Exercises

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The following are exercises from the book of Grinstead and Snell [2006].

Exercise 1 (Ex. 1, p. 392)**.** *Let $Z_1, Z_2, \ldots, Z_N$ describe a branching process in which each parent has j offspring with probability p_j. Find the probability d that the process eventually dies out if*

1. $p_0 = 1/2, p_1 = 1/4, p_2 = 1/4$.
2. $p_0 = 1/3, p_1 = 1/3, p_2 = 1/3$.
3. $p_0 = 1/3, p_1 = 0, p_2 = 2/3$.
4. $p_j = 1/2^{j+1}$, *for* $j = 0, 1, 2, \ldots$.
5. $p_j = (1/3)(2/3)^j$, *for* $j = 0, 1, 2, \ldots$.
6. $p_j = e^{-2}2^j/j!$, *for* $j = 0, 1, 2, \ldots$ *(estimate d numerically).*

Solution to 1.1. We have $m = \sum kp_k = 0(1/2) + 1(1/4) + 2(1/4) = 3/4 \leq 1$ so $d = 1$. $\square$

Solution to 1.2. Here, $m = \sum kp_k = 0(1/3) + 1(1/3) + 2(1/3) = 1 \leq 1$ so $d = 1$. $\square$

Solution to 1.3. Since $m = \sum kp_k = 0(1/3)+0(0)+2(2/3) = 4/3 > 1$, to find d we need to compute the roots of $h(x) = x$, where $h(x) = \sum p_k x^k$. In this case, $h(x) = (1/3) + (2/3)x^2$, and

$$\begin{aligned}
(1/3) + (2/3)x^2 = x &\Leftrightarrow 1 + 2x^2 = 3x && (1)\\
&\Leftrightarrow 2x^2 - 3x + 1 = 0 && (2)\\
&\Leftrightarrow (2x-1)(x-1) = 0 && (3)\\
&\Leftrightarrow x = 1/2, x = 1, && (4)
\end{aligned}$$

so $d = 1/2$. $\square$

Solution to 1.4. Knowing $\sum_{k=1}^{\infty} kx^k = \frac{x}{(1-x)^2}$ and $\sum_{k=1}^{\infty} x^k = \frac{x}{1-x}$ when $|x| < 1$, we see that

$$\begin{aligned}
m = \sum_{j=0}^{\infty} j\left(\frac{1}{2^{j+1}}\right) &= \sum_{j=1}^{\infty}(j-1)\left(\frac{1}{2^j}\right) && (5)\\
&= \sum_{j=1}^{\infty}\frac{j}{2^j} - \sum_{j=1}^{\infty}\frac{1}{2^j} && (6)\\
&= 2 - 1 = 1. && (7)
\end{aligned}$$

Given how it is equal to one, we conclude $d = 1$. $\square$

Solution to 1.5. Proceeding as before, we see that

$$\begin{aligned}
m &= \sum_{j=0}^{\infty} j(1/3)(2/3)^j && (8)\\
&= (1/3)\frac{2/3}{(1-(2/3))^2} && (9)\\
&= (1/3)\frac{(2/3)}{(1/3)^2} = \frac{(2/3)}{(1/3)} = 2. && (10)
\end{aligned}$$

Again, having $m = 2 > 1$, we must find x such that

$$\sum_{j=0}^{\infty}(1/3)(2/3)^j x^j = x. \tag{11}$$

Rearranging the terms, we must find x such that

$$(1/3)\sum_{j=0}^{\infty}(\frac{2}{3}x)^j = x. \tag{12}$$

Given that $|x| < 1$, then $|\frac{2}{3}x| < 1$ and

$$\sum_{j=0}^{\infty}(\frac{2}{3}x)^j = \frac{1}{1-(\frac{2}{3}x)}, \tag{13}$$

so we need x such that

$$(1/3)\frac{1}{1-(\frac{2}{3}x)} = (1/3)\left(\frac{3}{3-2x}\right) = \frac{1}{3-2x} = x. \tag{14}$$

This x is found by solving the equation $2x^2 - 3x + 1 = 0$, which we did in the solution to 1.3. Therefore $d = 1/2$. □

