

Two-Body Problem

Newton's Law: $\Sigma \vec{F} = \frac{d(m\vec{v})}{dt} = m\vec{a}$

Universal Law of Gravitation: $\vec{F}_g = -\frac{Gm_1m_2}{r^2} \frac{\vec{r}}{|\vec{r}|}$

Apply Newton's Laws to a two-body problem with the assumptions:

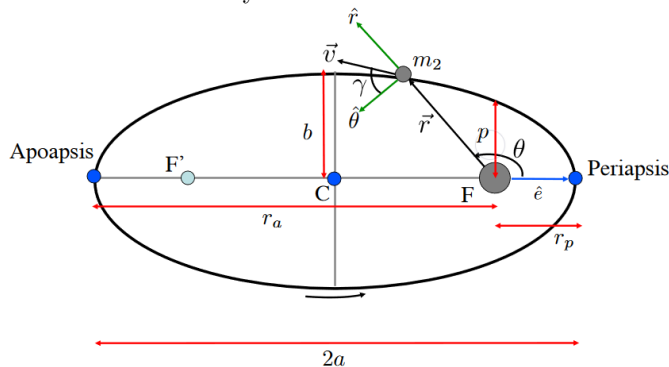
1. Only system force: Gravity \rightarrow acts along the line joining the centers of the bodies.
2. Mass of each body is constant.
3. Treat each body as a spherically symmetrical point mass with uniform density.

Orbits

Elliptical Orbits:

Orbital Properties:

- a = semimajor axis
- b = semiminor axis
- p = semiperimeter
- r_a/r_p = radii of apoapsis/periapsis
- \vec{e} = eccentricity



Useful Equations:

$$a = \frac{1}{2}(r_a + r_p)$$

$$p = \frac{b^2}{a} = a(1 - e^2) = \frac{h^2}{\mu}$$

$$r_a = \frac{p}{1-e} = a(1+e)$$

$$r_p = \frac{p}{1+e} = a(1-e)$$

$$e = \frac{r_a - r_p}{r_a + r_p} = \frac{c}{a} = \frac{\sqrt{a^2 - b^2}}{a} = \sqrt{1 + \frac{2h^2\varepsilon}{\mu^2}}$$

$$b = a\sqrt{1 - e^2}$$

$$\text{Angular Momentum: } \vec{h} = \vec{r} \times \vec{v} = \sqrt{\mu a(1 - e^2)}$$

$$\text{Eccentricity Vector: } \vec{e} = \frac{\vec{v} \times \vec{h}}{\mu} - \frac{\vec{r}}{r}$$

$$\text{Specific Energy: } \varepsilon = \frac{v^2}{2} - \frac{\mu}{r} = \frac{\mu^2(e^2 - 1)}{2h^2}$$

- $\varepsilon < 0$ Motion of Body 2 is bounded wrt Body 1
- $\varepsilon \geq 0$ Motion of Body 2 is unbounded wrt Body 1

$$\text{Conic Equation: } r = \frac{h^2/\mu}{1 + e \cos \theta} = \frac{p}{1 + e \cos \theta}$$

$$\text{Vis-Viva Equation: } v = \sqrt{\frac{2\mu}{r} - \frac{\mu}{a}}$$

True Anomaly:

- θ or ν

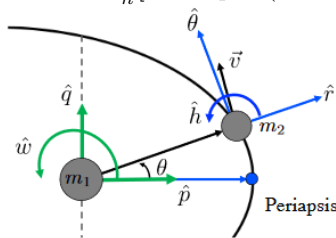
- Measured from periapsis, \vec{e} to radius, \vec{r}
- $\theta = 0$ at periapsis
- $0^\circ > \theta > 180^\circ \rightarrow m_2$ moving away from periapsis
- $180^\circ < \theta < 360^\circ m_2$ moving toward periapsis

Flight Path Angle:

