb{E}\left[Y_t\right]_{t\to\infty} = -2 + 0.4\sqrt{\frac{0.91}{\alpha}} = -\frac{a}{b}$$

as inferred above.

The variance of Y_t is,

$$\begin{aligned}
\mathbb{E}\left[(Y_t - \mathbb{E}[Y_t])^2\right] &= 0.16\alpha e^{-2\alpha t}\,\mathbb{E}\left[\left(\int_s^t e^{\alpha u}(-0.3dW_u + \sqrt{0.91}dZ_u)\right)\left(\int_s^t e^{\alpha v}(-0.3dW_v + \sqrt{0.91}dZ_v)\right)\right] \\
&= 0.16\alpha e^{-2\alpha t}\,\mathbb{E}\left[\int_s^t e^{2\alpha u}\left(0.09dW_u^2 + 0.91dZ_u^2\right)\right] \text{[1]} \\
&= 0.16\alpha e^{-2\alpha t}\int_s^t e^{2\alpha u}\left(0.09\,\underbrace{\mathbb{E}[dW_u^2]}_{=du} + 0.91\,\underbrace{\mathbb{E}[dZ_u^2]}_{=du}\right) \\
&= 0.16\alpha e^{-2\alpha t}\int_s^t e^{2\alpha u}du = 0.16\alpha e^{-2\alpha t}\left(\frac{1}{2\alpha}\right)\left(e^{2\alpha t} - e^{2\alpha s}\right) \\
&= 0.08\left(1 - e^{-2\alpha(t-s)}\right)
\end{aligned}$$

and the asymptotic ($t \to \infty$) variance is,

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$$\mathbb{E}\left[(Y_t - \mathbb{E}[Y_t])^2\right]_{t\to\infty} = 0.08 = \frac{c^2}{2|b|}$$

Part 2b

Given the model equation below for stability analysis,

$$dX_t = -\alpha X_t dt + \sqrt{\alpha}dW_t \tag{4}$$

where W is a Wiener process, solve the following:

(i) Compute expected value and variance of X_t and their corresponding limits at $t \to \infty$.

(ii) Compute expected value and variance of Forward Euler approximation to X_t and their corresponding limits at $t \to \infty$.

(iii) Compute expected value and variance of Backward Euler approximation to X_t and their corresponding limits at $t \to \infty$.

(iv) Interpret the results obtained in (i-iii).

Answer:
(i) Using the same approach as in Part 2a, first we introduce a substitute transient-free variable/process: $Z_t = e^{\alpha t}X_t$. Then,

$$\begin{aligned}
dZ_t &= \alpha e^{\alpha t}X_t dt + e^{\alpha t}dX_t \\
&= \alpha e^{\alpha t}X_t dt + e^{\alpha t}(-\alpha X_t dt + \sqrt{\alpha}dW_t) \\
&= \sqrt{\alpha}e^{\alpha t}dW_t
\end{aligned}$$

Next, integrating both sides of above:

$$\begin{aligned}
\int_s^t dZ_u &= \sqrt{\alpha}\int_s^t e^{\alpha u}dW_u \\
Z_t &= Z_s + \sqrt{\alpha}\int_s^t e^{\alpha u}dW_u
\end{aligned}$$

[1]All cross-terms vanishes due to independence of W and Z.

Returning back the transient component:

$$
\begin{aligned}
e^{\alpha t} X_t &= e^{\alpha s} X_s + \sqrt{\alpha} \int_s^t e^{\alpha u} dW_u \\
X_t &= e^{-\alpha(t-s)} X_s + \sqrt{\alpha} \int_s^t e^{-\alpha(t-u)} dW_u
\end{aligned}
$$

Hence, the mean of X_t,

$$
\begin{aligned}
\mathbb{E}\left[X_t\right] &= e^{-\alpha(t-s)} X_s + \sqrt{\alpha} e^{-\alpha t} \underbrace{\mathbb{E}\left[\int_s^t e^{\alpha u} dW_u\right]}_{=0} \\
&= e^{-\alpha(t-s)} X_s
\end{aligned}
$$

and it's asymptotic ($t \to \infty$) mean is,

$$
\mathbb{E}\left[X_t\right]_{t\to\infty} = 0
$$

The variance of X_t is,

$$
\begin{aligned}
\mathbb{E}\left[(X_t - \mathbb{E}[X_t])^2\right] &= \mathbb{E}\left[\left(\sqrt{\alpha} e^{-\alpha t} \int_s^t e^{\alpha u} dW_u\right)\left(\sqrt{\alpha} e^{-\alpha t} \int_s^t e^{\alpha v} dW_v\right)\right] \\
&= \alpha e^{-2\alpha t}\, \mathbb{E}\left[\int_s^t e^{2\alpha u} dW_u^2\right] = \alpha e^{-2\alpha t} \int_s^t e^{2\alpha u} \underbrace{\mathbb{E}[dW_u^2]}_{=du} \\
&= \alpha e^{-2\alpha t} \int_s^t e^{2\alpha u} du = \alpha e^{-2\alpha t} \left(\frac{1}{2\alpha}\right)\left(e^{2\alpha t} - e^{2\alpha s}\right) \\
&= \frac{1}{2}\left(1 - e^{-2\alpha(t-s)}\right)
\end{aligned}
$$

and it's asymptotic ($t \to \infty$) variance is,

$$
\mathbb{E}\left[(X_t - \mathbb{E}[X_t])^2\right]_{t\to\infty} = \frac{1}{2}
$$