Solution to 1.6. Starting by finding m one gets

$$\begin{aligned} m &= \sum_{j=0}^{\infty}\frac{e^{-2}2^j}{j!}j && (15)\\ &= e^{-2}\sum_{j=0}^{\infty}\frac{2^j}{j!}j && (16)\\ &= e^{-2}\sum_{j=1}^{\infty}2\frac{2^{j-1}}{(j-1)!} && (17)\\ &= 2e^{-2}\sum_{j=1}^{\infty}\frac{2^{j-1}}{(j-1)!} && (18)\\ &= 2e^{-2}\sum_{j=0}^{\infty}\frac{2^j}{j!} && (19)\\ &= 2e^{-2}e^2 = 2. && (20) \end{aligned}$$

Since $m > 1$, to obtain d we must find x such that

$$\sum_{j=0}^{\infty}\frac{e^{-2}2^j}{j!}x^j = x. \tag{21}$$

Given that

$$\sum_{j=0}^{\infty}\frac{e^{-2}2^j}{j!}x^j = e^{-2}\sum_{j=0}^{\infty}\frac{2^j}{j!}x^j = e^{-2}e^{2x} = e^{2x-2}, \tag{22}$$

we numerically estimate with Wolfram Alpha [Wolfram Research Inc.] that $e^{2x-2} = x$ for $x = 1$ and $x \approx 0.203$, so $d \approx 0.203$. □

Exercise 2 (Ex. 3, p. 392)**.** *In the chain letter problem (see Example 10.14) find your expected profit if*

1. $p_0 = 1/2, p_1 = 0,$ *and* $p_2 = 1/2$.
2. $p_0 = 1/6, p_1 = 1/2,$ *and* $p_2 = 1/3$.

Solution to 2.1. We see in Grinstead and Snell [2006] that the expected profit is $50m + 50m^{12} - 100$, where $m = p_1 + 2p_2$. Here, $m = 0 + 2(1/2) = 1$, so the expected profit is $50(1) + 50(1)^{12} - 100 = 100 - 100 = 0$. □

Solution to 2.2. Here, $m = (1/2) + 2(1/3) = 7/6$, so the expected profit is $50(7/6) + 50(7/6)^{12} - 100 \approx 276.26$. If $p_0 > 1/2$, then $p_1 + p_2 < 1/2$, so $2p_1 + 2p_2 < 1$ and $p_1 + 2p_2 < 1 - p_1 \leq 1$. Therefore, $50(p_1 + 2p_2) < 50$ and $50(p_1 + 2p_2)^{12} < 50$, so $50(p_1 + 2p_2) + 50(p_1 + 2p_2)^{12} < 100$, so the expected profit is negative. □

Exercise 3 (Ex. 3, p. 401)**.** *Let X be a continuous random variable with values in $[0,2]$ and density f_X. Find the moment generating function $g(t)$ for X if*

1. $f_X(x) = 1/2$.
2. $f_X(x) = (1/2)x$.
3. $f_X(x) = 1 - (1/2)x$.
4. $f_X(x) = |1 - x|$.
5. $f_X(x) = (3/8)x^2$.

Solution. We calculate these with the equation $g(t) = \int_{-\infty}^{\infty} e^{tx} f_X(x)\,dx$. In this particular case, since the random variable has values in $[0,2]$, the moment generating function will be $g(t) = \int_0^2 e^{tx} f_X(x)\,dx$. For each of the densities f_X from number 1. to 5., the corresponding generating function will be denoted by $g_k(t), k = 1, 2, \ldots, 5$. First,

$$\begin{aligned}
g_1(t) &= \int_0^2 e^{tx}(1/2)\,dx && (23)\\
&= (1/2)\int_0^2 e^{tx}\,dx && (24)\\
&= (1/2)[(1/t)e^{tx}\Big|_0^2] && (25)\\
&= (1/2)[(1/t)e^{2t} - (1/t)e^0] && (26)\\
&= (1/2)[\frac{e^{2t}-1}{t}] && (27)\\
&= \frac{1}{2t}(e^{2t} - 1) && (28)
\end{aligned}$$

