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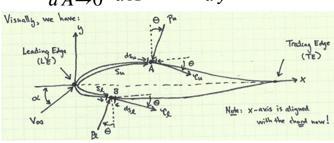
Concepts

Variables

Pressure - Normal force per unit area exerted on a surface due to the time rate of change of momentum of the gas particles impacing said surface.
$P = \lim_{dA\to 0} \frac{dF_n}{dA}$
Density - Mass per unit volume.
$\rho = \lim_{dV\to 0} \frac{dm}{dV}$
Temperature - Change in kinetic energy due to random molecular motion.
$\text{KE} = \frac{3}{2}kT$
Flow Velocity - Velocity due to organized motion.
$\vec{v} = \frac{d}{dt}\vec{x}$

Forces and Moments

Causes of forces and moments on an aerodynamic body:
Pressure Distribution over the body surface.
Shear Stress Distribution over the surface:
$\tau = \lim\limits_{dA\to 0} \frac{dF_f}{dA} = \mu\frac{dV}{dy}$



Lift and Drag per Unit Span:
$L' = N'\cos\alpha - A'\sin\alpha$
$D' = N'\sin\alpha + A'\cos\alpha$
Normal Force per Unit Span:
$N' = \int_{LE}^{TE}(P_u\cos\theta + \tau_u\sin\theta)ds_u + \int_{LE}^{TE}(P_l\cos\theta - \tau_l\sin\theta)ds_l$
Axial Force per Unit Span:
$A' = \int_{LE}^{TE}(-P_u\sin\theta + \tau_u\cos\theta)ds_u + \int_{LE}^{TE}(P_l\sin\theta + \tau_l\cos\theta)ds_l$
Moment per Unit Span:
$M'_* = \int_{LE}^{TE}[(P_u\cos\theta + \tau_u\sin\theta)(x - x^*) - (P_u\sin\theta - \tau_u\cos\theta)(y - y*)]ds_u + \int_{LE}^{TE}[(-P_l\cos\theta + \tau_u\sin\theta)(x - x^*) + (P_l\sin\theta + \tau_l\cos\theta)(y - y*)]ds_l$
When $x^* = y^* = 0$ we obtain the moment about the leading edge.
Center of Pressure:
$x_{cp} = -\frac{M'_{LE}}{N'}$

Force and Moment Coefficients

Dynamic Pressure: $q = \frac{1}{2}\rho_\infty v_\infty^2$
Reference Area: S
Reference Length: L or c
Lift Coefficient: $c_L = \frac{L}{q_\infty S}$
Drag Coefficient: $c_D = \frac{D}{q_\infty S}$
Moment Coefficient: $c_M = \frac{M}{q_\infty SL}$

Buckingham Pi Theorem

If we have a physically meaningful equation such as:
$f(P_1, P_2, ..., P_N) = 0$
where P_i are physical variables in terms of K independent physical units (mass, length, time, temperature) then it can be restated as:
$F(\Pi_1, \Pi_2, ..., \Pi_{N-K}) = 0$ or
$\Pi_{N-K} = G(\Pi_1, \Pi_2, ..., \Pi_{N-K-1})$
where Π_i are dimensionaless variables known as Pi Groups.

Basic Flow Equations

Fixed control volumes for basic flow equations.
Conservation of Mass
Net mass flow out of $\mathcal{V}$ = Time rate of decrease of mass inside $\mathcal{V}$.
Integral Continuity Equation:
$\frac{\partial}{\partial t}\iiint_{\mathcal{V}} \rho d\mathcal{V} + \oiint_S \rho\vec{V}\cdot d\vec{S} = 0$
Differential Continuity Equation:
$\frac{\partial\rho}{\partial t} + \vec{\nabla}\cdot(\rho\vec{V}) = 0$
Conservation of Momentum
Net force applied to $\mathcal{V}$ = Time rate of change of momentum of fluid in $\mathcal{V}$
$\frac{\partial}{\partial t}\iiint_{\mathcal{V}} \rho\vec{V}d\mathcal{V} + \oiint_S(\rho\vec{V}\cdot d\vec{S})\vec{V} = -\oiint_S pd\vec{S} + \iiint_{\mathcal{V}} \rho\vec{f}d\mathcal{V} + \vec{F}_{viscous}$