(ii) Forward Euler (F-E) approximation[2] to (4) is,

$$
X_{n+1} - X_n = -\alpha X_n \Delta t_n + \sqrt{\alpha} \Delta W_n
$$

Thus,

$$
\begin{aligned}
X_{n+1} &= X_n - \alpha X_n \Delta t_n + \sqrt{\alpha} \Delta W_n \\
&= (1 - \alpha \Delta t_n) X_n + \sqrt{\alpha} \Delta W_n
\end{aligned}
$$

and for "integrating"from n_0 to N, we have

$$
X_N = (1 - \alpha \Delta t_n)^{(N-n_0)} X_{n_0} + \sum_{k=n_0}^{N-1} (1 - \alpha \Delta t_n)^{(N-k-1)} \sqrt{\alpha} \Delta W_k
$$

The mean of this F-E approximation is,

$$
\begin{aligned}
\mathbb{E}\left[X_N\right] &= (1 - \alpha \Delta t_n)^{(N-n_0)} X_{n_0} + \underbrace{\mathbb{E}\left[\sum_{k=n_0}^{N-1} (1 - \alpha \Delta t_n)^{(N-k-1)} \sqrt{\alpha} \Delta W_k\right]}_{=0} \\
&= (1 - \alpha \Delta t_n)^{(N-n_0)} X_{n_0}
\end{aligned}
$$

[2]We assume a constant Δt which depends only on the size of partitions n; thus $\Delta t_k = \Delta t_n \ \forall k$

which is stable (and finite) only for $|1-\alpha\Delta t| < 1$. And in which case the asymptotic ($N \to \infty$) mean is,

$$\mathbb{E}\left[X_N\right]_{N\to\infty} = 0$$

The variance of this F-E approximation is,

$$\begin{aligned}
\mathbb{E}\left[(X_N-\mathbb{E}[X_N])^2\right] &= \mathbb{E}\left[\left(\sum_{k=n_0}^{N-1}(1-\alpha\Delta t_n)^{(N-k-1)}\sqrt{\alpha}\Delta W_k\right)\left(\sum_{l=n_0}^{N-1}(1-\alpha\Delta t_n)^{(N-l-1)}\sqrt{\alpha}\Delta W_l\right)\right] \\
&= \mathbb{E}\left[\sum_{k=l}(1-\alpha\Delta t_n)^{2(N-k-1)}\alpha\Delta W_k^2\right] + \underbrace{\mathbb{E}\left[\sum_{k<l}(1-\alpha\Delta t_n)^{(2N-(k+l)-2)}\Delta W_k\Delta W_l\right]}_{=0} \\
&= \sum_{k=l}(1-\alpha\Delta t_n)^{2(N-k-1)}\alpha\,\underbrace{\mathbb{E}[\Delta W_k^2]}_{=\Delta t_n} \\
&= \alpha\Delta t_n\sum_{k=n_0}^{N-1}(1-\alpha\Delta t_n)^{2(N-k-1)}
\end{aligned}$$

again, which is finite only if $|1-\alpha\Delta t| < 1$, and in which case the asymptotic ($N \to \infty$) variance is,

$$\mathbb{E}\left[(X_N-\mathbb{E}[X_N])^2\right]_{N\to\infty} = \alpha\Delta t_n\sum_{k}^{\infty}(1-\alpha\Delta t_n)^{2k}$$

is an infinite series with a ratio $r = (1-\alpha\Delta t_n)^2$. And thus,

$$\begin{aligned}
\mathbb{E}\left[(X_N-\mathbb{E}[X_N])^2\right]_{N\to\infty} &= \alpha\Delta t_n\left(\frac{1}{(1-(1-\alpha\Delta t_n)^2)}\right) \\
&= \frac{\alpha\Delta t_n}{\alpha\Delta t_n(2-\alpha\Delta t_n)} \\
&= \frac{1}{(2-\alpha\Delta t_n)}
\end{aligned}$$

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The asymptotic variance is finite for $0 \le \alpha\Delta t_n < 2$ (α must be positive for the approximation to have a finite mean; see above).

(iii) Backward Euler (B-E) approximation[3] to (4) is,

$$X_{n+1} - X_n = -\alpha X_{n+1}\Delta t_n + \sqrt{\alpha}\Delta W_n$$

Thus,

$$\begin{aligned}
X_{n+1} + \alpha X_{n+1}\Delta t_n &= X_n + \sqrt{\alpha}\Delta W_n \\
(1+\alpha\Delta t_n)X_{n+1} &= X_n + \sqrt{\alpha}\Delta W_n \\
X_{n+1} &= \frac{1}{(1+\alpha\Delta t_n)}X_n + \frac{\sqrt{\alpha}}{(1+\alpha\Delta t_n)}\Delta W_n
\end{aligned}$$

and for "integrating"from n_0 to N, we have

$$X_N = \left(\frac{1}{1+\alpha\Delta t_n}\right)^{(N-n_0)} X_{n_0} + \sum_{k=n_0}^{N-1}\left(\frac{1}{1+\alpha\Delta t_n}\right)^{(N-k)}\sqrt{\alpha}\Delta W_k$$