- $\tan \gamma = \frac{e \sin \theta}{1 + e \cos \theta} = \frac{v_r}{v_\theta}$
- $v_r = \frac{\mu}{h} e \sin \theta$ and $v_\theta = \frac{\mu}{h} (1 + e \cos \theta)$
- $\gamma > 0$ when $v_r > 0$ and $\theta > 0 \rightarrow m_2$ moving away from periapsis

Perifocal Frame:

- $\hat{p}, \hat{q}, \hat{w}$
- $\vec{r} = r\hat{r} = r \cos \theta \hat{p} + r \sin \theta \hat{q}$
- $\vec{v} = \frac{\mu}{h} [-\sin \theta \hat{p} + (e + \cos \theta) \hat{q}]$



Kepler's Law: A line joining a planet and the Sun sweep out equal areas during equal intervals of time.

Elliptical Orbits: $\mathbb{P} = 2\pi \sqrt{\frac{a^3}{\mu}}, \varepsilon < 0$

Mean Motion: $n = \sqrt{\frac{\mu}{a^3}}$ - mean angular rate of motion

Circular Orbits:

- $r = a, \vec{v} \perp \vec{r}$, and $\gamma = 0$ everywhere

$$v_c = \sqrt{\frac{\mu}{r}}$$

Parabolic Orbits:

- $e = 1, a = \inf, r_a = \text{undefined}$
- Conic equation applies still

$$p = \frac{h^2}{\mu}$$

$$\varepsilon = 0 \text{ everywhere}$$

$$v = \sqrt{\frac{2\mu}{r}} = v_{esc}$$

Hyperbolic Orbits:

- $v > v_{esc}, e > 1, \varepsilon > 0, a < 0$

$$r_p = |a|(e - 1)$$

$$p = |a|(e^2 - 1) = a(1 - e^2)$$

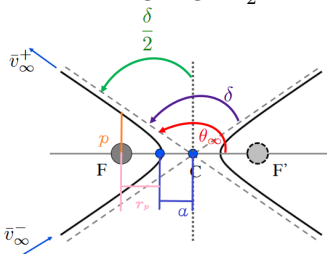
$$r = \frac{a(1 - e^2)}{1 + e \cos \theta} = \frac{|a|(e^2 - 1)}{1 + e \cos \theta}$$

$$\text{at } r = \infty, \varepsilon = \frac{\mu}{2a} = \frac{v_\infty^2}{2} \rightarrow v_\infty = \sqrt{\frac{\mu}{|a|}}$$

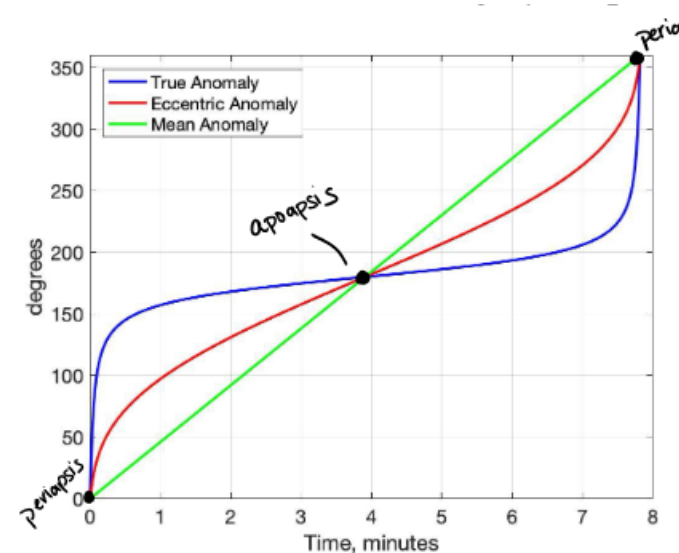
$$\theta_\infty = \pm \cos^{-1}(\frac{-1}{e})$$

$$v^2 = v_{esc}^2 + v_\infty^2$$

$$\text{turning angle: } \frac{\delta}{2} + 90^\circ = \theta_\infty, \delta = 2 \sin^{-1}(\frac{1}{e})$$



The Anomalies



- True anomaly
 - Advances quickly from periapsis
 - Advances slowly from apoapsis
- Mean anomaly, M
 - Computed from time and mean motion
 - $M = n(t - t_p)$
 - Advances at constant rate in elliptical orbit
- Eccentric anomaly, E
 - Angle that helps translate from true anomaly to mean anomaly
 - Advances at rate between True and Mean anomaly

Kepler's Equation: $M = n(t - t_p) = E - e \sin E$

Everything in rads!

