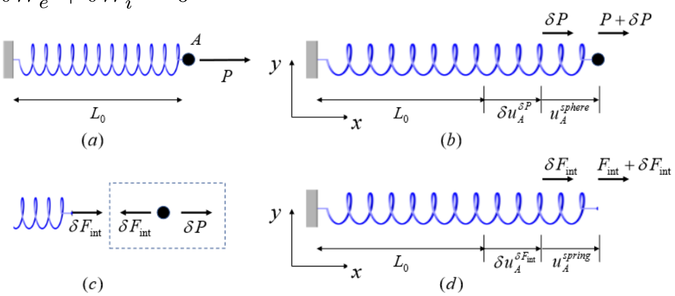
Marissa Palamara
ASEN 3112
Fall 2020

Virtual Force

$\delta W_e^* + \delta W_i^* = 0$



$\delta W_e^* = u_A^{sphere}\delta P \rightarrow \delta F_{int} = \delta P$

$\delta W_e^* = -u_A^{spring}\delta F_{int}$

$\left(u_A^{sphere} - u_A^{spring}\right)\delta P = 0$

This shows that the sum of the external and internal virtual work due to an external virtual force (or moment) vanishes for structure in static equilibrium, if the displacements and deformations are compatible.

Virtual Force Method for Computing Deflections

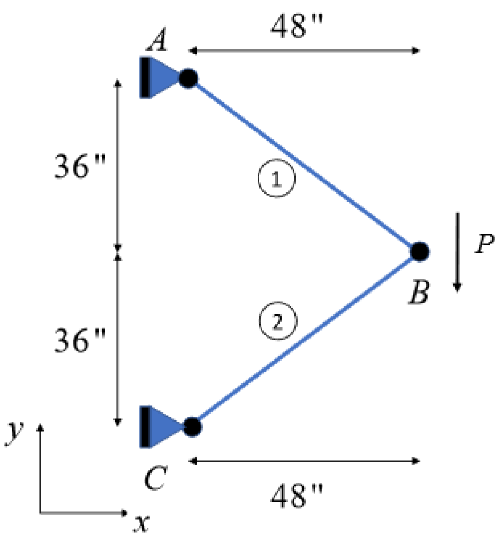
$\delta W_e^* = u\delta P$ or $\delta W_E^* = \theta\delta M$

Where u is the real displacement of the point at which the virtual force is applied.
The internal virtual work can be written for a multi-component member as follows:

$\delta W_{ie}^* = \sum_{N_m} \delta F_{int}\Delta$

Replace δP with 1, $\bar{1}u = \sum_N \bar{f}_{int}\Delta$

Truss Example



$\delta W_{ie,bar}^* = \int_L \varepsilon\bar{\sigma}Adx \rightarrow \delta W_{ie,bar}^* = \int_L \frac{\sigma}{E}\bar{\sigma}Adx$

For the case that the real and virtual stresses are constant in the bar, the above expression can be expressed in terms of the real internal force, N, and the virtual force, $\bar{n}$, as follows:

$\delta W_{ie,bar}^* = \frac{N\bar{n}L}{EA}$

For a truss composed of multiple bars:

$\delta W_{ie,truss}^* = \sum_{i=1}^{N_b} \frac{N_i\bar{n}_iL_i}{E_iA_i}$

Using the balance of external and internal virtual work, i.e. $W_e^* = \delta W_{ie,truss}^*$, and applying a dummy load in a particular direction, the displacement, d, at the chosen joint in this direction can be computed by:

$\bar{1}d = \sum_{i=1}^{N_b} \frac{N_i\bar{n}_iL_i}{E_iA_i}$

Applying to example above:

Step 1: Compute internal forces, N_i, in the bars due to real load P. Truss must be statically determinate.

$N_1 = \frac{5}{6}P$ and $N_2 = -\frac{5}{6}P$

Step 2: Compute internal forces, $\bar{n}_i$, in the bars due to the dummy loads $\bar{1}$. First, we apply a dummy load in the horizontal direction to compute the horizontal displacement.

$\bar{n}_1^u = \frac{5}{8}$ and $\bar{n}_2^u = \frac{5}{8}$

Do the same for a vertical dummy load at joint B.

$\bar{n}_1^v = -\frac{5}{6}$ and $\bar{n}_2^v = \frac{5}{6}$

Step 3: To evaluate the internal work, summarize in a table:

bar	N_i	$\bar{n}_i^u$	$\bar{n}_i^v$	A_i	L_i	E_i
1	$\frac{5}{6}P$	$\frac{5}{8}$	$-\frac{5}{6}$	0.15	60.0	$3 \cdot 10^6$
2	$-\frac{5}{6}P$	$\frac{5}{8}$	$\frac{5}{6}$	0.20	60.0	$3 \cdot 10^6$

Step 4: For each dummy load, evaluate the balance of external and internal forces.
For the displacement in horizontal:

$\bar{1}u = \sum_{i=1}^{2} \frac{N_i\bar{n}_i^uL_i}{E_iA_i} = 8.33 \cdot 10^{-3}in$

Vertical:

$\bar{1}v = \sum_{i=1}^{2} \frac{N_i\bar{n}_i^vL_i}{E_iA_i} = -44.4 \cdot 10^{-3}in$

