Asymptotic Notation: Definitions and Examples

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Definitions

Let f be a nonnegative function. Then we define the three most common asymptotic bounds as follows.

- We say that $f(n)$ is **Big-O** of $g(n)$, written as $f(n) = O(g(n))$, iff there are positive constants c and n_0 such that
$$0 \leq f(n) \leq c\, g(n) \text{ for all } n \geq n_0$$
If $f(n) = O(g(n))$, we say that $g(n)$ is an **upper bound** on $f(n)$.

- We say that $f(n)$ is **Big-Omega** of $g(n)$, written as $f(n) = \Omega(g(n))$, iff there are positive constants c and n_0 such that
$$0 \leq c\, g(n) \leq f(n) \text{ for all } n \geq n_0$$
If $f(n) = \Omega(g(n))$, we say that $g(n)$ is a **lower bound** on $f(n)$.

- We say that $f(n)$ is **Big-Theta** of $g(n)$, written as $f(n) = \Theta(g(n))$, iff there are positive constants c_1, c_2 and n_0 such that
$$0 \leq c_1\, g(n) \leq f(n) \leq c_2\, g(n) \text{ for all } n \geq n_0$$
Equivalently, $f(n) = \Theta(g(n))$ if and only if $f(n) = O(g(n))$ and $f(n) = \Omega(g(n))$. If $f(n) = \Theta(g(n))$, we say that $g(n)$ is a **tight bound** on $f(n)$.

Note: Sometimes the notation $f(n) \in O(g(n))$ is used instead of $f(n) = O(g(n))$ (similar for Ω and Θ). These mean essentially the same thing, and the use of either is generally personal preference.

Proving Bounds

There are two common ways of proving bounds. The first is according to the definitions above. The second, and much easier approach, uses the following theorem.

Theorem 1
Let $f(n)$ and $g(n)$ be functions such that
$$\lim_{x \to \infty} \frac{f(n)}{g(n)} = A.$$
Then

1. *If $A = 0$, then $f(n) = O(g(n))$, and $f(n) \neq \Theta(g(n))$.*
2. *If $A = \infty$, then $f(n) = \Omega(g(n))$, and $f(n) \neq \Theta(g(n))$.*
3. *If $A \neq 0$ is finite, then $f(n) = \Theta(g(n))$.*

Notice that if the above limit does not exist, then the first technique should be used. Luckily, in the analysis of algorithms the above approach works most of the time.
Now is probably a good time to recall a very useful theorem for computing limits, called **l'Hopital's Rule**.

Theorem 2: *l'Hopital's Rule*
Let $f(x)$ and $g(x)$ be differentiable functions. If $\lim_{x\to\infty} f(x) = \lim_{x\to\infty} g(x) = 0$ or $\lim_{x\to\infty} f(x) = \lim_{x\to\infty} g(x) = \infty$, then

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \frac{f'(x)}{g'(x)}$$

Examples

We present several examples of proving theorems about asymtotic bounds and proving bounds on several different functions.

1. Prove that if $f(x) = O(g(x))$, and $g(x) = O(f(x))$, then $f(x) = \Theta(g(x))$.
   Proof:
   If $f(x) = O(g(x))$, then there are positive constants c_2 and n_0' such that

   $$0 \le f(n) \le c_2\, g(n) \text{ for all } n \ge n_0'$$

   Similarly, if $g(x) = O(f(x))$, then there are positive constants c_1' and n_0'' such that

   $$0 \le g(n) \le c_1'\, f(n) \text{ for all } n \ge n_0''.$$

   We can divide this by c_1' to obtain

   $$0 \le \frac{1}{c_1'} g(n) \le f(n) \text{ for all } n \ge n_0''.$$

   Setting $c_1 = 1/c_1'$ and $n_0 = \max(n_0', n_0'')$, we have

   $$0 \le c_1 g(n) \le f(n) \le c_2\, g(n) \text{ for all } n \ge n_0.$$

   Thus, $f(x) = \Theta(g(x))$.

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2. Let $f(x) = O(g(x))$ and $g(x) = O(h(x))$. Show that $f(x) = O(h(x))$.
   Proof:
   If $f(x) = O(g(x))$, then there are positive constants c_1 and n_0' such that

   $$0 \le f(n) \le c_1\, g(n) \text{ for all } n \ge n_0',$$

   and if $g(x) = O(h(x))$, then there are positive constants c_2 and n_0'' such that

   $$0 \le g(n) \le c_2\, h(n) \text{ for all } n \ge n_0''.$$

   Set $n_0 = \max(n_0', n_0'')$ and $c_3 = c_1\, c_2$. Then

   $$0 \le f(n) \le c_1\, g(n) \le c_1\, c_2\, h(n) = c_3\, h(n) \text{ for all } n \ge n_0.$$

   Thus $f(x) = O(h(x))$.