Finding E:

- Choose an initial estimate of E:
 - $M_e < \pi \rightarrow E = M_e + e/2$ $M_e > \pi \rightarrow M_e - e/2$
- $f(E) = E - e \sin E - M_e$
- $f'(E) = 1 - e \cos E$
- Iterate and update E_i

$$E_{i+1} = E_i - \frac{f(E_i)}{f'(E_i)}$$

Switching between the Anomalies:

- $\cos E = \frac{e + \cos \theta}{1 + e \cos \theta}$
- $\tan \frac{E}{2} = \sqrt{\frac{1-e}{1+e}} \tan \frac{\theta}{2}$
- $\tan \frac{\theta}{2} = \sqrt{\frac{1-e}{1+e}} \tan \frac{E}{2}$
- $r = a(1 - e \cos E) = \frac{p}{1 + e \cos \theta} = \frac{a(1 - e^2)}{1 + e \cos \theta}$
- $\sin \theta = \frac{b}{r} \sin E$

To find time between A and B, with a, e , and θ of A and B known,

$$t_B - t_A = (t_B - t_p) - (t_A - t_p)$$

Hyperbolic Anomaly, H

Use an equilateral hyperbola to determine H: $e = \sqrt{2}$

- $M_h = \sqrt{\frac{\mu}{|a|^3}}(t - t_p) = e \sinh H - H$
- $\tanh \frac{H}{2} = \sqrt{\frac{e-1}{e+1}} \tan \frac{\theta^*}{2}$
- $\tanh \frac{\theta^*}{2} = \sqrt{\frac{e-1}{e+1}} \tan \frac{H}{2}$

Parabolic Orbits

Barkers Equation:

- $\sqrt{\frac{\mu}{p^3}}(t - t_p) = \frac{\mu^2}{h^2}(t - t_p) = M_p = \frac{1}{6} \tan^3 \frac{\theta}{2} + \frac{1}{2} \tan \frac{\theta}{2}$
- $\tan \frac{\theta}{2} = \left(3M_p + \sqrt{(3M_p)^2 + 1}\right)^{1/3}$
- $-\left(3M_p + \sqrt{(3M_p)^2 + 1}\right)^{-1/3}$

3D Orbits

How many variables required to completely describe the state of a satellite?

6: 3 position and 3 velocity

OR

Can also describe a satellite's states by set of orbital elements:

- Size and shape: a, e
- Orientation of the orbit plane: i, Ω
- Orientation of the orbit within the orbit plane: ω
- Location of the satellite on the orbit: $\theta (M, E, t - t_p)$

Inclination, i :

- Angular tilt of the orbital plane relative to $\hat{X}\hat{Y}$ and measured between orbit normal, \hat{h} and \hat{Z}
- Equatorial Orbit: $i = 0^\circ, 180^\circ$
- Polar Orbit: $i = 90^\circ$
- Prograde Orbit: $i = [0^\circ, 90^\circ]$
- Retrograde Orbit: $i = [90^\circ, 180^\circ]$
- $\cos i = \frac{\hat{Z} \cdot \hat{h}}{|\hat{Z}||\hat{h}|} \rightarrow \cos i = \frac{h_z}{h}$

Right Ascension of the Ascending Node, Ω :

