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ASEN 3200
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Two-Body Problem

Newton's Law: $\Sigma\vec{F} = \frac{d(m\vec{v})}{dt} = m\vec{a}$

Universal Law of Gravitation: $\vec{F}_g = -\frac{Gm_1m_2}{r^2}\frac{\vec{r}}{|\vec{r}|}$

Apply Newton's Laws to a two-body problem with the assumptions:

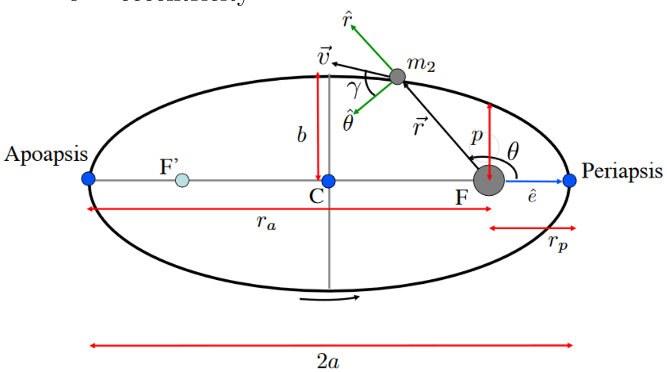
1. Only system force: Gravity $\rightarrow$ acts along the line joining the centers of the bodies.
2. Mass of each body is constant.
3. Treat each body as a spherically symmetrical point mass with uniform density.

Orbits

Elliptical Orbits:

Orbital Properties:

- a = semimajor axis
- b = semiminor axis
- p = semiperimeter
- r_a/r_p = radii of apoapsis/periapsis
- $\vec{e}$ = eccentricity



Useful Equations:

$a = \frac{1}{2}(r_a + r_p)$

$p = \frac{b^2}{a} = a(1-e^2) = \frac{h^2}{\mu}$

$r_a = \frac{p}{1-e} = a(1+e)$

$r_p = \frac{p}{1+e} = a(1-e)$

$e = \frac{r_a - r_p}{r_a + r_p} = \frac{c}{a} = \frac{\sqrt{a^2-b^2}}{a} = \sqrt{1 + \frac{2h^2\varepsilon}{\mu^2}}$

$b = a\sqrt{1-e^2}$

Angular Momentum: $\vec{h} = \vec{r}\times\vec{v} = \sqrt{\mu a(1-e^2)}$

Eccentricity Vector: $\vec{e} = \frac{\vec{v}\times\vec{h}}{\mu} - \frac{\vec{r}}{r}$

Specific Energy: $\varepsilon = \frac{v^2}{2} - \frac{\mu}{r} = \frac{\mu^2(e^2-1)}{2h^2}$

- $\varepsilon < 0$ Motion of Body 2 is bounded wrt Body 1
- $\varepsilon \geq 0$ Motion of Body 2 is unbounded wrt Body 1

Conic Equation: $r = \frac{h^2/\mu}{1+e\cos\theta} = \frac{p}{1+e\cos\theta}$

Vis-Viva Equation: $v = \sqrt{\frac{2\mu}{r} - \frac{\mu}{a}}$

True Anomaly:

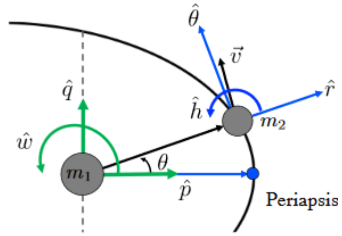
- θ or ν
- Measured from periapsis, $\vec{e}$ to radius, $\vec{r}$
- $\theta = 0$ at periapsis
- $0° > \theta > 180° \rightarrow m_2$ moving away from periapsis
- $180° < \theta < 360° m_2$ moving toward periapsis

Flight Path Angle:

- $\tan\gamma = \frac{e\sin\theta}{1+e\cos\theta} = \frac{v_r}{v_\theta}$
- $v_r = \frac{\mu}{h}e\sin\theta$ and $v_\theta = \frac{\mu}{h}(1+e\cos\theta)$
- $\gamma > 0$ when $v_r > 0$ and $\theta > 0 \rightarrow m_2$ moving away from periapsis

Perifocal Frame:

- $\hat{p}$, $\hat{q}$, $\hat{w}$
- $\vec{r} = r\hat{r} = r\cos\theta\hat{p} + r\sin\theta\hat{q}$
- $\vec{v} = \frac{\mu}{h}[-\sin\theta\hat{p} + (e+\cos\theta)\hat{q}]$



Kepler's Law: A line joining a planet and the Sun sweep out equal areas during equal intervals of time.

Elliptical Orbits: $\mathbb{P} = 2\pi\sqrt{\frac{a^3}{\mu}}$, $\varepsilon < 0$

Mean Motion: $n = \sqrt{\frac{\mu}{a^3}}$ - mean angular rate of motion

Circular Orbits:

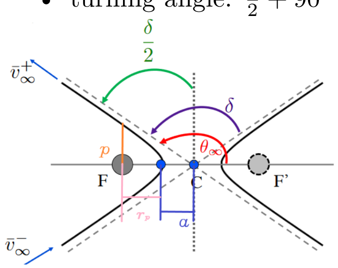
- $r = a$, $\vec{v} \perp \vec{r}$, and $\gamma = 0$ everywhere
- $v_c = \sqrt{\frac{\mu}{r}}$

Parabolic Orbits:

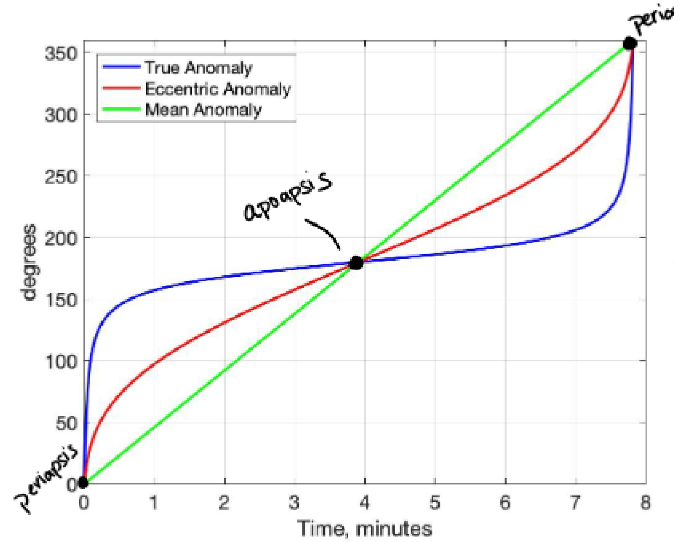
- $e = 1$, $a = \inf$, r_a = undefined
- Conic equation applies still
- $p = \frac{h^2}{\mu}$
- $\varepsilon = 0$ everywhere
- $v = \sqrt{\frac{2\mu}{r}} = v_{esc}$

Hyperbolic Orbits:

- $v > v_{esc}$, $e > 1$, $\varepsilon > 0$, $a < 0$
- $r_p = |a|(e-1)$
- $p = |a|(e^2-1) = a(1-e^2)$
- $r = \frac{a(1-e^2)}{1+e\cos\theta} = \frac{|a|(e^2-1)}{1+e\cos\theta}$
- at $r = \infty$, $\varepsilon = \frac{-\mu}{2a} = \frac{v_\infty^2}{2} \rightarrow v_\infty = \sqrt{\frac{\mu}{|a|}}$
- $\theta_\infty = \pm\cos^{-1}\left(\frac{-1}{e}\right)$
- $v^2 = v_{esc}^2 + v_\infty^2$
- turning angle: $\frac{\delta}{2} + 90° = \theta_\infty$, $\delta = 2\sin^{-1}(\frac{1}{e})$



The Anomalies



- True anomaly
  - Advances quickly from periapsis
  - Advances slowly from apoapsis
- Mean anomaly, M
  - Computed from time and mean motion
  - $M = n(t - t_p)$
  - Advances at constant rate in elliptical orbit
- Eccentric anomaly, E
  - Angle that helps translate from true anomaly to mean anomaly
  - Advances at rate between True and Mean anomaly

Kepler's Equation: $M = n(t-t_p) = E - e\sin E$

Everything in rads!