1. Body Forces (e.g. gravity) which act on the fluid inside $\mathcal{V}$
   $\iiint_{\mathcal{V}} \rho\vec{f}d\mathcal{V}$
2. Pressure and shear surface forces acting on S
   $-\oiint_S pd\vec{S} + \vec{F}_{viscous}$
3. Net flow of momentume out of volume across S
   $\oiint_S(\rho\vec{V}\cdot d\vec{S})\vec{V}$
4. Time rate of change of momentum due to unsteadiness in volume
   $\frac{\partial}{\partial t}\iiint_{\mathcal{V}} \rho\vec{V}d\mathcal{V}$

Navier-Stokes Equations:
$\frac{\partial}{\partial t}(\rho u) + \vec{\nabla}\cdot(\rho u\vec{V}) = -\frac{\partial p}{\partial x} + \rho f_x + (\mathcal{F}_x)_{viscous}$
$\frac{\partial}{\partial t}(\rho v) + \vec{\nabla}\cdot(\rho v\vec{V}) = -\frac{\partial p}{\partial y} + \rho f_y + (\mathcal{F}_y)_{viscous}$
Conservation of Energy
Integral:
$\frac{\partial}{\partial t}\iiint_{\mathcal{V}} \rho\left(e + \frac{V^2}{2}\right)d\mathcal{V} + \oiint_S \rho\left(e + \frac{V^2}{2}\right)\vec{V}\cdot d\vec{S} = \iiint_{\mathcal{V}} \dot{q}\rho d\mathcal{V} + \dot{Q}_{viscous} - \oiint_S p\vec{V}\cdot d\vec{S} + \iiint_{\mathcal{V}} \rho\vec{f}\cdot\vec{V}d\mathcal{V} + \dot{W}_{viscous}$
Net Time Rate of Change of Total E Due to Unsteadiness in $\mathcal{V}$ + Net Rate of Flow of Total E out of $\mathcal{V}$ = Rate of Volumetric Heating + Heat Addition due to Viscosity - Rate of Pressure work + Rate of Work Done by Body Forces + Rate of Viscous Work
Differential:
$\frac{\partial}{\partial t}\left(\rho\left(e + \frac{V^2}{2}\right)\right) + \vec{\nabla}\cdot\left(\rho\left(e + \frac{V^2}{2}\right)\right) = \rho\dot{q} + \dot{Q}'_{viscous} - \vec{\nabla}\cdot(p\vec{V}) + \rho\vec{f}\cdot\vec{V} + \dot{W}'_{viscous}$

Kinematics

$\vec{\nabla}\cdot\vec{V}$: time rate of change of volume of a fluid element per unit volume → **Dilation**
Stream Line: Curve everywhere tangent to the velocity.
$\frac{v}{u} = \frac{dy}{dx}$
Path Line: Trajectory of a particle released from a point in time.
Streak Line: Line connecting all particles that have passed through a given point.
For a steady flow the three lines coincide.

Substantial Derivative

Time rate of change of density of a given material fluid element as it moves through space and time.
$\frac{D\rho}{Dt} = \frac{\partial\rho}{\partial t} + \vec{V}\cdot\vec{\nabla}\rho$
Continuity Equation: $\frac{D\rho}{Dt} = -\rho\vec{\nabla}\cdot\vec{V}$
Material Acceleration: $(\frac{Du}{Dt}; \frac{Dv}{Dt}; \frac{Dz}{Dt})$
Navier-Stokes:
$\rho\frac{Du}{Dt} = -\frac{\partial p}{\partial x} + \rho f_x + (\mathcal{F}_x)_{viscous}$ → applicable to y and z as well.