is obtained. Then we compute

$$\begin{aligned}
g_2(t) &= \int_0^2 e^{tx}(1/2)x\,dx && (29)\\
&= (1/2)[\frac{1}{t}xe^{tx}\Big|_0^2 - \int_0^2 \frac{1}{t}e^{tx}\,dx\ ] && (30)\\
&= (1/2)[\frac{1}{t}xe^{tx}\Big|_0^2 - \frac{1}{t^2}e^{tx}\Big|_0^2] && (31)\\
&= (1/2)[(\frac{1}{t}2e^{2t} - \frac{1}{t^2}e^{2t}) - (0 - \frac{1}{t^2})] && (32)\\
&= (1/2)[\frac{2}{t}e^{2t} - \frac{1}{t^2}e^{2t} + \frac{1}{t^2}] && (33)\\
&= \frac{1}{t}e^{2t} - \frac{1}{2t^2}e^{2t} + \frac{1}{2t^2}\ . && (34)\\
& && (35)
\end{aligned}$$

For the third, one gets

$$\begin{aligned}
g_3(t) &= \int_0^2 e^{tx}(1 - (1/2)x)\,dx && (36)\\
&= \int_0^2 e^{tx}\,dx - \int_0^2 (1/2)xe^{tx}\,dx && (37)\\
&= \frac{1}{t}e^{tx}\Big|_0^2 - (\frac{1}{t}e^{2t} - \frac{1}{2t^2}e^{2t} + \frac{1}{2t^2}) && (38)\\
&= \frac{1}{t}e^{2t} - \frac{1}{t} - \frac{1}{t}e^{2t} + \frac{1}{2t^2}e^{2t} - \frac{1}{2t^2} && (39)\\
&= \frac{1}{2t^2}e^{2t} - \frac{1}{2t^2} - \frac{1}{t}. && (40)\\
& && (41)
\end{aligned}$$

Afterwards, we proceed to compute

$$g_4(t) = \int_0^2 |1 - x|e^{tx}\,dx = \int_0^1 (1 - x)e^{tx}\,dx + \int_1^2 (x - 1)e^{tx}\,dx. \qquad (42)$$

Computing the integrals one gets

$$
\begin{aligned}
\int_0^1 (1-x)e^{tx}\,dx &= \int_0^1 e^{tx}\,dx - \int_0^1 xe^{tx}\,dx && (43)\\
&= \frac{1}{t}e^{tx}\Big|_0^1 - [\frac{1}{t}xe^{tx} \ - \frac{1}{t^2}e^{tx}]\Big|_0^1 && (44)\\
&= (\frac{1}{t}e^t - \frac{1}{t}) - (\frac{1}{t}e^t - \frac{1}{t^2}e^t + \frac{1}{t^2}) && (45)\\
&= \frac{1}{t^2}e^t - \frac{1}{t^2} - \frac{1}{t}, && (46)
\end{aligned}
$$

and

$$
\begin{aligned}
\int_1^2 (x-1)e^{tx}\,dx &= \int_1^2 xe^{tx}\ \ dx - \int_1 2e^{tx}\,dx && (47)\\
&= (\frac{1}{t}xe^{tx} - \frac{1}{t^2}e^{tx})\Big|_1^2 - \frac{1}{t}e^{tx}\Big|_1^2 && (48)\\
&= \frac{2}{t}e^{2t} - \frac{1}{t^2}e^{2t} - \frac{1}{t}e^t + \frac{1}{t^2}e^t - \frac{1}{t}e^{2t} + \frac{1}{t}e^t && (49)\\
&= \frac{1}{t}e^{2t} - \frac{1}{t^2}e^{2t} + \frac{1}{t^2}e^t. && (50)\\
& && (51)
\end{aligned}
$$

Therefore

$$
\begin{aligned}
g(t) &= (\frac{1}{t^2}e^t - \frac{1}{t^2} - \frac{1}{t}) + (\frac{1}{t}\ e^{2t} - \frac{1}{t^2}e^{2t} + \frac{1}{t^2}e^t) && (52)\\
&= (\frac{1}{t} - \frac{1}{t^2})e^{2t} + \frac{2}{t^2}e^t - \frac{1}{t^2} - \frac{1}{t}. && (53)
\end{aligned}
$$