Finding E:

- Choose an initial estimate of E:
  - $M_e < \pi \rightarrow E = M_e + e/2 \quad M_e > \pi \rightarrow M_e - e/2$
- $f(E) = E - e\sin E - M_e$
- $f'(E) = 1 - e\cos E$
- Iterate and update E_i

  $E_{i+1} = E_i - \frac{f(E_i)}{f'(E_i)}$

Switching between the Anomalies:

- $\cos E = \frac{e+\cos\theta}{1+e\cos\theta}$
- $\tan\frac{E}{2} = \sqrt{\frac{1-e}{1+e}}\tan\frac{\theta}{2}$
- $\tan\frac{\theta}{2} = \sqrt{\frac{1-e}{1+e}}\tan\frac{E}{2}$
- $r = a(1-e\cos E) = \frac{p}{1+e\cos\theta} = \frac{a(1-e^2)}{1+e\cos\theta}$
- $\sin\theta = \frac{b}{r}\sin E$

To find time between A and B, with a,e,and θ of A and B known,

$t_B - t_A = (t_B - t_p) - (t_A - t_p)$

Hyperbolic Anomaly, H

Use an equilateral hyperbola to determine H: $e = \sqrt{2}$

- $M_h = \sqrt{\frac{\mu}{|a|^3}}(t - t_p) = e \sinh H - H$
- $\tanh \frac{H}{2} = \sqrt{\frac{e-1}{e+1}} \tan \frac{\theta^*}{2}$
- $\tanh \frac{\theta^*}{2} = \sqrt{\frac{e-1}{e+1}} \tan \frac{H}{2}$

Parabolic Orbits

Barkers Equation:

- $\sqrt{\frac{\mu}{p^3}}(t - t_p) = \frac{\mu^2}{h^3}(t - t_p) = M_p = \frac{1}{6}\tan^3\frac{\theta}{2} + \frac{1}{2}\tan\frac{\theta}{2}$
- $\tan\frac{\theta}{2} = \left(3M_p + \sqrt{(3M_p)^2 + 1}\right)^{1/3} - \left(3M_p + \sqrt{(3M_p)^2 + 1}\right)^{-1/3}$

3D Orbits

How many variables required to completely describe the state of a satellite?
6: 3 position and 3 velocity
OR
Can also describe a satellite's states by set of orbital elements:

- Size and shape: a, e
- Orientation of the orbit plane: i, Ω
- Orientation of the orbit within the orbit plane: ω
- Location of the satellite on the orbit: θ $(M, E, t - t_p)$

Inclination, i:

- Angular tile of the orbital plane relative to $\hat{X}\hat{Y}$ and measured between orbit normal, $\hat{h}$ and $\hat{Z}$
- Equatorial Orbit: $i = 0°, 180°$
- Polar Orbit: $i = 90°$
- Prograde Orbit: $i = [0°, 90°]$
- Retrograde Orbit: $i = [90°, 180°]$
- $\cos i = \frac{\hat{Z}\cdot\vec{h}}{|\hat{Z}||\vec{h}|} \rightarrow \cos i = \frac{h_z}{h}$

Right Ascension of the Ascending Node, Ω:

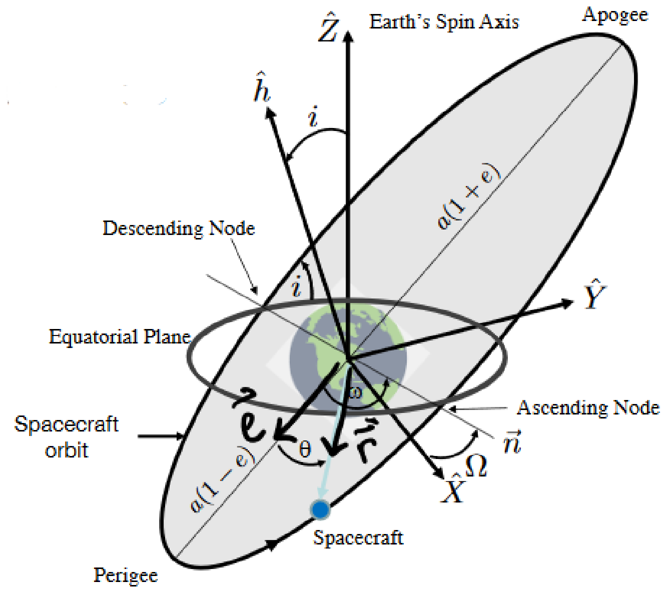
- Angle from the reference direction, $\hat{X}$, to the ascending node.
- Line of Nodes: $\vec{n} = \hat{Z} \times \vec{h}$
- $\cos\Omega = \frac{\hat{X}\cdot\vec{n}}{|\hat{X}||\vec{n}} = \frac{n_x}{n}$
- Quadrant Check:
  - $\vec{n}\cdot\hat{Y} > 0 \rightarrow 0 < \Omega < 180°$
  - $\vec{n}\cdot\hat{Y} < 0 \rightarrow 180° < \Omega < 360°$
  - Between $0 - 2\pi$

Argument of Periapsis, ω (AOP):

- Measured between the lines of nodes $\vec{n}$ and the eccentricity vector, $\vec{e}$
- Locates the closest point of the orbit
- Measured within the plane, varies from $0 - 2\pi$
- $\cos\omega = \frac{\vec{n}\cdot\vec{e}}{|\vec{n}||\vec{e}|}$
- Quadrant Check:
  - $\vec{e}\cdot\hat{z} > 0 \rightarrow 0 < \omega < 180°$
  - $\vec{e}\cdot\hat{z} < 0 \rightarrow 180° < \omega < 360°$

True Anomaly, θ:

- Location of the spacecraft within the orbit
- Varies from $0 - 2\pi$
- $\vec{r}\cdot\vec{e} = |\vec{r}||\vec{e}|\cos\theta$
- $\cos\theta = \frac{\vec{r}\cdot\vec{e}}{|\vec{r}||\vec{e}|}$
- Quadrant Check:
  - $\vec{r}\cdot\vec{v} > 0 \rightarrow 0 < \theta < 180°$
  - $\vec{r}\cdot\vec{v} < 0 \rightarrow 180° < \theta < 360°$



Converting Position and Velocity to Orbital Elements:

- Given $X = [\vec{r}, \vec{v}]$
- Compute vectors and their magnitudes:

$$\vec{h} = \vec{r}\times\vec{v} \quad \vec{n} = \hat{Z}\times\vec{h} \quad \vec{e} = \frac{\vec{v}\times\vec{h}}{\mu} - \frac{\vec{r}}{r}$$
$$h = |\vec{h}| \quad n = |\vec{n}| \quad e = |\vec{e}|$$

- Compute energy to get a:

$$\varepsilon = \frac{v^2}{2} - \frac{\mu}{r} \quad a = -\frac{\mu}{2\varepsilon}$$
$$p = \frac{h^2}{\mu} \quad a = \frac{p}{1-e^2}$$

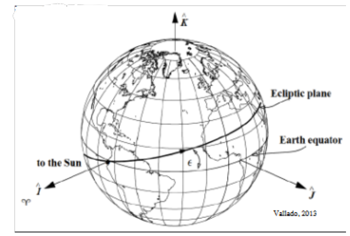
- Compute inclination & Orientation Angles from above
  - $\Omega \rightarrow$ If$(n_y < 0), \Omega = 360° - \Omega$
  - $\omega \rightarrow$ If$(e_z < 0), \omega = 360° - \omega$
  - $\theta \rightarrow$ If$(\vec{r}\cdot\vec{v} < 0), \theta = 360° - \theta$

What about...?