Thermal Loading

Assuming that the material properties are constant in the bar and expressing the virtual stress in terms of the internal force, $\bar{n}$, we obtain:

$\delta W_{ie,bar}^{*,thermal} = \alpha\Delta T\bar{n}L$

Defining an internal force due to differential heating/cooling as:

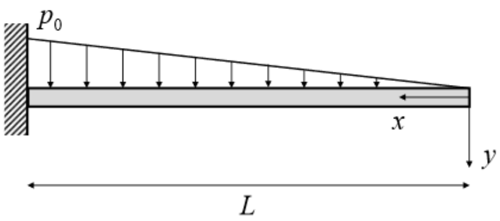
$N^{thermal} = EA\alpha\Delta T$

we can write the internal virtual work by replacing the internal force due to mechanical loading, N, with one for thermal loading, $N^{thermal}$.

$\delta W_{ie,bar}^* = \frac{N^{thermal}\bar{n}L}{EA}$

From here follow same procedure as previous.

Beam Example



First compute internal virtual work of the beam by virtual stress due to a unit dummy load.

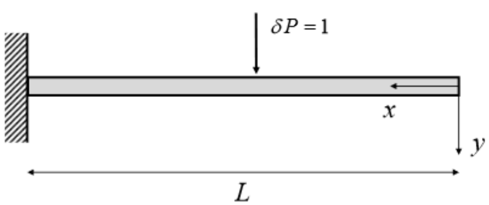
$\delta W^*_{ie,beam} = \int\int\int_V \varepsilon\bar{\sigma}dxdydz$

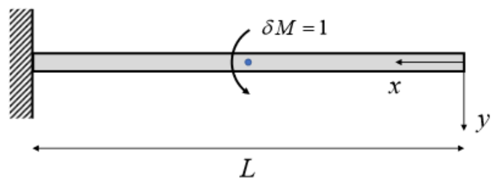
Use Hook's Law: $\delta W^*_{ie,beam} = \int\int\int_V \frac{\sigma}{E}\bar{\sigma}dxdydz$

Recall: $\sigma = -\frac{M}{I}y$

Finally:

$\delta W^*_{ie,beam} = \int_L \frac{M\bar{m}}{EI}dx$ where $\bar{m}$ is virtual bending moment due to unit dummy load.





Step 1: Compute the bending moment due to the real force.

$M = \frac{p_0 x^3}{6L}$

Step 2: Compute bending moments due to unit dummy foce and unit dummy moment.

$\bar{m}^v = 0$ for $0 \leq x < \frac{L}{2}$ and $\bar{m}^v = x - \frac{L}{2}$ for $\frac{L}{2} \leq x \leq L$

$\bar{m}^\phi = 0$ for $0 \leq x < \frac{L}{2}$ and $\bar{m}^\phi = -1$ for $\frac{L}{2} \leq x \leq L$

Step 3: To compute the displacement and the rotation in the middle of the beam, compute the internal work due to the two dummy load cases.

$v\left(x = \frac{L}{2}\right) = \int_L \frac{M\bar{m}^v}{EI}dx = \int_{l/2}^{L} \frac{(x-\frac{L}{2})p_0 x^3}{6EIL}dx$

$v\left(\frac{L}{2}\right) = \frac{49 p_0 L^4}{3840 EI}$

Then do the same thing for ϕ

The Direct Stiffness Method

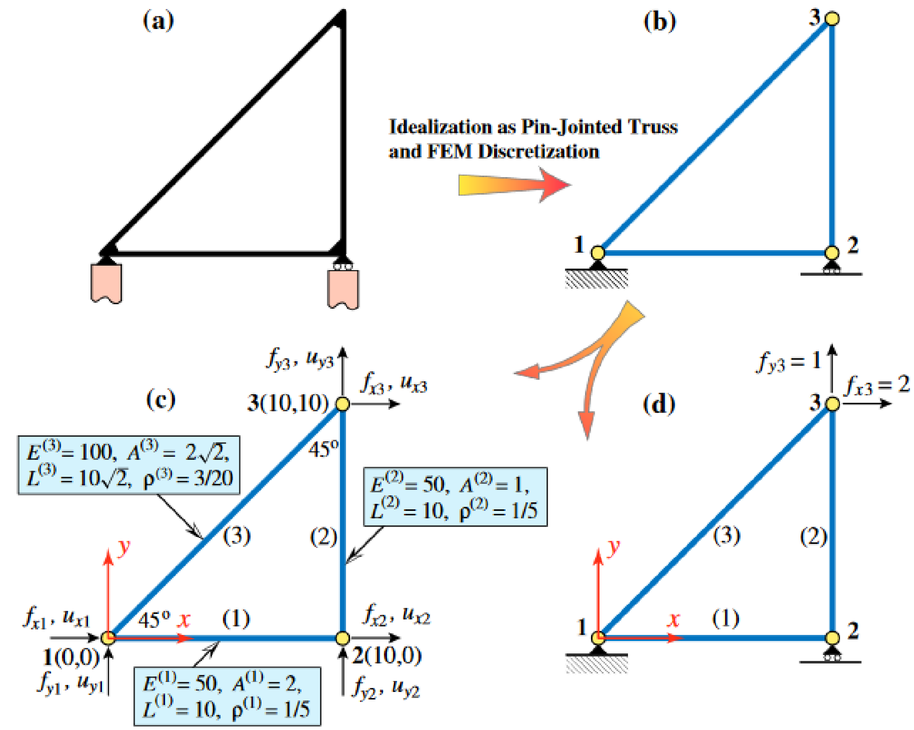


Figure 16.4: The three-member example truss: (a) physical structure; (b) idealization as a pin-jointed bar assemblage; (c) geometric, material and fabrication properties; (d) support conditions and applied loads.

