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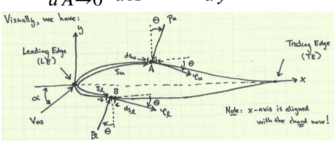
Concepts

Variables

Pressure - Normal force per unit area exerted on a surface due to the time rate of change of momentum of the gas particles impacing said surface.
$P = \lim_{dA \to 0} \frac{dF_n}{dA}$
Density - Mass per unit volume.
$\rho = \lim_{dV \to 0} \frac{dm}{dV}$
Temperature - Change in kinetic energy due to random molecular motion.
$\text{KE} = \frac{3}{2}kT$
Flow Velocity - Velocity due to organized motion.
$\vec{v} = \frac{d}{dt}\vec{x}$

Forces and Moments

Causes of forces and moments on an aerodynamic body:
Pressure Distribution over the body surface.
Shear Stress Distribution over the surface:
$\tau = \lim_{dA \to 0} \frac{dF_f}{dA} = \mu \frac{dV}{dy}$



Lift and Drag per Unit Span:
$L' = N' \cos\alpha - A' \sin\alpha$
$D' = N' \sin\alpha + A' \cos\alpha$
Normal Force per Unit Span:
$N' = \int_{LE}^{TE}(P_u \cos\theta + \tau_u \sin\theta)ds_u + \int_{LE}^{TE}(P_l \cos\theta - \tau_l \sin\theta)ds_l$
Axial Force per Unit Span:
$A' = \int_{LE}^{TE}(-P_u \sin\theta + \tau_u \cos\theta)ds_u + \int_{LE}^{TE}(P_l \sin\theta + \tau_l \cos\theta)ds_l$
Moment per Unit Span:
$M'_* = \int_{LE}^{TE}[(P_u \cos\theta + \tau_u \sin\theta)(x - x^*) - (P_u \sin\theta - \tau_u \cos\theta)(y - y*)]ds_u + \int_{LE}^{TE}[(-P_l \cos\theta + \tau_u \sin\theta)(x - x^*) + (P_l \sin\theta + \tau_l \cos\theta)(y - y*)]ds_l$
When $x^* = y^* = 0$ we obtain the moment about the leading edge.
Center of Pressure:
$x_{cp} = -\frac{M'_{LE}}{N'}$

Force and Moment Coefficients

Dynamic Pressure: $q = \frac{1}{2}\rho_\infty v_\infty^2$
Reference Area: S
Reference Length: L or c
Lift Coefficient: $c_L = \frac{L}{q_\infty S}$
Drag Coefficient: $c_D = \frac{D}{q_\infty S}$
Moment Coefficient: $c_M = \frac{M}{q_\infty SL}$

Buckingham Pi Theorem

If we have a physically meaningful equation such as:
$f(P_1, P_2, ..., P_N) = 0$
where P_i are physical variables in terms of K independent physical units (mass, length, time, temperature) then it can be restated as:
$F(\Pi_1, \Pi_2, ..., \Pi_{N-K}) = 0$ or
$\Pi_{N-K} = G(\Pi_1, \Pi_2, ..., \Pi_{N-K-1})$
where Π_i are dimensionaless variables known as Pi Groups.

Basic Flow Equations

Fixed control volumes for basic flow equations.

Conservation of Mass

Net mass flow out of $\mathcal{V}$ = Time rate of decrease of mass inside $\mathcal{V}$.
Integral Continuity Equation:
$\frac{\partial}{\partial t}\iiint_{\mathcal{V}} \rho d\mathcal{V} + \oiint_S \rho\vec{V} \cdot d\vec{S} = 0$
Differential Continuity Equation:
$\frac{\partial\rho}{\partial t} + \vec{\nabla} \cdot (\rho\vec{V}) = 0$

Conservation of Momentum

Net force applied to $\mathcal{V}$ = Time rate of change of momentum of fluid in $\mathcal{V}$
$\frac{\partial}{\partial t}\iiint_{\mathcal{V}} \rho\vec{V} d\mathcal{V} + \oiint_S(\rho\vec{V} \cdot d\vec{S})\vec{V} = -\oiint_S p d\vec{S} + \iiint_{\mathcal{V}} \rho\vec{f} d\mathcal{V} + \vec{F}_{viscous}$

1. Body Forces (e.g. gravity) which act on the fluid inside $\mathcal{V}$
   $\iiint_{\mathcal{V}} \rho\vec{f} d\mathcal{V}$
2. Pressure and shear surface forces acting on S
   $-\oiint_S p d\vec{S} + \vec{F}_{viscous}$
3. Net flow of momentume out of volume across S
   $\oiint_S(\rho\vec{V} \cdot d\vec{S})\vec{V}$
4. Time rate of change of momentum due to unsteadiness in volume
   $\frac{\partial}{\partial t}\iiint_{\mathcal{V}} \rho\vec{V} d\mathcal{V}$