- What is Ω for an elliptical equatorial orbit?
  - Undefined, used true longitude of periapse: $\tilde{\omega}_{true}$
  - $\cos\tilde{\omega}_{true} = \frac{\hat{X}\cdot\vec{e}}{|\hat{X}||\vec{e}|}$ where $\hat{X} = [1, 0, 0]$
  - If $e_y < 0 \rightarrow \tilde{\omega}_{true} = 360° - \tilde{\omega}_{true}$
- What is ω for a circular orbit? (No perigee) $\rightarrow$ undefined!
- Use arugment of latitude
  - $\cos u = \frac{\vec{n}\cdot\vec{r}}{|\vec{n}||\vec{r}|} \rightarrow$ If $(r_z < 0)$, then $u = 360° - u$
- Circular, equatorial orbits? Both ω and Ω undefined!
  - $\cos\lambda_{true} = \frac{\hat{X}\cdot\vec{r}}{|\hat{X}||\vec{r}|} \rightarrow$ If $(r_y < 0)$, then $\lambda_{true} = 360° - \lambda_{true}$

Coordinate Frames

Ecliptic Plane:



- Mean plane of the Earth's orbit around the Sun.
- Earth's equatorial plane is inclined about 23.5° relative to ecliptic.
- Vernal Equinox ♈:
  - intersection of Sun's path relative to the Earth with the equatorial plane, as Sun moves from South to North.
  - Occurs at ascending node of the Sun as viewed from Earth
  - Used as reference direction for defining intertial reference frames

ICRF: International Celestial Reference Frame

- Close representation of intertial frame (rotates a little)
- Nonrotating with respect to extragalactic radio sources (quasars)
- Center: Barycenter of the Solar System

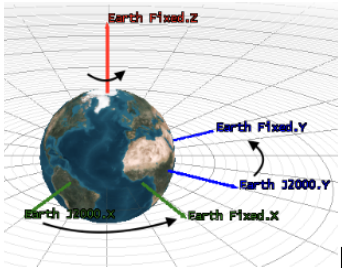
Earth-Centered Mean Equatorial J2000 System (EME2000):

- Origin: Earth Center
- $\hat{X}$: Vernal Equinox at 1/1/2000 12:00:00 TT
- $\hat{Z}$: Normal to the mean equatorial plane of Earth at the J2000 epoch (Spin axis of Earth)
- $\hat{Y}$: Completes right hand coordinate frame

Earth-Centered Mean Orbit and Equinox of J2000 (EMO2000):

- Origin: Earth Center
- $\hat{X}$: Vernal Equinox at 1/1/2000 12:00:00 TT
- $\hat{Z}$: Orbit normal vector at same time
- $\hat{Y}$: Completes right hand coordinate frame
- Differs from EME2000 by 23.4393°

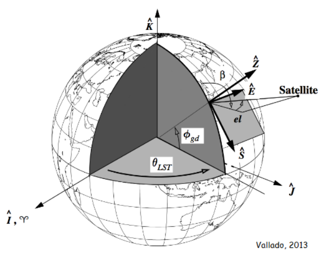
Earth-Centered Earth-Fixed:



[H]

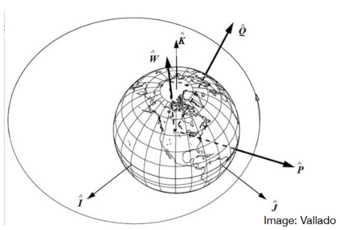
- Origin: Earth Center
- $\hat{X}$: Osculating vector from center of Earth toward the equator along the Prime Meridian (rotates with Earth)
- $\hat{Z}$: Aligned with Earth's spin axis
- $\hat{Y}$: Completes right hand coordinate frame

