Two-Body Problem

Newton's Law: $\Sigma\vec{F} = \frac{d(m\vec{v})}{dt} = m\vec{a}$

Universal Law of Gravitation: $\vec{F}_g = -\frac{Gm_1m_2}{r^2}\frac{\vec{r}}{|\vec{r}|}$

Apply Newton's Laws to a two-body problem with the assumptions:

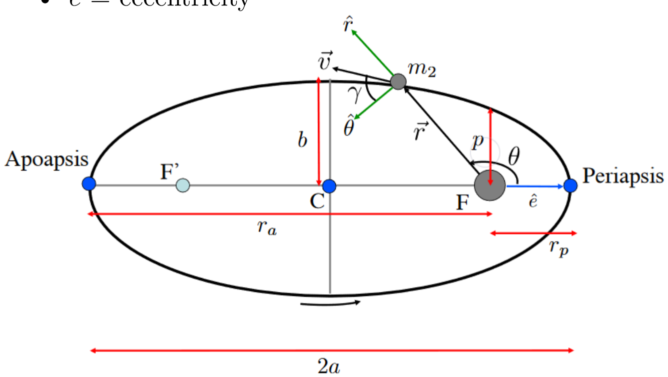
1. Only system force: Gravity $\rightarrow$ acts along the line joining the centers of the bodies.
2. Mass of each body is constant.
3. Treat each body as a spherically symmetrical point mass with uniform density.

Orbits

Elliptical Orbits:

Orbital Properties:

- a = semimajor axis
- b = semiminor axis
- p = semiperimeter
- r_a/r_p = radii of apoapsis/periapsis
- $\vec{e}$ = eccentricity



Useful Equations:

$a = \frac{1}{2}(r_a + r_p)$

$p = \frac{b^2}{a} = a(1-e^2) = \frac{h^2}{\mu}$

$r_a = \frac{p}{1-e} = a(1+e)$

$r_p = \frac{p}{1+e} = a(1-e)$

$e = \frac{r_a - r_p}{r_a + r_p} = \frac{c}{a} = \frac{\sqrt{a^2-b^2}}{a} = \sqrt{1 + \frac{2h^2\varepsilon}{\mu^2}}$

$b = a\sqrt{1-e^2}$

Angular Momentum: $\vec{h} = \vec{r} \times \vec{v} = \sqrt{\mu a(1-e^2)}$

Eccentricity Vector: $\vec{e} = \frac{\vec{v}\times\vec{h}}{\mu} - \frac{\vec{r}}{r}$

Specific Energy: $\varepsilon = \frac{v^2}{2} - \frac{\mu}{r} = \frac{\mu^2(e^2-1)}{2h^2}$

- $\varepsilon < 0$ Motion of Body 2 is bounded wrt Body 1
- $\varepsilon \geq 0$ Motion of Body 2 is unbounded wrt Body 1

Conic Equation: $r = \frac{h^2/\mu}{1+e\cos\theta} = \frac{p}{1+e\cos\theta}$

Vis-Viva Equation: $v = \sqrt{\frac{2\mu}{r} - \frac{\mu}{a}}$

True Anomaly:

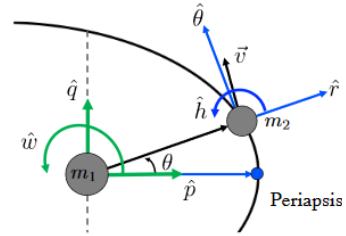
- θ or ν
- Measured from periapsis, $\vec{e}$ to radius, $\vec{r}$
- $\theta = 0$ at periapsis
- $0° > \theta > 180° \rightarrow m_2$ moving away from periapsis
- $180° < \theta < 360° m_2$ moving toward periapsis

Flight Path Angle:

- $\tan\gamma = \frac{e\sin\theta}{1+e\cos\theta} = \frac{v_r}{v_\theta}$
- $v_r = \frac{\mu}{h}e\sin\theta$ and $v_\theta = \frac{\mu}{h}(1+e\cos\theta)$
- $\gamma > 0$ when $v_r > 0$ and $\theta > 0 \rightarrow m_2$ moving away from periapsis

Perifocal Frame:

- $\hat{p}$, $\hat{q}$, $\hat{w}$
- $\vec{r} = r\hat{r} = r\cos\theta\hat{p} + r\sin\theta\hat{q}$
- $\vec{v} = \frac{\mu}{h}[-\sin\theta\hat{p} + (e+\cos\theta)\hat{q}]$



Kepler's Law: A line joining a planet and the Sun sweep out equal areas during equal intervals of time.

Elliptical Orbits: $\mathbb{P} = 2\pi\sqrt{\frac{a^3}{\mu}}$, $\varepsilon < 0$

Mean Motion: $n = \sqrt{\frac{\mu}{a^3}}$ - mean angular rate of motion

Circular Orbits:

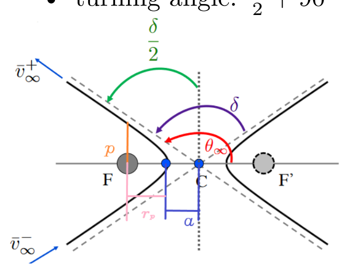
- $r = a$, $\vec{v} \perp \vec{r}$, and $\gamma = 0$ everywhere
- $v_c = \sqrt{\frac{\mu}{r}}$

Parabolic Orbits:

- $e = 1$, $a = \inf$, r_a = undefined
- Conic equation applies still
- $p = \frac{h^2}{\mu}$
- $\varepsilon = 0$ everywhere
- $v = \sqrt{\frac{2\mu}{r}} = v_{esc}$

Hyperbolic Orbits:

- $v > v_{esc}$, $e > 1$, $\varepsilon > 0$, $a < 0$
- $r_p = |a|(e-1)$
- $p = |a|(e^2-1) = a(1-e^2)$
- $r = \frac{a(1-e^2)}{1+e\cos\theta} = \frac{|a|(e^2-1)}{1+e\cos\theta}$
- at $r = \infty$, $\varepsilon = \frac{-\mu}{2a} = \frac{v_\infty^2}{2} \rightarrow v_\infty = \sqrt{\frac{\mu}{|a|}}$
- $\theta_\infty = \pm\cos^{-1}\left(\frac{-1}{e}\right)$
- $v^2 = v_{esc}^2 + v_\infty^2$
- turning angle: $\frac{\delta}{2} + 90° = \theta_\infty$, $\delta = 2\sin^{-1}(\frac{1}{e})$



The Anomalies