[3]Again, we assume a constant Δt; $\Delta t_k = \Delta t_n\ \forall k$

The mean of this B-E approximation is,

$$\mathbb{E}\left[X_N\right] = \left(\frac{1}{1+\alpha\Delta t_n}\right)^{(N-n_0)} X_{n_0} + \underbrace{\mathbb{E}\left[\sum_{k=n_0}^{N-1}\left(\frac{1}{1+\alpha\Delta t_n}\right)^{(N-k)}\sqrt{\alpha}\Delta W_k\right]}_{=0}$$

$$= \left(\frac{1}{1+\alpha\Delta t_n}\right)^{(N-n_0)} X_{n_0}$$

which is unconditionally stable (i.e. $\forall \Delta t_n$). The asymptotic ($N \to \infty$) mean is,

$$\mathbb{E}\left[X_N\right]_{N\to\infty} = 0$$

The variance of this B-E approximation is,

$$\mathbb{E}\left[(X_N - \mathbb{E}[X_N])^2\right] = \mathbb{E}\left[\left(\sum_{k=n_0}^{N-1}\left(\frac{1}{1+\alpha\Delta t_n}\right)^{(N-k)}\sqrt{\alpha}\Delta W_k\right)\left(\sum_{l=n_0}^{N-1}\left(\frac{1}{1+\alpha\Delta t_n}\right)^{(N-l)}\sqrt{\alpha}\Delta W_l\right)\right]$$

$$= \mathbb{E}\left[\sum_{k=l}\left(\frac{1}{1+\alpha\Delta t_n}\right)^{2(N-k)}\alpha\Delta W_k^2\right] + \underbrace{\mathbb{E}\left[\sum_{k<l}\left(\frac{1}{1+\alpha\Delta t_n}\right)^{(2N-(k+l))}\alpha\Delta W_k\Delta W_l\right]}_{=0}$$

$$= \sum_{k=l}\left(\frac{1}{1+\alpha\Delta t_n}\right)^{2(N-k)}\alpha\,\underbrace{\mathbb{E}[\Delta W_k^2]}_{=\Delta t_n}$$

$$= \alpha\Delta t_n\sum_{k=n_0}^{N-1}\left(\frac{1}{1+\alpha\Delta t_n}\right)^{2(N-k)}$$

which is unconditionally finite (i.e. $\forall \Delta t_n$). The asymptotic ($N \to \infty$) variance is,

$$\mathbb{E}\left[(X_N - \mathbb{E}[X_N])^2\right]_{N\to\infty} = \alpha\Delta t_n\sum_{k=1}^{\infty}\left(\frac{1}{1+\alpha\Delta t_n}\right)^{2k}$$ K

is an infinite series with a ratio $r = (1+\alpha\Delta t_n)^{-2}$ and $a = \frac{\alpha\Delta t_n}{1+\alpha\Delta t_n}$. And thus,

?

$$\mathbb{E}\left[(X_N - \mathbb{E}[X_N])^2\right]_{N\to\infty} = \frac{\frac{1}{1+\alpha\Delta t_n}}{1-\left(\frac{1}{1+\alpha\Delta t_n}\right)^2}\alpha\Delta t_n$$

$$= \frac{1+\alpha\Delta t_n}{\alpha\Delta t_n(2+\alpha\Delta t_n)}\alpha\Delta t_n$$

$$= \frac{1+\alpha\Delta t_n}{2+\alpha\Delta t_n}$$

The asymptotic variance is unconditionally finite and it converges to the analytical variance as $\Delta t_n \to 0$.

(iv) The table below compare (i-iii) in terms of mean and variance both in a finite sample and asymptotic cases.

	Analytical	Forward Euler	Backward Euler
$\mathbb{E}[X_t]$	$e^{-\alpha(t-s)}X_s$	$(1-\alpha\Delta t_n)^{(N-n_0)}X_{n_0}$	$\left(\frac{1}{1+\alpha\Delta t_n}\right)^{(N-n_0)}X_{n_0}$
$\mathbb{E}[X_t]_{t\to\infty}$	0	0	0
$Var[X_t]$	$\frac{1}{2}\left(1-e^{-2\alpha(t-s)}\right)$	$\alpha\Delta t_n\sum_{k=n_0}^{N-1}(1-\alpha\Delta t_n)^{2(N-k-1)}$	$\alpha\Delta t_n\sum_{k=n_0}^{N-1}\left(\frac{1}{1+\alpha\Delta t_n}\right)^{2(N-k)}$
$Var[X_t]_{t\to\infty}$	$\frac{1}{2}$	$\frac{1}{(2-\alpha\Delta t_n)}$	$\frac{1+\alpha\Delta t_n}{2+\alpha\Delta t_n}$?

A number of points can be concluded from the table:

1. Both F-E and B-E converge towards the analytical mean asymptotically (as the number of samples grows to infinity)

2. Both F-E and B-E also converge towards the analytical variance asymptotically (as the step length is decreased to infinitesimally small)

3. F-E is only conditionally stable for $|1-\alpha\Delta t_n| < 1$, while B-E is unconditionally stable $\forall\Delta t_n$

4. F-E only has a finite variance for $0 \leq \alpha\Delta t_n < 2$, while B-E always has a finite variance

5. Points 3 and 4 for F-E are equivalent, and thus defining the step sizes allowable for F-E, namely: $0 \leq \Delta t_n < \frac{2}{\alpha}$, where $\alpha > 0$. The efficiency of F-E method thus is dependent on the time constant of the decay term.