Topocentric Horizon System:



- Usefule for observing satellites and sensor systems
- Origin = site on Earth
- SEZ frame rotates with site
- Local Horizon forms fundamental plane:
  - $\hat{S}$: Points due South
  - $\hat{E}$: Points due East
  - $\hat{Z}$: Points radially outward
  - β: Azimuth, angle measured from North, clockwise to location beneath object of interest
  - el: Elevation, measured from local horizon, positive up to the object $[-90°, 90°]$

Perifocal Coordinate System:



- Fundamental Plane: Satellite Orbit
- Origin: Center of Earth
- $\hat{P}$: Points toward perigee
- $\hat{Q}$: $90°$ from $\hat{P}$ axis in direction of satellite motion
- $\hat{W}$: normal to the orbit

Coordinate Frame Transformations

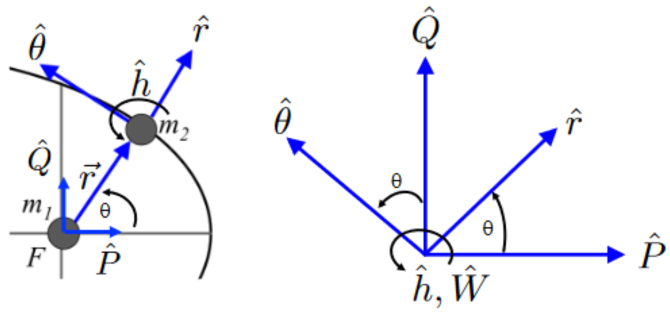
Generic:

$$X, I \quad R_1(\alpha) \quad ROT1(\alpha) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & c\alpha & s\alpha \\ 0 & -s\alpha & c\alpha \end{bmatrix}$$

$$Y, J \quad R_2(\alpha) \quad ROT2(\alpha) = \begin{bmatrix} c\alpha & 0 & -s\alpha \\ 0 & 1 & 0 \\ s\alpha & 0 & c\alpha \end{bmatrix}$$

$$Z, K \quad R_3(\alpha) \quad ROT3(\alpha) = \begin{bmatrix} c\alpha & s\alpha & 0 \\ -s\alpha & c\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Perifocal-Rotating Transformation:



$$\begin{bmatrix} r \\ \theta \\ h \end{bmatrix} = \begin{bmatrix} c\theta & s\theta & 0 \\ -s\theta & c\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} P \\ Q \\ W \end{bmatrix}$$

Express position and velocity in PQW frame to Cartesian:

$$\vec{r}_{PQW} = r\cos\theta\hat{P} + r\sin\theta\hat{Q} = \begin{bmatrix} \frac{p\cos\theta}{1+e\cos\theta} \\ \frac{p\sin\theta}{1+e\cos\theta} \\ 0 \end{bmatrix}$$

$$\vec{v}_{PQW} = \begin{bmatrix} \dot{r}\cos\theta - r\dot{\theta}\sin\theta \\ \dot{r}\sin\theta + r\dot{\theta}\cos\theta \\ 0 \end{bmatrix} = \begin{bmatrix} -\sqrt{\frac{\mu}{p}}\sin\theta \\ \sqrt{\frac{\mu}{p}}(e+\cos\theta) \\ 0 \end{bmatrix}$$

$[C] = R_3(-\Omega)R_1(-i)R_3(-\omega)$

$\vec{r}_{xyz} = [C]\vec{r}_{PQW} \quad \vec{v}_{xyz} = [C]\vec{v}_{PQW}$

To go from XYZ to PQW:

$\vec{r}_{PQW} = R_3(\omega)R_1(i)R_3(\Omega)\vec{r}_{xyz}$

Ground Track

Earth rotates $15.04°$/hour eastward beneath the satellite. Ground track advances westward at this rate.

We can relate the distance between successive ground track crossing to the orbital period using Earth's rotation: $\mathbb{P} = \frac{D}{15.04°/hour}$