$$\mathbf{f} = \begin{bmatrix} f_{x1} \\ f_{y1} \\ f_{x2} \\ f_{y2} \\ f_{x3} \\ f_{y3} \end{bmatrix} \quad \mathbf{u} = \begin{bmatrix} u_{x1} \\ u_{y1} \\ u_{x2} \\ u_{y2} \\ u_{x3} \\ u_{y3} \end{bmatrix}$$

Master Stiffness Equations

The master stiffness equations relate the joint forces **f** of the complete structure to the joint displacements **u** of the complete structure before specification of support conditions.

$$\begin{bmatrix} f_{x1} \\ f_{y1} \\ f_{x2} \\ f_{y2} \\ f_{x3} \\ f_{y3} \end{bmatrix} = \begin{bmatrix} K_{x1x1} & K_{x1y1} & K_{x1x2} & K_{x1y2} & K_{x1x3} & K_{x1y3} \\ K_{y1x1} & K_{y1y1} & K_{y1x2} & K_{y1y2} & K_{y1x3} & K_{y1y3} \\ K_{x2x1} & K_{x2y1} & K_{x2x2} & K_{x2y2} & K_{x2x3} & K_{x2y3} \\ K_{y2x1} & K_{y2y1} & K_{y2x2} & K_{y2y2} & K_{y2x3} & K_{y2y3} \\ K_{x3x1} & K_{x3y1} & K_{x3x2} & K_{x3y2} & K_{x3x3} & K_{x3y3} \\ K_{y3x1} & K_{y3y1} & K_{y3x2} & K_{y3y2} & K_{y3x3} & K_{y3y3} \end{bmatrix} \begin{bmatrix} u_{x1} \\ u_{y1} \\ u_{x2} \\ u_{y2} \\ u_{x3} \\ u_{y3} \end{bmatrix}$$

$$\mathbf{f} = \mathbf{K}\mathbf{u}$$

Where **K** is the master stiffness matrix or the global stiffness matrix.

Breakdown Stage

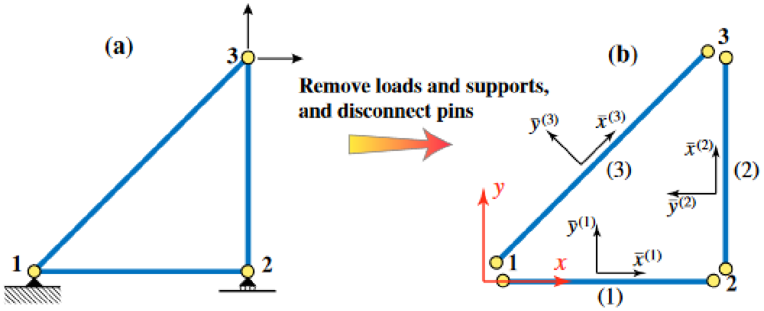


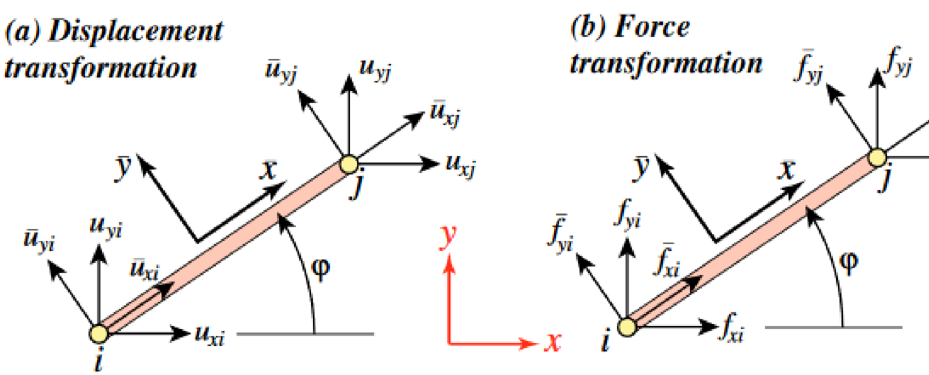
Figure 17.2: Disconnection step: (a) idealized example truss; (b) removal of loads and support, disconnection into members (1), (2) and (3), and selection of local coordinate systems. The latter are drawn offset from member axes for visualization convenience.