Good!

6. For B-E, $Var[X_N]_{N\to\infty} \xrightarrow{\Delta t\to\infty} 1$; that is, the asymptotic variance is upper-bounded to twice that of the analytical case even if we take extremely large steps. In contrast, for F-E, $Var[X_N]_{N\to\infty} \xrightarrow{\Delta t\to\frac{2}{\alpha}} \infty$; that is, the asymptotic variance will grow very large for step lengths close to $\frac{2}{\alpha}$ and is unbounded from above.

In summary, both F-E and B-E approximations are *consistent* estimators of the stochastic process X_t, but B-E approximation is by far a more *efficient* and robust (stable) estimator.

Parts 2c, d

Use the following F-E approximations,

$$S_{n+1} - S_n = rS_n\Delta t + e^{Y_n}S_n\Delta W_n \tag{5}$$

$$Y_{n+1} - Y_n = \left(-\alpha(2+Y_n) + 0.4\sqrt{\alpha}\sqrt{1-\rho^2}\right)\Delta t + 0.4\sqrt{\alpha}\Delta\hat{Z}_n \tag{6}$$

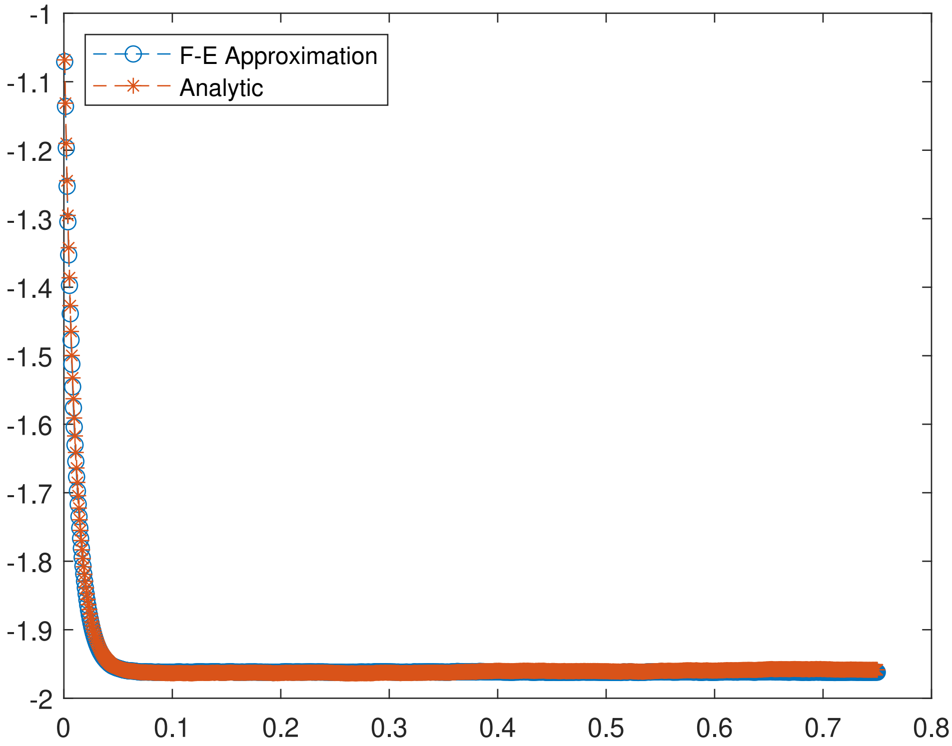
$$\hat{Z}_n = \rho W_n + \sqrt{1-\rho^2}Z_n \tag{7}$$

to compute the option value,

$$e^{-rT}\,\mathbb{E}[max(S(T)-K,0)]$$

where $\alpha = 100$, $r = 0.04$, $T = \frac{3}{4}$, $Y_0 = -1$, *and* $S_0 = K = 100$.

Answer: First of all, we run a simulation to understand the dynamics of Y(t) as well as the numerical behavior of its F-E approximation. Using a fixed step length of $\Delta t = \frac{1}{2^10} = \frac{1}{1024}$



Figur 2: Y_t as a function of T

and a sample size of $N = 10^6$, we obtained the result shown in Figure 2.

As predicted by hand-derived results in Part 2a, with $\Delta t \to 0$ (here, $\Delta t \approx 10^{-3}$), with $\alpha = 100$, both the mean ($\hat{\theta} = -1.9620$) and variance ($V\hat{a}r[Y_t]_{t\to\infty} = 0.0826$) of the approximation are very close to the asymptotic mean, $\theta = -1.9618$, and the asymptotic variance, $Var[Y_t]_{t\to\infty} = 0.08$, respectively.

Now that we are sure that our MATLAB code produced numerical behavior of Y_t as predicted by the hand-derived results, we run the simulation to compute the option price with the given stochastic volatility model using the F-E approximations in (5)-(7) above.