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3. Find a tight bound on $f(x) = x^8 + 7x^7 - 10x^5 - 2x^4 + 3x^2 - 17$.

Solution #1

We will prove that $f(x) = \Theta(x^8)$. First, we will prove an upper bound for $f(x)$. It is clear that when $x > 0$,

$$x^8 + 7x^7 - 10x^5 - 2x^4 + 3x^2 - 17 \leq x^8 + 7x^7 + 3x^2.$$

- *We can upper bound any function by removing the lower order terms with negative coefficients, as long as $x > 0$.*

Next, it is not hard to see that when $x \geq 1$,

$$x^8 + 7x^7 + 3x^2 \leq x^8 + 7x^8 + 3x^8 = 11x^8.$$

- *We can upper bound any function by replacing lower order terms with positive coefficients by the dominating term with the same coefficients. Here, we must make sure that the dominating term is larger than the given term for all values of x larger than some threshold x_0, and we must make note of the threshold value x_0.*

Thus, we have

$$f(x) = x^8 + 7x^7 - 10x^5 - 2x^4 + 3x^2 - 17 \leq 11x^8 \text{ for all } x \geq 1,$$

and we have proved that $f(x) = O(x^8)$.

Now, we will get a lower bound for $f(x)$. It is not hard to see that when $x \geq 0$,

$$x^8 + 7x^7 - 10x^5 - 2x^4 + 3x^2 - 17 \geq x^8 - 10x^5 - 2x^4 - 17.$$

- *We can lower bound any function by removing the lower order terms with positive coefficients, as long as $x > 0$.*

Next, we can see that when $x \geq 1$,

$$x^8 - 10x^5 - 2x^4 - 17 \geq x^8 - 10x^7 - 2x^7 - 17x^7 = x^8 - 29x^7.$$

- *We can lower bound any function by replacing lower order terms with negative coefficients by a sub-dominating term with the same coefficients. (By sub-dominating, I mean one which dominates all but the dominating term.) Here, we must make sure that the sub-dominating term is larger than the given term for all values of x larger than some threshold x_0, and we must make note of the threshold value x_0. Making a wise choice for which sub-dominating term to use is crucial in finishing the proof.*

Next, we need to find a value $c > 0$ such that $x^8 - 29x^7 \geq cx^8$. Doing a little arithmetic, we see that this is equivalent to $(1-c)x^8 \geq 29x^7$. When $x \geq 1$, we can divide by x^7 and obtain $(1-c)x \geq 29$. Solving for c we obtain

$$c \leq 1 - \frac{29}{x}.$$

If $x \geq 58$, then $c = 1/2$ suffices. We have just shown that if $x \geq 58$, then

$$f(x) = x^8 + 7x^7 - 10x^5 - 2x^4 + 3x^2 - 17 \geq \frac{1}{2}x^8.$$

Thus, $f(x) = \Omega(x^8)$. Since we have shown that $f(x) = \Omega(x^8)$ and that $f(x) = O(x^8)$, we have shown that $f(x) = \Theta(x^8)$.

Solution #2
We guess (or know, if we read Solution #1) that $f(x) = \Theta(x^8)$. To prove this, notice that

$$\begin{aligned}
\lim_{x\to\infty} \frac{x^8 + 7x^7 - 10x^5 - 2x^4 + 3x^2 - 17}{x^8} &= \lim_{x\to\infty} \frac{x^8}{x^8} + \frac{7x^7}{x^8} - \frac{10x^5}{x^8} - \frac{2x^4}{x^8} + \frac{3x^2}{x^8} - \frac{17}{x^8} \\
&= \lim_{x\to\infty} 1 + \frac{7}{x} - \frac{10}{x^3} - \frac{2}{x^4} + \frac{3}{x^6} - \frac{17}{x^8} \\
&= \lim_{x\to\infty} 1 + 0 - 0 - 0 + 0 - 0 = 1
\end{aligned}$$

Thus, $f(x) = \Theta(x^8)$ by the Theorem.

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4. Find a tight bound on $f(x) = x^4 - 23x^3 + 12x^2 + 15x - 21$.