Begin by discarding all loads and supports (boundary conditions). Then disconnect and disassemble into components. The local coordinate system is $\{\bar{x}, \bar{y}\}$

$$\begin{bmatrix} \bar{f}_{xi} \\ \bar{f}_{yi} \\ \bar{f}_{xj} \\ \bar{f}_{yj} \end{bmatrix} = \begin{bmatrix} \bar{K}_{xixi} & \bar{K}_{xiyi} & \bar{K}_{xixj} & \bar{K}_{xiyj} \\ \bar{K}_{yixi} & \bar{K}_{yiyi} & \bar{K}_{yixj} & \bar{K}_{yiyj} \\ \bar{K}_{xjxi} & \bar{K}_{xjyi} & \bar{K}_{xjxj} & \bar{K}_{xjyj} \\ \bar{K}_{yjxj} & \bar{K}_{yjyi} & \bar{K}_{yjxj} & \bar{K}_{yjyj} \end{bmatrix} \begin{bmatrix} \bar{u}_{xi} \\ \bar{u}_{yi} \\ \bar{u}_{xj} \\ \bar{u}_{yj} \end{bmatrix}$$

If membe properties are uniform along its length, $k_s = \frac{EA}{L}$ and, consequently, the force-displacement equation is $F = k_s d = \frac{EAd}{l}$ where F is the internal axial force and d is the relative axial displacement, which is physically the bar elongation.

$$\bar{\mathbf{K}} = \frac{EA}{L} \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$



$c = \cos\phi$ and $s = \sin\phi$ where ϕ is the angle formed by $\bar{x}$ and x, measured CCW from x.

$$\begin{bmatrix} \bar{u}_{xi} \\ \bar{u}_{yi} \\ \bar{u}_{xj} \\ \bar{u}_{xy} \end{bmatrix} = \begin{bmatrix} c & s & 0 & 0 \\ -s & c & 0 & 0 \\ 0 & 0 & c & s \\ 0 & 0 & -s & c \end{bmatrix} \begin{bmatrix} u_{xi} \\ u_{yi} \\ u_{xj} \\ u_{yj} \end{bmatrix} \qquad \begin{bmatrix} f_{xi} \\ f_{yi} \\ f_{xj} \\ f_{yj} \end{bmatrix} = \begin{bmatrix} c & -s & 0 & 0 \\ s & c & 0 & 0 \\ 0 & 0 & c & -s \\ 0 & 0 & s & c \end{bmatrix} \begin{bmatrix} \bar{f}_{xi} \\ \bar{f}_{yi} \\ \bar{f}_{xj} \\ \bar{f}_{yj} \end{bmatrix}$$

Global Member Stiffness Equations

$$\mathbf{K}^e = \frac{E^e A^e}{L^e} \begin{bmatrix} c^2 & sc & -c^2 & -sc \\ sc & s^2 & -sc & -s^2 \\ -c^2 & -sc & c^2 & sc \\ -sc & -s^2 & sc & s^2 \end{bmatrix}$$

Merging it All

1. *Compatibility of displacement:* The displacements of all members that meet are a joint are the same.
   $u_{x3}^{(2)} = u_{x3}^{(3)}$, $u_{y3}^{(2)} = u_{y3}^{(3)}$
2. *Force equilibrium:* The sum of internal forces exerted by all members that meet at a joint balances the external force applied to that joint.
   $f_{x3} = f_{x3}^{(2)} + f_{x3}^{(3)} = f_{x3}^{(1)} + f_{x3}^{(2)} + f_{x3}^{(3)}$, $f_{y3} = f_{y3}^{(2)} + f_{y3}^{(3)} = f_{y3}^{(1)} + f_{y3}^{(2)} + f_{y3}^{(3)}$
   The addition of $f_{x3}^{(1)}$ does nothing because member 1 is not connected to joint 3.

Finally,

$$\mathbf{f} = \mathbf{f}^{(1)} + \mathbf{f}^{(2)} + \mathbf{f}^{(3)} = \left(\mathbf{K}^{(1)} + \mathbf{K}^{(2)} + \mathbf{K}^{(3)}\right)\mathbf{u} = \mathbf{K}\mathbf{u}$$

Solution

The best way to account for support conditions is to remove equations that are associated with known zero joint displacements from the master system. This should result in a system of equations that is solveable for the displacements left because the forces at those displacements should be known.

Post Processing

Recovering Reaction Forces:
Multiply complete displacement solution found above by $\mathbf{K}$.
Recovery of Internal Forces & stresses:
The average axial stress σ^e is found by dividing F^e by A^e.
The axial force F^e in member e can be found by:

$$d^e = \bar{u}_{xj}^e - \bar{u}_{xi}^e \quad F^e = \frac{E^e A^e}{L^e} d^e$$

FEM Analysis of Plane Beam Structure

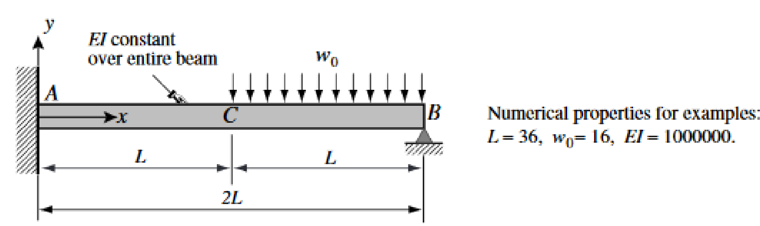
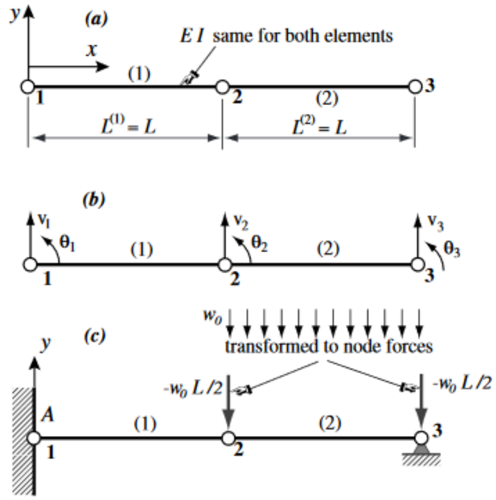


Figure 19.1: Plane beam problem.