First, we observed that with $\alpha = 100$, the F-E approximation will have a severe restriction on the allowable step lengths, namely $\Delta t < \frac{2}{\alpha} = \frac{2}{100} = 0.02$ (see point 5 in Part 2b(iv) above). Thus, we started the simulation with a step length smaller than this value, and systematically decreased it by halving it in each iteration for 4 times; thus we used $\Delta t \in [\frac{1}{64}, \frac{1}{128}, \frac{1}{256}, \frac{1}{512}]$. We also varied the sample sizes by increasing it by ten-fold in each iteration, also for 4 times, starting from 1000; thus, we used $N \in [10^3, 10^4, 10^5, 10^6]$. Furthermore, we calculated the ratio of sample size to sample variance, $\frac{N}{\sigma^2}$, needed to obtain a specific level of accuracy (approximation error), $|\epsilon_N|$, with a confidence level of 0.99 (or false positive rate of $\alpha = 0.01$) given $\frac{\sqrt{N}}{\sigma}\epsilon_N \sim N(0{,}1)$, as follows:

$$\begin{aligned}
Prob(a < \frac{\sqrt{N}}{\sigma}\epsilon_N < b) &= 0.99 \\
Prob(a < \frac{\sqrt{N}}{\sigma}\epsilon_N < b) &= Prob(-2.56 < \frac{\sqrt{N}}{\sigma}\epsilon_N < 2.56) \\
&\iff -2.56 < \frac{\sqrt{N}}{\sigma}\epsilon_N < 2.56 \\
&\iff \frac{N}{\sigma^2} > \frac{(2.56)^2}{\epsilon_N^2}
\end{aligned}$$

Thus, for $|\epsilon_N| = 0.05$, $\frac{N}{\sigma^2} > 2{,}622$, and for $|\epsilon_N| = 0.005$, $\frac{N}{\sigma^2} > 262{,}145$.

Now, for each of the 16 combinations of (step length, sample size), we computed the exact and F-E approximation of stock S_T and option $f_T(0{,}S_0 = 100)$ prices at the call time $T = 0.75$, along with $\mathbb{E}[\epsilon_N]$, $\hat{\sigma}^2$, computation time, and the ratio $\frac{N}{\sigma^2}$. The results are presented in table below.

Thus, it can be seen that for a sample size of $N = 1000$, a $\Delta t \geq \frac{1}{256}$ will guarantee an accuracy of $\epsilon_N = 0.005$ with a probability of 99%. For all the larger sample sizes that we tried here, $\Delta t \geq \frac{1}{64}$ will do the job. Moreover, for all the simulation parameters that we tried here, an accuracy of $\epsilon_N = 0.05$ should be guaranteed with a probability of 99%. Nonetheless, we observed an interesting pattern of increasing deviation of the terminal values at $T = 0.75$ between the 'exact' and F-E approximation. We suspected that this was due to the machine rounding-error introduced by making the step lengths smaller and smaller.

Δt	$\mathbb{E}[S_T]$				$f_T(0{,}100)$				$\frac{N}{\sigma^2}$ (10^3)	Run Time (sec)
	Exact	F-E	$\mathbb{E}[\epsilon_N]$	$\hat{\sigma}^2$	Exact	F-E	$\mathbb{E}[\epsilon_N]$	$\hat{\sigma}^2$		
$N = 1{,}000$										
$\frac{1}{64}$	103.770	103.800	0.012	$2.5*10^{-4}$	6.604	6.784	-0.003	0.022	46.4	0.028
$\frac{1}{128}$	102.836	102.849	0.002	$2.9*10^{-5}$	6.551	6.248	-0.001	0.011	92.6	0.085
$\frac{1}{256}$	103.047	103.047	-0.001	$1.5*10^{-7}$	6.389	6.386	-0.013	0.002	507.6	0.313
$\frac{1}{512}$	103.037	103.037	-0.001	$4.3*10^{-8}$	6.398	6.396	-0.014	0.002	558.2	3.516
$N = 10{,}000$										
$\frac{1}{64}$	103.666	103.696	-0.005	$3.9*10^{-4}$	6.242	6.710	0.157	0.015	674.6	0.041
$\frac{1}{128}$	102.950	102.953	0.008	$4.2*10^{-5}$	6.307	6.451	0.185	0.013	788.4	0.134
$\frac{1}{256}$	102.998	102.998	0.001	$8.5*10^{-8}$	6.431	6.389	0.048	0.002	6,854	0.571
$\frac{1}{512}$	103.053	103.053	0.000	$9.6*10^{-9}$	6.399	6.448	0.062	0.001	16,188	6.552
$N = 100{,}000$										
$\frac{1}{64}$	103.770	103.830	0.016	0.002	6.342	6.979	0.337	0.046	2,173	0.041
$\frac{1}{128}$	102.932	102.942	0.005	$6.0*10^{-6}$	6.301	6.414	0.275	0.006	18,089	0.162
$\frac{1}{256}$	103.065	103.063	0.001	$4.3*10^{-8}$	6.393	6.484	0.102	0.003	31,515	0.976
$\frac{1}{512}$	103.040	103.038	0.001	$1.3*10^{-8}$	6.408	6.464	0.119	0.002	47,885	12.938
$N = 1{,}000{,}000$										
$\frac{1}{64}$	103.882	103.903	0.017	$5.3*10^{-4}$	6.747	7.078	0.318	0.008	120,650	0.064
$\frac{1}{128}$	103.079	103.083	0.009	$1.6*10^{-5}$	6.233	6.514	0.331	0.006	169,400	0.291
$\frac{1}{256}$	102.947	103.945	-0.001	$1.6*10^{-7}$	6.394	6.418	0.141	0.004	259,560	1.952
$\frac{1}{512}$	103.057	103.057	0.001	$1.7*10^{-8}$	6.406	6.495	0.150	0.004	278,610	25.139