Navier-Stokes Equations:
$\frac{\partial}{\partial t}(\rho u) + \vec{\nabla} \cdot (\rho u\vec{V}) = -\frac{\partial p}{\partial x} + \rho f_x + (\mathcal{F}_x)_{viscous}$
$\frac{\partial}{\partial t}(\rho v) + \vec{\nabla} \cdot (\rho v\vec{V}) = -\frac{\partial p}{\partial y} + \rho f_y + (\mathcal{F}_y)_{viscous}$

Conservation of Energy

$$\underbrace{\frac{\partial}{\partial t}\iiint_{\mathcal{V}} \rho\left(e + \frac{V^2}{2}\right)d\mathcal{V}}_{\text{Net Time Rate of Change of Total Energy Due to Unsteadiness in } \mathcal{V}} + \underbrace{\oiint_S \rho\left(e + \frac{V^2}{2}\right)\vec{V} \cdot d\vec{S}}_{\text{Net Rate of Flow of Total Energy Out of } \mathcal{V}} =$$

$$\underbrace{\iiint_{\mathcal{V}} \dot{q}\rho\, d\mathcal{V}}_{\text{Rate of Volumetric Heating}} + \underbrace{\dot{Q}_{viscous}}_{\text{Heat Addition Due to Viscosity}}$$

$$\underbrace{-\oiint_S p\vec{V} \cdot d\vec{S}}_{\text{Rate of Pressure Work}} + \underbrace{\iiint_{\mathcal{V}} \rho\vec{f} \cdot \vec{V} d\mathcal{V}}_{\text{Rate of Work Done by Body Forces}}$$

$$+ \underbrace{\dot{W}_{viscous}}_{\text{Rate of Viscous Work}}$$

$$\frac{\partial}{\partial t}\left(\rho\left(e + V^2/2\right)\right) + \vec{\nabla} \cdot \left(\rho\left(e + V^2/2\right)\right) =$$

$$\rho\dot{q} + \dot{Q}'_{viscous} - \vec{\nabla} \cdot (p\vec{V}) + \rho\vec{f} \cdot \vec{V}$$

$$+ \dot{W}'_{viscous}$$

Kinematics

$\vec{\nabla} \cdot \vec{V}$: time rate of change of volume of a fluid element per unit volume → **Dilation**
Stream Line: Curve everywhere tangent to the velocity.
$\frac{v}{u} = \frac{dy}{dx}$
Path Line: Trajectory of a particle released from a point in time.
Streak Line: Line connecting all particles that have passed through a given point.
For a steady flow the three lines coincide.

Substantial Derivative

Time rate of change of density of a given material fluid element as it moves through space and time.
$\frac{D\rho}{Dt} = \frac{\partial\rho}{\partial t} + \vec{V} \cdot \vec{\nabla}\rho$
Continuity Equation: $\frac{D\rho}{Dt} = -\rho\vec{\nabla} \cdot \vec{V}$
Material Acceleration: $(\frac{Du}{Dt}; \frac{Dv}{Dt}; \frac{Dz}{Dt})$
Navier-Stokes:
$\rho\frac{Du}{Dt} = -\frac{\partial p}{\partial x} + \rho f_x + (\mathcal{F}_x)_{viscous}$ → applicable to y and z as well.

Vorticity and Strain

$\vec{\omega} = \frac{1}{2}\vec{\nabla} \times \vec{V}$ → angular velocity
$\vec{\xi} = \vec{\nabla} \times \vec{V}$ → Vorticity
$\varepsilon_{xy} = \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}$ → Rate of Strain in xy plane.

$\Gamma = -\oint_C \vec{V} \cdot \vec{ds} \rightarrow$ Circulation

Relation between circulation and vorticity: $\vec{xi} \cdot \vec{n} = \frac{d\Gamma}{dS}$

Stream Functions and Velocity Potentials

Stream Function: For every incompressible, two-dimensional flow, there exists a scalar stream function such that:

$u = \frac{\partial \psi}{\partial y}$, $v = -\frac{\partial \psi}{\partial x}$ DON'T FORGET +C

$\vec{\nabla}^2 \psi = 0$

Velocity Potential: For every irrotational flow, there exists a scalar velocity potential such that:

$\vec{V} = \vec{\nabla}\phi \rightarrow \frac{\partial \phi}{\partial x} = y, \frac{\partial \phi}{\partial y} = x$

$\vec{\nabla}^2 \phi = o$

Laplace's Equation: 2nd order linear partial differential equation. Allows **superposition** if both ψ_1 and ψ_2 are soluations to $\vec{\nabla}^2\phi = o$, then $\psi_3 = \psi_1 + \psi_2$ is also a solution.