Beam finite elements are obtained by subdividing beam members longitudinally.

$$\mathbf{u}^{(e)} = \begin{bmatrix} v_i^{(e)} \\ \theta_i^{(e)} \\ v_j^{(e)} \\ \theta_j^{(e)} \end{bmatrix}, \quad \mathbf{f}^{(e)} = \begin{bmatrix} f_i^{(e)} \\ m_i^{(e)} \\ f_j^{(e)} \\ m_j^{(e)} \end{bmatrix}$$



This beam can be solved like the truss from here on out. See CH 19 page 3 for more detailed steps.

Analytical Solution by Discontinuity Functions

$$\langle x-a\rangle^n = \begin{cases}(x-a)^n & x>a\\ 0 & x\le a\end{cases}$$

Begin with:
$q(x) = -w_0\langle x-L\rangle^0$
Then integrate 4 times in x:

$$\begin{aligned}
V(x) &= -w_0\langle x-L\rangle^1 + C_1\\
M(x) &= -\tfrac{1}{2}w_0\langle x-L\rangle^2 + C_1x + C_2\\
EI\theta(x) = EIv'(x) &= -\tfrac{1}{6}w_0\langle x-L\rangle^3 + \tfrac{1}{2}C_1x^2 + C_2x + C_3\\
EIv(x) &= -\tfrac{1}{24}w_0\langle x-L\rangle^4 + \tfrac{1}{6}C_1x^3 + \tfrac{1}{2}C_2x^2 + C_3x + C_4
\end{aligned}$$

Then apply boundary conditions and solve.

Free Single-DOF Oscillator

A weight of mass $m>0$ hangs from an extensional spring of stiffness $k/ge0$ under gravity acceleration g directed along the spring direction. The weight force is $W = mg$. Assume small displacements.

$$\delta_s = W/k = mg/k$$

The only degree of freedom is $u = u(t)$.

Undamped Oscillator

Remove the spring and replace it by force $F_s = k(\delta_s + u(t))$. The other two forces acting on the mass are its weight and the inertia force $m\ddot{u}(t)$, which acts in the opposite direction to the acceleration $\ddot{u} = \ddot{u}(t)$.
Equilibrium of forces in the x direction requires $m\ddot{u} = W - k(\delta_s + u) = mg - k\delta_s - ku$.
On cancelling $mg - k\delta_s = 0$, we get:

$$\boxed{m\ddot{u} + ku = 0}$$

This results in the formal form:

$$\boxed{\ddot{u} + \omega_n^2 u = 0,\ \text{in which } \omega_n = \sqrt{\frac{k}{m}}}$$

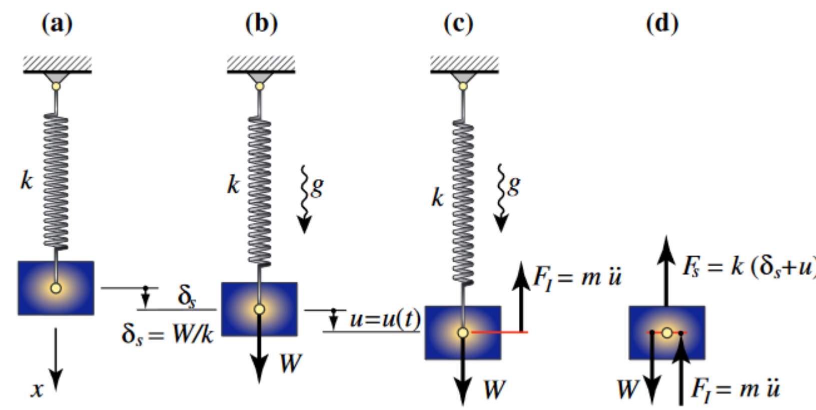


Figure 20.1: Undamped, unforced spring-mass SDOF oscillator undergoing free vibrations.

