

Law of large numbers

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Abstract

In this work, the law of large numbers is presented, along some applications of the same.

1 Introduction

Two theorems will be given, known as the weak and strong law of large numbers. Both of the statements, as well as the definitions preceding it, can be found in the work of [Casella and Berger \[2002\]](#).

Definition 1. A sequence of random variables X_1, X_2, \dots , converges in probability to a random variable X if, for every $\varepsilon > 0$,

$$\lim_{n \rightarrow \infty} \mathbb{P}(|X_n - X| \geq \varepsilon) = 0 \text{ or, equivalently } \lim_{n \rightarrow \infty} \mathbb{P}(|X_n - X| < \varepsilon) = 1 \quad (1)$$

Definition 2. A sequence of random variables X_1, X_2, \dots , converges almost surely to a random variable X if, for every $\varepsilon > 0$,

$$\mathbb{P}\left(\lim_{n \rightarrow \infty} |X_n - X| < \varepsilon\right) = 1 \quad (2)$$

Theorem 1 (Weak law of large numbers). Let X_1, X_2, \dots be independent, identically distributed random variables with $\mathbb{E}[X_i] = \mu$ and $\text{Var}(X_i) = \sigma^2 < \infty$. Then, $\bar{X}_n = (1/n) \sum_{i=1}^n X_i$ converges in probability to μ .

Theorem 2 (Strong law of large numbers). Let X_1, X_2, \dots be independent, identically distributed random variables with $\mathbb{E}[X_i] = \mu$ and $\text{Var}(X_i) = \sigma^2 < \infty$. Then, $\bar{X}_n = (1/n) \sum_{i=1}^n X_i$ converges almost surely to μ .

2 Renewal processes

One of the many applications of the law of large numbers is in a kind of stochastic process called renewal processes. The contents of this section can be found in the book of [Lawler \[2006\]](#).

Definition 3. Let T_1, T_2, \dots be independent, identically distributed, nonnegative random variables with distribution $F(x)$. The renewal process associated with $\{T_i\}$ is the process that count the number of events that have occurred by time t . More precisely, the renewal process N_t is defined by

$$N_t = \begin{cases} 0 & t < T_1; \\ \max\{n : T_1 + \dots + T_n \leq t\} & \text{else.} \end{cases} \quad (3)$$

The random variables T_i are thought of as being the lifetimes of a component, or as the times between occurrences of some event. Time 0 is assumed as the beginning of a lifetime. The random variables T_i are assumed to have finite, positive mean $\mu = \mathbb{E}[T_i]$.

Example 1. Consider the Poisson process with waiting times T_1, T_2, \dots . These are independent, exponential random variables with parameter λ and N_t is the Poisson process.

Example 2. Consider a queue with a single server where customers arrive according to a Poisson process with rate λ . Assume the service times for customers are independent, identically distributed random variables with mean μ . Let Y_t denote the number of people in the queue at time t . Let

$$R_1 = \inf\{t > 0 : Y_t = 1\}, \quad (4)$$

$$S_1 = \inf\{t > 0 : Y_{R_1+t} = 0\}, \quad (5)$$

$$T_1 = R_1 + S_1, \quad (6)$$

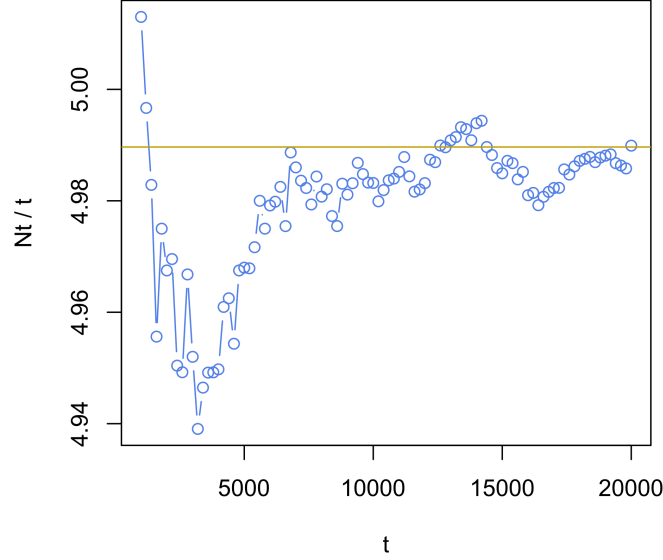


Figure 1: In blue, the value of N_t/t . In gold, the value $1/\mu$.