Finding Solutions:

1) Infinite Bounday Condition: far from the body, velocity is uniform and aligned with x-axis.

$\phi(x, y) = v_\infty x$ and $\psi(x, y) = v_\infty y$

2) Wall boundary: no flow of mass or momentum through solid boundaries.

$\frac{\partial \phi}{\partial n} = 0$ and $\frac{\partial \psi}{\partial n} = 0$

To Find L, D, and M:

1) Solve Laplace's eq. for ϕ w/ boundary conditions.

2) Obtain velocity field: $\vec{V} = \vec{\nabla}\phi$

3) Obtain P: Bernoulli's Equation **Euler's Equation**: $\frac{1}{2}d(V^2) = -\frac{1}{\rho}dp$

If flow is barotropic, $\rho = \rho(p)$ is a function of pressue only and can integrate the equation. Occurs in incompressible flow, isothermal flow, or isentropic (frictionless) flow.

Bernoulli's Equation: $p + \frac{1}{2}\rho V^2 =$ constant - along a streamline for steady, inviscid, incompressible flow with no body forces.

$c_P = 1 - \left(\frac{v}{v_\infty}\right)^2$

Flow in a Duct

$\rho_1 A_1 V_1 = \rho_2 A_2 V_2 \rightarrow$ Steady, inviscid flow in a duct with no body forces. For incompressible flow remove rho.

Elementary Flows & Combinations

Uniform Flow

$\phi(x, y) = v_\infty x$ and $\phi(r, \theta) = v_\infty r \cos\theta$

$\psi(x, y) = v_\infty y$ and $\phi(r, \theta) = v_\infty r \sin\theta$

$\Gamma = 0$

Source/Sink

$\vec{V} = \begin{bmatrix} \frac{\Lambda}{2\pi r} \\ 0 \end{bmatrix}$

$\phi(r, \theta) = \frac{\Lambda \ln(r)}{2\pi}$

$\psi(r, \theta) = \frac{\Lambda\theta}{2\pi}$

$\Gamma = 0$

Superposition

Uniform Flow + Source

$\psi(r, \theta) = v_\infty r \sin\theta + \frac{\Lambda\theta}{2\pi r}$

Equation for Body Shape: $r(\theta) = \frac{\Lambda(\pi - \theta)}{2\pi v_\infty \sin\theta}$ and $h = \frac{\Lambda}{v_\infty}$

Rankine Oval Flow

Uniform Flow + Source + Sink

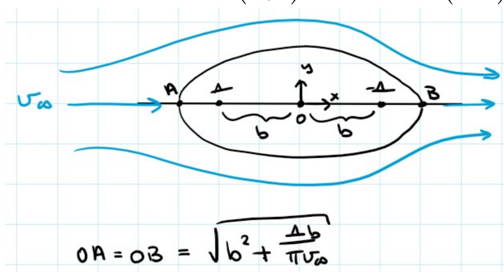
$\psi(r, \theta) = v_\infty r \sin\theta + \frac{\Lambda\theta_1}{2\pi} + \frac{-\Lambda\theta_2}{2\pi}$

$\psi(x, y) = v_\infty y + \frac{\Lambda}{2\pi}\tan^{-1}\left(\frac{y}{x+b}\right) - \frac{\Lambda}{2\pi}\tan^{-1}\left(\frac{y}{x-b}\right)$

$$\vec{V} = \begin{bmatrix} v_\infty + \frac{\Lambda}{2\pi}\frac{x+b}{(x+b)^2+y^2} - \frac{\Lambda}{2\pi}\frac{x-b}{(x-b)^2+y^2} \\ \frac{\Lambda}{2\pi}\frac{y}{(x+b)^2+y^2} - \frac{\Lambda}{2\pi}\frac{y}{(x-b)^2+y^2} \end{bmatrix}$$

Equation of a Rankine Oval:

$0 = v_\infty y + \frac{\Lambda}{2\pi}\tan^{-}1\left(\frac{y}{x+b}\right) - \frac{\Lambda}{2\pi}\tan^{-}1\left(\frac{y}{x-b}\right)$



Doublet

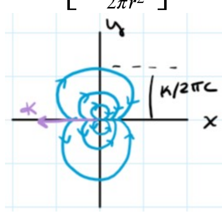
Source + Sink

Rankine Oval $b \rightarrow 0$

$\psi(r, \theta) = \frac{-\kappa \sin\theta}{2\pi r}$

$\phi(r, \theta) = \frac{\kappa\cos\theta}{2\pi r}$

$\vec{V} = \begin{bmatrix} -\frac{\kappa\cos\theta}{2\pi r^2} \\ -\frac{\kappa\sin\theta}{2\pi r^2} \end{bmatrix}$