Finally, we compute $g_5(t)$ as

$$
\begin{aligned}
g_5(t) &= \int_0^2 (3/8)x^2e^{tx}\,dx && (54)\\
&= (3/8)\int_0^2 x^2e^{tx}\,dx && (55)\\
&= (3/8)\left((\frac{x^2}{t}\ e^{tx})\Big|_0^2 - \int_0^2 2x(\frac{1}{t}e^{tx})\,dx \right) && (56)\\
&= (3/8)\left(\frac{4}{t}e^{2t} - \frac{2}{t}(\frac{2}{t}e^{2t} - \frac{1}{t^2}e^{2t} + \frac{1}{t^2}) \right) && (57)\\
&= (3/8)\left(\frac{4}{t}e^{2t} - \frac{4}{t^2}e^{2t} + \frac{2}{t^3}e^{2t} - \frac{2}{t^3} \right) && (58)\\
&= \frac{3}{2t}e^{2t} - \frac{3}{2t^2}e^{2t} + \frac{3}{4t^3}e^{2t} - \frac{3}{4t^3}. && (59)
\end{aligned}
$$

□

Exercise 4 (Ex. 6, p. 402)**.** *Let X be a continuous random variable whose characteristic function $k_X(\tau)$ is*

$$k_X(\tau) = e^{-|\tau|}, \quad -\infty < \tau < \tau.$$

Show directly that the density f_X of X is

$$f_X(x) = \frac{1}{\pi(1+x^2).}$$

Proof. We know that

$$f_X(x) = \frac{1}{2\ \pi}\int_{-\infty}^{\infty} e^{-i\tau x}k_X(\tau)\,d\tau, \tag{60}$$

where we have

$$k_X(\tau) = e^{-|\tau|} = \begin{cases} e^{\tau} & \tau \le 0; \\ e^{-\tau} & \tau > 0. \end{cases} \tag{61}$$

Additionally, we know that $e^{i\theta} = \cos\theta + i\sin\theta$. Thus, substituting this in Equation 60 we get

$$\begin{aligned}
f_X(x) &= \frac{1}{2\pi}\int_{-\infty}^{\infty}(\cos(-\tau x) + i\sin(-\tau x))k_X(\tau)\,d\tau && (62)\\
&= \frac{1}{2\pi}\left[\int_{-\infty}^{\infty}\cos(-\tau x)k_X(\tau)\,d\tau + i\int_{-\infty}^{\infty}\sin(-\tau x)k_X(\tau)\,d\tau\right] && (63)\\
&= \frac{1}{2\pi}\left[\left\{\int_{-\infty}^{0}\cos(-\tau x)e^{\tau}\,d\tau + \int_{0}^{\infty}\cos(-\tau x)e^{-\tau}\,d\tau\right\} + i\left\{\int_{-\infty}^{0}\sin(-\tau x)e^{\tau}\,d\tau + \int_{0}^{\infty}\sin(-\tau x)e^{-\tau}d\tau\right\}\right] && (64)
\end{aligned}$$

Using the fact that the cosine function is even, changing variables and rearranging the integration limits, we see that

$$\int_{-\infty}^{0}\cos(-\tau x)e^{\tau}\,d\tau = \int_{0}^{\infty}\cos(\tau x)e^{-\tau}\,d\tau \tag{65}$$

and

$$\int_{0}^{\infty}\cos(-\tau x)e^{-\tau}\,d\tau = \int_{0}^{\infty}\cos(\tau x)e^{-\tau}\,d\tau. \tag{66}$$

Similarly, seeing how sine is odd, we get

$$\int_{-\infty}^{0}\sin(-\tau x)e^{\tau}\,d\tau = \int_{0}^{\infty}\sin(\tau x)e^{-\tau}\,d\tau \tag{67}$$

and

$$\int_{0}^{\infty}\sin(-\tau x)e^{-\tau}\,d\tau = -\int_{0}^{\infty}\sin(\tau x)e^{-\tau}\,d\tau. \tag{68}$$