Solution #1
It is clear that when $x \geq 1$,

$$x^4 - 23x^3 + 12x^2 + 15x - 21 \leq x^4 + 12x^2 + 15x \leq x^4 + 12x^4 + 15x^4 = 28x^4.$$

Also,

$$x^4 - 23x^3 + 12x^2 + 15x - 21 \geq x^4 - 23x^3 - 21 \geq x^4 - 23x^3 - 21x^3 = x^4 - 44x^3 \geq \frac{1}{2}x^4,$$

whenever

$$\frac{1}{2}x^4 \geq 44x^3 \Leftrightarrow x \geq 88.$$

Thus

$$\frac{1}{2}x^4 \leq x^4 - 23x^3 + 12x^2 + 15x - 21 \leq 28x^4, \text{ for all } x \geq 88.$$

We have shown that $f(x) = x^4 - 23x^3 + 12x^2 + 15x - 21 = \Theta(x^4)$.

Solution #2
From Solution #1 we already know that $f(x) = \Theta(x^4)$. We verify this by noticing that

$$\begin{aligned}
\lim_{x\to\infty} x^4 - 23x^3 + 12x^2 + 15x - 21 &= \lim_{x\to\infty} \frac{x^4}{x^4} - \frac{23x^3}{x^4} + \frac{12x^2}{x^4} + \frac{15x}{x^4} - \frac{21}{x^4} \\
&= \lim_{x\to\infty} 1 - \frac{23}{x} + \frac{12}{x^2} + \frac{15}{x^3} - \frac{21}{x^4} \\
&= \lim_{x\to\infty} 1 - 0 + 0 + 0 - 0 = 1
\end{aligned}$$

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5. Show that $\log x = O(x)$.
Proof:

$$\lim_{x\to\infty} \frac{\log x}{x} = \lim_{x\to\infty} \frac{\frac{1}{x}}{1} = \lim_{x\to\infty} \frac{1}{x} = 0$$

Therefore, $\log n = O(n)$.

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6. Show that $n! = O(n^n)$
Proof:
Notice that when $n \geq 1$, $0 \leq n! = 1 \cdot 2 \cdot 3 \cdots n \leq n \cdot n \cdots n = n^n$. Therefore $n! = O(n^n)$ (Here $n_0 = 1$, and $c = 1$.)

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7. Show that $\log n! = O(n \log n)$
Proof:
In the previous problem, we showed that when $n \geq 1$, $n! \leq n^n$. Notice that $n! \geq 1$, so taking $\log$s of both sides, we obtain $0 \leq \log n! \leq \log n^n = n \log n$ for all $n \geq 1$. Therefore $\log n! = O(n \log n)$. (Here, $n_0 = 1$, and $c = 1$.)

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8. Find a good upper bound on $n \log(n^2 + 1) + n^2 \log n$.
Solution:
If $n > 1$,

$$\log(n^2 + 1) \leq \log(n^2 + n^2) = \log(2n^2) = (\log 2 + \log n^2) \leq (\log n + 2 \log n) = 3 \log n$$

Thus when $n > 1$,

$$0 \leq n \log(n^2 + 1) + n^2 \log n \leq n 3 \log n + n^2 \log n \leq 3n^2 \log n + n^2 \log n \leq 4n^2 \log n.$$

Thus, $n \log(n^2 + 1) + n^2 \log n = O(n^2 \log n)$.

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9. Show that $(\sqrt{2})^{\log n} = O(\sqrt{n})$, where $\log$ means $\log_2$.
Proof:
It is not too hard to see that

$$(\sqrt{2})^{\log n} = n^{\log \sqrt{2}} = n^{\log 2^{1/2}} = n^{\frac{1}{2} \log 2} = n^{\frac{1}{2}} = \sqrt{n}.$$

Thus it is clear that $(\sqrt{2})^{\log n} = O(\sqrt{n})$.

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10. Show that $2^x = O(3^x)$.
Proof #1:
This is easy to see since $\lim_{x \to \infty} \frac{2^x}{3^x} = \lim_{x \to \infty} \left(\frac{2}{3}\right)^x = \lim_{x \to \infty} 0.$

Proof #2:
If $x \geq 1$, then clearly $(3/2)^x \geq 1$, so

$$2^x \leq 2^x \left(\frac{3}{2}\right)^x = \left(\frac{2 \times 3}{2}\right)^x = 3^x.$$