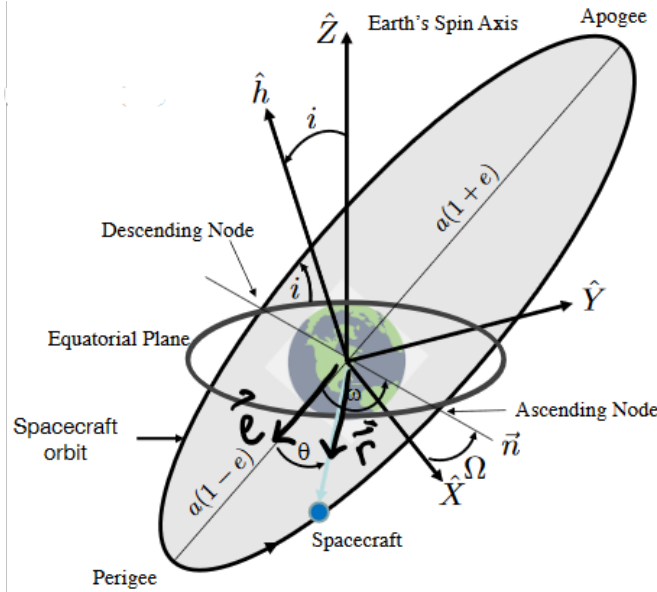
- Angle from the reference direction, \hat{X} , to the ascending node.
- Line of Nodes: $\vec{n} = \hat{Z} \times \vec{h}$
- $\cos \Omega = \frac{\hat{X} \cdot \vec{n}}{|\hat{X}||\vec{n}|} = \frac{n_x}{n}$
- Quadrant Check:
 - $\vec{n} \cdot \hat{Y} > 0 \rightarrow 0 < \Omega < 180^\circ$
 - $\vec{n} \cdot \hat{Y} < 0 \rightarrow 180^\circ < \Omega < 360^\circ$
 - Between $0 - 2\pi$

Argument of Periapsis, ω (AOP):

- Measured between the lines of nodes \vec{n} and the eccentricity vector, \vec{e}
- Locates the closest point of the orbit
- Measured within the plane, varies from $0 - 2\pi$
- $\cos \omega = \frac{\vec{n} \cdot \vec{e}}{|\vec{n}||\vec{e}|}$
- Quadrant Check:
 - $\vec{e} \cdot \hat{z} > 0 \rightarrow 0 < \omega < 180^\circ$
 - $\vec{e} \cdot \hat{z} < 0 \rightarrow 180^\circ < \omega < 360^\circ$

True Anomaly, θ :

- Location of the spacecraft within the orbit
- Varies from $0 - 2\pi$
- $\vec{r} \cdot \vec{e} = |\vec{r}||\vec{e}| \cos \theta$
- $\cos \theta = \frac{\vec{r} \cdot \vec{e}}{|\vec{r}||\vec{e}|}$
- Quadrant Check:
 - $\vec{r} \cdot \vec{v} > 0 \rightarrow 0 < \theta < 180^\circ$
 - $\vec{r} \cdot \vec{v} < 0 \rightarrow 180^\circ < \theta < 360^\circ$



Converting Position and Velocity to Orbital Elements:

- Given $X = [\vec{r}, \vec{v}]$
- Compute vectors and their magnitudes:

$$\begin{aligned} \vec{h} &= \vec{r} \times \vec{v} & \vec{n} &= \hat{Z} \times \vec{h} & \vec{e} &= \frac{\vec{v} \times \vec{h}}{\mu} - \frac{\vec{r}}{r} \\ h &= |\vec{h}| & n &= |\vec{n}| & e &= |\vec{e}| \end{aligned}$$

- Compute energy to get a :

$$\begin{aligned} \varepsilon &= \frac{v^2}{2} - \frac{\mu}{r} & a &= -\frac{\mu}{2\varepsilon} \\ p &= \frac{h^2}{\mu} & a &= \frac{p}{1-e^2} \end{aligned}$$

- Compute inclination & Orientation Angles from above
 - $\Omega \rightarrow \text{If}(n_y < 0), \Omega = 360^\circ - \Omega$
 - $\omega \rightarrow \text{If}(e_z < 0), \omega = 360^\circ - \omega$
 - $\theta \rightarrow \text{If}(\vec{r} \cdot \vec{v} < 0), \theta = 360^\circ - \theta$

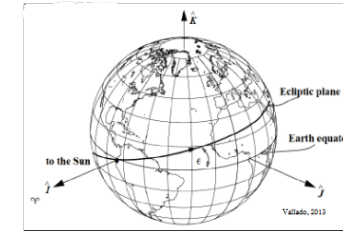
What about...?