6.94

Appendix: MATLAB Code

Problem 1

```
%% (a) Simulation of option price
clear

steps = 7; %number of different (equidistant) step lengths
samples = [3:7]; %number of different sample size


for i=1:length(steps)
    N = 2^steps(i) % number of timesteps
    randn('state',0);
    T = .5; dt = T/N; t = 0:dt:T;
    r = .04; sig = .2; S0 = 35; K = S0;

    S_bar_ks = {}; var_S_ks = {}; f_ks = {}; ft_ks = {}; St_ks = {};
    errorS_ks = {}; errorf_ks = {};
    S_bar_j = []; var_S_j = []; f_j = []; fT_j = [];
    errorf_j = []; MSE_S_j = []; MSE_S_ks = []; MSE_f_ks = []; err_S_ks =
        []; err_f_ks = [];
    for j=1:length(samples)
        M = 10^samples(j); % number of realisations
        S = S0*ones(M,1); % S(0) for all realizations
        W = zeros(M,1); % W(0) for all realizations
        S_bar = []; var_S = []; f = []; ST = []; fT = [];
        for k=1:N
            dW = sqrt(dt)*randn(M,1); % Wiener increments
            S = S + S.*(r*dt+sig*dW); % stock price at next time step
            W = W + dW; % Brownian paths at next step
            S_bar(k) = mean(S); % mean of S at next step
            var_S(k) = var(S); % variance of S at next step
            f(k) = exp(-r*k*dt)*mean(max(S - K, 0)); % a constant (mean)
                value of f at time step k

            Sk = S0*exp( (r-sig^2/2)*k*dt + sig*W ); % exact stock
                realizations at time k
            ft(k) = exp(-r*k*dt)*mean(max(Sk - K, 0));
            St(k) = mean(Sk);
        end
        S_bar_ks{j} = S_bar;
        var_S_ks{j} = var_S;
        f_ks{j} = f;
        St_ks{j} = St;
        ft_ks{j} = ft;

        errorS_ks{j} = S_bar - St;
```

```
        errorf_ks{j} = f - ft;

        MSE_S_ks(j) = var(errorS_ks{j});
        MSE_f_ks(j) = var(errorf_ks{j});
        err_S_ks(j) = mean(errorS_ks{j});
        err_f_ks(j) = mean(errorf_ks{j});

        S_bar_j(j) = S_bar(end);
        var_S_j(j) = var_S(end);
        f_j(j) = f(end);

        ST_j = S0*exp( (r-sig^2/2)*T + sig*W ); % exact final value;
        errorS_j = ST_j - S;
        MSE_S_j(j) = var(errorS_j);

        fT_j(j) = exp(-r*T)*mean(max(ST_j - K, 0));
        errorf_j(j) = fT_j(j) - f_j(j);

    end
end
%dt = T./2.^steps;
figure; % stock price over time
plot(t(2:end),St,'--',t(2:end),S_bar_ks{1},'o--',t(2:end),S_bar_ks{2},'*--',t(2:end),S
legend('Analytical','N=1,000','N=10,000','N=100,000','N=1,000,000','Location','northwe

figure; % option price over time
plot(t(2:end),ft,'--',t(2:end),f_ks{1},'o--',t(2:end),f_ks{2},'*--',t(2:end),f_ks{3},'
legend('Analytical','N=1,000','N=10,000','N=100,000','N=1,000,000','Location','northwe

figure; % variance of S over time
plot(t(2:end),var_S_ks{1},'o--',t(2:end),var_S_ks{2},'*--',t(2:end),var_S_ks{3},'--',t
legend('N=1,000','N=10,000','N=100,000','N=1,000,000','Location','northwest')

figure; % stock price error over time
plot(t(2:end),errorS_ks{1},'o--',t(2:end),errorS_ks{2},'*--',t(2:end),errorS_ks{3},'--
legend('N=1,000','N=10,000','N=100,000','N=1,000,000','Location','southwest')

figure; % option price error over time
plot(t(2:end),errorf_ks{1},'o--',t(2:end),errorf_ks{2},'*--',t(2:end),errorf_ks{3},'--
legend('N=1,000','N=10,000','N=100,000','N=1,000,000','Location','southwest')

figure; % log-variance of S and f errors
x = 10.^samples;
plot(log10(x),log10(MSE_S_ks),'o--',log10(x),log10(MSE_f_ks),'*--')
legend('Stock price Log-MSE','Option price Log-MSE','Location','southwest')

figure; % log-mean of S and f errors
x = 10.^samples;
plot(log10(x),log10(errorf_j),'o--',log10(x),log10(err_f_ks),'*--',log10(x),log10(err_
```

```
legend('Option price Log-Error (final time)','Option price Log-Error
    (mean)','Stock price Log-Error (mean)','Location','southwest')

%% (b) Computing the Sensitivity (delta)
delta_s = []; delta = []; fDev = [];
f_all = {}; fT_hat_all = []; fT = [];

N = 2^steps % number of timesteps
randn('state',0);
T = .5; dt = T/N; t = 0:dt:T;
r = .04; sig = .2;

%intervals = [.05, .25, .5, 1.25, 2.5, 5, 10, 15, 25];
intervals = linspace(-2,1,13);
for i=1:length(intervals)
    int = 10^intervals(i);
    S0s = linspace(35-int,35+int,3);

    for j=1:length(S0s)
        K = S0s(j); S0 = S0s(j);
        M = 10^samples(4); % number of realisations
        S = S0*ones(M,1); % S(0) for all realizations
        W = zeros(M,1); % W(0) for all realizations
        S_hat = []; sigma_sq = []; f = [];
        for k=1:N
            %rng('default');
            dW = sqrt(dt)*randn(M,1); % Wiener increments
            S = S + S.*(r*dt+sig*dW); % processes at next time step
            W = W + dW; % Brownian paths at next step
            f(k) = mean(max(S - K, 0));
        end
        f_all{j} = f;
        fT_hat_all(j) = f(N);
        ST = S0*exp( (r-sig^2/2)*T + sig*W ); % exact final value
        fT(j) = mean(max(ST - K, 0))*exp(-r*T);
        fDev(j) = fT(j) - fT_hat_all(j);
    end
    delta_s(i) = mean(diff(S0s));
    delta(i) = mean(diff(fT_hat_all)./diff(S0s));
end

figure;
plot(log10(delta_s),delta,'o--')
%print -deps epsFig

figure;
plot(log10(delta_s),abs(delta),'o--')
```