Vorticity and Strain

$\vec{\omega} = \frac{1}{2}\vec{\nabla}\times\vec{V}$ → angular velocity
$\vec{\xi} = \vec{\nabla}\times\vec{V}$ → Vorticity
$\varepsilon_{xy} = \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}$ → Rate of Strain in xy plane.
$\Gamma = -\oint_C \vec{V}\cdot d\vec{s}$ → Circulation
Relation between circulation and vorticity: $\vec{xi}\cdot\vec{n} = \frac{d\Gamma}{dS}$

Stream Functions and Velocity Potentials

Stream Function: For every incompressible, two-dimensional flow, there exists a scalar stream function such that:
$u = \frac{\partial\psi}{\partial y}$, $v = -\frac{\partial\psi}{\partial x}$ DON'T FORGET +C
$\vec{\nabla}^2\psi = 0$
Velocity Potential: For every irrotational flow, there exists a scalar velocity potential such that:
$\vec{V} = \vec{\nabla}\phi \to \frac{\partial\phi}{\partial x} = u$, $\frac{\partial\phi}{\partial y} = v$
$\vec{\nabla}^2\phi = 0$
Laplace's Equation: 2nd order linear partial differential equation. Allows **superposition** if both ψ_1 and ψ_2 are soluations to $\vec{\nabla}^2\psi = 0$, then $\psi_3 = \psi_1 + \psi_2$ is also a solution.
Finding Solutions:

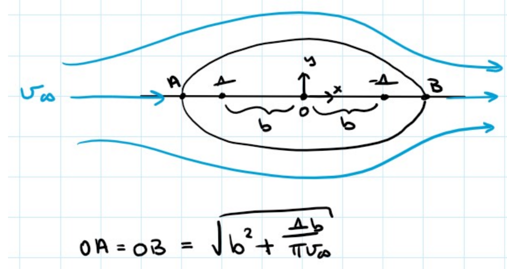
1) Infinite Bounday Condition: far from the body, velocity is uniform and aligned with x-axis.
$\phi(x, y) = v_\infty x$ and $\psi(x, y) = v_\infty y$
2) Wall boundary: no flow of mass or momentum through solid boundaries.
$\frac{\partial \phi}{\partial n} = 0$ and $\frac{\partial \psi}{\partial n} = 0$
To Find L, D, and M:
1) Solve Laplace's eq. for ϕ w/ boundary conditions.
2) Obtain velocity field: $\vec{V} = \vec{\nabla}\phi$
3) Obtain P: Bernoulli's Equation **Euler's Equation**: $\frac{1}{2}d(V^2) = -\frac{1}{\rho}dp$
If flow is barotropic, $\rho = \rho(p)$ is a function of pressue only and can integrate the equation. Occurs in incompressible flow, isothermal flow, or isentropic (frictionless) flow.
Bernoulli's Equation: $p + \frac{1}{2}\rho V^2 =$ constant - along a streamline for steady, inviscid, incompressible flow with no body forces.
$c_P = 1 - \left(\frac{v}{v_\infty}\right)^2$

Flow in a Duct

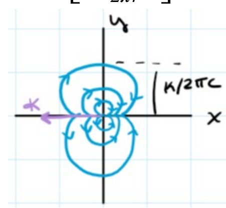
$\rho_1 A_1 V_1 = \rho_2 A_2 V_2 \rightarrow$ Steady, inviscid flow in a duct with no body forces. For incompressible flow remove rho.