- What is Ω for an elliptical equatorial orbit?
 - Undefined, used true longitude of periapse: $\tilde{\omega}_{true}$
 - $\cos \tilde{\omega}_{true} = \frac{\hat{X} \cdot \vec{e}}{|\hat{X}||\vec{e}|}$ where $\hat{X} = [1, 0, 0]$
 - If $e_y < 0 \rightarrow \tilde{\omega}_{true} = 360^\circ - \tilde{\omega}_{true}$
- What is ω for a circular orbit? (No perigee) \rightarrow undefined!

- Use argument of latitude
 - $\cos u = \frac{\vec{n} \cdot \vec{r}}{|\vec{n}||\vec{r}|} \rightarrow \text{If}(r_z < 0), \text{then } u = 360^\circ - u$
- Circular, equatorial orbits? Both ω and Ω undefined!
 - $\cos \lambda_{true} = \frac{\hat{X} \cdot \vec{r}}{|\hat{X}||\vec{r}|} \rightarrow \text{If}(r_y < 0), \text{then } \lambda_{true} = 360^\circ - \lambda_{true}$

Coordinate Frames

Ecliptic Plane:



- Mean plane of the Earth's orbit around the Sun.
- Earth's equatorial plane is inclined about 23.5° relative to ecliptic.
- Vernal Equinox Υ :
 - intersection of Sun's path relative to the Earth with the equatorial plane, as Sun moves from South to North.
 - Occurs at ascending node of the Sun as viewed from Earth
 - Used as reference direction for defining inertial reference frames

ICRF: International Celestial Reference Frame

- Close representation of inertial frame (rotates a little)
- Nonrotating with respect to extragalactic radio sources (quasars)
- Center: Barycenter of the Solar System

Earth-Centered Mean Equatorial J2000 System (EME2000):

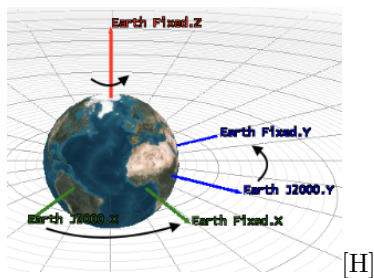
- Origin: Earth Center
- \hat{X} : Vernal Equinox at 1/1/2000 12:00:00 TT
- \hat{Z} : Normal to the mean equatorial plane of Earth at the J2000 epoch (Spin axis of Earth)

- \hat{Y} : Completes right hand coordinate frame

Earth-Centered Mean Orbit and Equinox of J2000 (EMO2000):

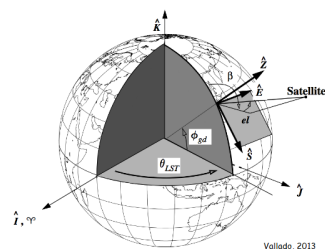
- Origin: Earth Center
- \hat{X} : Vernal Equinox at 1/1/2000 12:00:00 TT
- \hat{Z} : Orbit normal vector at same time
- \hat{Y} : Completes right hand coordinate frame
- Differs from EME2000 by 23.4393°

Earth-Centered Earth-Fixed:



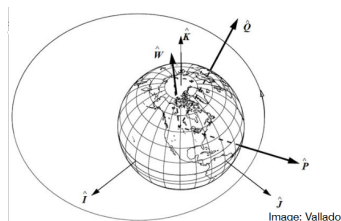
- Origin: Earth Center
- \hat{X} : Osculating vector from center of Earth toward the equator along the Prime Meridian (rotates with Earth)
- \hat{Z} : Aligned with Earth's spin axis
- \hat{Y} : Completes right hand coordinate frame