Therefore,

$$\begin{aligned}
f_X(x) &= \frac{1}{2\pi}\left[\left\{\int_{0}^{\infty}\cos(\tau x)e^{-\tau}\,d\tau + \int_{0}^{\infty}\cos(\tau x)e^{-\tau}\,d\tau\right\} + i\left\{\int_{0}^{\infty}\sin(\tau x)e^{-\tau}\,d\tau - \int_{0}^{\infty}\sin(\tau x)e^{-\tau}\,d\tau\right\}\right] && (69)\\
&= \frac{1}{2\pi}2\int_{0}^{\infty}\cos(\tau x)e^{-\tau}\,d\tau, && (70)
\end{aligned}$$

where integrating by parts twice we get

$$\begin{aligned}
\int_{0}^{\infty}\cos(\tau x)e^{-\tau}\,d\tau &= -e^{-\tau}\cos(\tau x)\Big|_{0}^{\infty} - \int_{0}^{\infty}x\sin(\tau x)e^{-\tau}\;\,d\tau && (71)\\
&= -e^{-\tau}\cos(\tau x)\Big|_{0}^{\infty} + x\sin(\tau x)e^{-\tau}\Big|_{0}^{\infty} - \int_{0}^{\infty}x^2\cos(\tau x)e^{-\tau}\,dt && (72)\\
&= \frac{x\sin(-\tau x) - \cos(\tau x)}{1+x^2}e^{-\tau}\Big|_{0}^{\infty} && (73)\\
&= 0 - \frac{x\sin 0 - \cos 0}{1+x^2} && (74)\\
&= \frac{1}{1+x^2}, && (75)
\end{aligned}$$

therefore

$$f_X(x) = \frac{1}{\pi(x^2+1)}. \tag{76}$$

□

Exercise 5 (Ex. 10, p. 403)**.** *Let $X_1, \ldots, X_n$ be an independent trials process with density*

$$f(x) = \frac{1}{2}e^{-|x|}, \quad -\infty < x < \infty.$$

1. *Find the mean and variance of $f(x)$.*
2. *Find the moment generating function for X_1, S_n, A_n, S_n^*.*
3. *What can you say about the moment generating function of S_n^* as $n \to \infty$.*
4. *What can you say about the moment generating function of A_n as $n \to \infty$.*

Solution to 5.1. We know from Grinstead and Snell [2006] that

$$k_X(\tau) = g_X(i\tau) = \int_{-\infty}^{\infty} e^{i\tau x} f_X(x)dx = \frac{1}{2}\int_{-\infty}^{\infty} e^{i\tau x} e^{-|x|}\, dx. \tag{77}$$

Making a $u = -x$ substitution in the integral from Exercise 4 one sees that

$$\int_{-\infty}^{\infty} e^{i\tau x} e^{-|x|}\, dx = \int_{-\infty}^{\infty} e^{-i\tau x} e^{-|x|}\, dx = \frac{2}{1+\tau^2}, \tag{78}$$

so $k_X(\tau) = \frac{1}{1+\tau^2}$, and from the relation $k_X(\tau) = g_X(i\tau)$, one sees $g_X(t) = \frac{1}{1-t^2}$. Knowing this, the mean is obtained as

$$\left.\frac{dg_X(t)}{dt}\right|_{t=0} = \left.\frac{2t}{(1-t^2)^2}\right|_{t=0} = 0, \tag{79}$$

and the variance as

$$\left.\frac{d^2 g_X(t)}{dt^2}\right|_{t=0} = \left.\left[\frac{2}{(1-t^2)^2} + \frac{8t^2}{(1-t^2)^3}\right]\right|_{t=0} = 2. \tag{80}$$

□

Solution to 5.2. The moment generating function for X_1, and the rest of the X_i, is as obtained in the solution to 5.1, and is $g_X(t) = \frac{1}{1-t^2}$. Then,

$$\begin{aligned}
g_{S_n}(t) = \mathbb{E}\left[e^{S_n t}\right] &= \mathbb{E}\left[e^{(X_1+\cdots+X_n)t}\right] &(81)\\
&= \mathbb{E}\left[e^{X_1 t} e^{X_2 t} \cdots e^{X_n t}\right] &(82)\\
&= \mathbb{E}\left[e^{X_1 t}\right] \cdots \mathbb{E}\left[e^{X_n t}\right] \quad \text{(because of independence)} &(83)\\
&= g_{X_1}(t) \cdots g_{X_n}(t) &(84)\\
&= \left(\frac{1}{1-t^2}\right)^n. &(85)
\end{aligned}$$