Solving for $u(t)$:

$$\begin{aligned}
u(t) &= C_1e^{i\omega_n t} + C_2e^{-i\omega_n t}\\
u(t) &= A_1\cos\omega_n t + A_2\sin\omega_n t\\
u(t) &= u_0\cos\omega_n t + \tfrac{v_0}{\omega_n}\sin\omega_n t\\
u(t) &= U\cos\omega_n t - \alpha\\
U &= \sqrt{u_0^2 + \left(\tfrac{v_o}{\omega_n}\right)^2}\\
\tan\alpha &= \tfrac{v_0}{\omega_n u_0}
\end{aligned}$$

If the mass is released from rest, and this $v_0 = 0$,

$$f_n = \tfrac{\omega_n}{2\pi} \text{ and } T_n = \tfrac{1}{f_n} = \tfrac{2\pi}{\omega_n}$$

Energy Conservation Property:

$$T = T_0 + \frac{1}{2}m\dot{u}^2 \text{ and } V = V_0 + \frac{1}{2}ku^2$$

$$H = T + V = T_0 + U_0 = \frac{1}{2}mv_o^2 + \frac{1}{2}ku_0^2$$

Free Vibrations of Viscous-Damped SDOF Oscillator

$$\boxed{m\ddot{u} + c\dot{u} + ku = 0}$$

$$w_n = \sqrt{\frac{k}{m}}\ , \ c = 2\xi\omega_n m$$

$$\boxed{\ddot{u} + 2\xi\omega_n\dot{u} + \omega_n^2 u = 0}$$

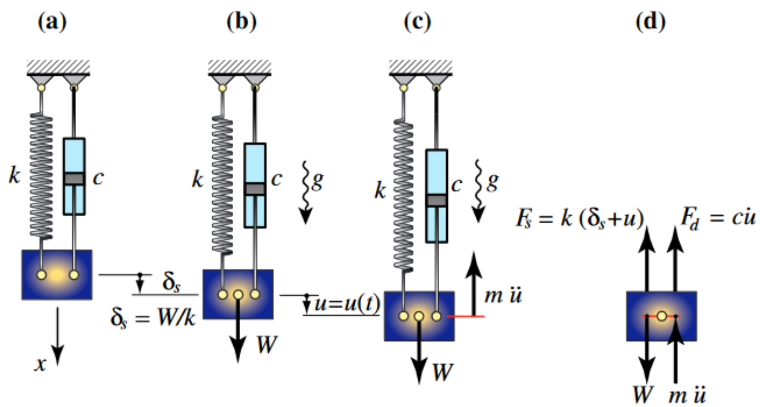


Figure 20.3: Damped spring-dashpot-mass oscillator undergoing free vibrations: (a) initial unloaded position; (b) static equilibrium position; (c) dynamic equilibrium position; (d) Dynamic Free Body Diagram (DFBD).

Solving for $u(t)$:

$$\begin{aligned} u(t) &= Ae^{\lambda t} \\ 0 &= \lambda^2 + 2\xi\omega_n\lambda + \omega_n^2 \\ \lambda_{1,2} &= -\xi\omega_n \pm \omega_n\sqrt{\zeta^2 - 1} \end{aligned}$$

$\xi < 1$ *Underdamped case:* Damping is subcritical. The roots $\lambda_{1,2}$ are complex conjugate. The motion is oscillatory with decreasing amplitude. This is the most common case in typical structures.
$\xi > 1$ *Overdampled case:* Damping is called overcritical. The roots $\lambda_{1,2}$ are negative real and distinct. The motion is non-oscillatory. Its amplitude decays monotonically except possibly for one zero crossing.
$\xi = 1$ *Critically Damped case:* The roots $\lambda_{1,2}$ are negative real and coalesce. The motion is non-oscillatory. Its amplitude decays monotonically except possibly for one zero crossing. Most rapid decay.

Underdamped Case: $\xi < 1$

$$\begin{aligned} \lambda_{1,2} &= -\xi\omega_n \pm i\omega_d \\ \omega_d &= \omega_n\sqrt{1-\xi^2} \\ T_d &= \tfrac{2\pi}{\omega_d} \\ u(t) &= e^{-\xi\omega_n t}\left(u_0\cos\omega_d t + \tfrac{v_0 + \xi\omega_n u_0}{\omega_d}\sin\omega_d t\right) \end{aligned}$$

Critically Damped: $\xi = 1$

$$u(t) = [u_0 + (v_0 + \omega_n u_0)t]e^{-\omega_n t}$$

Overdamped Case: $\xi > 1$

$$\begin{aligned} \omega^* &= \omega_n\sqrt{\xi^2 - 1} \\ u(t) &= e^{-\xi\omega_n t}\left[u_0\cosh\omega_* t + \tfrac{v_0 + \xi\omega_n u_0}{\omega^*}\sinh\omega^* t\right] \end{aligned}$$

Energy Dissipation: Page 10 of Ch 20.

Harmonically Forced SDOF Oscillator

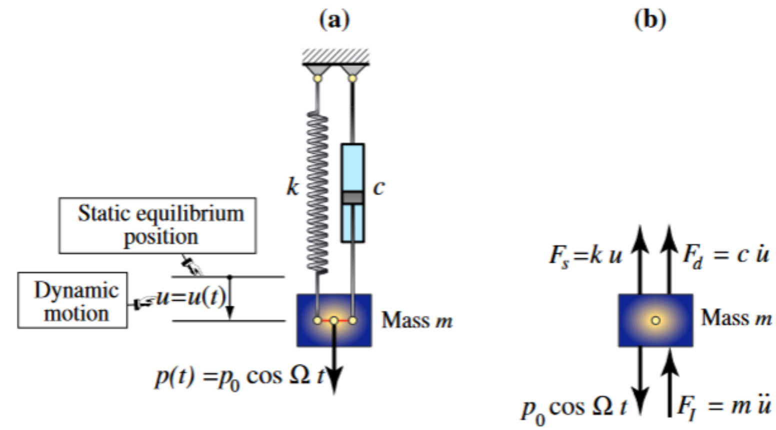


Figure 21.1: Damped spring-dashpot-mass oscillator under prescribed harmonic force excitation: (a) dynamic equilibrium position; (b) dynamic FBD.