Problem 2 (c, d)

```
%% understanding the numerical properties of F-E of Y(t)
clear

steps = [7:10]; %number of different (equidistant) step lengths
samples = [4:6]; %number of different sample size

Y_hat_js = []; YT_js = []; errorY_js = []; MSE_js = []; devY_js = [];
Y_Hat = {}; MSE = {}; dev_Y = {}; dev_VarY = {}; errorY = [];
for i=1:length(steps)
    for j=1:length(samples)
        N = 2^steps(i); % number of timesteps
        randn('state',0);
        T = .75; dt = T/N; t = 0:dt:T;
        alpha = 100; Y0 = -1; rho = -.3;
        Yinf = -2+0.4*sqrt(0.91/alpha);

        M = 10^samples(j); % number of realisations
        Y = Y0*ones(M,1); % Y(0) for all realizations
        W = zeros(M,1); % W(0) for all realizations
        Z_hat = zeros(M,1); % Z(0) for all realizations
        Y_hat = []; sigma_sq = []; YT = [];
        for k=1:N
            dW = sqrt(dt)*randn(M,1); % Wiener increments
            dZ = sqrt(dt)*randn(M,1); % Wiener increments
            dZ_hat = rho*dW + sqrt(1-rho^2)*dZ;
            Y = Y + (-alpha*(2+Y)+0.4*sqrt(alpha)*sqrt(1-rho^2)).*dt +
               0.4*sqrt(alpha)*dZ_hat;
            W = W + dW;
            Z_hat = Z_hat + dZ_hat; % Brownian paths at next step
            Y_hat(k) = mean(Y);
            sigma_sq(k) = var(Y); % variance of S at next step

            YT(k) = mean(-2+0.4*sqrt(0.91/alpha) +
               (Y0+2-0.4*sqrt(0.91/alpha))*exp(-alpha*k*dt) +
               0.4*sqrt(alpha)*Z_hat);
        end
        Y_hat_js(j) = Y_hat(end);
        YT_js(j) = YT(end);
        errorY_js(j) = Y_hat_js(j) - YT_js(j);
        MSE_js(j) = mean(sigma_sq);
        dev_Y_js(j) = Yinf - Y_hat_js(j);
        dev_VarY_js(j) = MSE_js(j) - 0.08;
    end
    %YT = (-2+0.4*sqrt(0.91/alpha)) +
        ((Y0+2-0.4*sqrt(0.91/alpha))*exp(-alpha*T)) +
        (0.4*sqrt(alpha)*Z_hat);
```

```
    Y_Hat{i} = Y_hat_js;
    errorY{i} = errorY_js;
    MSE{i} = MSE_js;
    dev_Y{i} = dev_Y_js;
    dev_VarY{i} = dev_VarY_js;
end

figure;
plot(t(2:end),Y_hat,'o--',t(2:end),YT,'*--')
legend('F-E Approximation','Analytic','Location','northwest')
print -depsc epsFig2

figure;
plot(t(2:end),sigma_sq,'.-')
legend('Error over time','Location','southeast')

%% (2c & d) Simulation of option price
clear;

steps = [6:9]; %number of different (equidistant) step lengths
samples = [3:6]; %number of different sample size

MSEs_S = {}; MSEs_f = {}; errs_S = {}; errs_f = {}; N_ov_ssqs = {};
durations = {}; all_Ss = {}; all_fs ={};
ST = []; fT = [];

rng('default')
s = rng;
for i=1:length(steps)

duration_j = []; S_j = []; f_j = [];

tStart = tic;
N = 2^steps(i) % number of timesteps
T = .75; dt = T/N; t = 0:dt:T;
r = .04; alpha = 100; Y0 = -1; rho = -.3; S0 = 100; K = S0;

errorS_ks = {}; errorf_ks = {};
MSE_S_j = []; MSE_f_j = []; err_S_j = []; err_f_j = []; N_ov_ssq_j = [];

for j=1:length(samples)
    M = 10^samples(j); % number of realisations
    S = S0*ones(M,1); % S(0) for all realizations
    Y = Y0*ones(M,1); % Y(0) for all realizations
    W = zeros(M,1); % W(0) for all realizations
    Z_hat = zeros(M,1); % Z(0) for all realizations

    S_hat = []; Y_hat = []; f = []; Yt = []; St = []; ft = [];