For $A_n = S_n/n$, see that

$$g_{A_n}(t) = g_{\frac{S_n}{n}} = \mathbb{E}\left[e^{\frac{S_n}{n} t}\right] = \mathbb{E}\left[e^{S_n \frac{t}{n}}\right] = g_{S_n}(t/n), \tag{86}$$

so

$$g_{A_n}(t) = \left(\frac{1}{1-(\frac{t}{n})^2}\right)^n. \tag{87}$$

For $S_n^* = \frac{S_n - n\mu}{\sqrt{n\sigma^2}} = \frac{S_n}{\sqrt{2n}}$ (substituting the mean and variance found), proceeding as was done with A_n, we obtain

$$g_{S_n^*}(t) = g_{\frac{S_n}{\sqrt{2n}}}(t) = g_{S_n}(t/\sqrt{2n}) = \left(\frac{1}{1-(\frac{t}{\sqrt{2n}})^2}\right)^n. \tag{88}$$

□

Solution to 5.3. Here, and in the solution of Exercise 5.4, the following result found in Casella and Berger [2002], will be used: *If a sequence* (a_n) *converges to* a*, then* $(1+\frac{a_n}{n})^n$ *converges to* e^a. To find the limit of $g_{S_n^*}$ as $n \longrightarrow \infty$, see that

$$\lim_{n\to\infty}\left(\frac{1}{1-(\frac{t}{\sqrt{2n}})^2}\right)^n = \frac{\lim_{n\to\infty} 1^n}{\lim_{n\to\infty}(1-(\frac{t}{\sqrt{2n}})^2)^n} \tag{89}$$

if both limits exist. Given how

$$1 - \left(\frac{t}{\sqrt{2n}}\right)^2 = 1 + \left(\frac{-t^2}{2n}\right) = 1 + \left(\frac{\frac{-t^2}{2}}{n}\right), \tag{90}$$

and the sequence $a_n = -t^2/2$ converges to $-t^2/2$ as n goes to infinity, the aforementioned theorem tells us

$$\lim_{n\to\infty}\left(1-(\frac{t}{\sqrt{2n}})^2\right)^n = \lim_{n\to\infty}\left(1+\left(\frac{\frac{-t^2}{2}}{n}\right)\right)^n = e^{\frac{-t^2}{2}}, \tag{91}$$

so

$$\lim_{n\to\infty} g_{S_n^*} = \lim_{n\to\infty}\left(\frac{1}{1-(\frac{t}{\sqrt{2n}})^2}\right)^n = \frac{\lim_{n\to\infty} 1^n}{\lim_{n\to\infty}(1-(\frac{t}{\sqrt{2n}})^2)^n} = \frac{1}{e^{\frac{-t^2}{2}}} = e^{\frac{t^2}{2}}. \tag{92}$$

□

Solution to 5.4. As before, we see that

$$\lim_{n\to\infty}\left(\frac{1}{1-(\frac{t}{n})^2}\right)^n = \frac{\lim_{n\to\infty} 1^n}{\lim_{n\to\infty}\left(1-(\frac{t}{n})^2\right)^n} \tag{93}$$

if both limits exist, and

$$1-(\frac{t}{n})^2 = 1+\left(\frac{-t^2}{n^2}\right) = 1+\frac{\frac{-t^2}{n}}{n}. \tag{94}$$

The sequence $a_n = -t^2/n$ converges to zero, so

$$\lim_{n\to\infty}\left(1-\left(\frac{t}{n}\right)^2\right)^n = \lim_{n\to\infty}\left(1+\frac{\frac{-t^2}{n}}{n}\right)^n = e^0 = 1, \tag{95}$$

which gives

$$\lim_{n\to\infty} g_{A_n} = \lim_{n\to\infty}\left(\frac{1}{1-(\frac{t}{n})^2}\right)^n = \frac{\lim_{n\to\infty} 1^n}{\lim_{n\to\infty}\left(1-(\frac{t}{n})^2\right)^n} = 1. \tag{96}$$

□

References

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