Elementary Flows & Combinations

Uniform Flow
$\phi(x, y) = v_\infty x$ and $\phi(r, \theta) = v_\infty r\cos\theta$
$\psi(x, y) = v_\infty y$ and $\phi(r, \theta) = v_\infty r\sin\theta$
$\Gamma = 0$
Source/Sink
$\vec{V} = \begin{bmatrix} \frac{\Lambda}{2\pi r} \\ 0 \end{bmatrix}$
$\phi(r, \theta) = \frac{\Lambda \ln(r)}{2\pi}$
$\psi(r, \theta) = \frac{\Lambda\theta}{2\pi}$
$\Gamma = 0$
Superposition
Uniform Flow + Source
$\psi(r, \theta) = v_\infty r\sin\theta + \frac{\Lambda\theta}{2\pi r}$
Equation for Body Shape: $r(\theta) = \frac{\Lambda(\pi-\theta)}{2\pi v_\infty \sin\theta}$ and $h = \frac{\Lambda}{v_\infty}$
Rankine Oval Flow
Uniform Flow + Source + Sink
$\psi(r, \theta) = v_\infty r\sin\theta + \frac{\Lambda\theta_1}{2\pi} + \frac{-\Lambda\theta_2}{2\pi}$
$\psi(x, y) = v_\infty y + \frac{\Lambda}{2\pi}\tan^{-1}\left(\frac{y}{x+b}\right) - \frac{\Lambda}{2\pi}\tan^{-1}\left(\frac{y}{x-b}\right)$
$\vec{V} = \begin{bmatrix} v_\infty + \frac{\Lambda}{2\pi}\frac{x+b}{(x+b)^2+y^2} - \frac{\Lambda}{2\pi}\frac{x-b}{(x-b)^2+y^2} \\ \frac{\Lambda}{2\pi}\frac{y}{(x+b)^2+y^2} - \frac{\Lambda}{2\pi}\frac{y}{(x-b)^2+y^2} \end{bmatrix}$
Equation of a Rankine Oval:
$0 = v_\infty y + \frac{\Lambda}{2\pi}\tan^-1\left(\frac{y}{x+b}\right) - \frac{\Lambda}{2\pi}\tan^-1\left(\frac{y}{x-b}\right)$



Doublet
Source + Sink
Rankine Oval $b \rightarrow 0$
$\psi(r, \theta) = \frac{-\kappa\sin\theta}{2\pi r}$
$\phi(r, \theta) = \frac{\kappa\cos\theta}{2\pi r}$
$\vec{V} = \begin{bmatrix} -\frac{\kappa\cos\theta}{2\pi r^2} \\ -\frac{\kappa\sin\theta}{2\pi r^2} \end{bmatrix}$



Non-Lifting Flow over Cylinder
Uniform Flow + Doublet
$\psi(r, \theta) = v_\infty r\sin\theta\left(1 - \frac{R^2}{r^2}\right)$
$\vec{V} = \begin{bmatrix} v_\infty\cos\theta\left(1 - \frac{R^2}{r^2}\right) \\ -v_\infty\sin\theta\left(1 + \frac{R^2}{r^2}\right) \end{bmatrix}$
Stagnation points at $v_r = v_\theta = 0$ or $(r, \theta) = (R, 0), (R, \pi)$
$L = D = 0$, when $D = 0$ it is called D'Alembert's Paradox
Vortex Flow
Rotation flow, but the fluid elements themselves do not rotate. They follow a streamline in a circle at a particular radius.
Must make $rv_\theta = c$ in order to prevent going to $r = 0$ where flow is not irrotational.
$\vec{V} = \begin{bmatrix} 0 \\ \frac{c}{r} = \frac{-\Gamma}{2\pi r} \end{bmatrix}$
$\phi(r, \theta) = \frac{-\Gamma\theta}{2\pi}$
$\psi(r, \theta) = \frac{\Gamma\ln r}{2\pi}$
Lifting Flow over Cylinder
Cyliner + Vortex
$\psi = (v_\infty r\sin\theta)\left(1 - \frac{R^2}{r^2}\right) + \frac{\Gamma}{2\pi}\ln\frac{r}{R}$
$\vec{V} = \begin{bmatrix} v_\infty\cos\theta\left(1 - \frac{R^2}{r^2}\right) \\ -\left(1 + \frac{R^2}{r^2}\right)v_\infty\sin\theta - \frac{\Gamma}{2\pi r} \end{bmatrix}$
Stagnation Points
$r = R, \theta = \sin^{-1}\frac{-\Gamma}{4\pi v_\infty R}$