```

```
    randn('state',0);
    for k=1:N
        dW = sqrt(dt)*randn(M,1); % Wiener increments
        dZ = sqrt(dt)*randn(M,1); % Wiener increments
        dZ_hat = rho*dW + sqrt(1-rho^2)*dZ;
        Y = Y + (-2*alpha+0.4*sqrt(alpha)*sqrt(1-rho^2))*dt -alpha*Y.*dt +
            0.4*sqrt(alpha)*dZ_hat;
        S = S + S.*(r*dt+exp(mean(Y))*dW); % processes at next time step
        W = W + dW;
        Z_hat = Z_hat + dZ_hat; % Brownian paths at next step
        Y_hat(k) = mean(Y);
        S_hat(k) = mean(S); % mean of S at next step
        f(k) = exp(-r*k*dt)*mean(max(S - K, 0));

        % analytical solution to S & f at time step k (if the call
        % terminated earlier)
        Yk = -2+0.4*sqrt(0.91/alpha) +
            (Y0+2-0.4*sqrt(0.91/alpha))*exp(-alpha*k*dt) +
            0.4*sqrt(alpha)*Z_hat;
        Yt(k) = mean(Yk);
        Sk = S0*exp( (r-exp(2*Yt(k))/2)*k*dt + exp(Yt(k))*W ); % exact
            stock realizations at time k
        St(k) = mean(Sk);
        ft(k) = exp(-r*k*dt)*mean(max(Sk - K, 0));
    end
    YT = mean(-2+0.4*sqrt(0.91/alpha) +
        (Y0+2-0.4*sqrt(0.91/alpha))*exp(-alpha*T) + 0.4*sqrt(alpha)*Z_hat);
    ST_j = S0*exp( (r-exp(2*YT)/2)*T + exp(YT)*W ); % exact stock
        realizations at final time
    ST((i-1)*length(steps)+j) = mean(ST_j);
    fT((i-1)*length(steps)+j) = exp(-r*T)*mean(max(ST_j - K, 0));

    errorS_ks{j} = S_hat - St;
    errorf_ks{j} = f - ft;

    MSE_S_j(j) = var(errorS_ks{j});
    MSE_f_j(j) = var(errorf_ks{j});
    err_S_j(j) = mean(errorS_ks{j});
    err_f_j(j) = mean(errorf_ks{j});
    N_ov_ssq_j(j) = (10^samples(j))/MSE_f_j(j)
    duration_j(j) = toc(tStart);
    S_j(j) = S_hat(end);
    f_j(j) = f(end);
end

MSEs_S{i} = MSE_S_j;
MSEs_f{i} = MSE_f_j;
errs_S{i} = err_S_j;
errs_f{i} = err_f_j;
```

```
N_ov_ssqs{i} = N_ov_ssq_j;
durations{i} = duration_j;
all_Ss{i} = S_j;
all_fs{i} = f_j;
end
toc(tStart)

figure;
plot(t(2:end),S_hat,'*-',t(2:end),St,'o-')
legend('Approx','Analyt','Location','northwest')
figure;
plot(t(2:end),f,'*-',t(2:end),ft,'o-')
legend('Approx','Analyt','Location','northwest')
figure;
plot(t(2:end),Y_hat,'*-',t(2:end),Yt,'o-')
legend('Approx','Analyt','Location','northwest')

figure; % stock price error over time (step size = 1/512)
plot(t(2:end),errorS_ks{1},'o--',t(2:end),errorS_ks{2},'*--',t(2:end),errorS_ks{3},'--
legend('N=1,000','N=10,000','N=100,000','N=1,000,000','Location','northwest')

figure; % option price error over time (step size = 1/512)
plot(t(2:end),errorf_ks{1},'o--',t(2:end),errorf_ks{2},'*--',t(2:end),errorf_ks{3},'--
legend('N=1,000','N=10,000','N=100,000','N=1,000,000','Location','northwest')

figure; % computational time over step & sample sizes
plot(samples,log(durations{1}),'o--',samples,log(durations{2}),'*--',samples,log(durat
legend('N=1,000','N=10,000','N=100,000','N=1,000,000','Location','northwest')

figure; % ratio of N to sample variance time over step & sample sizes
plot(samples,log10(N_ov_ssqs{1}),'o--',samples,log10(N_ov_ssqs{2}),'*--',samples,log10
legend('N=1,000','N=10,000','N=100,000','N=1,000,000','Location','northwest')
```