Topocentric Horizon System:



- Useful for observing satellites and sensor systems
- Origin = site on Earth
- SEX frame rotates with site
- Local Horizon forms fundamental plane:
 - \hat{S} : Points due South
 - \hat{E} : Points due East
 - \hat{Z} : Points radially outward
 - β : Azimuth, angle measured from North, clockwise to location beneath object of interest
 - el: Elevation, measured from local horizon, positive up to the object $[-90^\circ, 90^\circ]$

Perifocal Coordinate System:



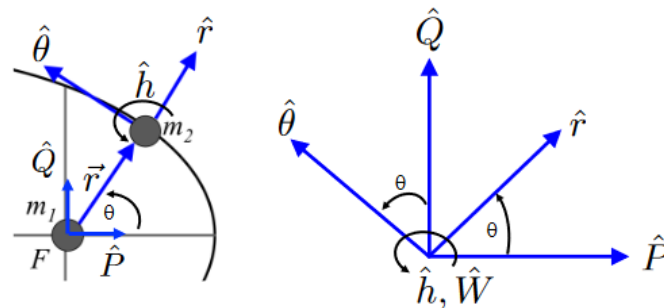
- Fundamental Plane: Satellite Orbit
- Origin: Center of Earth
- \hat{P} : Points toward perigee
- \hat{Q} : 90° from \hat{P} axis in direction of satellite motion
- \hat{W} : normal to the orbit

Coordinate Frame Transformations

Generic:

$$\begin{aligned} X, I \quad R_1(\alpha) \quad ROT1(\alpha) &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & c\alpha & s\alpha \\ 0 & -s\alpha & c\alpha \end{bmatrix} \\ Y, J \quad R_2(\alpha) \quad ROT2(\alpha) &= \begin{bmatrix} c\alpha & 0 & -s\alpha \\ 0 & 1 & 0 \\ s\alpha & 0 & c\alpha \end{bmatrix} \\ Z, K \quad R_3(\alpha) \quad ROT3(\alpha) &= \begin{bmatrix} c\alpha & s\alpha & 0 \\ -s\alpha & c\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix} \end{aligned}$$

Perifocal-Rotating Transformation:



$$\begin{bmatrix} r \\ \theta \\ h \end{bmatrix} = \begin{bmatrix} c\theta & s\theta & 0 \\ -s\theta & c\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} P \\ Q \\ W \end{bmatrix}$$

Express position and velocity in PQW frame to Cartesian:

$$\begin{aligned} \vec{r}_{PQW} &= r \cos \theta \hat{P} + r \sin \theta \hat{Q} = \begin{bmatrix} \frac{p \cos \theta}{1+e \cos \theta} \\ \frac{p \sin \theta}{1+e \cos \theta} \\ 0 \end{bmatrix} \\ \vec{v}_{PQW} &= \begin{bmatrix} \dot{r} \cos \theta - r \dot{\theta} \sin \theta \\ \dot{r} \sin \theta + r \dot{\theta} \cos \theta \\ 0 \end{bmatrix} = \begin{bmatrix} -\sqrt{\frac{\mu}{p}} \sin \theta \\ \sqrt{\frac{\mu}{p}} (e + \cos \theta) \\ 0 \end{bmatrix} \end{aligned}$$

$$\begin{aligned} [C] &= R_3(-\Omega) R_1(-i) R_3(-\omega) \\ \vec{r}_{xyz} &= [C] \vec{r}_{PQW} \quad \vec{v}_{xyz} = [C] \vec{v}_{PQW} \\ \text{To go from XYZ to PQW:} \\ \vec{r}_{PQW} &= R_3(\omega) R_1(i) R_3(\Omega) \vec{r}_{xyz} \end{aligned}$$

Ground Track

Earth rotates 15.04° /hour eastward beneath the satellite. Ground track advances westward at this rate.

We can relate the distance between successive ground track crossing to the orbital period using Earth's rotation:

$$\mathbb{P} = \frac{D}{15.04^\circ/\text{hour}}$$