Calculus Review

Vector Calculus

Magnitude: $|\vec{A}| = \sqrt{A_1^2 + A_2^2 + a_3^2}$
Dot Product: $\vec{A}\cdot\vec{B} = A_xB_x + A_yB_y + A_zB_z = |\vec{A}||\vec{B}|\cos\theta$
Cross Product: $\vec{A}\times\vec{B} = (A_yB_z - A_zB_y)\hat{i} + (A + zB_x - a_xB_z)\hat{j} + (A_xB_y - A_yB_x)\hat{k}$
$|\vec{A}\times\vec{B}| = |\vec{A}||\vec{B}|\sin\theta$
Gradient of Scalar Field: Vector Field
Cart.: $\vec{\nabla}p = \frac{\partial p}{\partial x}\hat{i} + \frac{\partial p}{\partial y}\hat{j} + \frac{\partial p}{\partial z}\hat{k}$
Cyl.: $\vec{\nabla}p = \frac{\partial p}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial p}{\partial\theta}\hat{\theta} + \frac{\partial p}{\partial z}\hat{z}$
Directional Derivative: $\frac{dp}{ds} = \vec{\nabla}p\cdot\vec{s}$-> unit vector in $\vec{s}$ direction.
Divergence of a Vector Field: Scalar Field
Cart.: div$\vec{V} = \vec{\nabla}\cdot\vec{V} = \frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} + \frac{\partial V_z}{\partial z}$
Cyl.: div$\vec{V} = \vec{\nabla}\cdot\vec{V} = \frac{1}{r}\frac{\partial(rV_r)}{\partial r} + \frac{1}{r}\frac{\partial V_\theta}{\partial\theta} + \frac{\partial V_z}{\partial z}$
Curl of a Vector Field: curl$\vec{V} = \vec{\nabla}\times\vec{V}$
Cart.: $= (\frac{\partial V_z}{\partial y} - \frac{\partial V_y}{\partial z})\hat{i} + (\frac{\partial V_x}{\partial z} - \frac{\partial V_z}{\partial x})\hat{j} + (\frac{\partial V_y}{\partial x} - \frac{\partial V_x}{\partial y})\hat{k}$
Cyl.: $= (\frac{1}{r}\frac{\partial V_z}{\partial\theta} - \frac{\partial V_\theta}{\partial z})\hat{r} + (\frac{\partial V_r}{\partial z} - \frac{\partial V_z}{\partial r})\hat{\theta} + (\frac{1}{r}\frac{\partial}{\partial r}(V_\theta r) - \frac{1}{r}\frac{\partial V_r}{\partial\theta})\hat{z}$
Laplacian: ∇^2
Cart.: $= \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$
Cyl.: $= \frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial f}{\partial r}) + \frac{1}{r^2}\frac{\partial^2 f}{\partial\theta^2} + \frac{\partial^2 f}{\partial z^2}$
Sph.: $= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial f}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial f}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 f}{\partial\psi^2}$

Integrals

Stokes' Theorem: $\oint_c \vec{A}\cdot d\vec{s} = \iint_S(\vec{\nabla}\times\vec{A})\cdot d\vec{S}$
Divergence Theorem: $\oiint_S \vec{A}\cdot d\vec{S} = \iiint_{\mathcal{V}}(\vec{\nabla}\cdot\vec{A}d\mathcal{V})$
Gradient Theorem: $\oiint_S p d\vec{S} = \iiint_{\mathcal{V}}\vec{\nabla}p d\mathcal{V}$