Non-Lifting Flow over Cylinder

Uniform Flow + Doublet

$\psi(r, \theta) = v_\infty r \sin\theta\left(1 - \frac{R^2}{r^2}\right)$

$$\vec{V} = \begin{bmatrix} v_\infty \cos\theta\left(1 - \frac{R^2}{r^2}\right) \\ -v_\infty \sin\theta\left(1 + \frac{R^2}{r^2}\right) \end{bmatrix}$$

Stagnation points at $v_r = v_\theta = 0$ or $(r, \theta) = (R, 0), (R, \pi)$

$L = D = 0$, when $D = 0$ it is called D'Alembert's Paradox

Vortex Flow

Rotation flow, but the fluid elements themselves do not rotate. They follow a streamline in a circle at a particular radius. Must make $rv_\theta = c$ in order to prevent going to $r = 0$ where flow is not irrotational.

$\vec{V} = \begin{bmatrix} 0 \\ \frac{c}{r} = \frac{-\Gamma}{2\pi r} \end{bmatrix}$

$\phi(r, \theta) = \frac{-\Gamma\theta}{2\pi}$

$\psi(r, \theta) = \frac{\Gamma \ln r}{2\pi}$

Calculus Review

Vector Calculus

Magnitude: $|\vec{A}| = \sqrt{A_1^2 + A_2^2 + a_3^2}$

Dot Product: $\vec{A} \cdot \vec{B} = A_x B_x + A_y B_y + A_z B_z = |\vec{A}||\vec{B}|\cos\theta$

Cross Product: $\vec{A} \times \vec{B} = (A_y B_z - A_z B_y)\hat{i} + (A + zB_x - a_x B_z)\hat{j} + (A_x B_y - A_y B_x)\hat{k}$

$|\vec{A} \times \vec{B}| = |\vec{A}||\vec{B}|\sin\theta$

Gradient of Scalar Field: Vector Field

Cart.: $\vec{\nabla}p = \frac{\partial p}{\partial x}\hat{i} + \frac{\partial p}{\partial y}\hat{j} + \frac{\partial p}{\partial z}\hat{k}$

Cyl.: $\vec{\nabla}p = \frac{\partial p}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial p}{\partial \theta}\hat{\theta} + \frac{\partial p}{\partial z}\hat{z}$

Directional Derivative: $\frac{dp}{ds} = \vec{\nabla}p \cdot \vec{s}$-> unit vector in $\vec{s}$ direction.

Divergence of a Vector Field: Scalar Field

Cart.: div$\vec{V} = \vec{\nabla} \cdot \vec{V} = \frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} + \frac{\partial V_z}{\partial z}$

Cyl.: div$\vec{V} = \vec{\nabla} \cdot \vec{V} = \frac{1}{r}\frac{\partial (rV_r)}{\partial r} + \frac{1}{r}\frac{\partial V_\theta}{\partial \theta} + \frac{\partial V_z}{\partial z}$

Curl of a Vector Field: curl$\vec{V} = \vec{\nabla} \times \vec{V}$

Cart.: $= (\frac{\partial V_z}{\partial y} - \frac{\partial V_y}{\partial z})\hat{i} + (\frac{\partial V_x}{\partial z} - \frac{\partial V_z}{\partial x})\hat{j} + (\frac{\partial V_y}{\partial x} - \frac{\partial V_x}{\partial y})\hat{k}$

Cyl.: $= (\frac{1}{r}\frac{\partial V_z}{\partial \theta} - \frac{\partial V_\theta}{\partial z})\hat{r} + (\frac{\partial V_r}{\partial z} - \frac{\partial V_z}{\partial r})\hat{\theta} + (\frac{1}{r}\frac{\partial}{\partial r}(V_\theta r) - \frac{1}{r}\frac{\partial V_r}{\partial \theta})\hat{z}$

Laplacian: ∇^2

Cart.: $= \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$

Cyl.: $= \frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial}{\partial r}) + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial z^2}$

Integrals

Stokes' Theorem: $\oint_c \vec{A} \cdot \vec{ds} = \iint_S (\vec{\nabla} \times \vec{A}) \cdot \vec{dS}$

Divergence Theorem: $\oiint_S \vec{A} \cdot \vec{dS} = \iiint_{\mathcal{V}} (\vec{\nabla} \cdot \vec{A} d\mathcal{V})$

Gradient Theorem: $\oiint_S p\vec{dS} = \iiint_{\mathcal{V}} \vec{\nabla}p d\mathcal{V}$