and for $i > 1$,

$$R_i = \inf\{t > 0 : Y_{T_1+\dots+T_{i-1}+t} = 1\}, \quad (7)$$

$$S_i = \inf\{t > 0 : Y_{T_1+\dots+T_{i-1}+R_i+t} = 0\}, \quad (8)$$

$$T_i = R_i + S_i. \quad (9)$$

The time represented by the R_i can be thought of as idle times, and the time represented by S_i as the busy times. The random variables T_1, T_2, \dots give rise to a renewal process N_t .

Using the Strong Law of Large Numbers (Theorem 2), the following analogue of the law of large numbers for renewal processes can be derived.

Proposition 1. For a renewal process N_t , with probability one,

$$\lim_{t \rightarrow \infty} \frac{N_t}{t} = \frac{1}{\mu}. \quad (10)$$

A Poisson process (see Example 1) was simulated¹ with R version 4.0.0 in a Jupyter notebook [R Core Team, 2020, Kluyver et al., 2016]. In Figure 1 it is observed how $N(t)/t \rightarrow 1/\mu$ as t grows.

3 Epidemics on networks

In the work of Janson et al. [2014], a law of large numbers is established for an epidemic process on a random graph. In particular, a random graph with a given degree sequence is considered. Given the graph, the epidemic evolves as a continuous-time Markov chain [Liggett, 2010] where each infectious vertex recovers at rate $\rho \geq 0$ and also infects each susceptible neighbor at a rate $\beta > 0$. The graph has n vertices, of which initially n_S, n_I, n_R are susceptible, infectious and recovered respectively. It is assumed the fractions of initially susceptible, infectious and recovered vertices converge to some $\alpha_S, \alpha_R, \alpha_I \in [0, 1]$, with $\alpha_S > 0$. Additionally, it is assumed the degree of a randomly chosen susceptible vertex converges to a probability distribution $(p_k)_0^\infty$, that is,

$$n_{S,k}/n \rightarrow p_k, \quad (11)$$

where $n_{S,k}$ is the number of susceptible vertices of degree k . Furthermore, the average degree over all vertices converges to $\mu > 0$. Let S_t, I_t, R_t be the number of susceptible, infectious and recovered vertices at time t , and T_0 the time in which

¹The code, as well as this report, can be found in <https://github.com/palafox794/AppliedProbabilityModels/tree/master/Assignment13>.

the fraction of susceptible individuals has fallen from about α_S to some fixed smaller s_0 . Janson et al. [2014] prove that when the quantity

$$R_0 := \left(\frac{\beta}{\beta + \rho} \right) \left(\frac{\alpha_S}{\mu} \right) \sum_{k=0}^{\infty} (k-1) k p_k, \quad (12)$$

interpreted as the basic reproduction number of the epidemic, is greater than one, for every $\varepsilon > 0$,

$$\mathbb{P}(T_0 = \infty \text{ and } S_0 - S_\infty > \varepsilon n) \rightarrow 0. \quad (13)$$

This represents a small outbreak. However, it is also proved that if $T_0 < \infty$, the outbreak is large.

3.1 Vaccination

In their work, Janson et al. [2014] also study the effects of vaccinations. A perfect vaccine is assumed, so a vaccinated vertex never becomes infected and in practice behaves as recovered. It is assumed each initially susceptible vertex of degree k is vaccinated with probability $\pi_k \in [0, 1)$, independently of the others. As an example, consider the *uniform vaccination* strategy. Here, every susceptible vertex is vaccinated with the same probability $v \in [0, 1)$, independently of all the others. Using the law of large numbers, it is proven that V/n_S converges in probability to v , where V is the total number V of vaccinations.

References

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