Harmonic Excitation Force: $p(t) = p_0\cos\Omega t$ where Ω is the excitation frequency. $u = u(t)$ is the dynamic motion measured from the static equilibrium position.

$$\boxed{m\ddot{u} + c\dot{u} + ku = p_0\cos\Omega t}$$

The complete response will be the sum of the transient (homogeneous) and steady-state (particular) components. If damping is present, after a while the transient part becomes unimportant.

$$\boxed{u + p(t) = U\cos(\Omega t - \alpha)}$$

$$\dot{u}_p(t) = -\Omega U\sin(\Omega t - \alpha),\ \ddot{u}_p(t) = -\Omega^2 U\cos(\Omega t - \alpha)$$

$$\begin{aligned} &(kU - m\Omega^2U)^2 + (c\Omega U)^2 = p_0^2,\ \tan\alpha = \tfrac{c\Omega}{k - m\Omega^2} \\ &U_0 = \tfrac{p_0}{k},\ r = \tfrac{\Omega}{\omega_n} \end{aligned}$$

U_0 is the displacement that the mass would undergo if a force of mangitude p_0 where to be applied statically.

$$\boxed{D_s(r) = \frac{U(r)}{U_0} = \frac{1}{\sqrt{(1-r^2)^2 + (2\xi r)^2}},\ \tan\alpha(r) = \frac{2\xi r}{1 - r^2}}$$

D_s is the steady-state magnification factor, also known as gain. The peaks below observed as r is near unity and $\xi << 1$ identify resonance.

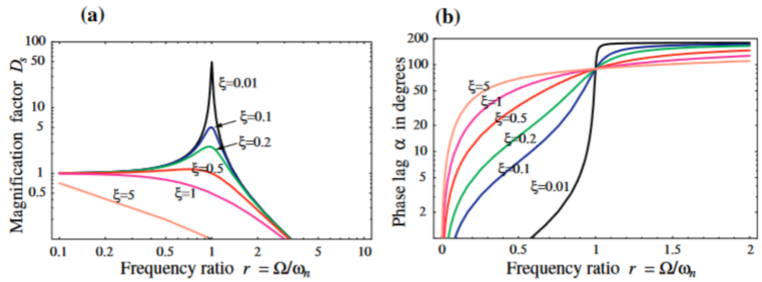


Figure 21.3: Frequency response log plots for harmonically forced viscous-damped oscillator: (a) log-log plot of magnification factor versus frequency ratio; (b) log plot of phase lag angle versus frequency ratio.

Total Response

$$\boxed{u(t) = U_0 D_s \cos \Omega t - \alpha + u_h(t), \text{ in which } u_h(t) = e^{-\xi\omega_n t}\left(A_1 \cos \omega_n t + A_2 \sin \omega_n t\right)}$$

Response to Harmonic Base Excitation

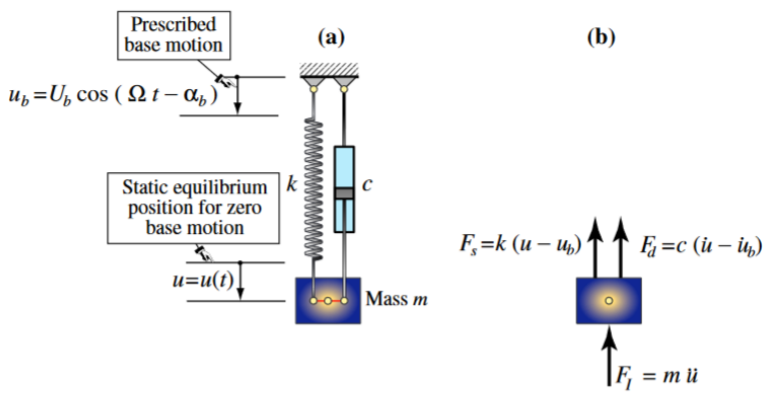


Figure 21.4: Damped spring-dashpot-mass oscillator under prescribed harmonic base motion: (a) dynamic equilibrium position; (d) dynamic FBD.

The base displaces by a prescribed harmonic motion specified as:

$$u_b(t) = U_b \cos \Omega t - \alpha$$

$$m\ddot{u} + k(u - u_b) + c(\dot{u} - \dot{u}_b) = 0$$

$$\boxed{m\ddot{u} + c\dot{u} + ku = ku_b + c\dot{u}_b}$$

If the base excitation is harmonic:

$$\boxed{m\ddot{u} + c\dot{u} + ku = U_b\left[k\cos\left(\Omega t - \alpha_b\right) + c\Omega \sin\left(\Omega t - \alpha_b\right)\right]}$$

If the damping vanishes, $c = 0$.

MDOF Dynamical Systems

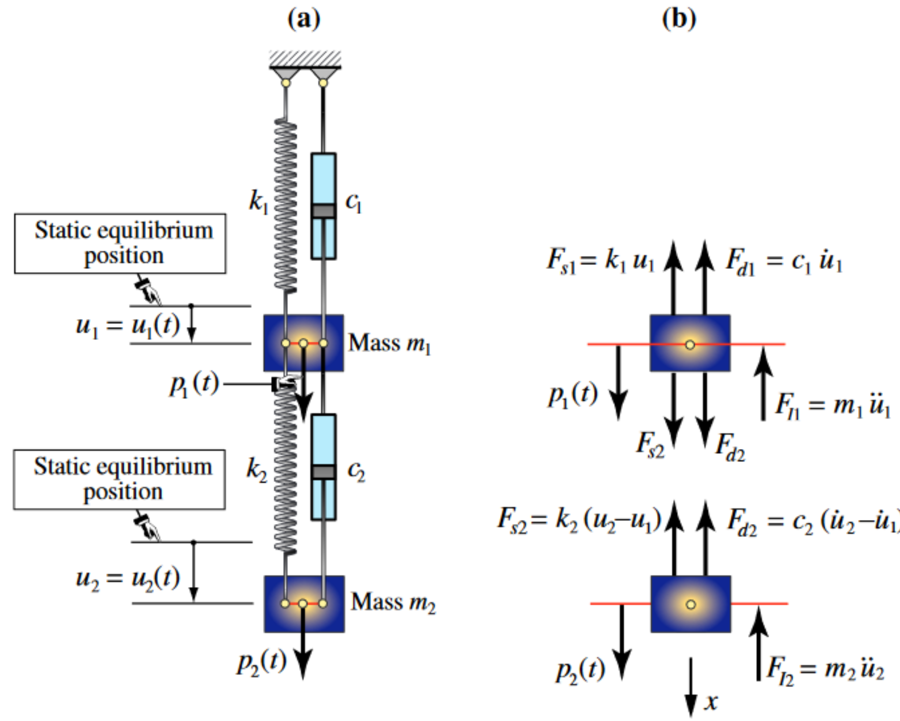


Figure 22.1: Two-DOF, lumped parameter, spring-mass-dashpot dynamic system.

$$\Sigma F_x \text{at mass 1:} \;\; -F_{I1} - F_{s1} - F_{d1} + F_{s2} + F_{d2} + p_1 = 0$$

$$\Sigma F_x \text{at mass 2:} \;\; -F_{I2} - F_{s2} - F_{d2} + p_2 = 0$$

Replace with quantities shown in figure above.

$$\boxed{\begin{array}{rcl} m_1\ddot{u}_1 + c_1\dot{u}_1 + c_2\dot{u}_2 + k_1u_1 + k_2u_1 - k_2u_2 & = & p_1 \\ m_2\ddot{u}_2 - c_2\dot{u}_1 + c_2\dot{u}_2 - k_2u_1 + k_2u_2 & = & p_2 \end{array}}$$

Matrix Form

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}\begin{bmatrix} \ddot{u}_1 \\ \ddot{u}_2 \end{bmatrix} + \begin{bmatrix} c_1 + c_2 & -c_2 \\ -c_2 & c_2 \end{bmatrix}\begin{bmatrix} \dot{u}_1 \\ \dot{u}_2 \end{bmatrix} + \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} p_1 \\ p_2 \end{bmatrix} \tag{22.5}$$

Passing to compact notation,

$$\boxed{\mathbf{M}\,\ddot{\mathbf{u}} + \mathbf{C}\,\dot{\mathbf{u}} + \mathbf{K}\,\mathbf{u} = \mathbf{p}.} \tag{22.6}$$

Undamped Natural Frequencies and Modes

Assuming undamped, $c_1 = c_2 = 0$, and unforced, $p_1 = p_2 = 0$, the matrix system reduces to:

$$\boxed{\mathbf{M\ddot{u}} + \mathbf{Ku} = \mathbf{0}}$$

Assuming the undamped, unforced system is undergoing in-phase harmonic motions of circular frequency ω.

$$\mathbf{u(t)} = \mathbf{U}\cos(\omega\mathbf{t} - \alpha), \text{ or } \mathbf{u(t)} = \mathbf{U}\mathbf{e}^{\mathbf{i}\omega\mathbf{t}}$$

$$\boxed{[\mathbf{K} - \omega^{\mathbf{2}}\mathbf{M}]\mathbf{U} = \mathbf{D}(\omega)\mathbf{U} = \mathbf{0}}$$

For nontrivial solution, $\mathbf{U} \neq \mathbf{0}$, the determinant of the dynamic matrix must vanish:

$$\boxed{C(\omega^2) = \det\mathbf{D}(\omega) = \det[\mathbf{K} - \omega^{\mathbf{2}}\mathbf{M}] = \mathbf{0}}$$

$$f_i = \frac{\omega_i}{2\pi},\ T_i = \frac{1}{f_i} = \frac{2\pi}{\omega_i}$$

How about the $\mathbf{U}$ vector? Insert $\omega^2 = \omega_i^2$ and solved for $\mathbf{U} \neq \mathbf{0}$. These two U vectors are eigenvectors (modes).
Use null function in MATLAB and divide by last index in matrix.